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Maximum Principle for Infinite-Horizon Optimal Control Problems under Weak Regularity Assumptions

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Abstract—The paper deals with first order necessary optimality conditions for a class of infinite-horizon optimal control problems that arise in economic applications. Neither convergence of the integral utility functional nor local boundedness of the optimal control is assumed. Using the classical needle variations technique we develop a normal form version of the Pontryagin maximum principle with an explicitly specified adjoint variable under weak regularity assumptions. The result generalizes some previous results in this direction. An illustrative economical example is presented.
Keywords: infinite horizon, Pontryagin maximum principle, transversality conditions, weak regularity assumptions.

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1. INTRODUCTION

Infinite-horizon optimal control problems arise in many fields of economics, in particular, in problems of optimization of economic growth. Typically, the initial state is fixed and the terminal state (at infinity) is free in such problems, while the utility functional to be maximized is given by an improper integral on the time interval $[0, \infty)$.

It is well known that the infinite time horizon may cause the appearance of various "pathological" phenomena in the relations of the corresponding general version of the Pontryagin maximum principle [15]. Although the state at infinity is not constrained, such problems could be abnormal ($\psi^0 = 0$ in this case) and the "standard" transversality conditions at infinity of the form

$$\lim_{t\to\infty} \psi(t) = 0 \tag{1.1}$$

or

$$\lim_{t\to\infty} \langle \psi(t), x_*(t)\rangle = 0 \tag{1.2}$$

may be inconsistent with the core conditions of the maximum principle (the adjoint system and the maximum condition). Here $x_*(\cdot)$ is an optimal trajectory and $(\psi^0, \psi(\cdot))$ is a pair of adjoint variables corresponding to the optimal pair $(x_*(\cdot), u_*(\cdot))$ according to the core conditions of the maximum principle. Examples exhibiting pathologies of these types are given in [4, 10, 15, 18, 21]. These

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examples clearly demonstrate that general complementary conditions on the adjoint variables (if such exists) must differ from (1.1) and (1.2).

A considerable progress in understanding the "right" form of such complementary conditions on the adjoint variables has been made in the last decade. In the case of autonomous problems with discounting it was proved in [3, 4] that if the discount rate is sufficiently large (this situation is referred to in the literature as "the case of dominating discount") then the maximum principle holds in normal form with the "right" adjoint variable $\psi(\cdot)$ specified by an explicit formula similar to the classical Cauchy formula for the solutions of systems of linear differential equations. In some situations this Cauchy type representation of $\psi(\cdot)$ implies transversality conditions at infinity of the form (1.1) or (1.2), and an even stronger exponential pointwise estimate for $\psi(\cdot)$ (see [4–6] for more details). Recently, the main constructions and results in [3, 4] were extended in [2, 9].

The approach used in [2–4, 9] is based on an appropriate *regularization* of the infinite-horizon problem, namely, on its explicit approximation by a family of standard finite-horizon problems. However, there are inherent limitations for the applicability of this approach. In particular, application of the approximation techniques typically needs some uniformity of the convergence of the improper integral utility functional for all admissible controls (see, e.g., condition (A3) in [2]) and boundedness of the optimal one (at least in a local sense). In many cases of interest regularity conditions of this type either fail or cannot be verified a priori. For instance, in problems without discounting and in models of endogenous economic growth (especially with declining discount rates) the corresponding integral utility functionals may diverge to infinity.

An alternative approach to derivation of first order necessary optimality conditions for infinite-horizon optimal problems, which is based on the classical needle variations technique, was recently developed in [6, 7]. The advantage of this approach is that typically it can be realized under less restrictive regularity assumptions than ones akin to the approximations based techniques. In particular, this approach can produce a complete set of necessary optimality conditions even in the case when the optimal objective value is infinite (see [6, 7]). A local modification of the notion of weakly overtaking optimality (see [10]) can be employed in this case. The normal form version of the maximum principle obtained in [6, 7] involves the same explicit single-valued representation for the adjoint variable $\psi(\cdot)$ as in [2–4, 9] but under weaker assumptions on the convergence of the improper integral utility functional. We mention that the same approach proved to be also productive for distributed control systems, as shown in [22] for a class of age-structured optimal control problems.

The main goal of the present paper is to extend the results obtained in [6, 7] to a more general class of infinite-horizon optimal control problems satisfying a weak regularity assumption. It should be emphasized that due to the economic nature of many optimal growth models the standard regularity assumptions that are common in the optimal control theory could be rather burdensome. In many cases of interest the improper integral utility functional could diverge to infinity, and natural admissible controls or the corresponding utility flows could not be a priori bounded (even locally). Notice that the validity of the Pontryagin maximum principle in finite-horizon problems under weak regularity assumptions is well known (see [11, Theorem 5.2.1] and [12, Theorem 22.17]).

Another extension of the results in [6, 7] is that the objective integrand and the right-hand side of the differential equation defining the problem (control system) need not be continuous with respect to the control variable u. The Lebesgue–Borel measurability in (t, u) is required instead, which is useful in several economic models, where the objective integrand and the control system are discontinuous in u (for example, models in which fixed costs are involved).

The proof of the main result — Pontryagin's maximum principle for infinite-horizon problems — adapts the one in [7] with some essential modifications that are needed because the weak regularity assumption in the present paper does not require local boundedness of the admissible (and the optimal) controls, and continuity of the data of the problem with respect to the control is also not required. The analysis is based on a modification of the classical needle variation technique.

The paper is organized as follows. In Section 2 we state the problem, formulate the weak regularity assumption and introduce the notion of optimality used in present paper. Section 3 presents the main result and its proof. In Section 4 we consider an illustrative economic example.

2. STATEMENT OF THE PROBLEM AND PRELIMINARY DISCUSSIONS

Let G be a nonempty open convex subset of $\mathbb{R}^n$ and let

$$f\colon [0,\infty)\times G\times\mathbb{R}^m\to\mathbb{R}^n \quad \text{and} \quad f^0\colon [0,\infty)\times G\times\mathbb{R}^m\to\mathbb{R}^1.$$

Throughout the paper we assume that for almost every $t\in[0,\infty)$ the derivatives $f_x(t,x,u)$ and $f^0_x(t,x,u)$ exist for all $(x,u)\in G\times\mathbb{R}^m$, and the functions $f(\cdot,\cdot,\cdot)$, $f^0(\cdot,\cdot,\cdot)$, $f_x(\cdot,\cdot,\cdot)$, and $f^0_x(\cdot,\cdot,\cdot)$ are Lebesgue–Borel (LB) measurable in (t,u) for every $x\in G$ and continuous in x for almost every $t\in[0,\infty)$ and every $u\in\mathbb{R}^m$.

The LB measurability in (t,u) [12, Definition 6.33] means that the functions (and the sets) to which the property applies are measurable with respect to the σ-algebra generated by the product of the Lebesgue σ-algebra on $[0,\infty)$ and the Borel σ-algebra on $\mathbb{R}^m$.

The LB measurability replaces the usual assumption of Lebesgue measurability in t and continuity in u of the functions involved in the assumption above. An important property is that for any function $g\colon[0,\infty)\times\mathbb{R}^m\to\mathbb{R}^n$ which is LB measurable, the superposition $t\mapsto g(t,u(t))$ with a Lebesgue measurable function $u\colon[0,\infty)\to\mathbb{R}^m$ is Lebesgue measurable [12, Proposition 6.34]. In particular, this implies that the functions $t\mapsto f(t,x(t),u(t))$, $t\mapsto f_x(t,x(t),u(t))$, etc., which appear below with a continuous $x\colon[0,\infty)\to\mathbb{R}^n$ and a Lebesgue measurable function $u\colon[0,\infty)\to\mathbb{R}^m$, are Lebesgue measurable on $[0,\infty)$.

Consider the following optimal control problem (P):

$$J(x(\cdot),u(\cdot))=\int\limits_0^\infty f^0(t,x(t),u(t))\,dt\to\max\,, \tag{2.1}$$

$$\dot{x}(t)=f(t,x(t),u(t)),\quad x(0)=x_0, \tag{2.2}$$

$$u(t)\in U(t), \tag{2.3}$$

where $x_0\in G$ is a given initial state of the system and $U\colon[0,\infty)\mapsto 2^{\mathbb{R}^m}$ is an LB measurable multivalued mapping with nonempty values $U(t)\subset\mathbb{R}^m$, $t\geq 0$. The LB measurability of $U(\cdot)$ means that its graph, i.e., the set $\operatorname{gr} U(\cdot)=\{(t,u)\in[0,\infty)\times\mathbb{R}^m : u\in U(t)\}$, is an LB measurable subset of $[0,\infty)\times\mathbb{R}^m$.

Since the utility functional (2.1) on an infinite horizon admits its values to be infinite, there are several concepts of optimality that can be used in the context of problem (P) (see, for example, [10]). The one that we use in the present paper will be specified at the end of this section. Before that we make some preliminary considerations.

We consider any Lebesgue measurable function $u\colon[0,\infty)\mapsto\mathbb{R}^m$ satisfying condition (2.3) for all $t\geq 0$ as a *control*. If $u(\cdot)$ is a control then the corresponding *trajectory* is a locally absolutely continuous solution $x(\cdot)$ of the initial value problem (2.2), which is defined on some finite or infinite time interval $[0,\tau)$, $\tau > 0$, in G (if such solution exists). The local absolute continuity of $x(\cdot)$ means that $x(\cdot)$ is absolutely continuous on any compact time interval $[0,T]$, $T > 0$, of its domain of definition $[0,\tau)$.

By definition, a pair $(x(\cdot),u(\cdot))$, where $u(\cdot)$ is a control and $x(\cdot)$ is the corresponding trajectory, is an *admissible pair* (in problem (P)), or a *process*, if the trajectory $x(\cdot)$ is defined on the whole infinite time interval $[0,\infty)$ and the function $t \mapsto f^0(t,x(t),u(t))$ is locally integrable on $[0,\infty)$ (i.e., integrable on any finite time interval $[0,T]$, $T > 0$). Thus, for an arbitrary admissible pair $(x(\cdot),u(\cdot))$ and any $T > 0$ the integral

$$J_T(x(\cdot),u(\cdot)) := \int_0^T f^0(t,x(t),u(t))\,dt$$

is finite.

Note that if all functions $f(\cdot,\cdot,\cdot)$, $f^0(\cdot,\cdot,\cdot)$, $f_x(\cdot,\cdot,\cdot)$, and $f^0_x(\cdot,\cdot,\cdot)$ are locally bounded,[3] then for any control $u(\cdot) \in L^\infty_{\mathrm{loc}}[0,\infty)$ the corresponding trajectory $x(\cdot)$ exists (and is unique) in G on some maximal time interval $[0,\tau)$, $\tau > 0$ (see [1, Chs. 2.5.1–2.5.3]), and in the case $\tau = \infty$ the pair $(x(\cdot),u(\cdot))$ is admissible.

Now we recall two basic concepts of optimality used in the literature (see [10]).

In the first one, the integral in (2.1) is understood in improper sense; i.e., for an arbitrary admissible pair $(x(\cdot),u(\cdot))$ by definition

$$J(x(\cdot),u(\cdot)) = \lim_{T\to\infty} \int_0^T f^0(t,x(t),u(t))\,dt,$$

if the limit exists.

Definition 2.1. *An admissible pair $(x_*(\cdot),u_*(\cdot))$ is called strongly optimal in problem (P) if the corresponding integral in (2.1) converges (to a finite number) and for any other admissible pair $(x(\cdot),u(\cdot))$ we have*

$$J(x_*(\cdot),u_*(\cdot)) \ge \limsup_{T\to\infty} \int_0^T f^0(t,x(t),u(t))\,dt.$$

In the second one, the integral in (2.1) is not necessarily finite.

Definition 2.2. *An admissible pair $(x_*(\cdot),u_*(\cdot))$ is called finitely optimal in problem (P) if for any $T > 0$ this pair (restricted to $[0,T]$) is optimal in the following optimal control problem (P_T) with fixed initial and final states*:

$$J_T(x(\cdot),u(\cdot)) = \int_0^T f^0(t,x(t),u(t))\,dt \to \max,$$

$$\dot{x}(t) = f(t,x(t),u(t)), \qquad x(0) = x_0, \quad x(T) = x_*(T), \quad u(t) \in U(t).$$

[3] The local boundedness of these functions of t, x, and u (take $\phi(\cdot,\cdot,\cdot)$ as a representative) means that for every $T > 0$, every compact $D \subset G$, and every bounded set $V \subset \mathbb{R}^m$ there exists M such that $\|\phi(t,x,u)\| \le M$ for almost all $t \in [0,T]$ and all $x \in D$ and $u \in V$.

It is easy to see that the strong optimality implies the finite one.

The following weak regularity assumption plays a key role for the validity of the general version of the Pontryagin maximum principle for a finitely optimal process $(x_*(\cdot), u_*(\cdot))$ in problem (P) (similar assumptions for problems with finite time horizons one can find in [11, Ch. 5; 12, Hypothesis 22.25]). In fact, this assumption will be used later for an arbitrarily fixed admissible pair $(x_*(\cdot), u_*(\cdot))$.

Assumption (A1). *There are a continuous function* $\gamma\colon [0,\infty) \mapsto (0,\infty)$ *and a locally integrable function* $\varphi\colon [0,\infty) \mapsto \mathbb{R}^1$ *such that* $\{x\colon \|x - x_*(t)\| \le \gamma(t)\} \subset G$ *for all* $t \ge 0$ *and for almost all* $t \in [0,\infty)$ *we have*

$$\max{}_{\{x\colon \|x-x_*(t)\|\le\gamma(t)\}}\big\{\|f_x(t,x,u_*(t))\| + \|f^0_x(t,x,u_*(t))\|\big\} \le \varphi(t). \tag{2.4}$$

Notice that due to the continuity of the functions $f_x(\cdot,\cdot,\cdot)$ and $f^0_x(\cdot,\cdot,\cdot)$ with respect to x, the maximum in (2.4) is achieved.

Assumption (A1) is implied by the following condition (see [12, Hypothesis 22.16])): there exist a constant $c \ge 0$, a locally integrable function $d\colon [0,\infty) \mapsto \mathbb{R}^1$, and a continuous function $\gamma\colon [0,\infty) \mapsto (0,\infty)$, $\big\{x\colon \|x - x_*(t)\| \le \gamma(t)\big\} \subset G$, $t \ge 0$, such that for almost every $t \in [0,\infty)$ and all $x\colon \|x - x_*(t)\| \le \gamma(t)$ we have

$$\|f_x(t,x,u_*(t))\| + \|f^0_x(t,x,u_*(t))\| \le c\,\big\{\|f(t,x,u_*(t))\| + |f^0(t,x,u_*(t))|\big\} + d(t).$$

We also mention that if $u_*(\cdot) \in L^\infty_{\mathrm{loc}}[0,\infty)$ and the functions $f_x(\cdot,\cdot,\cdot)$ and $f^0_x(\cdot,\cdot,\cdot)$ are measurable in t, continuous in (x,u), and locally bounded, as in [6,7], then assumption (A1) also holds true.

The following lemma will be used below.

Lemma 2.1. *Let* $(x_*(\cdot), u_*(\cdot))$ *be an admissible pair for which* (A1) *is fulfilled. Then the function* $k\colon [0,\infty) \times \mathbb{R}^m \mapsto \mathbb{R}^1$ *with values given for almost every* $t \ge 0$ *and all* $v \in \mathbb{R}^m$ *by the equality*

$$k(t,v) = \max{}_{\{x\colon \|x-x_*(t)\|\le\gamma(t)\}}\,\big\{\|f_x(t,x,v)\| + \|f^0_x(t,x,v)\|\big\} \tag{2.5}$$

is LB measurable. Moreover, the function $k(\cdot, u_*(\cdot))$ *is locally integrable.*

Proof. Since $\{x\colon \|x - x_*(t)\| \le \gamma(t)\} \subset G$, $t \ge 0$, the set-valued mapping $t \mapsto F(t) = \{x\colon \|x - x_*(t)\| \le \gamma(t)\}$ is compact, convex-valued, and continuous, with $F(t) \subset G$, $t \ge 0$. Hence, there is a countable family $\{\xi_i(\cdot)\}_{i=1}^\infty$ of continuous selectors of $F(\cdot)$ such that the set $\bigcup_{i=1}^\infty \xi_i(t)$ is dense in $F(t)$ for any $t \ge 0$. Hence, for almost every $t \ge 0$ and all $v \in \mathbb{R}^m$ we have

$$k(t,v) = \sup_{i\in\mathbb{N}}\,\big\{\|f_x(t,\xi_i(t),v)\| + \|f^0_x(t,\xi_i(t),v)\|\big\}.$$

Thus, the function $k(\cdot,\cdot)$ is LB measurable as a supremum of a countable family of LB measurable functions.

The latter claim follows directly from assumption (A1), since $k(t,u_*(t)) \le \varphi(t)$. $\square$

Define the Hamilton–Pontryagin function $\mathcal{H}\colon [0,\infty) \times G \times \mathbb{R}^m \times \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^1$ for problem (P) in the usual way:

$$\mathcal{H}(t,x,u,\psi^0,\psi) = \psi^0 f^0(t,x,u) + \langle f(t,x,u), \psi\rangle,$$

$$t \in [0,\infty), \quad x \in G, \quad u \in \mathbb{R}^m, \quad \psi \in \mathbb{R}^n, \quad \psi^0 \in \mathbb{R}^1.$$

In the normal case, where $\psi^0 = 1$, we simply write $\mathcal{H}(t,x,u,\psi)$ instead of $\mathcal{H}(t,x,u,1,\psi)$.

Any finitely optimal process $(x_*(\cdot), u_*(\cdot))$ satisfies the following general version of the maximum principle, which is proved in [15] under the *standard regularity conditions*. Namely, the proof given in [15] is valid if $u_*(\cdot) \in L^\infty_{\mathrm{loc}}[0,\infty)$, $U(t) \equiv U$, $t \geq 0$, and the functions $f(\cdot,\cdot,\cdot)$, $f^0(\cdot,\cdot,\cdot)$, $f_x(\cdot,\cdot,\cdot)$ and $f^0_x(\cdot,\cdot,\cdot)$ are measurable in t, continuous in (x,u), and locally bounded.

Theorem 2.1. *Let $(x_*(\cdot), u_*(\cdot))$ be a finitely optimal admissible pair in problem (P) and let* (A1) *be fulfilled. Then there is a nonvanishing pair of adjoint variables $(\psi^0, \psi(\cdot))$, with $\psi^0 \geq 0$ and a locally absolutely continuous $\psi(\cdot)\colon [0,\infty) \to \mathbb{R}^n$, such that the core conditions of the maximum principle hold*; *i.e.,*

(i) $\psi(\cdot)$ *is a solution to the adjoint system*

$$\dot{\psi}(t) = -\mathcal{H}_x(t, x_*(t), u_*(t), \psi^0, \psi(t)),$$

(ii) *the maximum condition takes place*:

$$\mathcal{H}(t, x_*(t), u_*(t), \psi^0, \psi(t)) \overset{a.e.}{=} \sup_{u \in U(t)} \mathcal{H}(t, x_*(t), u, \psi^0, \psi(t)).$$

The main points of the proof of this theorem coincide with those in the original proof of Halkin's result (see [15, Theorem 4.2]). Therefore we give only a sketch. Similarly to [15], the proof is based on the consideration of the family of auxiliary optimal control problems (P_T) on finite time intervals $[0,T]$, $T > 0$, appearing in Definition 2.2. The only difference is that the original proof of Halkin's theorem is based on the standard regularity assumptions. The finite optimality of the admissible pair $(x_*(\cdot), u_*(\cdot))$ in problem (P) implies that on any finite time interval $[0,T]$, $T > 0$, the core conditions of the Pontryagin maximum principle for the process $(x_*(\cdot), u_*(\cdot))$ hold with a corresponding nonvanishing pair of adjoint variables $\psi^0_T \geq 0$, $\psi_T(\cdot)$. This implies the validity of the core conditions of the infinite-horizon maximum principle after taking a limit in the conditions of the maximum principle for these auxiliary problems (P_T) as $T \to \infty$ (see details in [10,15]).

However, due to [11, Theorem 5.2.1], assumption (A1), together with Lemma 2.1, also implies the core conditions of the maximum principle for the process $(x_*(\cdot), u_*(\cdot))$ which is optimal in all corresponding finite-horizon problems (P_T) on time intervals $[0,T]$, $T > 0$, with fixed endpoints (see Definition 2.2). Thus, the scheme of the proof of Theorem 4.2 in [15] can be reproduced with some minor modifications also for Theorem 2.1.

The next concept of optimality takes an intermediate place between the strong and the finite ones (see [10,15]).

Definition 2.3. *An admissible pair $(x_*(\cdot), u_*(\cdot))$ is called weakly overtaking optimal if for arbitrary $\varepsilon > 0$ and $T > 0$ and any other admissible pair $(x(\cdot), u(\cdot))$ there is a $T' > T$ such that*

$$\int_0^{T'} f^0(t, x_*(t), u_*(t))\, dt \geq \int_0^{T'} f^0(t, x(t), u(t))\, dt - \varepsilon.$$

This concept of optimality appears to be the most useful among the numerous alternative definitions proposed in the context of economics (see [10]); therefore, we adopt it in this paper. Similarly to [6,7], because of the use of needle variations, it turns out that Pontryagin's necessary optimality conditions obtained in the next section are valid even for a local version of the weak overtaking optimality. Namely, it is enough to test the optimal pair $(x_*(\cdot), u_*(\cdot))$ against admissible pairs $(x(\cdot), u(\cdot))$ for which $u(\cdot)$ differs from $u_*(\cdot)$ only on a set of "small" measure.

Definition 2.4. *An admissible pair* $(x_*(\cdot), u_*(\cdot))$ *is called locally weakly overtaking optimal (LWOO) if there exists* $\delta > 0$ *such that for any other admissible pair* $(x(\cdot), u(\cdot))$ *satisfying*

$$\operatorname{meas}\{t \geq 0\colon u(t) \neq u_*(t)\} \leq \delta$$

and for arbitrary $\varepsilon > 0$, $T > 0$ *there is a* $T' > T$ *such that*

$$\int_0^{T'} f^0(t, x_*(t), u_*(t))\, dt \geq \int_0^{T'} f^0(t, x(t), u(t))\, dt - \varepsilon.$$

Obviously, the property of local weak overtaking optimality is weaker than the property of weak overtaking optimality but it does not imply the finite optimality, in general. The property of local weak overtaking optimality should be compared with the corresponding "local" version of the property of finite optimality. However, as it can be shown, any LWOO admissible pair $(x_*(\cdot), u_*(\cdot))$ is locally finite optimal and Theorem 2.1 holds true since it is valid also in the case of locally finite optimal admissible pair $(x_*(\cdot), u_*(\cdot))$.

Notice also that the concept of finite optimality is very weak. It can happen that even in simple situations this concept does not recognize strongly optimal pairs (which exist) in the set of all admissible ones (see discussion of Halkin's example in [6].) In the next section we show that the concept of weak overtaking optimality (see Definition 2.3 and its local modification given in Definition 2.4) provides a reasonable compromise between the concepts of strong optimality (Definition 2.1) and finite optimality (Definition 2.2). On the one hand, this concept of optimality is general enough and applicable even in the situation of infinite optimal utility value; on the other hand, this concept of optimality still admits the development of complete versions of the maximum principle.

3. MAXIMUM PRINCIPLE

This section presents the main result in the paper — a normal form version of the Pontryagin maximum principle with explicitly specified adjoint variable for the infinite-horizon problem (P). The analysis is based on the notion of simple needle variation (see, for example, [1, Ch. 1.5.4]), using the weak regularity assumption (A1) and the following growth assumption for a given process $(x_*(\cdot), u_*(\cdot))$:

Assumption (A2). *There exist a number* $\beta > 0$ *and a nonnegative integrable function* $\lambda : [0, \infty) \mapsto \mathbb{R}^1$ *such that for every* $\zeta \in G$ *with* $\|\zeta - x_0\| < \beta$ *equation* (2.2) *with* $u(\cdot) = u_*(\cdot)$ *and initial condition* $x(0) = \zeta$ (*instead of* $x(0) = x_0$) *has a solution* $x(\zeta; \cdot)$ *on* $[0, \infty)$ *in* G, *and*

$$\max\nolimits_{x \in [x(\zeta;t), x_*(t)]} \left| \langle f_x^0(t, x, u_*(t)), x(\zeta; t) - x_*(t) \rangle \right| \overset{\text{a.e.}}{\leq} \|\zeta - x_0\| \lambda(t).$$

Here $[x(\zeta; t), x_*(t)] = \operatorname{co}\{x(\zeta; t), x_*(t)\}$ *denotes the line segment with vertices* $x(\zeta; t)$ *and* $x_*(t)$.

This assumption was introduced in [7] as an invariant counterpart of the dominating discount condition in [3, 4, 6]. Notice that a locally bounded function $\lambda(\cdot)$ satisfying the inequality in (A2) always exists. The essence of (A2) is that such an *integrable* $\lambda(\cdot)$ does exist. Thus (A2) is an asymptotic assumption which complements in the infinite horizon case the local assumption (A1). It should be noted also that due to (A2) for any initial state $\zeta \in G$, $\|\zeta - x_0\| < \beta$, the function $t \mapsto f^0(t, x(\zeta; t), u_*(t))$ is locally integrable on $[0, \infty)$.

The following auxiliary statement is needed in order to apply theorems on existence, continuous dependence, and differentiability with respect to initial data of the solution of a differential equation under assumption (A1) (see [1, Chs. 2.5.1–2.5.6]).

Lemma 3.1. *If* $(x_*(\cdot), u_*(\cdot))$ *is a process and* (A1) *holds, then for every* $x \in G$ *for which the set* $G_x := \{t \geq 0\colon \|x - x_*(t)\| \leq \gamma(t)\}$ *is nonempty, the function* $t \mapsto f(t, x, u_*(t))$ *is locally integrable on* G_x.

Proof. Since $(x_*(\cdot), u_*(\cdot))$ is a process, $t \mapsto f(t, x_*(t), u_*(t))$ is locally integrable. For $t \in G_x$ define $\xi(t) = x - x_*(t)$. Then the function $\xi\colon G_x \to \mathbb{R}^n$ is continuous, $\|\xi(t)\| \leq \gamma(t)$, $t \in G_x$, and

$$f(t, x, u_*(t)) = f(t, x_*(t) + \xi(t), u_*(t)) = f(t, x_*(t), u_*(t)) + \langle f_x(t, x_*(t) + \tilde{\xi}(t), u_*(t)), \xi(t)\rangle,$$

where $\tilde{\xi}\colon G_x \to \mathbb{R}^n$ is measurable and $\|\tilde{\xi}(t)\| \leq \|\xi(t)\| \leq \gamma(t)$, $t \in G_x$. Now the statement follows from the local integrability of $t \mapsto f(t, x_*(t), u_*(t))$ on G_x and (A1). $\square$

Consider the following linear differential equation (the linearization of (2.2) along $(x_*(\cdot), u_*(\cdot))$):

$$\dot{y}(t) = f_x(t, x_*(t), u_*(t))\, y(t), \quad t \geq 0. \tag{3.1}$$

Due to (A1), the partial derivative $f_x(\cdot, x_*(\cdot), u_*(\cdot))$ is measurable and locally integrable. Hence, for any given time $\tau \geq 0$ and vector $y_*(\tau) \in \mathbb{R}^n$ there is a unique (Carathéodory) solution $y(\cdot)$ of equation (3.1) with $y(\tau) = y_*(\tau)$, which is defined on the whole time interval $[0, \infty)$. Moreover,

$$y(t) = Y_*(t)\, Y_*^{-1}(\tau)\, y_*(\tau), \quad t \geq 0, \tag{3.2}$$

where $Y_*(\cdot)$ is the fundamental matrix solution of (3.1) normalized at $t = 0$. This means (see, for example, [16, Ch. 4]) that the columns $\xi_i(\cdot)$, $i = 1, \dots, n$, of the $(n \times n)$-matrix function $Y_*(\cdot)$ are (linearly independent) solutions of (3.1) on $[0, \infty)$ that satisfy the initial conditions

$$\xi_i^j(0) = \delta_{i,j}, \quad i, j = 1, \dots, n,$$

where

$$\delta_{i,i} = 1, \quad i = 1, \dots, n, \quad \text{and} \quad \delta_{i,j} = 0, \quad i \neq j, \quad i, j = 1, \dots, n.$$

Analogously, consider the fundamental matrix solution $Z_*(\cdot)$ (normalized at $t = 0$) of the linear adjoint equation

$$\dot{z}(t) = -\left[f_x(t, x_*(t), u_*(t))\right]^* z(t).$$

Then

$$Z_*^{-1}(t) = \left[Y_*(t)\right]^*, \quad t \geq 0. \tag{3.3}$$

Lemma 3.2. *Let* (A1) *and* (A2) *be satisfied. Then the following estimate holds:*

$$\left\| \left[Y_*(t)\right]^* f_x^0(t, x_*(t), u_*(t)) \right\| \leq \sqrt{n}\lambda(t) \quad \textit{for a.e.} \quad t \geq 0. \tag{3.4}$$

Proof. Define $\zeta_i \in \mathbb{R}^n$ as the vector with components $\zeta_i^j = \delta_{i,j}$, $i, j = 1, \dots n$. Due to (A2) for every $\alpha \in (0, \beta)$ the solution $x(x_0 + \alpha\zeta_i; \cdot)$ of equation (2.2) with $u(\cdot) = u_*(\cdot)$ and initial condition $x(0) = x_0 + \alpha\zeta_i$ (instead of x_0) exists on $[0, \infty)$ and

$$\left|\langle f_x^0(t, x_*(t), u_*(t)),\, x(x_0 + \alpha\zeta_i; t) - x_*(t)\rangle\right| \overset{\text{a.e.}}{\leq} \alpha\lambda(t). \tag{3.5}$$

Due to Lemma 3.1 and the theorem on differentiation of the solution of a differential equation with respect to the initial conditions (see, e.g., Ch. 2.5.6 in [1]), we get the equality

$$x(x_0 + \alpha\zeta_i; t) = x_*(t) + \alpha\xi_i(t) + \mathrm{o}_i(\alpha, t), \quad i = 1, \dots, n, \quad t \geq 0.$$

Here the vector functions $\xi_i(\cdot)$, $i = 1, \dots, n$ are columns of $Y_*(\cdot)$, and for any $i = 1, \dots, n$ we have $\|\mathrm{o}_i(\alpha, t)\|/\alpha \to 0$ as $\alpha \to 0$ uniformly with respect to t on any finite time interval $[0, T]$, $T > 0$. Then in view of (3.5) we get

$$\left|\left\langle f_x^0(t, x_*(t), u_*(t)), \xi_i(t) + \frac{\mathrm{o}_i(\alpha, t)}{\alpha}\right\rangle\right| \overset{\text{a.e.}}{\leq} \lambda(t), \quad i = 1, \dots, n, \quad t \geq 0.$$

Passing to the limit as $\alpha \to 0$ in the last inequality for a.e. $t \geq 0$ and $i = 1, \dots, n$ we get

$$\left|\langle f_x^0(t, x_*(t), u_*(t)), \xi_i(t)\rangle\right| \overset{\text{a.e.}}{\leq} \lambda(t), \quad i = 1, \dots, n, \quad t \geq 0.$$

This implies (3.4). □

Due to (3.3) and Lemma 3.2, assumption (A2) implies that the function $\psi\colon [0, \infty) \to \mathbb{R}^n$ defined as

$$\psi(t) = Z_*(t) \int_t^\infty [Z_*(s)]^{-1} f_x^0(s, x_*(s), u_*(s))\, ds, \quad t \geq 0, \tag{3.6}$$

is locally absolutely continuous. Indeed, the integral in (3.6) converges absolutely for any $t \geq 0$ due to the integrability of $\lambda(\cdot)$.

By a direct differentiation we verify that the so defined function $\psi(\cdot)$ satisfies on $[0, \infty)$ the adjoint system

$$\dot\psi(t) = -\mathcal{H}_x(t, x_*(t), u_*(t), \psi(t)).$$

(Recall that in the case $\psi^0 = 1$ we omit this variable in the Hamilton–Pontryagin function.)

The following result is a version of the Pontryagin maximum principle for infinite-horizon problem (P) under regularity assumption (A1) and growth assumption (A2).

Theorem 3.1. *Let $(x_*(\cdot), u_*(\cdot))$ be a* LWOO *admissible pair in problem (P). Assume that* (A1) *and* (A2) *are satisfied. Then the vector function $\psi\colon [0, \infty) \mapsto \mathbb{R}^n$ defined by* (3.6) *is* (*locally*) *absolutely continuous and satisfies the core conditions of the normal form maximum principle*; *i.e.,*

(i) $\psi(\cdot)$ *is a solution to the adjoint system*

$$\dot\psi(t) = -\mathcal{H}_x(t, x_*(t), u_*(t), \psi(t)), \tag{3.7}$$

(ii) *the maximum condition takes place*:

$$\mathcal{H}(t, x_*(t), u_*(t), \psi(t)) \overset{\text{a.e.}}{=} \sup_{u \in U(t)} \mathcal{H}(t, x_*(t), u, \psi(t)).$$

Proof. It has been prove already that the vector function $\psi\colon [0, \infty) \to \mathbb{R}^n$ defined by (3.6) is locally absolutely continuous and satisfies (3.7). We shall prove condition (ii) ad absurdum by using a modified form of the simple needle variations of the control $u_*(\cdot)$.

Assume that condition (ii) fails. Then there are a set $\Omega \subset [0, \infty)$ of positive measure and an $\varepsilon > 0$ such that the multivalued mapping $\Gamma\colon \Omega \to 2^{\mathbb{R}^m}$ defined by the equality

$$\Gamma(t) = \{u \in U(t)\colon \mathcal{H}(t, x_*(t), u_*(t), \psi(t)) \leq \mathcal{H}(t, x_*(t), u, \psi(t)) - \varepsilon\}, \quad t \in \Omega,$$

has nonempty values and its graph, i.e., the set

$$\operatorname{gr}\Gamma(\cdot) = \{(t,u)\colon t \in \Omega,\ u \in \Gamma(t)\}$$

is an LB measurable subset of $[0,\infty)\times\mathbb{R}^m$ since the function $t \mapsto \mathcal{H}(t, x_*(t), u_*(t), \psi(t))$ is Lebesgue measurable and the function $(t,u) \mapsto \mathcal{H}(t, x_*(t), u, \psi(t))$ is LB measurable. Due to the Yankov–von Neumann–Aumann selection theorem (see, for example, [11, Theorem 4.1.1.; 17, Theorem 2.14]) there is a Lebesgue measurable selection of $\Gamma(\cdot)$, i.e., a Lebesgue measurable function $v\colon \Omega \to \mathbb{R}^m$, such that $v(t) \in \Gamma(t)$ for all $t \in \Omega$.

Due to [19, Ch. 9, Theorem 2] there is a $\tau \in \Omega$ which is a point of approximate continuity of the Lebesgue measurable functions $f(\cdot, x_*(\cdot),\ u_*(\cdot))$, $f^0(\cdot, x_*(\cdot), u_*(\cdot))$, $f(\cdot, x_*(\cdot), v(\cdot))$, $f^0(\cdot, x_*(\cdot), v(\cdot))$, $k(\cdot, v(\cdot))$, and $v(\cdot)$. Here $k(\cdot,\cdot)$ is the function defined in Lemma 2.1, and according to this lemma $k(\cdot, v(\cdot))$ is a measurable function.

This means (see [19, Ch. 9, Sect. 5]) that for each of these functions of t (take $\phi(\cdot)$ as a representative) there is a measurable set $\mathfrak{M} \subset [0,\tau]$ such that $\tau \in \mathfrak{M}$, $\phi(\cdot)$ is continuous at τ along $\mathfrak{M}$, and

$$\lim_{\alpha\to 0+} \frac{\operatorname{meas}\{\mathfrak{M} \cap (\tau-\alpha, \tau]\}}{\alpha} = 1.$$

Clearly, we may assume that $\mathfrak{M}$ is the same for all the above functions.

Further in the proof, we denote $v = v(\tau)$. Then

$$\mathcal{H}(\tau, x_*(\tau), u_*(\tau), \psi(\tau)) \le \mathcal{H}(\tau, x_*(\tau), v, \psi(\tau)) - \varepsilon. \tag{3.8}$$

For any $0 < \alpha \le \tau$ define

$$u_\alpha(t) := \begin{cases} u_*(t), & t \notin (\tau-\alpha,\tau] \cap \mathfrak{M}, \\ v(t), & t \in (\tau-\alpha,\tau] \cap \mathfrak{M}. \end{cases} \tag{3.9}$$

The above defined control $u_\alpha(\cdot)$ is a modified version of the standard *simple variation* of $u_*(\cdot)$ commonly used in optimal control (see, for example, [1, Ch. 1.5.4]). Denote by $x_\alpha(\cdot)$ the trajectory that corresponds to $u_\alpha(\cdot)$ (notice that $x_\alpha(\cdot)$ coincides with $x_*(\cdot)$ on $[0, \tau-\alpha]$).

According to Lemma 2.1, the function $t \mapsto k(t, u_*(t))$ is locally integrable. Moreover, the function $t \mapsto k(t, v(t))$ is continuous at τ along $\mathfrak{M}$. Hence, by construction (see (3.9)) for all sufficiently small $\alpha > 0$ the function $t \mapsto k(t, u_\alpha(t))$ is locally integrable on $[0,\infty)$, since $k(\cdot, u_\alpha(\cdot))$ coincides with $k(\cdot, v(\cdot))$ on the set $(\tau-\alpha,\tau] \cap \mathfrak{M}$ and coincides with $k(\cdot, u_*(\cdot))$ on its complement in $[0,\infty)$. Moreover, due to the continuity of $k(\cdot, v(\cdot))$ at τ along $\mathfrak{M}$ and $v(\tau) = v$, for all sufficiently small $\alpha > 0$ we have

$$k(t, v(t)) \le k(\tau, v) + 1, \quad t \in (\tau-\alpha,\tau] \cap \mathfrak{M}.$$

Hence, for all sufficiently small $\alpha > 0$ we can estimate $k(\cdot, u_\alpha(\cdot))$ in the following way:

$$k(t, u_\alpha(t)) \le k(t) := \max\{k(t, u_*(t)), k(\tau, v) + 1\}, \quad t \ge 0, \tag{3.10}$$

where $k(\cdot)$ is locally integrable on $[0,\infty)$.

The latter estimate implies that for all sufficiently small $\alpha > 0$ the function $x_\alpha(\cdot)$ is defined at least on the time interval $[0,\tau]$. This fact follows from Lemma 3.1, the local existence theorem [1, Theorem 2.5.2.], and estimate (3.10). Due to this circumstance; estimate (3.10); the property that τ

is a point of approximate continuity of the functions $f(\cdot, x_*(\cdot), u_*(\cdot))$, $f(\cdot, x_*(\cdot), v(\cdot))$, $k(\cdot, v(\cdot))$, and $v(\cdot)$; and the equality $v(\tau) = v$, we obviously have

$$x_\alpha(\tau) - x_*(\tau) = \alpha\,[f(\tau, x_*(\tau), v) - f(\tau, x_*(\tau), u_*(\tau))] + o(\alpha), \tag{3.11}$$

where here and further $\mathrm{o}(\alpha)$ denotes a function of $\alpha > 0$ that satisfies $\|\mathrm{o}(\alpha)\|/\alpha \to 0$ as $\alpha \to 0$. Note that $\mathrm{o}(\alpha)$ may depend on v and τ (which are fixed in the present consideration).

Denote by $y_*(\cdot)$ the solution of the linear equation (3.1) on $[0, \infty)$ with the condition

$$y(\tau) = f(\tau, x_*(\tau), v) - f(\tau, x_*(\tau), u_*(\tau)). \tag{3.12}$$

As argued above, for all sufficiently small $\alpha > 0$ the trajectory $x_\alpha(\cdot)$ corresponding to $u_\alpha(\cdot)$ exists at least on $[0, \tau]$ (and equals $x_*(t)$ for $t \in [0, \tau - \alpha]$) and from (3.11) we conclude that $\|x_*(\tau) - x_\alpha(\tau)\| \le c'\,\alpha$ with some constant c'.

The following lemma provides the key tool for proving the maximum principle.

Lemma 3.3. *There is a number $\alpha_0 > 0$ such that for every $\alpha \in (0, \alpha_0]$ the following two properties hold*:

(i) *for the control function $u_\alpha(\cdot)$ defined in* (3.9) *the corresponding trajectory $x_\alpha(\cdot)$ exists on $[0, \infty)$ and the pair $(x_\alpha(\cdot), u_\alpha(\cdot))$ is admissible*;

(ii) *there is a constant $c \ge 0$ and a function $\sigma\colon (0, \alpha_0] \times [\tau, \infty) \to [0, \infty)$ with $\lim_{\alpha\to 0} \sigma(\alpha, t) \to 0$ for any fixed $t \ge \tau$, such that for every $\alpha \in (0, \alpha_0]$ and $T > \tau$*

$$\begin{aligned} &\frac{J_T(x_\alpha(\cdot), u_\alpha(\cdot)) - J_T(x_*(\cdot), u_*(\cdot))}{\alpha} \\ &= \mathcal{H}(\tau, x_*(\tau), v, \psi(\tau)) - \mathcal{H}(\tau, x_*(\tau), u_*(\tau), \psi(\tau)) \;+\; \eta(\alpha, T), \end{aligned} \tag{3.13}$$

where the function $\eta(\alpha, T)$ satisfies the following inequality for every $\tilde{T} \in [\tau, T]$:

$$|\eta(\alpha, T)| \le \sigma(\alpha, \tilde{T}) + c \int_{\tilde{T}}^{\infty} \lambda(t)\, dt. \tag{3.14}$$

Proof of Lemma 3.3. Consider the Cauchy problem

$$\dot{x}(t) = f(t, x(t), u_*(t)), \quad x(\tau) = x_\alpha(\tau). \tag{3.15}$$

Due to Lemma 3.1 and the continuous dependence of the solution of a differential equation on the initial condition (see, e.g., Chapter 2.5.5 in [1]), there is a sufficiently small $\alpha_0 > 0$ such that for all $\alpha \in (0, \alpha_0]$ the solution $\tilde{x}_\alpha(\cdot)$ of (3.15) exists on $[0, \tau]$ and $\|\tilde{x}_\alpha(0) - x_*(0)\| < \beta$. Then the first part of (A2) implies that the solution $\tilde{x}_\alpha(\cdot)$ exists in G on $[0, \infty)$. Thus for all $\alpha \in (0, \alpha_0]$ the solution $x_\alpha(\cdot)$ also exists on $[0, \infty)$, since $x_\alpha(t) = \tilde{x}_\alpha(t)$ for $t \ge \tau$. Due to (A2) the function $t \mapsto f^0(t, x_\alpha(t), u_\alpha(t))$ is locally integrable. Hence, $(x_\alpha(\cdot), u_\alpha(\cdot))$ is an admissible pair.

Due to Lemma 3.1 and the theorem on differentiability of the solution of a differential equation with respect to the initial conditions (see [1, Ch. 2.5.6]) the following representation holds:

$$\tilde{x}_\alpha(t) = x_*(t) + \alpha y_*(t) + \mathrm{o}(\alpha, t), \quad t \ge 0, \tag{3.16}$$

where $y_*(\cdot)$ is the solution of the Cauchy problem (3.1), (3.12). Here $\|\mathrm{o}(\alpha, t)\|/\alpha \to 0$ as $\alpha \to 0$ and the convergence is uniform in t on every finite interval $[\tau, T]$, $T > \tau$.

Let us prove that for any sufficiently small $\alpha > 0$ the following estimate holds:

$$\max_{x\in[x_\alpha(t),x_*(t)]}\left|\left\langle f^0_x(t,x,u_*(t)),y_*(t)+\frac{\mathrm{o}(\alpha,t)}{\alpha}\right\rangle\right| \overset{\text{a.e.}}{\le} c_1\,\lambda(t),\quad t\ge\tau, \tag{3.17}$$

where $c_1 \ge 0$ is independent of α and t.

Due to (A2)

$$\max_{x\in[\tilde{x}_\alpha(t),x_*(t)]}\left|\langle f^0_x(t,x,u_*(t)),\tilde{x}_\alpha(t)-x_*(t)\rangle\right| \overset{\text{a.e.}}{\le} \|\tilde{x}_\alpha(0)-x_*(0)\|\lambda(t),\quad t\ge 0.$$

Then using (3.16) we obtain

$$\max_{x\in[\tilde{x}_\alpha(t),x_*(t)]}\left|\langle f^0_x(t,x,u_*(t)),\alpha y_*(t)+o(\alpha,t)\rangle\right| \overset{\text{a.e.}}{\le} \|\alpha y_*(0)+o(\alpha,0)\|\lambda(t).$$

Choosing $c_1 \ge \|y_*(0)\|+1$, dividing by α, and taking into account that $\tilde{x}_\alpha(t)=x_\alpha(t)$ for $t\ge\tau$ we obtain (3.17).

It is clear that τ is a point of approximate continuity also for the function $t\mapsto f(t,x_*(t),u_\alpha(t))$ (with the same set $\mathfrak{M}$ and with $u_\alpha(\tau)=v(\tau)=v$). Therefore, we can represent

$$\int\limits_{\tau-\alpha}^{\tau} f^0(t,x_\alpha(t),u_\alpha(t))\,dt=\alpha f^0(\tau,x_\alpha(\tau),v)+\int\limits_{\tau-\alpha}^{\tau}\left[f^0(t,x_\alpha(t),u_\alpha(t))-f^0(t,x_*(t),u_\alpha(t))\right]\,dt+o(\alpha).$$

The integral on the right-hand side can be estimated in absolute value by

$$\int\limits_{\tau-\alpha}^{\tau} k(t,u_\alpha(t))\,\|x_\alpha(t)-x_*(t)\|\,dt\le\int\limits_{\tau-\alpha}^{\tau} k(t)\,c'\alpha\,dt\le o(\alpha),$$

where we use the fact that $\|x_\alpha(t)-x_*(t)\|\le c'\alpha$ for $t\in[\tau-\alpha,\tau]$ with an appropriate constant c'.

Using this and (3.16) (where $\tilde{x}_\alpha(t)=x_\alpha(t)$ for $t\ge\tau$) for all $\alpha\in(0,\alpha_0]$ we get

$$\frac{1}{\alpha}\big[J_T(x_\alpha(\cdot),u_\alpha(\cdot))-J_T(x_*(\cdot),u_*(\cdot))\big]=\frac{1}{\alpha}\int\limits_{\tau-\alpha}^{\tau} f^0(t,x_\alpha(t),u_\alpha(t))\,dt-\frac{1}{\alpha}\int\limits_{\tau-\alpha}^{\tau} f^0(t,x_*(t),u_*(t))\,dt$$

$$+\frac{1}{\alpha}\int\limits_{\tau}^{T}\left[f^0(t,x_\alpha(t),u_*(t))-f^0(t,x_*(t),u_*(t))\right]\,dt=f^0(\tau,x_*(\tau),v)-f^0(\tau,x_*(\tau),u_*(\tau))+\frac{\mathrm{o}(\alpha)}{\alpha}$$

$$+\int\limits_{\tau}^{T}\left\langle\int\limits_{0}^{1} f^0_x(t,x_*(t)+s(x_\alpha(t)-x_*(t)),u_*(t))\,ds,\;y_*(t)+\frac{\mathrm{o}(\alpha,t)}{\alpha}\right\rangle dt. \tag{3.18}$$

On the other hand, according to (3.2), (3.3), (3.12), and (3.6)

$$\int\limits_{\tau}^{\infty}\langle f^0_x(t,x_*(t),u_*(t)),\;y_*(t)\rangle\,dt$$

$$=\left\langle Z_*(\tau)\int\limits_{\tau}^{\infty}\big[Z_*(t)\big]^{-1}f^0_x(t,x_*(t),u_*(t))\,dt,\;f(\tau,x_*(\tau),v)-f(\tau,x_*(\tau),u_*(\tau))\right\rangle$$

$$= \big\langle \psi(\tau), f(\tau, x_*(\tau), v) - f(\tau, x_*(\tau), u_*(\tau)) \big\rangle.$$

Using this equality in (3.18) we obtain (3.13) with

$$\eta(\alpha, T) := \int\limits_{\tau}^{T} \Big\langle \int\limits_{0}^{1} f_x^0(t, x_*(t) + s(x_\alpha(t) - x_*(t)), u_*(t))\, ds,\ y_*(t) + \frac{\mathrm{o}(\alpha, t)}{\alpha} \Big\rangle\, dt$$

$$- \int\limits_{\tau}^{\infty} \big\langle f_x^0(t, x_*(t), u_*(t)),\ y_*(t) \big\rangle\, dt + \frac{\mathrm{o}(\alpha)}{\alpha}.$$

Let $\tilde{T}$ be any number between τ and T. Define

$$\sigma(\alpha, \tilde{T}) := \Bigg| \int\limits_{\tau}^{\tilde{T}} \Big\langle \int\limits_{0}^{1} f_x^0(t, x_*(t) + s(x_\alpha(t) - x_*(t)), u_*(t))\, ds,\ y_*(t) + \frac{\mathrm{o}(\alpha, t)}{\alpha} \Big\rangle\, dt$$

$$- \int\limits_{\tau}^{\tilde{T}} \langle f_x^0(t, x_*(t), u_*(t)),\ y_*(t) \rangle\, dt + \frac{\mathrm{o}(\alpha)}{\alpha} \Bigg|.$$

Due to (A1), we apparently have for a fixed $\tilde{T}$ that $\sigma(\alpha, \tilde{T}) \to 0$ as $\alpha \to 0$. Moreover, due to (3.17)

$$\Bigg| \int\limits_{\tilde{T}}^{T} \Big\langle \int\limits_{0}^{1} f_x^0(t, x_*(t) + s(x_\alpha(t) - x_*(t)), u_*(t))\, ds,\ y_*(t) + \frac{\mathrm{o}(\alpha, t)}{\alpha} \Big\rangle\, dt \Bigg| \le c_1 \int\limits_{\tilde{T}}^{\infty} \lambda(t)\, dt.$$

In addition,

$$\Bigg| \int\limits_{\tilde{T}}^{\infty} \langle f_x^0(t, x_*(t), u_*(t)),\ y_*(t) \rangle\, dt \Bigg|$$

$$= \Bigg| \Big\langle Z_*(\tau) \int\limits_{\tilde{T}}^{\infty} \big[Z_*(t)\big]^{-1} f_x^0(t, x_*(t), u_*(t))\, dt,\ f(\tau, x_*(\tau), v) - f(\tau, x_*(\tau), u_*(\tau)) \Big\rangle \Bigg|$$

$$\le \|Z_*(\tau)\| \, \Bigg\| \int\limits_{\tilde{T}}^{\infty} [Y_*(t)]^* f_x^0(t, x_*(t), u_*(t))\, dt \Bigg\| \, \|f(\tau, x_*(\tau), v) - f(\tau, x_*(\tau), u_*(\tau))\| \le c_2 \int\limits_{\tilde{T}}^{\infty} \lambda(t)\, dt,$$

where in the last inequality we use Lemma 3.2.

Combining the above two inequalities and the definition of $\sigma(\alpha, \tilde{T})$ we obtain (3.14) with $c := c_1 + c_2$. □

Now we continue with the proof of the theorem. Let us fix an arbitrary number $\varepsilon_0 > 0$, and let us choose the number $\tilde{T} > \tau$ in such a way that $\int_{\tilde{T}}^{\infty} \lambda(t)\, dt \le \varepsilon_0$. According to Definition 2.4 for every $\alpha \in (0, \alpha_0] \cap (0, \delta]$, for $\varepsilon := \alpha^2$, and for the number $T = \tilde{T}$ there exists $T_\alpha \ge \tilde{T}$ such that

$$J_{T_\alpha}(x_\alpha(\cdot), u_\alpha(\cdot)) - J_{T_\alpha}(x_*(\cdot), u_*(\cdot)) \le \alpha^2.$$

Then from (3.13) we obtain

$$\mathcal{H}(\tau, x_*(\tau), v, \psi(\tau)) - \mathcal{H}(\tau, x_*(\tau), u_*(\tau), \psi(\tau)) \le \alpha - \eta(\alpha, T_\alpha).$$

Since $\tilde{T} \in [\tau, T_\alpha]$, we obtain from (3.14)

$$\mathcal{H}(\tau, x_*(\tau), v, \psi(\tau)) - \mathcal{H}(\tau, x_*(\tau), u_*(\tau), \psi(\tau)) \le \alpha + \sigma(\alpha, \tilde{T}) + c\int_{\tilde{T}}^{\infty} \lambda(t)\,dt \le \alpha + \sigma(\alpha, \tilde{T}) + \varepsilon_0.$$

Passing to the limit as $\alpha \to 0$ and then taking into account that ε_0 was chosen arbitrarily, we obtain

$$\mathcal{H}(\tau, x_*(\tau), u_*(\tau), \psi(\tau)) \ge \mathcal{H}(\tau, x_*(\tau), v, \psi(\tau)).$$

This inequality contradicts (3.8), which completes the proof of the theorem. □

4. EXAMPLE

Here we apply Theorem 3.1 to a stylized (micro-level) economic model studied earlier in [6] by means of another version of the maximum principle.

Consider the following problem $(P1)$:

$$J(K(\cdot), I(\cdot)) = \int_0^\infty e^{-\theta t}\Big[e^{pt}(K(t))^\sigma - \frac{b}{2}(I(t))^2\Big]\,dt \to \max,$$

$$\dot{K}(t) = -\nu K(t) + I(t), \qquad K(0) = K_0, \quad I(t) \ge 0.$$

Here $K(t)$ is the capital stock at time t, $I(t)$ is the investment, $\nu > 0$ is the depreciation rate, $K_0 > 0$ is a given initial state, $\theta \ge 0$ is the discount rate, $p \ge 0$ is the (exogenous) exponential rate of technological advancement, $bI^2(t)$ $(b > 0)$ is the cost of investment $I(t)$, and $\sigma \in (0, 1]$ defines the "production function." We set $G = (0, \infty)$ and $U(t) \equiv [0, \infty)$, $t \ge 0$. As far as the utility functional admits its values to be infinite, the optimality of an admissible pair $(K_*(\cdot), I_*(\cdot))$ in problem $(P1)$ is understood in the sense of Definition 2.4.

Following [6], we transform the problem $(P1)$ to an equivalent one by introducing the variables

$$x(t) = e^{-\alpha t}K(t), \quad u(t) = e^{-\alpha t}I(t), \quad t \ge 0, \quad \text{with} \quad \alpha = \frac{p}{2-\sigma}.$$

In terms of the new variables $x(\cdot)$ and $u(\cdot)$ the model takes the form of the following problem $(\tilde{P1})$:

$$J(x(\cdot), u(\cdot)) = \int_0^\infty e^{-\rho t}\Big[(x(t))^\sigma - \frac{b}{2}(u(t))^2\Big]\,dt \to \max,$$

$$\dot{x}(t) = -(\nu + \alpha)x(t) + u(t), \quad x(0) = K_0, \tag{4.1}$$

$$u(t) \ge 0.$$

Here $\rho = \theta - 2\alpha$ is a new "discount rate," which can be even nonpositive. As above we set $G = (0, \infty)$ and $U(t) \equiv [0, \infty)$, $t \ge 0$, and we are looking for an admissible LWOO control $u_*(\cdot)$ in problem $(\tilde{P1})$. Obviously, $(\tilde{P1})$ is a particular case of problem (P).

Let $(x_*(\cdot), u_*(\cdot))$ be an LWOO admissible pair (if it exists) in $(\tilde{P}1)$.

It can be directly shown that the optimal trajectory $x_*(\cdot)$ is uniformly positive, so there is a sufficiently small number $\eta > 0$ such that $x_*(t) \geq \eta$, $t \geq 0$.

In problem $(\tilde{P}1)$ the functions $f(\cdot,\cdot,\cdot)$ and $f^0(\cdot,\cdot,\cdot)$ are the following:

$$f(t,x,u) = -(\nu+\alpha)x + u, \quad f^0(t,x,u) = e^{-\rho t}\Big[x^\sigma - \frac{b}{2}u^2\Big], \quad t \geq 0, \quad x \in G, \quad u \in \mathbb{R}^1.$$

Hence,

$$f_x(t,x,u) = -(\nu+\alpha), \quad f_x^0(t,x,u) = \frac{\sigma e^{-\rho t}}{x^{1-\sigma}}, \quad t \geq 0, \quad x \in G, \quad u \in \mathbb{R}^1, \tag{4.2}$$

and assumption (A1) is satisfied with continuous function $\gamma(\cdot)$ and locally integrable function $\varphi(\cdot)$ defined as follows:

$$\gamma(t) \equiv \frac{\eta}{2}, \quad \varphi(t) = \nu + \alpha + \frac{2\sigma e^{-\rho t}}{\eta^{1-\sigma}}, \quad t \geq 0.$$

Consider condition (A2). For arbitrary $\zeta > 0$ the solution $x(\zeta;\cdot)$ of equation (4.1) with $u(\cdot) = u_*(\cdot)$ and initial condition $x(0) = \zeta$ (instead of $x(0) = x_0$) is defined on $(0,\infty)$ by

$$x(\zeta;t) = e^{-(\nu+\alpha)t}\zeta + e^{-(\nu+\alpha)t}\int_0^t e^{(\nu+\alpha)s}u_*(s)\,ds, \quad t \geq 0. \tag{4.3}$$

Here the integral in (4.3) is finite since $(x_*(\cdot), u_*(\cdot))$ is a process and the integral appears also in the similar formula for $x_*(\cdot)$.

Set $\beta = \eta/2$. Then due to (4.2) and (4.3) for arbitrary $\zeta\colon |\zeta - x_0| < \beta$ we have

$$\max{}_{x \in [x(\zeta;t), x_*(t)]} \Big|\langle f_x^0(t,x,u_*(t)), x(\zeta;t) - x_*(t)\rangle\Big| \overset{\text{a.e.}}{\leq} \|\zeta - x_0\| \frac{2\sigma e^{-\rho t}e^{-(\nu+\alpha)t}}{\eta^{1-\sigma}}$$
$$= \frac{2\sigma\|\zeta - x_0\|}{\eta^{1-\sigma}}e^{-(\theta+\nu-\alpha)t}, \quad t \geq 0.$$

Hence, if

$$\theta + \nu > \alpha \quad \Big(= \frac{p}{2-\sigma}\Big), \tag{4.4}$$

then assumption (A2) is satisfied. In what follows we assume that this condition is fulfilled.[4]

Due to Theorem 3.1, the LWOO process $(x_*(\cdot), u_*(\cdot))$ satisfies the core conditions of the maximum principle together with the adjoint variable $\psi(\cdot)$ defined as (see (3.6) and (4.2))

$$\psi(t) = e^{(\nu+\alpha)t}\int_t^\infty e^{-(\nu+\alpha)s}\frac{\sigma e^{-\rho s}}{(x_*(s))^{1-\sigma}}\,ds = \sigma e^{(\nu+\alpha)t}\int_t^\infty \frac{e^{-(\nu+\theta-\alpha)s}}{(x_*(s))^{1-\sigma}}\,ds, \quad t \geq 0. \tag{4.5}$$

As far as $x_*(t) \geq \eta$, $t \geq 0$, equality (4.5) implies

$$\psi(t) \leq \frac{\sigma}{\eta^{1-\sigma}}e^{(\nu+\alpha)t}\int_t^\infty e^{-(\nu+\theta-\alpha)s}\,ds = \frac{\sigma}{\eta^{1-\sigma}(\nu+\theta-\alpha)}e^{-\rho t}, \quad t \geq 0. \tag{4.6}$$

[4]In the opposite case, i.e., when $\theta + \nu \leq \alpha$, there is no LWOO admissible pair $(x_*(\cdot), u_*(\cdot))$ in problem $(\tilde{P}1)$, as it can be shown directly.

On the other hand, as shown in [6], the following estimate holds for the LWOO control $u_*(t)$:

$$u_*(t) \overset{\text{a.e.}}{\le} M = \frac{4}{b}\frac{\sigma}{\eta^{1-\sigma}}\Big(\frac{1}{\theta+\nu-\alpha}+2\Big), \quad t \ge 0.$$

Due to (4.1), this estimate immediately implies that $x_*(\cdot)$ is uniformly bounded from above:

$$x_*(t) \le \kappa = \max\Big\{K_0, \frac{M}{\nu+\alpha}\Big\}, \quad t \ge 0.$$

Hence, due to (4.5) we get the opposite estimate for $\psi(\cdot)$:

$$\psi(t) \ge \frac{\sigma}{\kappa^{1-\sigma}}e^{(\nu+\alpha)t}\int\limits_t^\infty e^{-(\nu+\theta-\alpha)s}\,ds = \frac{\sigma}{\kappa^{1-\sigma}(\nu+\theta-\alpha)}e^{-\rho t}, \quad t \ge 0. \tag{4.7}$$

Combining estimates (4.6) and (4.7) we get the following characterization of the asymptotic behavior of the adjoint variable $\psi(\cdot)$:

$$\frac{\sigma}{\kappa^{1-\sigma}(\nu+\theta-\alpha)}e^{-\rho t} \le \psi(t) \le \frac{\sigma}{\eta^{1-\sigma}(\nu+\theta-\alpha)}e^{-\rho t}, \quad t \ge 0. \tag{4.8}$$

Note that due to (4.8) both the standard transversality conditions $\psi(t) \to 0$ as $t \to \infty$ and $\psi(t)x_*(t) \to 0$ as $t \to \infty$ are valid only if the discount rate $\rho = \theta - 2\alpha$ is positive, i.e., if $\alpha < \theta/2$.

In the case

$$\frac{\theta}{2} \le \alpha < \theta + \nu,$$

the discount rate $\rho = \theta - 2\alpha$ is nonpositive, and (4.8) implies that both these standard transversality conditions fail. In this case a solution with a finite objective value does not exist, although an LWOO solution exists, as it is shown in [6].

As far as $(\tilde{P}1)$ is an autonomous problem with exponential discounting $e^{-\rho t}$, $t \ge 0$, one can reformulate Theorem 3.1 in an equivalent way in terms of the current value adjoint variable $\xi(\cdot)$: $\xi(t) = e^{\rho t}\psi(t)$, $t \ge 0$.

Due to (4.5) the current value the adjoint variable $\xi(\cdot)$ is defined by the following equality:

$$\xi(t) = \sigma e^{(\rho+\nu+\alpha)t}\int\limits_t^\infty \frac{e^{-(\nu+\theta-\alpha)s}}{(x_*(s))^{1-\sigma}}\,ds, \quad t \ge 0, \tag{4.9}$$

and (see (4.8))

$$\frac{\sigma}{\kappa^{1-\sigma}(\nu+\theta-\alpha)} \le \xi(t) \le \frac{\sigma}{\eta^{1-\sigma}(\nu+\theta-\alpha)}, \quad t \ge 0.$$

Thus, although the function $\psi(\cdot)$ can be unbounded (if $\rho = \theta - 2\alpha$ is negative, i.e., in the case $\alpha > \theta/2$), the corresponding current value adjoint variable $\xi(\cdot)$ is always bounded for all admissible values of the parameters (see (4.4)).

The current value adjoint system for problem $(\tilde{P}1)$ reads as

$$\dot{\xi}(t) = (\rho+\lambda)\xi(t) - \sigma x_*(t)^{\sigma-1} = (\nu+\theta-\alpha)\xi - \sigma x_*(t)^{\sigma-1}, \tag{4.10}$$

and the maximum condition takes the form

$$u_*(t) \overset{\text{a.e.}}{=} \frac{1}{b}\xi(t), \quad t \ge 0. \tag{4.11}$$

Due to Theorem 3.1 and (4.4) the current value adjoint variable $\xi(\cdot)$ defined by (4.9) is the unique bounded solution of (4.10) while the LWOO control $u_*(\cdot)$ satisfies (4.11).

Thus we come up with the following system of equations determining the LWOO solution in problem $(\tilde{P}1)$:

$$\dot{x}(t) = -(\nu+\alpha)x(t) + \frac{1}{b}\xi(t), \quad x(0) = K_0,$$
$$\dot{\xi}(t) = -\sigma x(t)^{\sigma-1} + (\nu+\theta-\alpha)\xi(t), \quad \xi(\cdot) \text{ is bounded.}$$

According to Theorem 3.1 this specific "boundary value problem" has a unique solution. This property makes it possible to apply standard methods of investigation, based on the fact that $(x(0),\xi(0)) = (K_0,\xi(0))$ must belong to the stable invariant manifold of the above system (see, e.g., [14]).

In the particular case $\sigma = 1$ the solution is explicit. Noticing that $\alpha = p$ in this case we obtain:

$$\xi(t) \equiv \frac{1}{\nu+\theta-p}, \quad t \geq 0.$$

Hence, the LWOO optimal control for the original problem is

$$I(t) = \frac{e^{pt}}{b(\nu+\theta-p)}, \quad t \geq 0,$$

provided that $\theta+\nu > p$. We stress again that in the case $\theta+\nu \leq p$ an LWOO solution does not exist and that in the case $\frac{\theta}{2} \leq p < \theta+\nu$ the LWOO solution produces infinite objective value, thus it has no "classical" meaning.

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