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On Some Properties of Nash Equilibrium Points in Two-Person Games

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Abstract—In modern game theory, a lot of attention is paid to the concept of Nash equilibrium. The paper is devoted to the study of some properties of the set $\mathfrak{A}$ of Nash equilibrium points in two-person games. In particular, the character of possible complexity of the set $\mathfrak{A}$ is investigated, and the stability of the set $\mathfrak{A}$ under small perturbations of payoff functions is analyzed.

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Denote by $\mathbb{R}^k$ $(k \geq 1)$ the k-dimensional Euclidean real arithmetic space with standard inner product and with elements identified with ordered sets (columns) of k real numbers. For a nonempty set $M \subset \mathbb{R}^k$, denote by $\operatorname{comp}(M)$ the set of nonempty compact sets that belong to M. We will use the notions of a set-valued map (s.m.), an upper semicontinuous set-valued map (u.s.c. s.m.), a lower semicontinuous set-valued map, and a continuous set-valued map (see, for example, [1]). If a scalar function $h(a)$ is defined and continuous on a nonempty compact set $A \subset \mathbb{R}^k$, then we denote by $\operatorname{Arg\,max}_{a\in A} h(a)$ and $\operatorname{Arg\,min}_{a\in A} h(a)$ the set of maximizers and the set of minimizers of the function $h(a)$ on the set A.

Consider the following two-person (two-player) game (see, for example, [2–6]). Let X and Y be nonempty compact sets in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively. Put $Z = X \times Y$. Suppose that continuous scalar functions $f(x, y)$ and $g(x, y)$ are fixed on Z.

The aim of the first player is to maximize the function $f(x, y)$ by choosing $x \in X$, and the aim of the second player is to maximize the function $g(x, y)$ by choosing $y \in Y$. So the functions $f(x, y)$ and $g(x, y)$ represent the criteria corresponding to the first and second player, respectively.

Definition. A point $(x_0, y_0) \in Z$ is called a *Nash equilibrium point* if the following two conditions are satisfied simultaneously:

(1) the inequality $f(x, y_0) \leq f(x_0, y_0)$ holds for all $x \in X$;

(2) the inequality $g(x_0, y) \leq g(x_0, y_0)$ holds for all $y \in Y$.

Denote the set of all Nash equilibrium points in the game by $\mathfrak{A}$. It is known that in the general case the set $\mathfrak{A}$ may be empty. There is a well-known general theorem (see, for example, [2, Theorem 7.2.2]) that gives sufficient conditions for $\mathfrak{A}$ to be nonempty. Below (see Theorem 2), we present other known sufficient conditions under which the set $\mathfrak{A}$ is nonempty.

To find and study the set $\mathfrak{A}$, it is useful to consider two marginal maps

$$\Omega_1(y) = \operatorname*{Arg\,max}_{x\in X} f(x, y), \qquad y \in Y, \tag{1}$$

$$\Omega_2(x) = \operatorname*{Arg\,max}_{y\in Y} g(x, y), \qquad x \in X. \tag{2}$$

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It is clear that $\Omega_1(y) \neq \varnothing$ for $y \in Y$ and $\Omega_2(x) \neq \varnothing$ for $x \in X$ and that $\Omega_1(y) \in \operatorname{comp}(X)$ for $y \in Y$ and $\Omega_2(x) \in \operatorname{comp}(Y)$ for $x \in X$. Thus, the game under consideration is assigned two s.m.'s

$$\Omega_1\colon\ Y \to \operatorname{comp}(X) \qquad \text{and} \qquad \Omega_2\colon\ X \to \operatorname{comp}(Y).$$

It is easy to show that the s.m. Ω_1 is u.s.c. on Y and the s.m. Ω_2 is u.s.c. on X. Next, we put

$$\operatorname{Gr}\Omega_1 = \{(x,y)\colon\ x \in \Omega_1(y),\ y \in Y\} \qquad \text{and} \qquad \operatorname{Gr}\Omega_2 = \{(x,y)\colon\ x \in X,\ y \in \Omega_2(x)\}.$$

Using the fact that the s.m. $\Omega_1(y)$ is u.s.c. on Y and s.m. $\Omega_2(x)$ is u.s.c. on X, one can easily check that the sets $\operatorname{Gr}\Omega_1$ and $\operatorname{Gr}\Omega_2$ are compact.

Let $\mathfrak{B} = \operatorname{Gr}\Omega_1 \cap \operatorname{Gr}\Omega_2$. The following theorem is well-known in game theory.

Theorem 1. *The equality* $\mathfrak{A} = \mathfrak{B}$ *holds.*

The following lemma may be useful in studying the properties of the s.m.'s $\Omega_1(y)$ and $\Omega_2(x)$.

Lemma 1. *Suppose that* $P \subset \mathbb{R}^k$ $(k \geq 1)$ *and* $Q \subset \mathbb{R}^l$ $(l \geq 1)$ *are nonempty compact sets and* $\Omega\colon P \to \operatorname{comp}(Q)$ *is an s.m. If* Ω *is u.s.c. at a point* $\xi \in P$ *and* $\Omega(\xi)$ *is a one-point set, then the s.m.* Ω *is continuous at the point* ξ.

Proof. It suffices to prove the lower semicontinuity of the s.m. Ω at the point ξ. This fact is proved by contradiction. □

Remark 1. Note that if the hypothesis of Lemma 1 holds for all $\xi \in P$, then the single-valued function $\omega(\xi) = \Omega(\xi)$ is continuous (in the ordinary sense) on P.

Using this remark and the well-known theorem of Brouwer (see, for example, [7]), we arrive at the following well-known theorem (see, for example, [6]).

Theorem 2. *Suppose that the marginal maps* $\Omega_1(y)$ *and* $\Omega_2(x)$ (*see* (1) *and* (2)) *in the game are single-valued for all* $y \in Y$ *and* $x \in X$, *respectively, and the compact sets* X *and* Y *are convex. Then the set* $\mathfrak{A}$ *in the game is nonempty.*

Remark 2. This theorem is interesting because it imposes no explicit requirements on the concavity of the function $f(x,y)$ with respect to $x \in X$ for $y \in Y$ and on the concavity of the function $g(x,y)$ with respect to $y \in Y$ for $x \in X$ (here X and Y are convex compact sets), as is usually done in the literature (see, for example, [2, Theorem 7.2.2]).

In connection with Theorem 1, a question arises as to what extent the compact set $\mathfrak{A}$ may be arbitrary, provided that $\mathfrak{A} \neq \varnothing$, as a function of the pair of continuous payoff functions f and g defined on the compact set $Z = X \times Y$.

Theorem 3. *In the nonempty compact set* Z, *we fix an arbitrary nonempty compact subset* M_1 *with the following property*: *for every* $y \in Y$, *the intersection of the set* $\bigcup_{x \in X}\{(x,y)\}$ *with* M_1 *is nonempty. Then there exists a continuous scalar function* $f(x,y)$ *on* Z *such that* $\operatorname{Gr}\Omega_1 = M_1$.

Proof. According to the recipe from [7, Ch. 2, Sect. 3, Theorem 1], one can explicitly construct a smooth scalar function $F(x,y)$ on $\mathbb{R}^p \times \mathbb{R}^q$ such that $F(x,y) = 0$ for $(x,y) \in M_1$ and $F(x,y) < 0$ for $(x,y) \notin M_1$. As the sought function $f(x,y)$, one can take the restriction of the function $F(x,y)$ to Z. □

The next theorem can be proved in a similar way.

Theorem 4. *In the nonempty compact set* Z, *let* M_2 *be an arbitrary fixed compact subset with the following property*: *for every* $x \in X$, *the intersection of the set* $\bigcup_{y \in Y}\{(x,y)\}$ *with* M_2 *is nonempty. Then there exists a continuous scalar function* $g(x,y)$ *on* Z *such that* $\operatorname{Gr}\Omega_2 = M_2$.

The following theorem is related to Theorems 3 and 4.

Theorem 5. *Suppose that an arbitrary nonempty compact subset* M *is fixed in the nonempty compact set* Z. *Then there exist continuous scalar functions* $f(x,y)$ *and* $g(x,y)$ *on* Z *such that* $\mathfrak{A} = M$.

Proof. Fix a point $(\xi, \eta) \in M$, where $\xi \in X$ and $\eta \in Y$. Consider the sets $\mathfrak{M}_1 = M \cup L_1$ and $\mathfrak{M}_2 = M \cup L_2$ with

$$L_1 = \{(x, y) \in Z\colon\ x = \xi,\ y \in Y\} \qquad \text{and} \qquad L_2 = \{(x, y) \in Z\colon\ x \in X,\ y = \eta\}.$$

One can easily prove that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are compact sets; moreover,

$$\mathfrak{M}_1 \cap \mathfrak{M}_2 = M. \tag{3}$$

According to Theorem 3, for the set $\mathfrak{M}_1$ one can construct a continuous function $f(x, y)$ such that

$$\operatorname{Gr} \Omega_1 = \mathfrak{M}_1. \tag{4}$$

According to Theorem 4, for the set $\mathfrak{M}_2$ one can construct a continuous function $g(x, y)$ such that

$$\operatorname{Gr} \Omega_2 = \mathfrak{M}_2. \tag{5}$$

Taking into account relations (3)–(5) and using Theorem 1, we find that the equality $\mathfrak{A} = M$ is satisfied for the functions $f(x, y)$ and $g(x, y)$. $\square$

Now, we examine separately the case of a two-person zero-sum game with

$$g(x, y) = -f(x, y) \qquad \text{for} \quad (x, y) \in Z. \tag{6}$$

Here we are interested in the saddle points of the game. It is known that when the functions $f(x, y)$ and $g(x, y)$ are continuous on the compact set $Z = X \times Y$ and condition (6) is satisfied, the concept of a Nash equilibrium point is equivalent to the concept of a saddle point. Therefore, the equality $\mathfrak{A}_1 = \mathfrak{A}$ holds, where $\mathfrak{A}_1$ is the set of saddle points in the game. The specific feature of a zero-sum game is that if $\mathfrak{A}_1$ is nonempty, it is a rectangular set, i.e., $\mathfrak{A}_1 = K_1 \times K_2$, where K_1 is a nonempty compact set in X and K_2 is a nonempty compact set in Y; moreover,

$$K_1 = \operatorname*{Arg\,max}_{x \in X} \Bigl(\min_{y \in Y} f(x, y) \Bigr) \qquad \text{and} \qquad K_2 = \operatorname*{Arg\,min}_{y \in Y} \Bigl(\max_{x \in X} f(x, y) \Bigr).$$

Theorem 6. *Suppose that an arbitrary nonempty rectangular compact subset M in the nonempty compact set Z is fixed*; *i.e., $M = M_1 \times M_2$, where M_1 is a nonempty compact subset of X and M_2 is a nonempty compact subset of Y. Then there exists a continuous scalar function $f(x, y)$ on Z such that the equality $\mathfrak{A}_1 = M$ holds for the corresponding zero-sum game* (*see* (6)).

Proof. Using the arguments from [7, Ch. 2, Sect. 3, Theorem 1], we construct a continuous function $f_1(x)$ on X such that

$$\operatorname*{Arg\,max}_{x \in X} f_1 = M_1.$$

Using the same arguments, we construct a continuous function $f_2(y)$ on Y such that

$$\operatorname*{Arg\,min}_{y \in Y} f_2 = M_2.$$

After constructing the functions $f_1(x)$ and $f_2(y)$, one can take the function $f(x, y) = f_1(x) + f_2(y)$ as the sought function $f(x, y)$. $\square$

To study further the set $\mathfrak{A}$, consider the function

$$\Delta(p, q) = \bigl(f(x, \widetilde{y}\,) - f(\widetilde{x}, \widetilde{y}\,)\bigr) + \bigl(g(\widetilde{x}, y) - g(\widetilde{x}, \widetilde{y}\,)\bigr), \tag{7}$$

where $x, \widetilde{x} \in X$, $y, \widetilde{y} \in Y$, $p = (x, y)$, and $q = (\widetilde{x}, \widetilde{y})$, and introduce the function

$$L(q) = \max_{p \in Z} \Delta(p, q). \tag{8}$$

Note that under the assumptions made above, the function $L(q)$ is continuous on Z. It is easy to show (see (7)) that

$$L(q) \geq 0 \qquad \forall q \in Z. \tag{9}$$

Note that functions of the type (8) have been considered in some previous studies in N-person game theory (see, for example, [8]).

Lemma 2. *Let*

$$\min_{q \in Z} L(q) = 0. \tag{10}$$

Then $\mathfrak{A} \neq \varnothing$.

Proof. Suppose that a point $q_0 = (x_0, y_0)$ in Z is such that (see (10))

$$L(q_0) = 0.$$

Then (see (7), (8), (10))

$$\big(f(x, y_0) - f(x_0, y_0)\big) + \big(g(x_0, y) - g(x_0, y_0)\big) \leq 0 \qquad \forall (x, y) \in Z.$$

Substituting successively $y = y_0$ and $x = x_0$ into this inequality, we obtain

$$f(x, y_0) \leq f(x_0, y_0) \quad \forall x \in X, \qquad g(x_0, y) \leq g(x_0, y_0) \quad \forall y \in Y, \tag{11}$$

i.e., $q_0 \in \mathfrak{A}$. □

Lemma 3. *If* $\mathfrak{A} \neq \varnothing$, *then relation* (10) *is satisfied.*

Proof. Let $q_0 = (x_0, y_0) \in \mathfrak{A}$. Then relations (11) hold. Hence (see (7)),

$$\Delta(p, q_0) \leq 0 \qquad \forall p \in Z. \tag{12}$$

Relations (8), (9), and (12) imply equality (10). □

The following lemma is a corollary to Lemmas 2 and 3.

Lemma 4. 1. *The set* $\mathfrak{A}$ *coincides with the set of points* $q \in Z$ *that* (*see* (8)) *satisfy the equality* $L(q) = 0$.

2. *The set* $\mathfrak{A}$ *coincides with the set of points* $\operatorname{Arg\,min}_{q \in Z} L(q)$ *provided that relation* (10) *is valid.*

3. *The set* $\mathfrak{A}$ *is empty if and only if* $\min_{q \in Z} L(q) > 0$.

Lemma 4 opens up certain possibilities for calculating the points of the set $\mathfrak{A}$ on the basis of the calculation of the roots of the function $L(q)$ (see (8)) on the set Z (see [9]). Note that in view of formulas (7) and (8), the analytic properties of $L(q)$ may deteriorate compared with the analytic properties of the function $\Delta(p, q)$ due to the presence of a maximum-type operation. Note also that there are a fairly large number of publications devoted to the numerical aspects of calculating the points of the set $\mathfrak{A}$. We mention [10] among these publications.

In conclusion, we examine how the set $\mathfrak{A}$ depends on perturbations of the functions $f(x, y)$ and $g(x, y)$. To this end, we consider a two-person game with payoff functions $f(x, y, \varepsilon)$ and $g(x, y, \varepsilon)$ that are defined and continuous on $X \times Y \times \mathcal{E}$, where the sets $X \subset \mathbb{R}^p$, $Y \subset \mathbb{R}^q$, and $\mathcal{E} \subset \mathbb{R}^r$ are nonempty and compact and the point 0 is an interior point of the set $\mathcal{E}$. For a fixed $\varepsilon \in \mathcal{E}$, we have an ordinary two-player game considered above. To every such game, there corresponds a set of Nash equilibrium points $\mathfrak{A}(\varepsilon)$. By analogy with formulas (7) and (8), we define functions $\Delta(p,q,\varepsilon)$ and $L(q,\varepsilon)$. These functions are continuous on $Z \times Z \times \mathcal{E}$ and $Z \times \mathcal{E}$, respectively. Suppose that $\min_{q\in Z} L(q,0) > 0$, i.e., $\mathfrak{A}(0) = \varnothing$. Then, for sufficiently small $|\varepsilon|$, we have $\min_{q\in Z} L(q,\varepsilon) > 0$; i.e., $\mathfrak{A}(\varepsilon) = \varnothing$ for such ε. Now, suppose that $\mathfrak{A}(\varepsilon) \neq \varnothing$ in a neighborhood $V \subset \mathcal{E} \subset \mathbb{R}^r$ of the point $0 \in \mathcal{E} \subset \mathbb{R}^r$. Then, according to the aforesaid, in this neighborhood V the set $\mathfrak{A}(\varepsilon)$ consists of those points $q \in Z$ for which

$$L(q,\varepsilon) = 0.$$

Hence, the s.m. $\mathfrak{A}(\varepsilon)$ is u.s.c. at the point $\varepsilon = 0$. If $\mathfrak{A}(0)$ is a one-point set, then (see Lemma 1) the s.m. $\mathfrak{A}(\varepsilon)$ is continuous at the point $\varepsilon = 0$. Thus, in this case, the set $\mathfrak{A}(0)$ is stable under small perturbations of the payoff functions.

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