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On Sum Sets of Sets Having Small Product Set

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Abstract—We improve the sum–product result of Solymosi in $\mathbb{R}$; namely, we prove that $\max\{|A+A|,|AA|\} \gg |A|^{4/3+c}$, where $c>0$ is an absolute constant. New lower bounds for sums of sets with small product set are found. Previous results are improved effectively for sets $A\subset\mathbb{R}$ with $|AA|\le|A|^{4/3}$.

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1. INTRODUCTION

Let $A,B\subset\mathbb{R}$ be finite sets. Define the *sum set*, *product set*, and *quotient set* of A and B as

$$A+B := \{a+b\colon\ a\in A,\ b\in B\},\qquad AB := \{ab\colon\ a\in A,\ b\in B\},$$

and

$$A/B := \{a/b\colon\ a\in A,\ b\in B,\ b\ne 0\},$$

respectively. The Erdős–Szemerédi conjecture [3] says that for any $\epsilon>0$ one has

$$\max\{|A+A|,|AA|\} \gg |A|^{2-\epsilon}.$$

Roughly speaking, it states that an arbitrary subset of real numbers (or integers) cannot have good additive and multiplicative structure simultaneously. At present the best result in this direction is due to Solymosi [10].

Theorem 1 (Solymosi)**.** *Let $A\subset\mathbb{R}$ be an arbitrary set. Then*

$$|A+A|^2|A/A| \ge \frac{|A|^4}{4\lceil\log|A|\rceil},\qquad |A+A|^2|AA| \ge \frac{|A|^4}{4\lceil\log|A|\rceil}. \tag{1.1}$$

In particular,

$$\max\{|A+A|,|AA|\} \gg \frac{|A|^{4/3}}{\log^{1/3}|A|}. \tag{1.2}$$

Here and below we suppose that $|A|\ge 2$.

It is easy to see that bound (1.1) is tight up to logarithmic factors if the size of $A+A$ is small relative to A. The first part of the paper concerns the case where the product AA is small. We will write $a\lesssim b$ or $b\gtrsim a$ if $a=O(b\log^c|A|)$, $c>0$. In these terms inequality (1.1) implies the following.

Corollary 2. *Let $A\subset\mathbb{R}$ be a finite set and $K\ge 1$ a real number. Suppose that $|A/A|\le K|A|$ or $|AA|\le K|A|$. Then*

$$|A+A| \gtrsim |A|^{3/2}K^{-1/2}. \tag{1.3}$$

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Estimate (1.3) was improved for small K (see, e.g., references in [9]; sharper bounds for the *difference* of two sets having small multiplicative doubling can be found in [8]). Here we cite a result from [9].

Theorem 3. *Let $A \subset \mathbb{R}$ be a finite set and $K \geq 1$ a real number. Suppose that $|A/A| \leq K|A|$ or $|AA| \leq K|A|$. Then*

$$|A+A| \gtrsim |A|^{58/37} K^{-42/37}. \tag{1.4}$$

It is easy to check that the bound of Theorem 3 is better than Corollary 2 for $K \lesssim |A|^{5/47}$.

Let us formulate our first result (its refined version is contained in Theorems 11 and 13 below).

Theorem 4. *Let $A \subset \mathbb{R}$ be a finite set and $K \geq 1$ a real number. Suppose that $|A/A| \leq K|A|$ or $|AA| \leq K|A|$. Then*

$$|A+A| \gtrsim |A|^{19/12} K^{-5/6} \qquad \text{and} \qquad |A+A| \gtrsim |A|^{49/32} K^{-19/32}.$$

Theorem 4 is stronger than Theorem 3 and refines estimate (1.3) for $K \lesssim |A|^{1/3}$.

In the next theorem we improve bound (1.2) (its refined version is contained in Theorem 15).

Theorem 5. *Let $A \subset \mathbb{R}$ be an arbitrary set. Then*

$$\max\{|A+A|, |AA|\} \gg |A|^{4/3+c},$$

where $c > 0$ is an absolute constant.

In addition, we consider a "critical" case of Solymosi's theorem, i.e., the situation where the reverse inequality to (1.1) holds (see Proposition 14).

We use a combination of methods from [10] and [7] in our arguments.

2. DEFINITIONS AND PRELIMINARY RESULTS

The *additive energy* $\mathsf{E}^+(A,B)$ between two sets A and B is the number of the solutions of the equation $a_1 + b_1 = a_2 + b_2$ for $a_1, a_2 \in A$ and $b_1, b_2 \in B$ (see [11]):

$$\mathsf{E}^+(A,B) = \big|\{a_1 + b_1 = a_2 + b_2 \colon\ a_1, a_2 \in A,\ b_1, b_2 \in B\}\big|.$$

The *multiplicative energy* $\mathsf{E}^\times(A,B)$ between two sets A and B is the number of the solutions of the equation $a_1 b_1 = a_2 b_2$ for $a_1, a_2 \in A$ and $b_1, b_2 \in B$ (see [11]):

$$\mathsf{E}^\times(A,B) = \big|\{a_1 b_1 = a_2 b_2 \colon\ a_1, a_2 \in A,\ b_1, b_2 \in B\}\big|.$$

In the case $A = B$ we write $\mathsf{E}^+(A)$ for $\mathsf{E}^+(A,A)$ and $\mathsf{E}^\times(A)$ for $\mathsf{E}^\times(A,A)$. Having $\lambda \in A/A$, we put $A_\lambda = A \cap \lambda A$. Clearly, if $0 \notin A$, then

$$\mathsf{E}^\times(A) = \sum_{\lambda \in A/A} |A_\lambda|^2 \tag{2.1}$$

and similarly for the energy $\mathsf{E}^+(A)$. Finally, the Cauchy–Schwarz inequality implies for $0 \notin A$, $A_1 \subset A$, and $A_2 \subset A$ that

$$\mathsf{E}^\times(A_1,A_2)|A/A| \geq |A_1|^2|A_2|^2, \qquad \mathsf{E}^\times(A_1,A_2)|AA| \geq |A_1|^2|A_2|^2. \tag{2.2}$$

In particular,

$$\mathsf{E}^\times(A)|A/A| \geq |A|^4, \qquad \mathsf{E}^\times(A)|AA| \geq |A|^4. \tag{2.3}$$

Solymosi's Theorem 1 can be derived from a slightly more delicate result on an upper bound for the multiplicative energy of a set in terms of its sum set (see [10]). Estimation of the cardinality of the set on the left-hand side of (2.4) is the main goal of our crucial Lemma 10.

Theorem 6. *Let* $A, B \subseteq \mathbb{R}$ *be finite sets with* $\min\{|A|, |B|\} \geq 2$ *and* $\tau \geq 1$ *be a real number. Then*

$$|\{x\colon |A \cap xB| \geq \tau\}| \ll \frac{|A+A| \cdot |B+B|}{\tau^2}. \tag{2.4}$$

In particular,

$$\mathsf{E}^{\times}(A, B) \ll |A+A| \cdot |B+B| \log(\min\{|A|, |B|\}). \tag{2.5}$$

We need the assertion from [5, Lemma 7]. In [7, Lemma 27] the same result was obtained with the redundant factor $\log^2 d(A)$.

Lemma 7. *Let* $A \subset \mathbb{R}$ *be a finite set. Then for any finite set* $B \subset \mathbb{R}$ *and an arbitrary real number* $\tau \geq 1$ *one has*

$$\big|\big\{x \in A+B\colon |A \cap (x-B)| \geq \tau\big\}\big| \ll d(A) \frac{|A| \cdot |B|^2}{\tau^3}, \tag{2.6}$$

where

$$d(A) := \min_{C \neq \varnothing} \frac{|AC|^2}{|A| \cdot |C|}.$$

Obviously, if $|A/A| \leq K|A|$ or $|AA| \leq K|A|$, then $d(A)$ does not exceed K^2. The quantity $d(A)$ is a more delicate characteristic of a set than $|A/A|/|A|$ or $|AA|/|A|$. For example, the rough estimate (1.4) can be derived from the stronger one

$$|A+A| \gtrsim |A|^{58/37} (d(A))^{-21/37} \tag{2.7}$$

(see [9]).

Lemma 7 implies the following result.

Corollary 8. *Let* $A_1, A_2, A_3 \subset \mathbb{R}$ *be any finite sets and* α_1, α_2, α_3 *be arbitrary nonzero numbers. Then the number*

$$\sigma(\alpha_1 A_1, \alpha_2 A_2, \alpha_3 A_3) := \big|\big\{\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 = 0\colon\ a_1 \in A_1,\ a_2 \in A_2,\ a_3 \in A_3\big\}\big| \tag{2.8}$$

does not exceed $O((d(A_1))^{1/3} |A_1|^{1/3} |A_2|^{2/3} |A_3|^{2/3})$.

Proof. Without loss of generality, we can suppose that $\alpha_1 = 1$. Then the number (2.8) is

$$\sigma := \sum_{x \in (-\alpha_3 A_3)} |A_1 \cap (x - \alpha_2 A_2)|. \tag{2.9}$$

Let us arrange the values of $|A_1 \cap (x - \alpha_2 A_2)|$ in decreasing order, that is,

$$|A_1 \cap (x_1 - \alpha_2 A_2)| \geq |A_1 \cap (x_2 - \alpha_2 A_2)| \geq \dots .$$

Using Lemma 7, we obtain $|A_1 \cap (x_j - \alpha_2 A_2)| \ll (d(A_1))^{1/3} |A_1|^{1/3} |A_2|^{2/3} j^{-1/3}$. Substituting the last bound into (2.9), we get

$$\sigma \ll (d(A_1))^{1/3} |A_1|^{1/3} |A_2|^{2/3} |A_3|^{2/3},$$

as required. $\square$

The last result of the section connects the quantity $\mathsf{E}^+(A)$ with $|A/A|$ and $|AA|$. We follow the arguments from [2] in the proof.

Theorem 9. *Let $A \subset \mathbb{R}$ be a finite set. Then*

$$|A/A| \cdot |A|^{10} \log|A| \gg (\mathsf{E}^+(A))^4, \qquad |AA| \cdot |A|^{10} \log|A| \gg (\mathsf{E}^+(A))^4. \tag{2.10}$$

Proof. Without loss of generality, we can suppose that all elements of A are positive. For $x \in \mathbb{R}$ put

$$N(x) = |A \cap (x - A)|.$$

We have

$$\sum_{x \in A+A} N(x) = |A|^2, \qquad \sum_{x \in A+A} N^2(x) = \mathsf{E}^+(A). \tag{2.11}$$

Let

$$F = \left\{ x \in A + A\colon\ N(x) > \frac{\mathsf{E}^+(A)}{2|A|^2} \right\}.$$

Then

$$\sum_{x \notin F} N^2(x) \le \sum_{x \notin F} N(x) \frac{\mathsf{E}^+(A)}{2|A|^2}.$$

Using this and the first formula of (2.11), we obtain

$$\sum_{x \notin F} N^2(x) \le |A|^2 \frac{\mathsf{E}^+(A)}{2|A|^2} = \frac{\mathsf{E}^+(A)}{2}.$$

Applying (2.11) once again, we get

$$\sum_{x \in F} N^2(x) \ge \frac{\mathsf{E}^+(A)}{2}. \tag{2.12}$$

Put

$$U = \sum_{x \in F} N(x).$$

Because of (2.12) and the trivial bound $N(x) \le |A|$, we have

$$U \ge \frac{\mathsf{E}^+(A)}{2|A|}. \tag{2.13}$$

Further, by the definition of the set F

$$|F| \le \frac{2|A|^2 U}{\mathsf{E}^+(A)}.$$

Using this and inequality (2.13), we obtain

$$|F| + |A| \le \frac{4|A|^2 U}{\mathsf{E}^+(A)}. \tag{2.14}$$

Let us consider the set

$$P = (A \cup F) \times (A \cup F)$$

of points in $\mathbb{R}^2$ and estimate the number T of collinear triples in P (points in a triple are not necessarily distinct). On the one hand, a general upper bound for the number of such triples in Cartesian products [11, Corollary 8.9] yields

$$T \ll |A \cup F|^4 \log|A|.$$

Because of (2.14), it implies

$$T \ll \frac{|A|^8 U^4 \log|A|}{(\mathsf{E}^+(A))^4}. \tag{2.15}$$

On the other hand, for $x \in A$ put

$$F(x) = \{y \in A\colon\ x + y \in F\}.$$

Fixing $e, f \in A$, we find from (2.2) that there are at least

$$T(e,f) = \frac{F^2(e)F^2(f)}{\min\{|AA|, |A/A|\}}$$

quadruples (a,b,c,d) such that $ab = cd$, $a, c \in F(e)$, and $b, d \in F(f)$. They form at least $T(e,f)$ collinear triples

$$(e,f),\ (e+a, f+d),\ (e+c, f+b).$$

It follows that

$$T \geq \frac{1}{\min\{|AA|, |A/A|\}} \sum_{e,f\in A} F^2(e)F^2(f) = \frac{1}{\min\{|AA|, |A/A|\}} \left(\sum_{e\in A} F^2(e) \right)^2.$$

By the Cauchy–Schwarz inequality,

$$\sum_{e\in A} F^2(e) \geq \left(\sum_{e\in A} F(e) \right)^2 |A|^{-1} = U^2 |A|^{-1}.$$

Therefore,

$$T \geq \frac{1}{\min\{|AA|, |A/A|\}} U^4 |A|^{-2}. \tag{2.16}$$

Combining estimates (2.15) and (2.16), we obtain the required result. □

3. PROOF OF THE MAIN RESULTS

We begin with a key technical lemma.

Let $A \subset \mathbb{R}$, $0 \notin A$, be a finite set and $\tau > 0$ a real number. Let also S'_τ be a set such that

$$S'_\tau \subset S_\tau := \{\lambda\colon\ \tau < |A_\lambda| \leq 2\tau\} \subseteq A/A$$

and for any nonzero α_1, α_2, α_3 and different $\lambda_1, \lambda_2, \lambda_3 \in S'_\tau$ one has

$$\sigma(\alpha_1 A_{\lambda_1}, \alpha_2 A_{\lambda_2}, \alpha_3 A_{\lambda_3}) \leq \sigma.$$

Lemma 10. *Let $A \subset \mathbb{R}$, $0 \notin A$, be a finite set, $\tau > 0$ be a real number,*

$$32\sigma \leq \tau^2 \leq |A + A|\sqrt{\sigma}, \tag{3.1}$$

and S'_τ and σ be as defined above. Then

$$|A + A|^2 \geq \frac{\tau^3 |S'_\tau|}{128\sqrt{\sigma}}. \tag{3.2}$$

Proof. We follow the arguments from [10]. Without loss of generality, one can suppose that $A \subset \mathbb{R}^+$. Consider the Cartesian product $A \times A$ and the lines l_λ of the form $y = \lambda x$, where $\lambda \in A/A$. Clearly, any line l_λ intersects $A \times A$ at the points $(x, \lambda x)$, $x \in A_\lambda$. Put $\mathcal{A}_\lambda = l_\lambda \cap (A \times A)$.

Let $2 \le M \le |S'_\tau|$ be an integer parameter, which will be chosen later. Arrange the elements of the set S'_τ in increasing order and split them into groups of size M of consecutive elements. We get $k \ge \lfloor |S'_\tau|/M \rfloor \ge |S'_\tau|/(2M)$ such groups U_j. For the lines l_λ in each of these groups, we take the points lying in the sets $\mathcal{A}_\lambda$ and consider all their sums. Clearly, the sums belong $(A + A) \times (A + A)$ and thus their total number does not exceed $|A + A|^2$. On the other hand, by the inclusion–exclusion principle the number of such sums in any fixed group U_j is at least

$$\rho_j := \tau^2 \binom{M}{2} - \sum_{\substack{\lambda_1,\dots,\lambda_4 \in U_j \\ \lambda_1 \ne \lambda_2,\ \lambda_3 \ne \lambda_4,\ \{\lambda_1,\lambda_2\} \ne \{\lambda_3,\lambda_4\}}} \bigl|\{z\colon\ z \in (\mathcal{A}_{\lambda_1} + \mathcal{A}_{\lambda_2}) \cap (\mathcal{A}_{\lambda_3} + \mathcal{A}_{\lambda_4})\}\bigr|$$

$$= \tau^2 \binom{M}{2} - \sum_{\substack{\lambda_1,\dots,\lambda_4 \in U_j \\ \lambda_1 \ne \lambda_2,\ \lambda_3 \ne \lambda_4,\ \{\lambda_1,\lambda_2\} \ne \{\lambda_3,\lambda_4\}}} \mathcal{E}(\lambda_1, \dots, \lambda_4). \tag{3.3}$$

Fix $\lambda_1, \dots, \lambda_4$ and prove that the quantity $\mathcal{E}(\lambda_1, \dots, \lambda_4)$ does not exceed σ.

Either all the numbers $\lambda_1, \dots, \lambda_4$ are distinct or two of them coincide but the other two are different and differ from the first two numbers. In any case one of these numbers differs from the other three. Without loss of generality, we can assume that it is λ_4. If

$$z = (z_1, z_2) \in (\mathcal{A}_{\lambda_1} + \mathcal{A}_{\lambda_2}) \cap (\mathcal{A}_{\lambda_3} + \mathcal{A}_{\lambda_4}),$$

then $z_1 = a_1 + a_2 = a_3 + a_4$ and $z_2 = \lambda_1 a_1 + \lambda_2 a_2 = \lambda_3 a_3 + \lambda_4 a_4$ for some $a_j \in A_{\lambda_j}$ $(j = 1, 2, 3, 4)$. It follows that

$$0 = \lambda_1 a_1 + \lambda_2 a_2 - \lambda_3 a_3 - \lambda_4 a_4 - \lambda_4(a_1 + a_2 - a_3 - a_4);$$

hence

$$(\lambda_1 - \lambda_4) a_1 + (\lambda_2 - \lambda_4) a_2 - (\lambda_3 - \lambda_4) a_3 = 0.$$

The number of triples (a_1, a_2, a_3) satisfying this equation is

$$\sigma\bigl((\lambda_1 - \lambda_4) A_{\lambda_1}, (\lambda_2 - \lambda_4) A_{\lambda_2}, (\lambda_4 - \lambda_3) A_{\lambda_3}\bigr) \le \sigma.$$

Returning to formula (3.3) and using the bound $\mathcal{E}(\lambda_1, \dots, \lambda_4) \le \sigma$, we get

$$\rho_j \ge \tau^2 \binom{M}{2} - \sigma M^4.$$

Hence

$$|A + A|^2 \ge \frac{|S'_\tau|}{2M} \left(\tau^2 \binom{M}{2} - \sigma M^4 \right) \ge \frac{|S'_\tau|}{2M} \left(\frac{\tau^2 M^2}{4} - \sigma M^4 \right).$$

Put $M = [\sqrt{\tau^2/(8\sigma)}\,]$. The required inequality $M \ge 2$ follows from the first condition of (3.1). In addition, if we have $M \le |S'_\tau|$, then

$$|A + A|^2 \ge \frac{M \tau^2 |S'_\tau|}{16} \ge \frac{\tau^3 |S'_\tau|}{128 \sqrt{\sigma}},$$

as required. Suppose that $M > |S'_\tau|$ and assume that inequality (3.2) fails. Then

$$|A + A|^2 < \frac{\tau^3 |S'_\tau|}{128 \sqrt{\sigma}} < \frac{\tau^3 M}{128 \sqrt{\sigma}} < \frac{\tau^4}{256 \sigma},$$

which contradicts the right inequality in (3.1). This concludes the proof of the lemma. □

Let us prove the first part of Theorem 4, which is our main result on sets with small product set. It is easy to see that the theorem below refines Solymosi's estimate (1.3) for $K \lesssim |A|^{1/4}$.

Theorem 11. *Let $A \subset \mathbb{R}$ be a finite set and $K \geq 1$ a real number. Suppose that $|AA| \leq K|A|$ or $|A/A| \leq K|A|$. Then*

$$\mathsf{E}^{\times}(A) \ll K^{1/4}|A|^{5/8}|A+A|^{3/2}\log^{3/4}|A|. \tag{3.4}$$

In particular,

$$|A+A| \gg |A|^{19/12}K^{-5/6}\log^{-1/2}|A|. \tag{3.5}$$

Proof. Estimate (3.5) follows from (3.4) via inequality (2.3); thus it is sufficient to prove (3.4).

Without loss of generality, we can suppose that $0 \notin A$. Let $L = \log|A|$. In the light of inequality (2.5) it is sufficient to check bound (3.4) just for $K^2 \leq L^2|A+A|^4|A|^{-5}$. From this bound and Solymosi's estimate (1.1), we derive

$$|A+A| \gg |A|^{11/8}L^{-1/2}. \tag{3.6}$$

Further, because of $d(A) \leq K^2$, we have

$$d(A) \ll L^2|A+A|^4|A|^{-5}. \tag{3.7}$$

Take a parameter $\Delta = CL^{3/4}(d(A))^{1/8}|A+A|^{3/2}|A|^{-11/8}$, where $C > 0$ is an absolute constant that will be chosen later. The constant C depends on another constant $C_1 > 0$ that will be chosen later as well. By (3.7)

$$d(A)|A| \ll L^{3/2}(d(A))^{1/4}|A+A|^3|A|^{-11/4},$$

and for sufficiently large C we have

$$C_1 d(A)|A| \leq \Delta^2. \tag{3.8}$$

Further,

$$\mathsf{E}^{\times}(A) = \sum_x |A \cap xA|^2 \leq \Delta|A|^2 + \sum_{j\geq 1}\ \sum_{x:\, 2^{j-1}\Delta < |A\cap xA| \leq 2^j\Delta} |A \cap xA|^2.$$

Note that in this formula, for $|A|$ large enough, it is sufficient to consider j satisfying the inequality

$$2^j \leq |A|^{11/8}|A+A|^{-3/4}. \tag{3.9}$$

Indeed, suppose on the contrary that $2^j > |A|^{11/8}|A+A|^{-3/4}$. Then by inequality (3.6) we get

$$\begin{aligned}|A| \geq 2^j\Delta &> CL^{3/4}(d(A))^{1/8}|A+A|^{3/2}|A|^{11/8}|A|^{-11/8}|A+A|^{-3/4}\\ &= CL^{3/4}(d(A))^{1/8}|A+A|^{3/4} \geq CL^{3/4}|A+A|^{3/4} \gg_C |A|^{33/32}L^{3/8},\end{aligned}$$

which is impossible for large $|A|$. Let $\tau = 2^{j-1}\Delta$ and $\sigma = \sigma(S_\tau)$. Take an arbitrary $\lambda \in S_\tau$. By the definition of the set S_τ, we get $d(A_\lambda) \leq |A|\tau^{-1}d(A)$. Applying Corollary 8 and using the definition of the set S_τ once again, we get

$$\sigma(\alpha_1 A_\lambda, \alpha_2 A_\lambda, \alpha_3 A_\lambda) \leq \sigma$$

for any nonzero numbers α_1, α_2, and α_3, where

$$\sigma \ll (|A|\tau^{-1}d(A))^{1/3}\tau^{5/3}, \tag{3.10}$$

and we can take $\sigma = M(d(A))^{1/3}|A|^{1/3}\tau^{4/3}$, where $M > 0$ is some constant. Put $C_1 = (32M)^3$ and choose the constant C in such a way that inequality (3.8) holds. It follows that

$$\Delta^{2/3} \ge 32M(d(A))^{1/3}|A|^{1/3}.$$

Hence for $\tau \ge \Delta$ we have

$$\tau^2 \ge 32M(d(A))^{1/3}|A|^{1/3}\tau^{4/3}.$$

Thus, the first condition of (3.1) is valid.

For any j and sufficiently large $|A|$, in view of inequality (3.9), we obtain

$$\tau = 2^{j-1}\Delta \le CL^{3/4}(d(A))^{1/8}|A+A|^{3/4} \le M^{3/8}|A|^{1/8}(d(A))^{1/8}|A+A|^{3/4}.$$

It follows that

$$\tau^2 \le M^{1/2}|A|^{1/6}(d(A))^{1/6}|A+A|\tau^{2/3},$$

and thus the second inequality of (3.1) holds.

So, both conditions in (3.1) for $\tau = 2^{j-1}\Delta$ are satisfied and we can apply inequality (3.2) of Lemma 10 to estimate the cardinality of the set $S_{2^{j-1}\Delta}$. Using (3.2) and (3.10), we get

$$\mathsf{E}^{\times}(A) \ll \Delta|A|^2 + \sum_{j\ge 1} \frac{(d(A))^{1/6}|A|^{1/6}|A+A|^2}{2^{j/3}\Delta^{1/3}} \ll \Delta|A|^2.$$

It follows that

$$\mathsf{E}^{\times}(A) \ll L^{3/4}(d(A))^{1/8}|A|^{5/8}|A+A|^{3/2} \le L^{3/4}K^{1/4}|A|^{5/8}|A+A|^{3/2}.$$

This completes the proof of the theorem. $\square$

In the next result we suppose that Solymosi's inequality (1.1) cannot be improved. We will show that this assumption implies a lower bound for the additive energy of a set and its product set AA.

Lemma 12. *Let $A \subset \mathbb{R}$, $0 \notin A$, be a finite set and $L \ge 1$ a real number. Suppose that*

$$|A+A|^2|A/A| \le L|A|^4. \tag{3.11}$$

Then there exist $\tau \ge \mathsf{E}^{\times}(A)/(2|A|^2)$ and sets $S'_\tau \subseteq S_\tau \subseteq A/A$, $|S_\tau|\tau^2 \gtrsim \mathsf{E}^{\times}(A)$, $|S'_\tau| \ge |S_\tau|/2$, such that for any element λ in S'_τ one has

$$\mathsf{E}^{+}(A_\lambda) \gtrsim \tau^3 L^{-4} \tag{3.12}$$

and

$$|A_\lambda/A_\lambda| \gtrsim \tau^2 L^{-16}. \tag{3.13}$$

Similarly, if

$$|A+A|^2|AA| \le L|A|^4, \tag{3.14}$$

then there exist $\tau \ge \mathsf{E}^{\times}(A)/(2|A|^2)$ and sets $S'_\tau \subseteq S_\tau \subseteq A/A$, $|S_\tau|\tau^2 \gtrsim \mathsf{E}^{\times}(A)$, $|S'_\tau| \ge |S_\tau|/2$, such that for any $\lambda \in S'_\tau$ one has (3.12) *and*

$$|A_\lambda A_\lambda| \gtrsim \tau^2 L^{-16}. \tag{3.15}$$

Proof. We only consider the set A/A because the arguments in the case of the set AA are similar. One can assume

$$L = \max\{1, |A+A|^2|A/A| \cdot |A|^{-4}\}.$$

By the Dirichlet principle there is a $\tau \geq \mathsf{E}^{\times}(A)/(2|A|^2)$ such that $|S_\tau|\tau^2 \gtrsim \mathsf{E}^{\times}(A)$. From (2.3), we have

$$|S_\tau|\tau^2 \gtrsim \frac{|A|^4}{|A/A|}. \tag{3.16}$$

If $|S_\tau| \geq 2$, then by S''_τ we denote the set of cardinality $[|S_\tau|/2]$ consisting of all $\lambda \in S_\tau$ with the minimal additive energy $\mathsf{E}^+(A_\lambda)$, and put $S'_\tau = S_\tau \setminus S''_\tau$. It is sufficient to check that for some $\lambda \in S''_\tau$ one has

$$\mathsf{E}^+(A_\lambda) \gtrsim \tau^3 L^{-4}. \tag{3.17}$$

In the case $|S_\tau| = 1$ we put $S'_\tau = S''_\tau = S_\tau$, and it is again sufficient to check inequality (3.17).

Put $\sigma := \max_{\lambda \in S''_\tau} \sqrt{2\tau \mathsf{E}^+(A_\lambda)}$. Bound (3.12) follows from the inequality

$$\sigma \gtrsim \tau^2 L^{-2}, \tag{3.18}$$

which is the aim of our proof.

By the Cauchy–Schwarz inequality, for any $\alpha, \beta \neq 0$ and arbitrary sets A_{λ_1}, A_{λ_2}, and A_{λ_3}, $\lambda_1, \lambda_2, \lambda_3 \in S''_\tau$, one has

$$\sigma(A_{\lambda_1}, \alpha A_{\lambda_2}, \beta A_{\lambda_3}) \leq |A_{\lambda_2}|^{1/2}(\mathsf{E}^+(A_{\lambda_1}, \beta A_{\lambda_3}))^{1/2} \leq (2\tau)^{1/2}\mathsf{E}^+(A_{\lambda_1})^{1/4}\mathsf{E}^+(A_{\lambda_3})^{1/4} \leq \sigma.$$

If both conditions (3.1) of Lemma 10 (with S''_τ instead of S'_τ) are satisfied, then we have

$$|A+A|^2 \geq \frac{\tau^3|S''_\tau|}{128\sqrt{\sigma}} \geq \frac{\tau^3|S_\tau|}{384\sqrt{\sigma}}.$$

Using condition (3.11), we obtain

$$\sigma^{1/2} \gg \frac{|S_\tau|\tau^3|A/A|}{|A|^4 L}. \tag{3.19}$$

Substituting inequality (3.16) into (3.19), we get (3.18).

If the first condition (3.1) does not hold, then we obtain (3.18) immediately. Suppose that the second condition (3.1) fails, that is, $\tau^2 > |A+A|\sqrt{\sigma}$. By inequality (2.3) for sums, we have a lower bound for σ, namely, $\sigma^2 \geq 2\tau^5|A+A|^{-1}$. But then

$$\tau^8 > |A+A|^4 \cdot 2\tau^5|A+A|^{-1},$$

which is impossible because, clearly, the parameter τ does not exceed the size of A.

Thus, we have proved inequality (3.12). Using Theorem 9, we obtain inequality (3.13). This concludes the proof of the lemma. ☐

Now let us obtain the second main result of the paper concerning the sets with small product set. It is easy to see that we improve inequality (1.3) for $K \lesssim |A|^{1/3}$.

Theorem 13. *Let $A \subset \mathbb{R}$ be a finite set and $K \geq 1$ a real number. Suppose that $|AA| \leq K|A|$ or $|A/A| \leq K|A|$. Then*

$$|A+A| \gtrsim |A|^{49/32}K^{-19/32}. \tag{3.20}$$

Proof. Consider the situation where $|A/A| \leq K|A|$. The case $|AA| \leq K|A|$ is similar. One can suppose that $0 \notin A$. Let us apply Lemma 12, where

$$L = \max\{1, |A+A|^2|A/A| \cdot |A|^{-4}\}.$$

Take any λ in S'_τ and use inequality (3.13) combined with the lower bound for τ. It yields

$$|A/A| \ge |A_\lambda/A_\lambda| \gtrsim \tau^2 L^{-16} \ge (\mathsf{E}^\times(A))^2 |A|^{-4} L^{-16}.$$

Further, because of (2.3), we have

$$|A/A| \gtrsim |A|^4 |A/A|^{-2} L^{-16}.$$

It follows that

$$L \gtrsim |A|^{1/4} |A/A|^{-3/16}.$$

After some simple calculations we obtain the result. □

Theorem 13 improves Theorem 11 for $K \gtrsim |A|^{5/23}$.

Let us obtain a result on the multiplicative energies of A/A and AA in the "critical case."

Proposition 14. *Let $A \subset \mathbb{R}$ be a finite set. If condition* (3.11) *holds, then*

$$\mathsf{E}^\times(A/A) \gtrsim \frac{(\mathsf{E}^\times(A))^3}{L^{32}|A|^4}. \tag{3.21}$$

If condition (3.14) *holds, then*

$$\mathsf{E}^\times(AA) \gtrsim \frac{(\mathsf{E}^\times(A))^3}{L^{32}|A|^4}. \tag{3.22}$$

Proof. Without loss of generality, we can suppose that $0 \notin A$. Let us begin with inequality (3.21). Put $\Pi = A/A$. Using Lemma 12, we find a number τ and a set S'_τ satisfying all implications of the lemma. By the Katz–Koester inclusion (see [4]), namely, $A_\lambda/A_\lambda \subseteq \Pi \cap \lambda\Pi$, we see that for all $\lambda \in S'_\tau$ the following holds:

$$|\Pi \cap \lambda\Pi| \ge |A_\lambda/A_\lambda| \gtrsim \tau^2 L^{-16}.$$

Hence

$$\sum_{\lambda \in S'_\tau} |\Pi \cap \lambda\Pi| \gtrsim L^{-16}\tau^2|S_\tau| \gtrsim L^{-16}\mathsf{E}^\times(A). \tag{3.23}$$

Using the last bound as well as the Cauchy–Schwarz inequality, we get (3.21).

Now put $\Pi' = AA$. Then, by the Katz–Koester inclusion, we have $A_\lambda A_\lambda \subseteq \Pi' \cap \lambda\Pi'$ and the previous arguments can be applied. This completes the proof of the proposition. □

Thus, if $|A/A| \lesssim |A|^{4/3}$ and $L \lesssim 1$, then inequality (3.21) and bound (2.3) imply $\mathsf{E}^\times(A/A) \gtrsim L^{-32}|A/A|^3 \gtrsim |A/A|^3$. In other words, the multiplicative energy of the set A/A is close to its maximum possible value. We use this observation in the proof of the final result of the paper.

Theorem 15. *Let $A \subset \mathbb{R}$ be an arbitrary set. Then for any $c < 1/20\,598$ one has*

$$\max\{|A+A|, |A/A|\} \gg |A|^{4/3+c} \tag{3.24}$$

and

$$\max\{|A+A|, |AA|\} \gg |A|^{4/3+c}. \tag{3.25}$$

Proof. We prove estimate (3.24) because inequality (3.25) can be obtained similarly. Without loss of generality, suppose that $0 \notin A$. Now assume that inequality (3.11) holds with some parameter L. Let also $|A/A|^3 \le L'|A|^4$. Our goal is to find lower bounds for the quantities L and L'. Using Lemma 12, we have $\tau \ge \mathsf{E}^\times(A)/(2|A|^2)$ and sets $S'_\tau \subseteq S_\tau \subseteq A/A$, $|S_\tau|\tau^2 \gtrsim \mathsf{E}^\times(A)$, $|S'_\tau| \gtrsim |S_\tau|$, such that for any element λ in S'_τ one has $|A_\lambda/A_\lambda| \gtrsim L^{-16}\tau^2$. Using this as well as the Katz–Koester inclusion, we obtain

$$\sum_{x\in A/A} |S'_\tau \cap x(A/A)| = \sum_{\lambda\in S'_\tau} |A/A\cap \lambda(A/A)| \ge \sum_{\lambda\in S'_\tau} |A_\lambda/A_\lambda| \gtrsim L^{-16}\tau^2|S_\tau|.$$

In view of the last bound and the Cauchy–Schwarz inequality, we get

$$|A/A|\,\mathsf{E}^\times(S'_\tau, A/A) = |A/A| \sum_{x\in AA/AA} |S'_\tau\cap x(A/A)|^2 \ge |A/A| \sum_{x\in A/A} |S'_\tau \cap x(A/A)|^2 \gtrsim L^{-32}\tau^4|S_\tau|^2.$$

Applying the Cauchy–Schwarz inequality once again, we obtain

$$\mathsf{E}^\times(S'_\tau) \gtrsim L^{-64}\tau^8|S_\tau|^4|A/A|^{-2}(\mathsf{E}^\times(A/A))^{-1} \gtrsim L^{-64}\mathsf{E}^\times(A)\tau^6|A/A|^{-5}|S_\tau|^3 = \eta|S_\tau|^3,$$

where $\eta = L^{-64}\mathsf{E}^\times(A)\tau^6|A/A|^{-5}$. We have

$$\begin{aligned}\eta &\gg L^{-64}\mathsf{E}^\times(A)\big(\mathsf{E}^\times(A)|A|^{-2}\big)^6|A/A|^{-5} = L^{-64}\mathsf{E}^\times(A)^7|A|^{-12}|A/A|^{-5}\\ &\ge L^{-64}\big(|A|^4|A/A|^{-1}\big)^7|A|^{-12}|A/A|^{-5} = L^{-64}|A|^{16}|A/A|^{-12} \ge L^{-64}(L')^{-4}.\end{aligned}$$

In other words,

$$\mathsf{E}^\times(S'_\tau) \gtrsim L^{-64}(L')^{-4}|S_\tau|^3.$$

By the Balog–Szemerédi–Gowers theorem from [1] (see also [6]), there exists a set $S''_\tau \subseteq S'_\tau$, $|S''_\tau| \gtrsim \eta|S_\tau|$, such that $|S''_\tau/S''_\tau| \lesssim \eta^{-4}|S''_\tau|^3|S'_\tau|^{-2}$. Since $S''_\tau\subseteq S_\tau$, we obtain

$$\sum_{a\in A} |A\cap aS''_\tau| = \sum_{\lambda\in S''_\tau} |A\cap\lambda A| \gg \tau|S''_\tau|$$

and hence there is an $a\in A$ such that for the set $A' := A\cap aS''_\tau$ one has

$$|A'| \gg \tau|S''_\tau|\cdot|A|^{-1}. \tag{3.26}$$

It follows that

$$d(A') \le \frac{|A'/S''_\tau|^2}{|A'|\cdot|S''_\tau|} \ll \frac{|S''_\tau/S''_\tau|^2|A|}{\tau|S''_\tau|^2} \lesssim \eta^{-8}\frac{|A|}{\tau}\,\frac{|S''_\tau|^4}{|S_\tau|^4}.$$

Using inequalities (2.3), (2.7) and the estimate for $d(A')$, we get

$$\begin{aligned}|A+A| &\ge |A'+A'| \gtrsim |A'|^{58/37}(d(A'))^{-21/37} \gtrsim (\tau|S''_\tau|\cdot|A|^{-1})^{58/37}\big(\eta^8\tau|A|^{-1}|S_\tau|^4|S''_\tau|^{-4}\big)^{21/37}\\ &\gtrsim |S_\tau|^{58/37}(\tau|A|^{-1})^{79/37}\eta^{168/37} \gtrsim (\mathsf{E}^\times(A))^{58/37}|A|^{-79/37}\eta^{168/37}\tau^{-1}\\ &= (\mathsf{E}^\times(A))^{58/37}|A|^{-79/37}\eta^{971/222}\big(L^{-64}\mathsf{E}^\times(A)|A/A|^{-5}\big)^{1/6}\\ &\gtrsim (\mathsf{E}^\times(A))^{58/37}|A|^{-79/37}(L^{-64}(L')^{-4})^{971/222}\big(L^{-64}\mathsf{E}^\times(A)|A/A|^{-5}\big)^{1/6}\\ &= L^{-10\,752/37}(\mathsf{E}^\times(A))^{385/222}|A|^{-79/37}(L')^{-1942/111}|A/A|^{-5/6}\\ &\gtrsim L^{-10\,752/37}(|A|^4|A/A|^{-1})^{385/222}|A|^{-79/37}(L')^{-1942/111}|A/A|^{-5/6}\\ &= L^{-10\,752/37}(L')^{-1942/111}|A|^{533/111}|A/A|^{-95/37}\\ &\gtrsim L^{-10\,752/37}(L')^{-1942/111}|A|^{533/111}\big((L')^{1/3}|A|^{4/3}\big)^{-95/37}\\ &= |A|^{51/37}L^{-10\,752/37}(L')^{-679/37}.\end{aligned}$$

The last estimate is greater than $|A|^{4/3}$ by a factor of some power of $|A|$. Easy calculations show that one can take any number less than $1/20\,598$ for the constant c. This concludes the proof. $\square$

Remark 16. It seems likely that the arguments of the proof of Theorem 15 allow one to slightly improve the lower bound for the size of $A+A$ in Theorem 13 when $K \lesssim |A|^{1/3}$. We have not performed such calculations.

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