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# On Lebesgue Constants of Local Parabolic Splines

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**Abstract**—Lebesgue constants (the norms of linear operators from $C$ to $C$) are calculated exactly for local parabolic splines with an arbitrary arrangement of knots, which were constructed by the second author in 2005, and for N.P. Korneichuk's local parabolic splines, which are exact on quadratic functions. Both constants are smaller than the constants for interpolating parabolic splines.

**Keywords:** Lebesgue constants, local parabolic splines, arbitrary knots.

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## INTRODUCTION

For a function $f\colon \mathbb{R} \to \mathbb{R}$, we consider a linear method $S(x) = S(f,x)$ of approximating this function on the axis $\mathbb{R}$ by polynomial splines of minimal defect of degree $r$ (of order $r+1$) with arbitrary knots. One of the stability characteristics of the method $S$ is the behavior of the uniform norm of the operator $S$ (as an operator acting from the space $C = C(\mathbb{R})$ of functions continuous on the axis to $C$); this is the value

$$L = \|S\|_C^C = \sup_{\|f\|_C \le 1} \|S(f,\cdot)\|_C.$$

The number $L$ is called the Lebesgue constant of the method $S$. The smaller this constant, the greater the stability of the method with respect to a change in approximation conditions.

Various issues related to Lebesgue constants for interpolating polynomial splines (and their generalizations) were studied by Schurer and Cheney [1], Richards [2], Zhensykbaev [3], Tzimbalario [4], Morsche [5], Subbotin and Telyakovskii [6], Kim [8, 9], and many others. A fundamental result in this area belongs to Subbotin and Telyakovskii [6], who proved that Lebesgue constants $L$ of interpolating $N$-periodic polynomial splines $S_{r,N}(x)$ of degree $r$ with uniform knots have asymptotic behavior

$$L = \|S_{r,N}\|_C^C = \frac{2}{\pi}\ln(\min(N,r)) + O(1), \tag{0.1}$$

where the term $O(1)$ is independent of $N$ and $r$. Note that it is not always possible to calculate the constants $L$ exactly, and even finding orders of $L$ in different parameters is of great interest. It is natural to pose the question of studying Lebesgue constants for noninterpolating polynomial splines (and their generalizations), which approximate (in some sense) continuous functions on a closed interval of the real line $\mathbb{R}$ and on the whole real axis. Let us give more exact formulations.

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Let values $\{y_j\}_{j\in\mathbb{Z}}$ of some function $f(x)$ be given at the nodes of a uniform grid $\{jh\}_{j\in\mathbb{Z}}$ of the real line with step $h>0$: $y_j=f(jh)$, $j\in\mathbb{Z}$. Denote by $\widetilde{B}_{r+1}$ the normalized (in $C$) polynomial basis spline ($B$-spline) of degree $r$ (of order $r+1$) with support $\operatorname{supp}\widetilde{B}_{r+1}=[0;(r+1)h]$ and uniform knots $0$, $h$, $2h$, …, $(r+1)h$ (see, for example, [10, Ch. 1]). In 1975, Lyche and Schumaker [11] (see also [10, Ch. 9]) constructed for any function $f:\ \mathbb{R}\to\mathbb{R}$ local polynomial splines of order $(r+1)$ of the form

$$S_{r+1}(x)=S_{r+1}(f,x)=\sum_{j\in\mathbb{Z}}\sum_{s=-k}\gamma_s f((j+s)h)\widetilde{B}_{r+1}\Big(x-jh-\frac{r+1}{2}h\Big)\qquad(x\in\mathbb{R}),\tag{0.2}$$

where $k=[r/2]$ and the real coefficients $\gamma_s$ were chosen from the condition that the formula $S_{r+1}(f,x)=f(x)$ is exact for algebraic polynomials of degree $r$. It was proved that the choice of the coefficients is unique. A local spline of form (0.2) is not interpolating, since $S_{r+1}(jh)\neq y_j$ ($j\in\mathbb{Z}$), and its value at a fixed point $x\in\mathbb{R}$ depends only on several values $y_j=f(jh)$ defined by the supports of the shifts of the $B$-spline that contain the point $x$. Lyche and Schumaker's results [11] were developed and generalized in different directions (see, for example, references in the authors' paper [12]). Methods of local spline approximation (with uniform and nonuniform knots) became an effective tool for solving various problems in function approximation theory and numerical analysis as a useful alternative for the interpolation method. It turned out (see, for example, [10, 13]) that the orders of approximation by local polynomial splines of order $(r+1)$ with uniform knots of the classical Sobolev classes $W^r_\infty$ of $r$ times almost everywhere differentiable functions in the uniform metric coincide with the orders of approximation of these classes by the corresponding interpolation splines and are equal to the orders of the Kolmogorov widths of these function classes. Recall that the function class $W^r_\infty$ is defined as follows:

$$W^r_\infty=\{f\colon f^{(r-1)}\in AC,\ \|f^{(r)}\|_{L_\infty}\le 1\}.$$

Here, $AC$ is the class of locally absolutely continuous functions and $\|g\|_{L_\infty}=\|g\|_\infty=\operatorname*{ess\,sup}_{x\in\mathbb{R}}|g(x)|$. In addition to the simplicity of construction, methods of local spline approximation (in contrast to interpolation methods) possess useful shape-preserving and smoothing properties (see, for example, [14–16] and references therein). It is natural to consider the question of comparing local and interpolating splines in the sense of their stability to a change in the initial data (i.e., the numbers $y_j=f(jh)$). It is interesting to find which of these splines have smaller Lebesgue constants. Let us pose the problem of calculating (or estimating) the Lebesgue constants

$$L=\|S_{r+1}\|_C^C=\sup_{\|f\|_C\le 1}\|S_{r+1}(f,\cdot)\|_C$$

for Lyche and Schumaker's local polynomial splines of form (0.2) $S_{r+1}(f,x)$ [11]. At present, no approaches to finding these values in the case of arbitrary $r$ are known. First, it would be desirable to obtain an asymptotic equality of type (0.1).

For parabolic splines (i.e., for $r=2$) preserving quadratic function, the following equality will be shown later (Theorem 1):

$$\|S_3\|_C^C=1.25.$$

Kim [8] proved that the Lebesgue constant for interpolating parabolic splines with a knot grid shifted half-step (i.e., by $h/2$) with respect to the grid of interpolation nodes is $L=\sqrt{2}\approx 1.41$. The comparison of these results shows that, in the question of stability, local parabolic splines of form (0.2) (their coefficients $\gamma_{-1} = -1/8$, $\gamma_2 = 5/4$, and $\gamma_3 = -1/8$ were found by Korneichuk [17]) have an advantage over corresponding interpolating splines.

On the class of functions $W^2_\infty$ given on a uniform grid $\{jh\}_{j\in\mathbb{Z}}$, Subbotin [14] constructed in 1993 one more (noninterpolating) method of local parabolic approximation, which used parabolic splines with additional knots and preserved certain locally geometric properties (monotonicity and convexity) of the initial data $y_j = f(jh)$ $(j \in \mathbb{Z})$. In the periodic case, this method turned out to be extremal in the sense of Kolmogorov's and Konovalov's widths. In 2005, one of the authors of the present paper Shevaldin [15] extended this method to parabolic splines with an arbitrary arrangement of knots. We have proved (see Theorem 2 below) that, for any grid of spline knots, Lebesgue constants of such splines are equal to 1.

## 1. KORNEICHUK'S SPLINES

Let $B_3(x) = \widetilde{B}_3(x + 3h/2)$ (see, for example, [10]) be the normalized parabolic $B$-spline with uniform knots $-3h/2$, $-h/2$, $h/2$, and $3h/2$. It can be written in the form

$$B_3(x) = \frac{1}{2h^2}\begin{cases} \Big(x + \dfrac{3h}{2}\Big)^2, & x \in \Big[-\dfrac{3h}{2}; -\dfrac{h}{2}\Big], \\ \dfrac{3h^2}{2} - 2x^2, & x \in \Big[-\dfrac{h}{2}; \dfrac{h}{2}\Big], \\ \Big(\dfrac{3h}{2} - x\Big)^2, & x \in \Big[\dfrac{h}{2}; \dfrac{3h}{2}\Big], \\ 0, & x \notin \Big[-\dfrac{3h}{2}; \dfrac{3h}{2}\Big]. \end{cases} \tag{1.1}$$

For a function $f\colon \mathbb{R} \to \mathbb{R}$, define $y_j = f(jh)$ $(j \in \mathbb{Z})$ and consider the sequence of linear functionals

$$I_j = \Big(-\frac{1}{8}\Big)y_{j-1} + \frac{5}{4}y_j + \Big(-\frac{1}{8}\Big)y_{j+1} \quad (j \in \mathbb{Z}). \tag{1.2}$$

Consider the local parabolic spline

$$S_3(x) = S_3(f, x) = \sum_{j\in\mathbb{Z}} I_j\; B_3(x - jh) \quad (x \in \mathbb{R}). \tag{1.3}$$

Formula (1.3) is a special case of formula (0.2) for $r = 2$. Such local splines were studied by Korneichuk [17]. He proved that, for any quadratic polynomial $p_2(x) \in P_2$,

$$S_3(p_2(\cdot), x) = p_2(x) \quad (x \in \mathbb{R}). \tag{1.4}$$

Here, $P_2$ is the space of algebraic polynomials of second degree with real coefficients. Equality (1.4) means that local approximation scheme (1.2), (1.3) preserves the space $P_2$. It is easy to verify that $S_3(jh) \neq y_j$ $(j \in \mathbb{Z})$; i.e., the constructed local splines are not interpolating. In addition, Korneichuk proved that

$$\sup_{f\in W^2_\infty} \|f - S_3\|_C = \frac{9}{32}h^2. \tag{1.5}$$

Let

$$d_m(W^2_\infty)_C = \inf_{\dim M_m \le m}\ \sup_{f\in W^2_\infty}\ \inf_{g\in M_m} \|f - g\|_C \tag{1.6}$$

be the Kolmogorov width of order $m$ of the function class $W^2_\infty = W^2_\infty(\mathbb{R})$. It is known (see, for example, [13]) that, for 1-periodic functions, the equality

$$d_{2n-1}(W^2_\infty)_C = d_{2n}(W^2_\infty)_C = \frac{h^2}{8} \qquad \Big(h = \frac{1}{n}\Big)$$

holds, and an extremal subspace $M_m$ (which implements the outer infimum in (1.6)) for even $m = 2n$ is the space of interpolating parabolic splines with uniform knots (their interpolation nodes are shifted half-step with respect to spline knots) and Subbotin's space of local splines [14]. Equality (1.5) means that, though Korneichuk's splines approximate the class of functions $W^2_\infty$ with the same order $h^2$, they do not form an extremal subspace (in the sense of Kolmogorov widths). Consider now the Lebesgue constant of Korneichuk's method [17]. We are interested in the answer to the following question. Let all the numbers $y_j$ be such that $|y_j| \le 1$ $(j \in \mathbb{Z})$. In this case, what is the numerical value of the expression

$$L_1 = \max_{x\in\mathbb{R}} \big\{|S_3(x)|\colon |y_j| \le 1 \ (j \in \mathbb{Z})\big\}\ ?$$

**Theorem 1.** *The following equality hods*:

$$L_1 = 1.25.$$

**Proof.** For $x \in [(l-1/2)h; (l+1/2)h]$ $(l \in \mathbb{Z})$, the spline $S_3(x)$ defined by formulas (1.2) and (1.3), in view of equality (1.1), can be represented in the form

$$S_3(x) = \frac{1}{16h^2}\Big[(-y_{l-2} + 10y_{l-1} - y_l)\Big(t - \frac{h}{2}\Big)^2 + (-y_{l-1} + 10y_l - y_{l+1})\Big(\frac{3h^2}{2} - 2t^2\Big)$$
$$+ (-y_l + 10y_{l+1} - y_{l+2})\Big(t + \frac{h}{2}\Big)^2\Big] = \frac{1}{16h^2}\sum_{s=1}^{5} y_{l+s-1}q_s(t), \tag{1.7}$$

where $t = x - lh \in [-h/2; h/2]$ and

$$q_1(t) = -\Big(t - \frac{h}{2}\Big)^2, \qquad q_2(t) = 12t^2 - 10th + h^2, \qquad q_3(t) = -22t^2 + \frac{29}{2}h^2,$$
$$q_4(t) = 12t^2 + 10th + h^2, \qquad q_5(t) = -\Big(t + \frac{h}{2}\Big)^2.$$

Without loss of generality, we can assume that $l = 0$. From (1.7), we have

$$|S_3(x)| \le \frac{1}{16h^2} q(t) \max_{-2\le j\le 2} |y_j|, \tag{1.8}$$

where $q(t) = \sum_{s=1}^{5} |q_s(t)|$. Analyzing zeros of the quadratic polynomials $q_s(t)$ $(s = \overline{1,5})$, we find that

$$q(t) = \begin{cases} 5(-4t^2 - 4th + 3h^2), & -\dfrac{h}{2} \le t \le \dfrac{-5+\sqrt{13}}{12}h, \\[2ex] 4t^2 + 17h^2, & \dfrac{-5+\sqrt{13}}{12}h \le t \le \dfrac{5-\sqrt{13}}{12}h, \\[2ex] 5(-4t^2 + 4th + 3h^2), & \dfrac{5-\sqrt{13}}{12}h \le t \le \dfrac{h}{2}. \end{cases} \tag{1.9}$$

From (1.9), we derive the equality

$$\max_{t\in[-h/2;h/2]} q(t) = q\Big(-\frac{h}{2}\Big) = q\Big(\frac{h}{2}\Big) = 20h^2; \tag{1.10}$$

moreover, the equality in (1.8) is realized for $y_{-2} = y_1 = y_2 = -1$ and $y_{-1} = y_0 = 1$ in the case $t = -h/2$ and for $y_{-2} = y_{-1} = y_2 = -1$ and $y_0 = y_1 = 1$ in the case $t = h/2$. The statement of Theorem 1 follows from (1.8)–(1.10).

## 2. LOCAL PARABOLIC SPLINES WITH ARBITRARY KNOTS PRESERVING LINEAR FUNCTIONS

Consider on the axis $\mathbb{R}$ the node grid $\ldots < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \ldots$, which is infinite on either side, with $h_j = x_{j+1} - x_j$ and $x_{j+1/2} = 0.5(x_j + x_{j+1})$ $(j \in \mathbb{Z})$. For a function $f \in W^2_\infty(\mathbb{R})$, we construct a second-order divided difference using the values of the function $y = f(x)$ at the points $x_j$, $x_{j+1}$, and $x_{j+2}$:

$$[y_j, y_{j+1}, y_{j+2}] = f[x_j, x_{j+1}, x_{j+2}] = \frac{y_{j+2}}{h_{j+1}(h_{j+1}+h_j)} - \frac{y_{j+1}}{h_{j+1}h_j} + \frac{y_j}{h_j(h_{j+1}+h_j)} \qquad (j \in \mathbb{Z}).$$

A function $f \in W^2_\infty(\mathbb{R})$ is associated (see [15]) with the local parabolic spline

$$\widetilde{S}_3(x) = \widetilde{S}_3(f,x) = f(x_j) + \frac{h_{j-1}h_j}{4} f[x_{j-1}, x_j, x_{j+1}] + \frac{f(x_{j+1}) - f(x_{j-1})}{h_j + h_{j-1}}(x - x_j)$$
$$+ \frac{h_{j-1}}{h_j}(x - x_j)^2 f[x_{j-1}, x_j, x_{j+1}] + \Big(\frac{h_{j+1}}{h_j} f[x_j, x_{j+1}, x_{j+2}] - \frac{h_{j-1}}{h_j} f[x_{j-1}, x_j, x_{j+1}]\Big)$$
$$\times (x - x_{j+1/2})^2_+, \qquad x \in [x_j; x_{j+1}] \qquad (j \in \mathbb{Z}), \tag{2.1}$$

where $(x - x_{j+1/2})^2_+ = \max\{0; (x - x_{j+1/2})\}^2$. For a uniform node grid $h_j = h$ $(j \in \mathbb{Z})$, spline (2.1) was constructed by Subbotin [14]. The spline $\widetilde{S}_3(x)$ has shape-preserving and smoothing properties (see [15, Theorem 1]) and preserves linear functions. On the class of functions $W^2_\infty(\mathbb{R})$, the approximation errors

$$\sup_{f\in W^2_\infty} \|f - \widetilde{S}_3\|_C, \qquad \sup_{f\in W^2_\infty} \|f' - \widetilde{S}_3'\|_C$$

were calculated for this spline in [15]. In the case of a uniform grid, Subbotin showed [14] that the first value is $h^2/8$ and the second is $h/2$.

In the present paper, we study the Lebesgue constant

$$L_2 = \max_{x\in\mathbb{R}} \big\{|\widetilde{S}_3(x)| \colon |y_j| \le 1 \ (j \in \mathbb{Z})\big\}.$$

**Theorem 2.** *The following equality holds*:

$$L_2 = 1.$$

**Proof.** Collecting similar terms in(2.1) for $x \in [x_j; x_{j+1/2}]$, we obtain

$$\widetilde{S}_3(x) = y_{j-1}r_1(x) + y_j r_2(x) + y_{j+1} r_3(x), \tag{2.2}$$

where

$$r_1(x) = \frac{1}{h_{j-1}+h_j}\Big(\frac{h_j}{4} - (x-x_j) + \frac{(x-x_j)^2}{h_j}\Big), \qquad r_2(x) = \frac{3}{4} - \frac{(x-x_j)^2}{h_j^2},$$

$$r_3(x) = \frac{1}{h_{j-1}+h_j}\Big(\frac{h_{j-1}}{4} + x - x_j + \frac{(x-x_j)^2 h_{j-1}}{h_j^2}\Big).$$

It follows from (2.2) that

$$|\widetilde{S}_3(x)| \le \max_{j-1\le s\le j+1} |y_s|\big\{|r_1(x)| + |r_2(x)| + |r_3(x)|\big\}. \tag{2.3}$$

Let us show that the polynomials $r_1(x)$, $r_2(x)$, and $r_3(x)$ are nonnegative for $x \in [x_j; x_{j+1/2}]$. Indeed, $r_1(x_j) > 0$, $r_1(x_{j+1/2}) = 0$, and $r_1'(x_{j+1/2}) = 0$; hence, $r_1(x) \ge 0$ for $x \in [x_j; x_{j+1/2}]$. The coefficient at $x^2$ in the quadratic polynomial $r_2(x)$ is negative, $r_2(x_j) = 3/4$, and $r_2(x_{j+1/2}) = 1/2$; then, $r_2(x) > 0$ for $x \in [x_j; x_{j+1/2}]$. Further, $r_3(x_j) > 0$ and $r_3'(x) > 0$ for $x > x_j$. Consequently, $r_3(x) > 0$ for $x \in [x_j; x_{j+1/2}]$. From inequality (2.3), we derive the estimate

$$|\widetilde{S}_3(x)| \le \max_{j-1\le s\le j+1} |y_s|\big\{r_1(x) + r_2(x) + r_3(x)\big\}. \tag{2.4}$$

After simple transformations, we find that

$$r_1(x) + r_2(x) + r_3(x) = 1.$$

Since $|y_j| \le 1$ for all $j \in \mathbb{Z}$, we derive the following estimate from (2.4) for $x \in [x_j; x_{j+1/2}]$:

$$|\widetilde{S}_3(x)| \le 1;$$

the equality here is realized for $y_j = 1$ ($j = s-1, s, s+1$). Since formula (2.1) for the spline $\widetilde{S}_3(x)$ on the interval $[x_j; x_{j+1}]$ is symmetric with respect to the middle of this interval (i.e., the point $x = x_{j+1/2}$), we have the inequality $|\widetilde{S}_3(x)| \le 1$ for $x \in [x_{j+1/2}; x_{j+1}]$. Theorem 2 is proved.

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