ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2015, Vol. 289, pp. 227–285. © Pleiades Publishing, Ltd., 2015.
Published in Russian in Trudy Matematicheskogo Instituta imeni V.A. Steklova, 2015, Vol. 289, pp. 242–303.

Adiabatic Limit in the Ginzburg–Landau and Seiberg–Witten Equations

A. G. Sergeev[a]

Received January 15, 2015

Abstract—Hyperbolic Ginzburg–Landau equations arise in gauge field theory as the Euler–Lagrange equations for the $(2+1)$-dimensional Abelian Higgs model. The moduli space of their static solutions, called vortices, was described by Taubes; however, little is known about the moduli space of dynamic solutions. Manton proposed to study dynamic solutions with small kinetic energy with the help of the adiabatic limit by introducing the "slow time" on solution trajectories. In this limit the dynamic solutions converge to geodesics in the space of vortices with respect to the metric generated by the kinetic energy functional. So, the original equations reduce to Euler geodesic equations, and by solving them one can describe the behavior of slowly moving dynamic solutions. It turns out that this procedure has a 4-dimensional analog. Namely, for the Seiberg–Witten equations on 4-dimensional symplectic manifolds it is possible to introduce an analog of the adiabatic limit. In this limit, solutions of the Seiberg–Witten equations reduce to families of vortices in normal planes to pseudoholomorphic curves, which can be considered as complex analogs of geodesics parameterized by "complex time." The study of the adiabatic limit for the equations indicated in the title is the main content of this paper.

DOI: 10.1134/S008154381504015X

[a] Steklov Mathematical Institute of Russian Academy of Sciences, ul. Gubkina 8, Moscow, 119991 Russia.
E-mail address: sergeev@mi.ras.ru

FOREWORD

In their papers [19, 20], Seiberg and Witten proposed a new kind of invariants of symplectic 4-dimensional manifolds. The invariants are constructed by using the equations which are now called the *Seiberg–Witten equations*. In contrast with the well-known *Yang–Mills duality equations*, the Seiberg–Witten equations are Abelian but, just as the duality equations, they can be obtained from the *supersymmetric Yang–Mills theory* in a certain limit. (Namely, the duality equations correspond to the ultraviolet limit, while the Seiberg–Witten equations arise in the infrared limit of this theory.) So one can expect that any information obtained from the duality equations can be also extracted from the Seiberg–Witten equations with less effort.

Moreover, it turned out that the new invariants of symplectic 4-dimensional manifolds introduced by Seiberg and Witten are closely related to their *Gromov invariant*, which counts the number of pseudoholomorphic curves in a given homology class. Taubes has even proposed the following mnemonic "equation":

$$\mathrm{Gr} = \mathrm{SW},$$

which expresses a simple relation between the Seiberg–Witten and Gromov invariants of a symplectic 4-dimensional manifold. This *Taubes "equation"* is based on a remarkable construction that associates a pseudoholomorphic curve with a solution of the Seiberg–Witten equations. This pseudoholomorphic curve arises in the so-called *adiabatic limit* of the Seiberg–Witten equations.

It turns out that the construction has a nontrivial 3-dimensional analog related to the adiabatic limit in the *hyperbolic Ginzburg–Landau equations* arising in the *Abelian* $(2+1)$*-dimensional Higgs model*. Moreover, we show that the adiabatic limit in the Seiberg–Witten equations can be regarded as a complex version of the adiabatic limit in the hyperbolic Ginzburg–Landau equations.

The study of the adiabatic limit is the main subject of this paper.

We start our analysis from the two-dimensional case considered in Section 1. The section begins with the physical introduction, in which we explain the relation between the vortex equations and superconductivity theory.

Then we turn to the vortex equations on the complex plane. The main result of this part is the *Taubes theorem* that gives a complete description of the moduli space of solutions of vortex equations on the complex plane. The proof of this theorem is based on a reduction to the case of a *Liouville-type equation* whose solvability was studied by Kazdan and Warner [8].

In the second part of Section 1 (Subsections 1.4 and 1.5) we extend the results to vortex equations on compact Riemann surfaces. In contrast to the case of the complex plane, these equations can be solved only under an additional condition analogous to the stability condition for holomorphic bundles. The description of the moduli space of solutions of vortex equations in this case is given by the *Bradlow theorem*.

In Section 2 we turn to dimension 3, where the third variable is treated either as another space variable (this variant of the theory describes the so-called *Abrikosov strings*) or as the time variable (in this version we obtain a description of the vortex dynamics).

We mainly pay attention to the second case, in which the central place is occupied by the construction of the *adiabatic limit*. The vortex dynamics is described by the *hyperbolic Ginzburg–Landau equations*, which are not invariant under changes of scale unlike the conformally invariant Yang–Mills equations. For this reason, in order to extract useful information from these equations, one should consider the limit of these equations as the scale parameter tends to infinity. Taking this limit, one should simultaneously change the time scaling by introducing the so-called slow time. Such a limit is called *adiabatic*.

The Ginzburg–Landau equations in this limit turn into the *adiabatic equations*. Their solutions, called *adiabatic trajectories*, are given by the geodesics on the moduli space of vortex solutions with respect to the metric generated by the kinetic energy functional. Solving the Euler equation for these geodesics, we can approximately describe solutions of the original Ginzburg–Landau equations with small kinetic energy. In Subsection 2.2 we give some particular examples of dynamic solutions: scattering of vortices after a head-on collision, system of two periodic vortices on the sphere, and so on.

However, there are situations in which the adiabatic limit cannot be used. This applies, in particular, to the "vortex–antivortex" system. In the static case, according to the Taubes theorem, any solution of the Ginzburg–Landau equations consists either only of vortices or only of antivortices. Hence, a static solution of the vortex–antivortex type does not exist. However, can a *dynamic* solution of this type exist (such a solution occurs, for example, in hydrodynamics)? Unfortunately, an answer to this question, which still remains open, cannot be given in the framework of the adiabatic approach (such a solution, if it exists, should have the velocity exceeding a certain nonzero threshold).

Section 3 is a digression on Clifford algebras and spinor geometry. This subject is considered in detail in the book by Lawson and Michelsohn [12]. In this section, in particular, we introduce the notion of Spin^c-*structure*, which plays an important role in the next section devoted to the theory of Seiberg–Witten equations.

We conclude our paper with Section 4 dealing with dimension 4, in which we study the *Seiberg–Witten equations* on compact 4-dimensional Riemannian manifolds. A model example is given by the Seiberg–Witten equations on a compact Kähler surface. In this case it is possible to obtain a complete description of the moduli space of solutions of these equations with the help of the already mentioned Kazdan–Warner theorem. As in the Bradlow theorem, the solvability of these equations takes place under an additional condition similar to that in the Bradlow theorem. Under this condition the moduli space of Seiberg–Witten solutions coincides with the space of holomorphic curves on the surface under consideration that lie in a given topological class.

However, the main content of Section 4 is the study of the Seiberg–Witten equations on a compact 4-dimensional *symplectic* manifold. A key point is Taubes's construction of the adiabatic limit (note, however, that he did not use this term). Namely, with a sequence of solutions of the Seiberg–Witten equations depending on a scale parameter, Taubes associated a pseudoholomorphic curve (replacing the holomorphic curve in the Kähler case), which can be regarded as a complex analog of the adiabatic trajectory in the $(2+1)$-dimensional case. A parameter along this limiting curve plays the role of the "*complex time*." The Seiberg–Witten equations in this limit reduce to a family of vortex equations defined in the normal planes to the limiting pseudoholomorphic curve. The limiting curve and a family of vortex solutions along this curve must satisfy the *adiabatic equation*, which is deduced in Subsection 4.5 from arguments similar to those in the $(2+1)$-dimensional case. Note that Taubes obtained a similar equation from other considerations (discussed in Subsection 4.6.1).

Conversely, if we have a pseudoholomorphic curve and a family of vortex equations in the normal planes that satisfy the adiabatic equation, then from these data we can reconstruct a solution of the Seiberg–Witten equations that tends in the adiabatic limit to the original pseudoholomorphic curve and given family of vortex solutions. This is the *converse Taubes construction*, which is briefly presented in Subsection 4.6.

It turns out that even the one-dimensional variant of the adiabatic limit makes sense. This variant, proposed by Andrei Domrin, is presented in the supplement.

A few words on how this text has appeared. In 2002, in Nagoya University, I delivered a lecture course entitled "Vortices and Seiberg–Witten Equations" (see [21]). This course was written down by Yuuji Tanaka and published in the Nagoya University Lecture Series.

While preparing this text, I first intended to simply translate the Nagoya lectures into Russian and publish them at the Steklov Mathematical Institute. However, during the preparation work I have realized that the original text needs to be substantially revised. The necessity of such a revision has arisen for two reasons. First of all, since the first publication, several new important results have appeared that are directly related to the topic in question (such as the justification of the adiabatic principle for the hyperbolic Ginzburg–Landau equations) but are not mentioned in the Nagoya lectures. Second, it has become clear that some parts of the original text (especially those related to the adiabatic limit in the 4-dimensional case) should be presented in a completely different way. In particular, a central place in the new text is occupied by the notion of the *adiabatic limit* (which is reflected in the new title).

1. DIMENSION TWO: VORTEX EQUATIONS

In this section we study the vortex equations arising in superconductivity theory. In Subsection 1.1 we give necessary information from this theory (book [11] can serve as a general reference here). In Subsection 1.2 we introduce the vortex equations on the complex plane. A full description of the moduli space of their solutions is given by the Taubes theorems presented in Subsection 1.3. In Subsection 1.4 we extend the constructed theory to compact Riemann surfaces; in particular, we discuss obstructions to the solvability of the vortex equations on such surfaces. The moduli space of solutions of vortex equations is described by the Bradlow theorem proved in Subsection 1.5.

1.1. Physical introduction: Superconductivity and Ginzburg–Landau Lagrangian.

1.1.1. *Superconductivity.* Superconductivity was discovered in 1911 by Kamerlingh Onnes while he was studying the variation of the mercury resistance under low temperatures. It was known that the resistance of metals should decrease when they are cooled down. However, to Kamerlingh Onnes's surprise, this resistance has completely disappeared at temperature 4.15 K. This phenomenon was called *superconductivity*, and, as it turned out during subsequent investigations, many metals and alloys have the same property under temperatures close to the absolute zero.

According to the modern superconductivity theory proposed by Bardeen, Cooper, and Schriffer, the superconductivity phenomenon is explained by the fact that under very low temperatures it becomes more energetically profitable for free electrons to unite in pairs and form a new kind of quasiparticles, called *Cooper pairs*. Unlike fermionic electrons, Cooper pairs are *bosons* with charge equal to the double charge of the electron and with zero spin. Precisely their current is superconducting.

1.1.2. *Flux tubes and Abrikosov strings.* Let us place a superconductor in an external magnetic field $\vec{H}$. Another remarkable property of superconductors was discovered in 1933 by Meissner and was called the *Meissner effect*. According to it, the magnetic field $\vec{H}$ is pushed away from the superconductor; i.e., $\vec{H}$ vanishes inside the superconductor. (This effect is used as a practical tool for detecting superconductivity.)

If we increase the level of the external magnetic field, then at some critical value H^1_{cr} the superconductivity will start to break down and the magnetic field will start to penetrate into the body of the superconductor. More precisely, certain tube-like zones of intermediate conductivity will start to appear inside the superconductor, which are called *flux tubes*. They are oriented along the direction of the external magnetic field, and along their axes, called *Abrikosov strings*, the conductivity is already normal. In the other part of the tubes, i.e., away from the axes, the conductivity has an intermediate character, while outside the tubes the superconductivity still persists. If we increase the level of the magnetic field further on, then the number of the tubes and their diameters will also increase until, at the second critical value H^2_{cr}, these tubes fill up the whole body of the superconductor, thus making it into a normal conductor.

1.1.3. *Ginzburg–Landau Lagrangian.* In order to describe mathematically the intermediate state of the superconductor that arises inside the flux tubes, consider the following idealized model in which the superconductor coincides with the whole space $\mathbb{R}^3_{(x_1,x_2,x_3)}$ with coordinates (x_1,x_2,x_3) and is in the intermediate state everywhere outside a finite number of Abrikosov strings, while the pure superconductivity persists only at infinity.

Consider the horizontal section of our superconductor by the plane $\mathbb{R}^2_{(x_1,x_2)}$, assuming that the external magnetic field $\vec{H}$ is directed along the axis (x_3).

The *Ginzburg–Landau Lagrangian* defined on this plane is given by the formula

$$\mathcal{L}(A,\Phi) = |F_A|^2 + |d_A\Phi|^2 + \frac{\lambda}{4}(1-|\Phi|^2)^2. \tag{1.1}$$

From the physical point of view, the variable A represents the *electromagnetic vector potential*, while mathematically it is a U(1)-connection on $\mathbb{R}^2_{(x_1,x_2)}$ given by the 1-form

$$A = A_1\,dx_1 + A_2\,dx_2$$

whose coefficients are smooth pure imaginary functions on $\mathbb{R}^2_{(x_1,x_2)}$.

The curvature F_A of this connection is given by the 2-form

$$F_A = dA = \sum_{i,j=1}^{2} F_{ij}\, dx_i \wedge dx_j = \widetilde{F}_{12}\, dx_1 \wedge dx_2$$

with $F_{ij} = \partial_i A_j - \partial_j A_i$ and $\partial_i := \partial/\partial x_i$. Physically, it is interpreted as the *stress tensor of the electromagnetic field* (or simply as the *electromagnetic field*), so the term $|F_A|^2$ is nothing else but the *Maxwell Lagrangian.*

The second variable Φ is called the *Higgs field* or *order parameter* and is a smooth complex-valued function $\Phi = \Phi_1 + i\Phi_2$ on $\mathbb{R}^2_{(x_1,x_2)}$. From the physical point of view, it can be considered as a scalar field interacting with the electromagnetic field defined by the potential A and is interpreted as the *wave function of Cooper pairs.*

The exterior covariant derivative in the second term of the Ginzburg–Landau Lagrangian is defined by the formula

$$d_A\Phi = d\Phi + A\Phi = \sum_{i=1}^{2} (\partial_i + A_i)\Phi\, dx_i,$$

and the term $|d_A\Phi|^2$ is responsible for the interaction of the electromagnetic field generated by the potential A with the Higgs field Φ.

The most important term in (1.1) is

$$\frac{\lambda}{4}(1 - |\Phi|^2)^2$$

with a parameter $\lambda > 0$. From the physical point of view, it describes the nonlinear "*self-action*" of the field Φ.

We define the *potential energy* of our model as the integral

$$U(A,\Phi) := \frac{1}{2}\int\limits_{\mathbb{R}^2} \mathcal{L}(A,\Phi)\, d^2x. \tag{1.2}$$

By the *solutions* of our model we mean the pairs (A,Φ) minimizing the potential energy.

1.1.4. *Vortices. Superconductors of the first and second kind.* Consider the behavior of the function Φ in more detail. The zeros of this function correspond to the points of intersection of the Abrikosov strings with the plane $\mathbb{R}^2_{(x_1,x_2)}$. We require that $|\Phi| \to 1$ as $\sqrt{x_1^2 + x_2^2} \to \infty$, which means physically that the pure superconductivity is preserved at infinity. In a neighborhood of a zero of $\Phi = \rho e^{i\theta}$, the vector field $\vec{v} := \nabla\theta$ behaves like a *hydrodynamical vortex.* For this reason the solutions of our model, which minimize the potential energy (1.2), are called *vortices.*

The only parameter λ entering the Ginzburg–Landau Lagrangian has the following physical meaning. For $\lambda < 1$ the solutions of our model, i.e., vortices, are attracted to each other. Superconductors with such values of λ are called *superconductors of the first kind.* On the contrary, for $\lambda > 1$ the vortices are repelled from each other, and such values of λ are characteristic for *superconductors of the second kind.*

In the critical case of $\lambda = 1$, the vortices do not interact with each other, so we can expect that for this value of λ any configuration of vortices, i.e., zeros of Φ, is possible. For that reason the critical case is most interesting from the mathematical point of view, and we mainly deal with this case in our paper.

1.2. Vortices.

1.2.1. *Vortex number.* Recall that the *potential energy* of our model is defined by the formula

$$U(A,\Phi)=\frac{1}{2}\int\mathcal{L}(A,\Phi)\,d^2x,$$

where $\mathcal{L}(A,\Phi)$ is the Ginzburg–Landau Lagrangian (1.1).

The Euler–Lagrange equations $\delta U(A,\Phi)=0$ for the potential energy functional are otherwise called the *static Ginzburg–Landau equations* and have the form

$$\begin{cases}\partial_i F_{ij}=0, \quad j=1,2,\\ \nabla_A^2\Phi=\dfrac{\lambda}{2}\,\Phi(|\Phi|^2-1).\end{cases}$$

To satisfy the condition $U(A,\Phi)<\infty$, we will require that $\Phi\to 1$ as $|x|\to\infty$. It follows from this asymptotic condition that our model has a topological invariant defined as the *rotation number* d of the map

$$\Phi\colon\ S^1_R\to\{|\Phi|\approx 1\}\cong S^1$$

for sufficiently large R. This invariant takes integer values and is called the *vortex number*.

If $|d_A\Phi|$ decreases at infinity faster than $1/|x|^{1+\delta}$, then the following relation holds:

$$d=\frac{i}{2\pi}\int F_A,$$

so d can be interpreted as the *full magnetic flux through the plane* (x_1,x_2).

1.2.2. *Vortex equations.* Now we can define the vortices in a more formal way. Namely, by *d-vortices* we will call the local minima of the potential energy functional $U(A,\Phi)<\infty$ with a given vortex number d.

Let us deduce the equations for these solutions, assuming here and in the sequel that $\lambda=1$. Consider first the case $d\ge 0$.

Introduce the complex coordinate $z=x_1+ix_2$ and derivations

$$\bar\partial=\frac{\partial_1+i\partial_2}{2},\qquad \bar\partial_A:=\bar\partial+A^{0,1},$$

where $A=A^{1,0}+A^{0,1}$ is the representation of A in the complex form, i.e., as the sum of a form $A^{1,0}$ of type $(1,0)$ and a form $A^{0,1}$ of type $(0,1)$, with $A^{0,1}=-\overline{A^{1,0}}$.

We transform the potential energy functional $U(A,\Phi)$ by using the following *Bogomolny formula*:

$$U(A,\Phi)=\frac{1}{2}\int\left\{2|\bar\partial_A\Phi|^2+\left|iF_{12}+\frac{1}{2}(|\Phi|^2-1)\right|^2\right\}+\frac{i}{2}\int F_A.$$

In other words, we represent the potential energy $U(A,\Phi)$ as a sum of nonnegative terms and the topological term $(i/2)\int F_A$ (equal to πd).

The Bogomolny formula implies the following lower estimate for the energy:

$$U(A,\Phi)\ge\pi d$$

for the fixed vortex number d. The equality in this estimate is attained only on solutions of the system of equations

$$\begin{cases}\bar\partial_A\Phi=0,\\ iF_{12}=\dfrac{1}{2}(1-|\Phi|^2),\end{cases}\tag{1.3}$$

which are called the *vortex equations.* Note that the second of them is equivalent to the equation

$$iF_A = *\frac{1}{2}(1 - |\Phi|^2),$$

where $*$ denotes the Hodge operator on $\mathbb{R}^2$.

One of the most important features of this system is its invariance under the *gauge transformations* of the form

$$A \mapsto A + id\chi, \qquad \Phi \mapsto e^{-i\chi}\Phi,$$

where χ is an arbitrary smooth real-valued function on $\mathbb{C}$. The potential energy functional $U(A,\Phi)$ is also invariant under these transformations.

For $d < 0$ there is an analogous Bogomolny transform, which implies the inequality

$$U(A,\Phi) \geq -\pi d,$$

in which the equality is attained only on solutions of the system of equations

$$\begin{cases} \partial_A \Phi = 0, \\ iF_{12} = \dfrac{1}{2}(|\Phi|^2 - 1), \end{cases}$$

called the *antivortex equations.* These equations are also invariant under the gauge transformations.

1.3. Theorems of Taubes.

1.3.1. *Formulation of the theorems.* The Taubes theorems give a description of the *moduli space of solutions* of the static Ginzburg–Landau equations, i.e., the space of all solutions of these equations modulo gauge transformations.

In [24] (see also [7]) Taubes proved the following theorem.

Theorem 1 (Taubes)**.** *For any positive integer d and an arbitrary collection $\{z_1, z_2, \dots, z_k\}$ of different points in the complex plane $\mathbb{C}$ taken with multiplicities $d_1, d_2, \dots, d_k$ such that $\sum_{j=1}^k d_j = d$, there exists a unique* (*up to gauge transformations*) *d-vortex solution (A,Φ) with $U(A,\Phi) < \infty$ such that the divisor of zeros of the function Φ coincides with $\sum_{j=1}^k d_j z_j$.*

An analogous theorem is true for $d < 0$ because the complex conjugation sends solutions of the antivortex equations to vortex solutions. The solution of the antivortex equations that is obtained in this way from a $|d|$-vortex is called a $|d|$-*antivortex.*

Note that for $d = 0$ any solution is gauge equivalent to the trivial one with $A \equiv 0$ and $\Phi \equiv 1$.

In addition to the first theorem Taubes proved the following result in [25].

Theorem 2 (Taubes)**.** *Any critical point (A,Φ) of the potential energy functional $U(A,\Phi)$ with $U(A,\Phi) < \infty$* (*or, equivalently, any solution of the Euler–Lagrange equations with $U(A,\Phi) < \infty$*) *and vortex number $d > 0$ is gauge equivalent to some d-vortex solution.*

Remark 1. It follows that under the hypothesis of Theorem 2 any solution of the Euler–Lagrange equations is either a d-vortex or a $|d|$-antivortex. In particular, there exists no solution of these equations of the vortex–antivortex type. Physically this means that all solutions of these equations are stable and have minimal energy in a given topological class.

Remark 2. The Euler–Lagrange equations for critical points of the functional $U(A,\Phi)$ are of the second order in the variables (A,Φ); however, under the condition $U(A,\Phi) < \infty$ they have the same solutions as the vortex equations for local minima of $U(A,\Phi)$, which have the first order in (A,Φ). This is a rare phenomenon in gauge field theories; more often such theories admit nonminimal (or physically unstable) solutions. This holds, for example, for the Bogomolny–Prasad–Sommerfield monopole equations on $\mathbb{R}^3$ and Yang–Mills equations on $\mathbb{R}^4$.

Recall that the *moduli space of d-vortices* is the quotient

$$\mathcal{M}_d = \frac{\{d\text{-vortices } (A,\Phi)\}}{\{\text{gauge transformations}\}}.$$

Everywhere below we restrict ourselves to the case $d > 0$.

Theorem 1 implies that the moduli space of d-vortices coincides with the set of unordered collections of d points in the complex plane $\mathbb{C}$, i.e., with the dth *symmetric power* of $\mathbb{C}$:

$$\mathcal{M}_d = \operatorname{Sym}^d \mathbb{C}.$$

Note that the symmetric power $\operatorname{Sym}^d \mathbb{C}$ can be identified with the space $\mathbb{C}^d$ in the following way: a collection of d points in the complex plane $\mathbb{C}$ is assigned the monic polynomial with zeros at the given points.

1.3.2. *Proof of the first Taubes theorem.* Let us explain the strategy of the proof of this theorem. We start from an approximate solution satisfying the first vortex equation together with the condition on zeros and correct asymptotics at infinity. Substituting this approximate solution into the second vortex equation, we obtain a nonlinear elliptic equation for the remainder term and solve it by the implicit function theorem.

To construct the approximate solution, we use the fact that the vortex equations linearize for $|\Phi| \to 1$. So, as such an approximation, we can take the solution of these linear equations given by the superposition of radial solutions with one zero.

More precisely, we consider the *ansatz* of the form

$$\Phi = e^{(u+i\theta)/2}$$

where u and θ are real-valued functions. Since the function Φ has zeros at the points $z_1, \dots, z_k$, the function $u(z)$ should tend to $-\infty$ as $z \to z_j$, while $\theta(z)$ should be a multivalued function with ramification points z_j of order d_j.

The first vortex equation (1.3) implies that outside the zeros of Φ the condition $A^{0,1} = -\bar\partial \log \Phi$ holds. Since the function $A^{0,1}$ is smooth, this equality should also hold at the zeros of Φ. Taking into account that $A^{1,0} = -\bar{A}^{0,1} = \partial \log \bar\Phi$, we have

$$A^{0,1} = -\bar\partial(u + i\theta), \qquad A^{1,0} = \partial(u - i\theta).$$

Now we fix the *gauge* by setting

$$\theta(z) = 2\sum_{j=1}^{k} d_j \operatorname{Arg}(z - z_j).$$

Plugging this function into the second vortex equation (1.3), we get

$$\Delta u = e^u - 1 + 4\pi \sum_{j=1}^{k} \delta(z - z_j),$$

where $\delta(z)$ is the Dirac delta function.

In order to solve this equation, we introduce the function

$$u_0(z) = -2\sum_{j=1}^{k} \log\biggl(1 + \frac{\mu}{|z - z_j|^2}\biggr)^{d_j}$$

with $\mu > 4d$. Note that the function u_0 satisfies the equation

$$\Delta u_0 = 4\pi \sum_{j=1}^{k} d_j \delta(z - z_j) - 4 \sum_{j=1}^{k} \frac{\mu d_j}{(\mu + |z - z_j|^2)^2}.$$

Hence, defining $v := u - u_0$, we obtain the following *Liouville-type equation* for the function v:

$$\Delta v(z) = -1 + g(z) f_1 + h(z) e^v f_2$$

with the boundary condition $v(z) \to 0$ as $|z| \to \infty$. Here

$$h(z) = e^{u_0(z)}, \qquad g(z) = 4 \sum_{j=1}^{k} \frac{\mu d_j}{(\mu + |z - z_j|^2)^2},$$

where $0 < g(z) < 1$ due to the inequality $\mu > 4d$.

According to the *Kazdan–Warner theorem* (see Subsection 1.5.3), the equation

$$\Delta v = f_1 + f_2 e^v$$

with $f_1 < 0$, $f_2 > 0$, and $v(z) \to 0$ as $|z| \to \infty$ has a unique real-analytic solution. This solution determines the desired d-vortex solution (A, Φ) of the original vortex equations.

Remark 3. Note that the above Liouville-type equation arises in differential geometry in the solution of the following problem: Given a Riemannian metric g with Gaussian curvature k, find a conformally equivalent Riemannian metric G with given Gaussian curvature K. Setting $G := ge^{2v}$, we get the following Liouville-type equation for v:

$$-\Delta_g v = k - K e^{2v},$$

where Δ_g is the Laplace–Beltrami operator associated with g.

1.4. Vortex equations on compact Riemann surfaces. Here we generalize the results of Subsection 1.3 to compact Riemann surfaces.

1.4.1. *Preliminary considerations.* Let X be a compact Riemann surface equipped with a Riemannian metric g and a Kähler form ω. We fix a complex Hermitian line bundle $L \to X$ with Hermitian metric h and define the energy functional by analogy with the complex plane case:

$$U(A, \Phi) = \frac{1}{2} \int_X \left\{ |F_A|^2 + |d_A \Phi|^2 + \frac{1}{4}(1 - |\Phi|^2)^2 \right\} \omega.$$

Here, A is a U(1)-connection on L, $F_A = dA$ is its curvature, d_A is the covariant exterior derivative generated by A, and Φ is a section of the bundle $L \to X$, with its norm $|\Phi|$ computed with respect to the metric h. As in the complex plane case, this functional is invariant under gauge transformations given by the maps $u \in C^\infty(X, U(1))$.

To the functional $U(A, \Phi)$ we apply the *Bogomolny transform* to get

$$U(A, \Phi) = \int_X \left\{ |\bar{\partial}_A \Phi|^2 + \frac{1}{2} |i F_A^\omega| + \frac{1}{2}(1 - |\Phi|^2)^2 \right\} \omega + \frac{i}{2} \int_X F_A,$$

where $F_A^\omega = \omega \lrcorner F_A$ is the $(1,1)$-component of F_A parallel to ω.

The above Bogomolny formula follows from the relation

$$\int_X iF_A\Phi = -\int_X |\bar\partial_A\Phi|^2\omega + \int_X |\partial_A\Phi|^2\omega$$

and the identities

$$i\omega \lrcorner \bar\partial_A\alpha = -\partial_A^*\alpha, \qquad i\omega \lrcorner \partial_A\beta = \bar\partial_A^*\beta,$$

which hold for arbitrary $(1,0)$-form α and $(0,1)$-form β. According to the *Gauss–Bonnet formula*, the last term in the Bogomolny formula can be rewritten as

$$\frac{i}{2\pi}\int_X F_A = c_1(L).$$

Hence, assuming that $c_1(L) > 0$, we arrive at a lower estimate for the energy:

$$U(A,\Phi) \ge \pi c_1(L),$$

where the equality is attained only on solutions of the equations

$$\begin{cases} \bar\partial_A\Phi = 0, \\ iF_A^\omega = \dfrac{1}{2}(1-|\Phi|^2). \end{cases}$$

The obtained equations look like the vortex equations on the complex plane. However, in the case of a compact Riemann surface we have an evident obstruction to their solvability. Namely, integrating the second equation over X, we get

$$\frac{i}{2\pi}\int_X F_A = \frac{1}{4\pi}\int_X \omega - \frac{1}{4\pi}\int_X |\Phi|^2\omega,$$

which can be rewritten in the form

$$c_1(L) = \frac{1}{4\pi}\operatorname{Vol}_g(X) - \frac{1}{4\pi}\|\Phi\|_{L^2}^2.$$

So we arrive at a *necessary condition for the solvability* of the above equations:

$$c_1(L) \le \frac{1}{4\pi}\operatorname{Vol}_g(X).$$

As we will see below, this condition arises because the energy is not invariant with respect to the scale transformation.

1.4.2. *Vortex equations.* The *scale transformation* increases the linear scales by a factor of $t > 0$ and converts the metric g into the metric $g_t := t^2g$. Simultaneously, the Kähler form and volume change to

$$\omega_t = t^2\omega, \qquad \operatorname{Vol}_{g_t}(X) = t^2\operatorname{Vol}_g(X).$$

The necessary solvability condition for the rescaled metric g_t reads as follows:

$$c_1(L) \le \frac{t^2}{4\pi}\operatorname{Vol}_g(X).$$

This condition is evidently satisfied for sufficiently large t. So we can always guarantee the fulfillment of the necessary solvability condition of the above equations by rescaling the original metric g.

It is, however, more convenient to fix the metric and introduce the scaling into the definition of the functional $U(A,\Phi)$. Namely, we replace the energy functional $U(A,\Phi)$ by its *rescaled version*

$$U_\tau(A,\Phi)=\frac{1}{2}\int\limits_X\left\{|F_A|^2+|d_A\Phi|^2+\frac{1}{2}(\tau-|\Phi|^2)^2\right\},$$

where $\tau>0$ is the scaling parameter.

Applying the Bogomolny transform to the rescaled energy functional, we obtain the following lower estimate for the energy:

$$U_\tau(A,\Phi)\ge\pi c_1(L),$$

in which the equality is attained only on solutions of the equations

$$\begin{cases}\bar\partial_A\Phi=0,\\ iF_A^\omega=\dfrac{1}{2}(\tau-|\Phi|^2).\end{cases}\tag{1.4}$$

These are the correct *vortex equations on a compact Riemann surface.* The necessary solvability condition for them takes the form

$$c_1(L)\le\frac{\tau}{4\pi}\operatorname{Vol}_g(X).$$

1.5. Bradlow theorem.

1.5.1. *Formulation.* In [2] Bradlow proved the following theorem:

Theorem 3 (Bradlow)**.** *Let $d:=c_1(L)>0$ and D be an effective divisor on X of degree d, i.e., $D=\sum_{j=1}^k d_jz_j$ with $\sum_{j=1}^k d_j=d$. Then the condition*

$$c_1(L)<\frac{\tau}{4\pi}\operatorname{Vol}(X)$$

is necessary and sufficient for the existence of a unique (*up to gauge equivalence*) *d-vortex solution (A,Φ) such that the zero divisor of Φ coincides with D.*

Moreover, the holomorphic line bundle L equipped with the complex structure determined by the operator $\bar\partial_A$ is isomorphic to the holomorphic line bundle $[D]$ defined by the divisor D.

Note that the first vortex equation $\bar\partial_A\Phi=0$ means, in other words, that Φ is a holomorphic section of the Hermitian line bundle $(L,\bar\partial_A)$, where A is a holomorphic Hermitian connection on $(L,\bar\partial_A)$. Recall that such a connection is uniquely determined by the Hermitian metric.

1.5.2. *Reformulation of the original problem.* We now change our original point of view and, instead of the Hermitian metric, fix the holomorphic structure on L determined by the $\bar\partial$-operator $\bar\partial_L$. Given a holomorphic section Φ of the bundle $(L,\bar\partial_L)$, we want to construct a Hermitian metric H on L such that the holomorphic connection A associated with this metric satisfies the second vortex equation.

In other words, the original problem is formulated as follows:

Problem 1. Given a Hermitian line bundle (L,h), find a Hermitian connection A on L and a holomorphic section Φ of the bundle $(L,\bar\partial_A)$ that satisfy the second vortex equation.

Instead of this problem, we consider the following

Problem 2. Given a Hermitian holomorphic line bundle $(L,h,\bar\partial_L)$ and a holomorphic section Φ of the bundle $(L,\bar\partial_L)$, find a Hermitian metric H on L that is conformally equivalent to the metric h and is such that the connection A_H compatible with H and $\bar\partial_L$ satisfies the second vortex equation.

There is an action of the group $\mathcal{G} = C^\infty(X, U(1))$ of gauge transformations on solutions of Problem 1. On the other hand, there is a natural action of the complexified group $\mathcal{G}_{\mathbb{C}} = C^\infty(X, \mathbb{C}^*)$ of gauge transformations on solutions of Problem 2. The latter action is given by the gauge transformations of the form

$$\bar\partial_L \mapsto g(\bar\partial_L) = g \circ \bar\partial_L \circ g^{-1}, \qquad \Phi \mapsto g\Phi, \qquad H \mapsto |g^{-1}|^2 H,$$

where $g \in \mathcal{G}_{\mathbb{C}}$.

Proposition 1. *There exists a bijective correspondence between the sets*

$$\{\textit{solutions } (A,\Phi) \textit{ of Problem } 1\}/\mathcal{G} \qquad \textit{and} \qquad \{\textit{solutions } (\bar\partial_L, H, \Phi) \textit{ of Problem } 2\}/\mathcal{G}_{\mathbb{C}}.$$

In order to construct a solution of Problem 1 from a solution of Problem 2, we write H in the form $H = he^{2v} = hg^2$ and equip the bundle L with the new holomorphic structure

$$g(\bar\partial_L) := g \circ \bar\partial_L \circ g^{-1}.$$

Denote by A_g the connection on L compatible with h and $g(\bar\partial_L)$ and put $\Phi_g := g\Phi$. Then the pair (A_g, Φ_g) will give a solution of Problem 1.

1.5.3. *Solution of Problem* 2. Suppose that $(L, h, \bar\partial_L)$ is a holomorphic Hermitian line bundle together with a given holomorphic section Φ. We want to find a Hermitian metric $H = he^{2u}$ with $u \in C^\infty(X, \mathbb{R})$ such that

$$iF^\omega_{A_H} = \frac{1}{2}(\tau - |\Phi|^2_H)$$

for the holomorphic connection A_H compatible with H. This equation is equivalent to the following Liouville-type equation for the conformal factor u:

$$-\Delta u = iF^\omega_{A_h} - \frac{\tau}{2} + \frac{1}{2}|\Phi|^2_h e^{2u},$$

where A_h is the connection compatible with $\bar\partial_L$ and h. If we define

$$f_1 := iF^\omega_A - \frac{\tau}{2}, \qquad f_2 := \frac{1}{2}|\Phi|^2_h,$$

then the latter equation will be rewritten in the form

$$-\Delta u = f_1 + f_2 e^{2u}.$$

In addition, we can get rid of one of the coefficients by setting

$$c := 2\int_X f_1\omega = 2i\int_X F_A - \tau\int_X \omega = 4\pi c_1(L) - \tau\,\mathrm{Vol}(X).$$

Denoting by v the unique (up to a constant) solution of the Laplace equation

$$-\Delta v = f_1 - \bar f_1$$

with $\bar f_1 = \int_X f_1\omega$, we will get the following Liouville-type equation for $w := 2(u - v)$:

$$-\Delta w = c - fe^w,$$

where $f := -|\Phi|^2_h e^{2v}$ is a smooth nonpositive function.

Now we apply the Kazdan–Warner theorem from [8].

Theorem 4 (Kazdan–Warner). *Let X be a compact Riemann surface. Suppose that a function $f \in C^\infty(X,\mathbb{R})$ does not vanish identically and $c \in \mathbb{R}$. Consider the Liouville-type equation*

$$-\Delta w = c - fe^w \tag{1.5}$$

with $w \in C^\infty(X,\mathbb{R})$. Then

(1) *if $c = 0$, then a solution of equation* (1.5) *exists if and only if $\bar f := \int_X f\omega < 0$ and $f > 0$ somewhere on X;*

(2) *if $c < 0$, then*

  (a) *the condition $\bar f < 0$ is necessary for the solvability of equation* (1.5)*;*

  (b) *under the condition $\bar f < 0$ there exists a constant $c_-(f)$ satisfying the inequality $-\infty \le c_-(f) < 0$ such that a solution of equation* (1.5) *exists if and only if $c > c_-(f)$;*

  (c) *the equality $c_-(f) = -\infty$ holds if and only if $f \le 0$ everywhere on X; in this case the solution of equation* (1.5) *is unique;*

(3) *if $c > 0$, then*

  (a) *the condition that $f > 0$ somewhere on X is necessary for the solvability of equation* (1.5)*;*

  (b) *under this necessary condition there exists a constant $c_+(f)$ satisfying the inequality $0 < c_+(f) \le +\infty$ such that a solution of equation* (1.5) *exists for $c < c_+(f)$.*

In our case, $f \le 0$ everywhere, so the condition $c < 0$ is necessary and sufficient for the existence of a solution. Moreover, this solution is unique according to the Kazdan–Warner theorem. The inequality $c < 0$ is equivalent to the condition $4\pi c_1(L) < \tau \operatorname{Vol}(X)$, which is satisfied by the hypothesis of the Bradlow theorem.

1.5.4. *End of the proof of the Bradlow theorem.* To finish the proof of the Bradlow theorem, for a given effective divisor D of degree d consider the associated holomorphic line bundle $(L, \bar\partial_L) = [D]$ and its canonical holomorphic section Φ such that the zero divisor of Φ coincides with D. Then the Kazdan–Warner theorem will imply that there exists a unique Hermitian metric H giving a solution of Problem 2, which is equivalent to the existence of a unique vortex solution.

Consider now the remaining *critical case* of the solvability condition, in which

$$c_1(L) = \frac{\tau}{4\pi} \operatorname{Vol}(X).$$

Integrating the second vortex equation (1.4), we get

$$c_1(L) = \frac{\tau}{4\pi} \operatorname{Vol}(X) - \frac{1}{4\pi} \|\Phi\|^2,$$

which implies that $\Phi \equiv 0$.

Recall that the problem of solvability of vortex equations (up to the gauge action of the group $\mathcal{G}$) is equivalent to finding, on a given holomorphic line bundle $(L, \bar\partial_L)$, a Hermitian metric H that is conformally equivalent to the metric h and satisfies the second vortex equation (up to the gauge action of the group $\mathcal{G}_\mathbb{C}$).

Since $\tau = 4\pi c_1(L)/\operatorname{Vol}(X)$ and $\|\Phi\|^2 \equiv 0$, the second vortex equation takes the form

$$iF^\omega_{A_H} = \frac{2\pi c_1(L)}{\operatorname{Vol}(X)}.$$

This is an equation of the *Einstein–Hermite* type. If we look for the metric H in the form $H = he^{2u}$ with $u \in C^\infty(X, \mathbb{R})$, then the function u must satisfy the Laplace equation

$$-\Delta u = iF^\omega_{A_h} - \frac{2\pi c_1(L)}{\mathrm{Vol}(X)},$$

where A_h is compatible with $\bar\partial_L$ and h. This equation has a unique solution (up to a constant).

Hence, in the critical case we have a bijective correspondence between the sets

$$\{\text{connections } A = A_H \text{ on } (L, \bar\partial_L) \text{ satisfying the Einstein–Hermite equation}\}/\mathcal{G}$$

and

$$\{\text{holomorphic line bundles } (L, \bar\partial_L)\}/\mathcal{G}_{\mathbb{C}} =: \mathrm{Pic}(X).$$

Remark 4. According to the Bradlow theorem, in the case

$$c_1(L) < \frac{\tau}{4\pi}\,\mathrm{Vol}(X)$$

we have a bijective correspondence between the sets

$$\{d\text{-vortex solutions } (A, \Phi)\}/\mathcal{G} \qquad \text{and} \qquad \{\text{effective divisors } D \text{ of degree } d = c_1(L)\}.$$

So the moduli space of d-vortex solutions coincides with the symmetric power $\mathrm{Sym}^d X$.

Remark 5. The inequality

$$\tau > \frac{4\pi c_1(L)}{\mathrm{Vol}(X)}$$

coincides with the *stability condition for the pair* (L, Φ) (see [2]). Accordingly, the *semi-stability condition for the pair* (L, Φ) is equivalent to the inequality

$$\tau \ge \frac{4\pi c_1(L)}{\mathrm{Vol}(X)}.$$

Remark 6. There is another proof of the Bradlow theorem, due to García-Prada [5], based on the use of the *moment map.*

2. DIMENSION THREE: ADIABATIC LIMIT IN THE GINZBURG–LANDAU EQUATIONS

In this section we pass from the vortices in dimension 2, considered in Section 1, to the three-dimensional case. We can add the third variable in two different ways. One of them is related to the Euclidean setting and leads to the Euclidean Ginzburg–Landau equations in $\mathbb{R}^3$, which describe the Abrikosov strings. Another way is related to the Lorentz setting and leads to the hyperbolic Ginzburg–Landau equations in $\mathbb{R}^{1+2}$, which describe the vortex dynamics in $\mathbb{R}^2$. In the main part of this section we study the vortex dynamics, while the Euclidean model is considered in the last Subsection 2.2.5.

2.1. Adiabatic limit.

2.1.1. *Hyperbolic Ginzburg–Landau equations.* We switch on the time in our model by adding the variable $x_0 = t$. In this case the Higgs field $\Phi = \Phi(t, x_1, x_2)$ is given by a smooth complex-valued function on the space $\mathbb{R}^{1+2}$ with coordinates (t, x_1, x_2) and the form A is replaced by the form

$$\mathcal{A} = A_0\,dt + A_1\,dx_1 + A_2\,dx_2$$

whose coefficients $A_\mu = A_\mu(t, x_1, x_2)$, $\mu = 0, 1, 2$, are smooth functions with pure imaginary values on the space $\mathbb{R}^{1+2}$. Denote the time component of the form $\mathcal{A}$ by $A^0 := A_0\,dt$ and its space component, as before, by $A = A_1\,dx_1 + A_2\,dx_2$.

Then the *potential energy* of the system is given by the same formula as before; in other words, $U(\mathcal{A}, \Phi) = U(A, \Phi)$.

The *kinetic energy* is defined by

$$T(\mathcal{A}, \Phi) = \frac{1}{2}\int \bigl\{|F_{01}|^2 + |F_{02}|^2 + |d_{A^0}\Phi|^2\bigr\}\,dx_1\,dx_2,$$

where F_{0j}, $j = 1, 2$, are given, as before, by the formula

$$F_{0j} = \partial_0 A_j - \partial_j A_0$$

and $d_{A^0}\Phi = d\Phi + A_0\,dt$. (Note that the formula for the kinetic energy contains terms similar to those in the formula for the potential energy, but they involve the time derivative.)

Introduce the *Ginzburg–Landau action* functional

$$S(\mathcal{A}, \Phi) = \int_0^{T_0} \bigl(T(\mathcal{A}, \Phi) - U(\mathcal{A}, \Phi)\bigr)\,dt.$$

The *Euler–Lagrange equations* for this functional, $\delta S(\mathcal{A}, \Phi) = 0$, also called the *hyperbolic Ginzburg–Landau equations*, have the form

$$\begin{cases} \partial_1 F_{01} + \partial_2 F_{02} = i\,\mathrm{Im}(\bar\Phi \nabla_{A,0}\Phi), \\ \partial_0 F_{0j} + \displaystyle\sum_{k=1}^{2} \varepsilon_{jk}\partial_k F_{12} = i\,\mathrm{Im}(\bar\Phi \nabla_{A,j}\Phi), \quad j = 1, 2, \\ \bigl(\nabla_{A,0}^2 - \nabla_{A,1}^2 - \nabla_{A,2}^2\bigr)\Phi = \dfrac{\lambda}{2}\Phi(1 - |\Phi|^2), \end{cases}$$

where $\nabla_{A,\mu} = \partial_\mu + A_\mu$, $\mu = 0, 1, 2$, $\varepsilon_{12} = -\varepsilon_{21} = 1$, and $\varepsilon_{11} = \varepsilon_{22} = 0$.

The first of these equations is of *constraint type*, which means that it holds for any t if it is satisfied at the initial moment of time. The latter equation, containing the *covariant D'Alembertian* on its left-hand side, is a nonlinear wave equation.

These equations are invariant under the *gauge transformations* of the form

$$A \mapsto A + id\chi, \qquad \Phi \mapsto e^{-i\chi}\Phi,$$

where $\chi = \chi(t, x_1, x_2)$ is a smooth real-valued function on $\mathbb{R}^{1+2}$.

2.1.2. *Temporal gauge.* We can choose the gauge function χ so that $A_0 = 0$; such a choice is called the *temporal gauge*. (Note that after fixing the temporal gauge we still have the gauge freedom with respect to static gauge transformations given by gauge functions χ that do not depend on the time t.)

In the temporal gauge the kinetic energy is written in the form

$$T(A, \Phi) = \frac{1}{2}\{\|\dot\Phi\|^2 + \|\dot A\|^2\},$$

where the dot denotes the time derivative $\partial/\partial t = \partial/\partial x_0$ and $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^2)}$ is the norm in the space $L^2(\mathbb{R}^2)$.

The Euler–Lagrange equations in the temporal gauge take the form

$$\begin{cases} \partial_1 \dot{A}_1 + \partial_2 \dot{A}_2 = i\,\mathrm{Im}(\bar{\Phi}\dot{\Phi}), \\ \ddot{A}_j + \sum \varepsilon_{jk}\partial_k F_{12} = i\,\mathrm{Im}(\bar{\Phi}\nabla_{A,j}\Phi), \quad j = 1, 2, \\ \ddot{\Phi} - \Delta_A \Phi = \dfrac{\lambda}{2}\,\Phi(1 - |\Phi|^2), \end{cases}$$

where $\Delta_A = \nabla_{A,1}^2 + \nabla_{A,2}^2$.

2.1.3. *Heuristic considerations.* Our goal is to describe the space of solutions of the hyperbolic Ginzburg–Landau equations modulo dynamic gauge transformations. For brevity, the solutions of these equations will be called *dynamic solutions*, and the quotient of the space of dynamic solutions modulo gauge transformations will be called the *moduli space of dynamic solutions*.

In contrast to the moduli space of static solutions, whose structure is completely described by the Taubes theorems, we cannot expect to get anything similar in the dynamic case. However, we can hope to obtain an approximate description of at least some classes of dynamic solutions. In this subsection we will present Manton's heuristic approach to the approximate description of "slowly moving" dynamic solutions (see [13]).

In the temporal gauge the dynamic solutions of the hyperbolic Ginzburg–Landau equations are given by smooth trajectories

$$\gamma\colon\ t \mapsto [A(t), \Phi(t)]$$

in the *static configuration space*

$$\mathcal{N}_d = \frac{\{\text{smooth data } (A,\Phi) \text{ with } U(A,\Phi) < \infty \text{ and vortex number } d\}}{\{\text{static gauge transformations}\}},$$

where $[A(t), \Phi(t)]$ denotes the gauge class of the pair $(A(t), \Phi(t))$ modulo static gauge transformations. This space contains, in particular, the moduli space $\mathcal{M}_d$ of d-vortex solutions.

In other words, we represent a dynamic solution by a family of vortex data (A, Φ) on $\mathbb{R}^2$ that depend on t as a parameter and are defined up to static gauge transformations.

The configuration space $\mathcal{N}_d$ can be thought of as a horizontal gutter with a small ball rolling along the trajectory $\gamma(t)$ inside it. The moduli space of d-vortex solutions $\mathcal{M}_d$, for which the potential energy is minimal, corresponds to the bottom of this gutter. The lower the kinetic energy of the ball, the closer its trajectory to the bottom of the gutter. The ball can even hit this bottom but, having a nonzero kinetic energy, cannot stop there and is forced to climb the wall of the gutter.

Define the *kinetic energy of the trajectory* $\gamma(t) = [A(t), \Phi(t)]$ as

$$T(\gamma) := \frac{1}{2}\{\|\dot{A}\|^2 + \|\dot{\Phi}\|^2\}.$$

Consider a family of trajectories $\gamma_\varepsilon(t)$ that depend on the parameter $\varepsilon > 0$ and have the kinetic energy $\|T(\gamma_\varepsilon)\|$ proportional to ε, which is thus tending to zero as $\varepsilon \to 0$. For small ε the trajectories $\gamma_\varepsilon(t)$ will lie close to the static moduli space of vortices $\mathcal{M}_d$, and in the limit as $\varepsilon \to 0$ they will converge to a static solution, i.e., to a point on $\mathcal{M}_d$.

However, if we introduce the *slow time* $\tau := \varepsilon t$ on the trajectory γ_ε, then in the limit as $\varepsilon \to 0$ the "rescaled" trajectories $\gamma_\varepsilon(\tau)$ will converge not to a point but rather to some trajectory γ_0 lying in $\mathcal{M}_d$.

The indicated limit is called *adiabatic*, and the equations to which the original Ginzburg–Landau equations reduce in this limit are called the *adiabatic equations.* We call their solutions the *adiabatic trajectories.*

The adiabatic trajectories admit the following intrinsic description in terms of the moduli space $\mathcal{M}_d$.

Theorem 5. *On the vortex space $\mathcal{M}_d$, the kinetic energy functional determines a Riemannian metric, called the* kinetic metric *or T-metric. The geodesics of this metric coincide precisely with the adiabatic trajectories.*

Since any point of an adiabatic trajectory γ_0 is a static solution, the trajectory itself cannot be a dynamic solution. However, such trajectories describe approximately dynamic solutions with small kinetic energy.

Manton also formulated the following *adiabatic principle* [13]:

> *For any adiabatic trajectory γ_0 on the moduli space $\mathcal{M}_d$ there should exist a sequence $\{\gamma_\varepsilon\}$ of dynamic trajectories (solutions of the hyperbolic Ginzburg–Landau equations) that converges as $\varepsilon \to 0$ to γ_0 in the adiabatic limit.*

The justification of the above theorem and adiabatic principle will be given below after we introduce some necessary notions.

2.1.4. *Tangent structure of the moduli space $\mathcal{M}_d$.* In order to study the structure of the tangent bundle $T\mathcal{M}_d$ to the moduli space of vortex solutions $\mathcal{M}_d$, it is necessary to introduce the *Sobolev version* of the space $\mathcal{M}_d$.

Denote by $\mathcal{V}^s := \mathcal{V}^s_d$ the space of d-vortex solutions of the vortex equations

$$\begin{cases} \bar\partial_A \Phi = 0, \\ 2idA = *(1 - |\Phi|^2), \end{cases}$$

where A is a 1-form with coefficients in the Sobolev space $H_s(\mathbb{C}, i\mathbb{R})$, $s \ge 1$, i.e.,

$$A \in H_s(\mathbb{C}, i\mathbb{R}) \otimes \Omega^1(\mathbb{C}) = \Omega^1_s(\mathbb{C}, i\mathbb{R}) =: \Omega^1_s,$$

and $\Phi \in H_s(\mathbb{C}, \mathbb{C}) =: H_s$ so that $(A, \Phi) \in \Omega^1_s \times H_s$. The group of *Sobolev gauge transformations* is defined as

$$\mathcal{G}_s := \{\text{gauge transformations generated by functions } \chi \in H_s(\mathbb{C}, \mathbb{R})\}.$$

The *Sobolev version $\mathcal{M}^s_d$ of the space $\mathcal{M}_d$* is defined as

$$\mathcal{M}^s_d := \mathcal{V}^s_d / \mathcal{G}_{s+1}.$$

It can be shown that the space $\mathcal{M}^s_d$ coincides with $\operatorname{Sym}^d \mathbb{C}$ and so does not depend on $s \ge 1$.

By varying the vortex equations in A and Φ at some fixed solution (A, Φ), we obtain the *linearized vortex equations*

$$\begin{cases} \bar\partial_A \varphi + a^{0,1}\Phi = 0, \\ *i(da) + \operatorname{Re}(\varphi\bar\Phi) = 0, \end{cases}$$

where $(a, \varphi) \in \Omega^1_s \times H_s$.

Introduce the *linearized vortex operator*

$$\mathcal{D}_{A,\Phi}\colon\ \Omega^1_s \times H_s \to \Omega^{0,1}_{s-1} \times H_{s-1}(\mathbb{C}, \mathbb{R})$$

given by the left-hand side of the linearized vortex equations:

$$\mathcal{D}_{A,\Phi}\colon\ (a, \varphi) \mapsto \big(\bar\partial_A \varphi + a^{0,1}\Phi, *i(da) + \operatorname{Re}(\varphi\bar\Phi)\big).$$

With the help of this operator we can define the tangent space to $\mathcal{V}_d^s$ at (A,Φ) as

$$T_{(A,\Phi)}\mathcal{V}_d^s = \operatorname{Ker}\mathcal{D}_{(A,\Phi)} = \big\{(a,\varphi)\in\Omega_s^1\times H_s\colon\ \mathcal{D}_{(A,\Phi)}(a,\varphi)=0\big\}.$$

The linearized vortex equations are invariant under the *infinitesimal gauge transformations* of class $H_{s+1}(\mathbb{C},\mathbb{R})$ given by the maps

$$a\mapsto a+id\chi,\qquad \varphi\mapsto\varphi-i\Phi\chi$$

with $\chi\in H_{s+1}(\mathbb{C},\mathbb{R})$. The orbit passing through the origin consists of the pairs $(id\chi,-i\Phi\chi)$. Taking this into account, we introduce the *tangent gauge operator*

$$\delta_{(A,\Phi)}\colon\ H_{s+1}(\mathbb{C},\mathbb{R})\to\Omega_s^1\times H_s(\mathbb{C},\mathbb{C})$$

by the formula $\chi\mapsto(id\chi,-i\Phi\chi)$. The *adjoint operator*

$$\delta^*_{(A,\Phi)}\colon\ \Omega_s^1\times H_s(\mathbb{C},\mathbb{C})\to H_{s-1}(\mathbb{C},\mathbb{R})$$

is given by the map

$$(a,\varphi)\mapsto(d^*a+\operatorname{Im}(\bar\Phi\varphi)).$$

Since

$$\Omega_s^1\times H_s = T_{(A,\Phi)}(\mathcal{G}_{s+1}(A,\varphi))\oplus\operatorname{Ker}\delta^*_{(A,\Phi)},$$

we can fix the infinitesimal gauge by the following *gauge-fixing condition*:

$$\delta^*_{(A,\Phi)}(a,\varphi)=0. \tag{2.1}$$

Then the tangent space to $\mathcal{M}_d^s$ will be given by

$$T_{(A,\Phi)}\mathcal{M}_d^s=\operatorname{Ker}\mathcal{D}_{(A,\Phi)}\cap\operatorname{Ker}\delta^*_{(A,\Phi)}=\big\{(a,\varphi)\in\Omega_s^1\times H_s\colon\ \mathcal{D}_{(A,\Phi)}(a,\varphi)=\delta^*_{(A,\Phi)}(a,\varphi)=0\big\}.$$

2.1.5. *Vortex trajectories.* Consider again the trajectories in the moduli space $\mathcal{M}_d=\mathcal{M}_d^s$ for some $s\ge 1$. We can describe such vortex trajectories by using the Taubes theorem. Namely, by this theorem any trajectory $t\mapsto q(t)$ in the space $\operatorname{Sym}^d\mathbb{C}\simeq\mathbb{C}^d$ uniquely determines a vortex trajectory

$$\gamma\colon\ t\mapsto[A(q(t)),\Phi(q(t))]$$

in the space $\mathcal{M}_d$. We can also consider it as a trajectory

$$\gamma\colon\ t\mapsto(A(q(t)),\Phi(q(t)))$$

in $\mathcal{V}_d$ that satisfies the gauge-fixing condition

$$\delta^*_{(A,\Phi)}(\dot A,\dot\Phi)=0$$

for any t, where the dot denotes, as before, the derivative with respect to t.

Consider a perturbation $\widetilde\gamma$ of a vortex trajectory $\gamma=[A(q),\Phi(q)]$ in the configuration space $\mathcal{N}_d$ of the form

$$\widetilde\gamma(t)=[\widetilde A(t),\widetilde\Phi(t)],$$

where

$$\widetilde A(t)=A(q(t))+a(t),\qquad\widetilde\Phi(t)=\Phi(q(t))+\varphi(t).$$

We impose the following natural *orthogonality condition* on the pairs (a,φ) under consideration:

$$(a,\varphi) \perp T_{(A,\Phi)}\mathcal{M}_d, \tag{2.2}$$

thus excluding the deformations in the directions tangent to $T_{(A,\Phi)}\mathcal{M}_d$.

The indicated orthogonality condition can also be obtained by the *least squares method.* Namely, for a given trajectory $\widetilde{\gamma} = [\widetilde{A}, \widetilde{\Phi}]$ in $\mathcal{N}_d$ corresponding to a dynamic solution, we want to find a trajectory $t \mapsto q(t)$ in the space $\mathrm{Sym}^d\,\mathbb{C}$ such that the corresponding d-vortex trajectory $\gamma\colon t \mapsto [A(q(t)), \Phi(q(t))]$ is "closest" to $\widetilde{\gamma}$. By the least squares method such a trajectory γ should minimize the functional

$$\frac{1}{2}\int \bigl\{\|\widetilde{A}(t) - A(q(t))\|^2_{L^2} + \|\widetilde{\varphi}(t) - \Phi(q(t))\|^2_{L^2}\bigr\}\,dt.$$

The critical points of this functional satisfy the Euler–Lagrange equation of the form

$$\langle a, \delta A\rangle + \langle \varphi, \delta\Phi\rangle = 0,$$

where $(\delta A, \delta\Phi)$ is the variation of $(A(q), \Phi(q))$ with respect to q. So, $(\delta A, \delta\Phi) \in T_{(A,\Phi)}\mathcal{M}_d$ and

$$(a,\varphi) \perp T_{(A,\Phi)}\mathcal{M}_d = \operatorname{Ker}\mathcal{D}_{(A,\Phi)} \cap \operatorname{Ker}\delta^*_{(A,\Phi)}.$$

Assuming that the gauge-fixing condition $\delta^*_{(A,\Phi)}(a,\varphi) = 0$ is fulfilled, we can write the orthogonality condition in the form

$$(a,\varphi) \perp \operatorname{Ker}\mathcal{D}_{(A,\Phi)}. \tag{2.3}$$

If we choose an L^2-basis $\{n_\mu\}$ in the space $\operatorname{Ker}\mathcal{D}_{(A,\Phi)}$ or, in other words, a basis of solutions of the equation

$$\mathcal{D}_{(A,\Phi)} n_\mu = 0, \qquad \mu = 0, 1, \dots, 2d,$$

then condition (2.3) can be rewritten in the form

$$\langle (a,\varphi), n_\mu\rangle = 0 \tag{2.4}$$

for $\mu = 1, 2, \dots, 2d$.

2.1.6. *Adiabatic equations.* Consider a sequence of dynamic solutions γ_ε that depend on a small parameter $\varepsilon > 0$ and are written in the form

$$\gamma_\varepsilon\colon\ t \mapsto [A^\varepsilon(t), \Phi^\varepsilon(t)],$$

where

$$A^\varepsilon(t) = A(q(t)) + \varepsilon^2 a(t), \qquad \Phi^\varepsilon(t) = \Phi(q(t)) + \varepsilon^2\varphi(t). \tag{2.5}$$

We assume that γ_ε satisfies the gauge-fixing condition (2.1) and the orthogonality condition (2.2).

Now we introduce the slow time variable $\tau := \varepsilon t$ and substitute $(A^\varepsilon, \Phi^\varepsilon)$ into the Ginzburg–Landau equations. Dividing both sides of the obtained equation by ε^2, we arrive at the equality

$$\varepsilon^2\partial^2_\tau(a,\varphi) + \mathcal{D}^*_{(A,\Phi)}\mathcal{D}_{(A,\Phi)}(a,\varphi) = -\partial^2_\tau(A,\Phi) + j, \tag{2.6}$$

where j is the sum of nonlinear current-type terms of order less than or equal to ε. (To obtain this formula, we have used the fact that the pair $(A(q(t)), \Phi(q(t)))$ satisfies the vortex equations for any t and, hence, the static Euler–Lagrange equations as well.)

On the other hand, differentiating the orthogonality condition (2.4) twice with respect to τ, we get

$$\langle \partial_\tau^2(a,\varphi), n_\mu\rangle = -\langle (a,\varphi), \partial_\tau^2 n_\mu\rangle - 2\langle \partial_\tau(a,\varphi), \partial_\tau n_\mu\rangle, \qquad \mu = 1,\dots,2d. \tag{2.7}$$

Note that the right-hand side of this equation, in which $\tau = \varepsilon t$, tends to zero as $\varepsilon \to 0$ after the multiplication by ε^2. (This follows from the condition $T(\gamma_\varepsilon - \gamma_0) \to 0$ as $\varepsilon \to 0$.)

We take now the inner product of both sides of (2.6) with n_μ and use the equation $\mathcal{D}_{(A,\Phi)} n_\mu = 0$. Replacing the inner product $\langle \partial_t^2(a,\varphi), n_\mu\rangle$ with the right-hand side of equation (2.7), we obtain the relation

$$\langle (\partial_\tau^2 A, \partial_\tau^2 \Phi), n_\mu\rangle = j_1, \qquad \mu = 1,\dots,2d,$$

in which j_1 denotes the sum of the term j from (2.6) and the terms containing the derivatives of n_μ with respect to τ. Note that j_1 tends to zero as $\varepsilon \to 0$. The latter relation implies that if the trajectory γ is the adiabatic limit of dynamic solutions γ_ε as $\varepsilon \to 0$, then it should satisfy the following *adiabatic equations*:

$$\langle (\partial_\tau^2 A, \partial_\tau^2 \Phi), n_\mu\rangle = 0 \qquad \text{for} \quad \mu = 1,\dots,2d. \tag{2.8}$$

2.1.7. *Geometric interpretation.* Let us show that the adiabatic equations (2.8) coincide with the Euler equations for the geodesics in the space $\mathcal{M}_d$ equipped with the Riemannian T-metric generated by the kinetic energy functional. This will justify Manton's heuristic approach.

Recall that the geodesics γ in the metric defined by the kinetic energy T are extremals of the functional

$$\int\limits_\gamma T(A,\Phi)\,d\tau = \frac{1}{2}\int\limits_\gamma \{\|\dot A\|^2 + \|\dot\Phi\|^2\}\,d\tau$$

defined on the trajectories $\gamma\colon \tau \to [A(\tau),\Phi(\tau)]$ in the space $\mathcal{M}_d$. The Euler–Lagrange equation for this functional has the form

$$\int \{\langle \dot A, \delta\dot A\rangle + \langle \dot\Phi, \delta\dot\Phi\rangle\}\,d\tau = -\int \{\langle \ddot A, \delta A\rangle + \langle \ddot\Phi, \delta\Phi\rangle\}\,d\tau = 0.$$

This integral equality is equivalent to the adiabatic equation

$$\langle \partial_\tau^2(A,\Phi), n_\mu\rangle = 0, \qquad \mu = 1,\dots,2d,$$

because the pairs $(\delta A, \delta\Phi)$ satisfying the gauge-fixing condition generate the whole space $\operatorname{Ker}\mathcal{D}_{(A,\Phi)}$.

2.1.8. *Rigorous formulation of the adiabatic principle.*

Theorem 6 (Palvelev [16]). *Suppose that a trajectory*

$$\gamma_0 = [A_0,\Phi_0]\colon\ [0,\tau_0] \to \mathcal{M}_d$$

is a geodesic of the space $\mathcal{M}_d$ in the kinetic T-metric. Then it has a pull-back

$$(A_0,\Phi_0)\colon\ [0,\tau_0] \to \mathcal{V}_d$$

defined by a smooth trajectory (A_0,Φ_0) in the space $\mathcal{V}_d$ of static d-vortex solutions, and there are positive constants $\tau_1 \le \tau_0$, ε_0, and K such that for any $\varepsilon < \varepsilon_0$ there exists a dynamic solution $(\mathcal{A}^\varepsilon, \Phi^\varepsilon)$ of the Ginzburg–Landau equations on the interval $[0,\tau_1/\varepsilon]$ of the form

$$\begin{cases} A_0^\varepsilon = \varepsilon^3 a_0, \\ A^\varepsilon(t) = A_0(\varepsilon t) + \varepsilon^2 a(t) \equiv A(\varepsilon t) + \varepsilon^2 a(t), \\ \Phi^\varepsilon(t) = \Phi_0(\varepsilon t) + \varepsilon^2 \varphi(t) \equiv \Phi(\varepsilon t) + \varepsilon^2 \varphi(t) \end{cases} \tag{2.9}$$

that satisfies the estimate

$$\max\{\|a_0(t)\|_{H^3}, \|a(t)\|_{H^3}, \|\varphi(t)\|_{H^3}\} \le K \tag{2.10}$$

for any $t \in [0, \tau_1/\varepsilon]$. *The functions* $a(t) = (a_1(t), a_2(t))$ *and* $\varphi(t) = (\varphi_1(t), \varphi_2(t))$ *belong to the class*

$$C\big([0,\tau_1/\varepsilon], H^3(\mathbb{R}^2)\big) \cap C^1\big([0,\tau_1/\varepsilon], H^2(\mathbb{R}^2)\big) \cap C^2\big([0,\tau_1/\varepsilon], H^1(\mathbb{R}^2)\big),$$

and the function $a_0(t)$ *belongs to the class* $C^1([0,\tau_1/\varepsilon], H^3(\mathbb{R}^2))$. *The norm* $\|\cdot\|_{H^3}$ *in* (2.10) *denotes the Sobolev* H^3*-norm on the space* $\mathbb{R}^2$.

A detailed proof of this theorem is given in [16], and its idea is also presented in [17].

2.1.9. *Adiabatic equations in the case* $\lambda \neq 1$. Up to this moment we have supposed in our arguments that $\lambda = 1$. Let us see which of them can be extended to the general case $\lambda \neq 1$. We again want to obtain the adiabatic equations from the extremality condition for the action functional restricted to the trajectories in $\mathcal{M}_d$.

So we say that a vortex trajectory $\tau \to [A(\tau), \Phi(\tau)]$ in the space $\mathcal{M}_d$ is *adiabatic* if it is extremal for the action $S(A,\Phi)$ restricted to the trajectories γ lying in $\mathcal{M}_d$.

The *action functional* in this case has the form

$$S(\gamma) = S(A,\Phi) = \int\limits_\gamma \{T(A,\Phi) - U(A,\Phi)\}\, d\tau,$$

where

$$T(A,\Phi) = T(\gamma) = \frac{1}{2}\{\|\dot{A}\|^2 + \|\dot{\Phi}\|^2\},$$
$$U(A,\Phi) = U(\gamma) = \frac{1}{2}\left\{\|dA\|^2 + \|d_A\Phi\|^2 + \frac{\lambda}{4}\|1 - |\Phi|^2\|^2\right\}.$$

The first variation of the functionals T and U is given by the formulas

$$\begin{cases} \delta T(A,\Phi) = -\langle \ddot{A}, \delta A\rangle - \langle \ddot{\Phi}, \delta\Phi\rangle, \\ \delta U(A,\Phi) = -\langle d^*dA + i\,\mathrm{Im}(\bar{\Phi} d_A\Phi), \delta A\rangle - \left\langle d_A^* d_A\Phi - \dfrac{\lambda}{2}\Phi(1-|\Phi|^2), \delta\Phi\right\rangle. \end{cases}$$

Since the pair (A,Φ) satisfies the vortex equations for any τ, it also satisfies the Ginzburg–Landau equations for $\lambda = 1$. So the equation $\delta S(A,\Phi) = 0$ is equivalent to the condition

$$\left(-\ddot{A}, -\ddot{\Phi} + \frac{\lambda - 1}{2}\Phi(1-|\Phi|^2)\right) \perp T_{(A,\Phi)}\mathcal{M}_d.$$

This relation (provided that the gauge-fixing condition is satisfied) is equivalent to the following system of equations written in terms of an L^2-basis $\{n_\mu\}$ of the space $\operatorname{Ker}\mathcal{D}_{(A,\Phi)}$:

$$\left\langle \left(-\ddot{A}, -\ddot{\Phi} + \frac{\lambda - 1}{2}\Phi(1-|\Phi|^2)\right), n_\mu\right\rangle = 0, \qquad \mu = 1,\dots,2d.$$

These are precisely the *adiabatic equations* for $\lambda \neq 1$.

The adiabatic equations

$$\langle \partial_t^2(A,\Phi), n_\mu\rangle = \frac{\lambda - 1}{2}\langle \Phi(1-|\Phi|^2), n_\mu\rangle, \qquad \mu = 1,2,\dots,2d,$$

have the *Newtonian form*; in other words, the left-hand side of these equations can be interpreted as the "acceleration multiplied by mass," while the right-hand side can be considered as the "force." This observation indicates that the above equations are, in fact, *Hamiltonian equations* on $T^*\mathcal{M}_d$ governed by some *adiabatic Hamiltonian*

$$H_{\mathrm{ad}} = T_{\mathrm{ad}} + U_{\mathrm{ad}}.$$

We will obtain an explicit expression for the Hamiltonian H_{ad} in local coordinates on $T^*\mathcal{M}_d$.

Let $\{q_\mu\}$ be the local coordinates on the space $\mathcal{M}_d$ in a neighborhood of a point $q = [A,\Phi] \in \mathcal{M}_d$, and let $\{\dot{q}_\mu\}$ be the local coordinates on $T_q\mathcal{M}_d$. Denote, as above, by $\{n_\mu\}$ the basis of solutions of the equation $\mathcal{D}_{(A,\Phi)}n_\mu = 0$. Then the T-metric on $T_q\mathcal{M}_d$ will be given by the formula

$$T_q(\dot{q},\dot{q}) := \sum_{\mu,\nu=1}^{2d} \langle n_\mu, n_\nu\rangle \dot{q}_\mu \dot{q}_\nu.$$

Denote by $\{p_\mu\}$ the momenta, i.e., the coordinates in the fiber of the bundle $T_q^*\mathcal{M}_d$ that are given by the *Legendre transform*

$$p_\mu := \sum_{\mu=1}^{2d} \langle n_\mu, n_\nu\rangle \dot{q}_\mu.$$

We equip $T_q^*\mathcal{M}_d$ with the dual metric

$$T_q(p,p) := T_q(\dot{q},\dot{q}).$$

Then the *adiabatic Hamiltonian* is given by the formula

$$H_{\mathrm{ad}} := \frac{1}{2}\, T_q(p,p) + U_{\mathrm{ad}}(q),$$

where

$$U_{\mathrm{ad}}(q) := \frac{|\lambda - 1|}{8} \int (1 - |\Phi|^2)^2\, d^2x.$$

The corresponding *Hamiltonian equations* have the form

$$\begin{cases} \dfrac{dp_\mu}{d\tau} = -\dfrac{\partial H_{\mathrm{ad}}}{\partial q_\mu} & \text{(Newton law)}, \\[2ex] \dfrac{dq_\mu}{d\tau} = \dfrac{\partial H_{\mathrm{ad}}}{\partial p_\mu} & \text{(definition of momentum)}. \end{cases}$$

In this case the adiabatic principle states that *any solution of the adiabatic Hamiltonian equations can be approximated with any prescribed accuracy by solutions of the dynamic Ginzburg–Landau equations.*

2.2. Vortex dynamics. Here we show how one can apply the adiabatic principle from the previous subsection to the description of vortex dynamics.

2.2.1. *Scattering of two vortices.* Consider first the scattering problem for two vortices on the complex plane $\mathbb{C}$ in the critical case $\lambda = 1$, following paper [22]. In the adiabatic limit this problem is reduced to the description of the moduli space of two-vortex solutions $\mathcal{M}_2 = \mathrm{Sym}^2\,\mathbb{C}$ equipped with the T-metric.

The natural coordinates on $\mathrm{Sym}^2\,\mathbb{C}$ are provided by the following identification of $\mathrm{Sym}^2\,\mathbb{C}$ with $\mathbb{C}^2$:

$$\mathrm{Sym}^2\,\mathbb{C} \ni (z_1,z_2) \mapsto (z_1+z_2, z_1z_2) \in \mathbb{C}^2.$$

At the center of mass we have the relations

$$z_1 + z_2 = 0, \qquad z_1 z_2 = a^2$$

satisfied for some $a \in \mathbb{C}$. Consider the geodesic $[A(t), \Phi(t)]$ on $\mathcal{M}_2$ written in the form

$$\Phi(z) = (z-a)(z+a)f(z),$$

where a and f depend on t and, in addition, f satisfies the following conditions:

(1) $f > 0$ everywhere on $\mathbb{C}$ (gauge-fixing condition);

(2) $|f(z)| \sim 1/|z|^2$ as $|z| \to \infty$ (asymptotic condition).

The kinetic energy

$$T(A,\Phi) = \int \bigl\{|\dot{A}_1|^2 + |\dot{A}_2|^2 + |\dot{\Phi}|^2\bigr\}\, |dz \wedge d\bar{z}|$$

can be rewritten in the form

$$T = \frac{1}{2}\bigl(\dot{\rho}^2 m_\parallel + \rho^2 \dot{\theta} m_\perp\bigr),$$

where $a = \rho e^{i\theta}$ and

$$m_\parallel = m_\parallel(\rho,\theta) = \int \left\{4\rho^2 f^2 + \frac{1}{4}\frac{\partial f^2}{\partial \rho}\frac{\partial g^2}{\partial \rho}\right\} |dz \wedge d\bar{z}|,$$

$$m_\perp = m_\perp(\rho,\theta) = \int \left\{4\rho^2 f^2 + \frac{1}{4\rho^2}\frac{\partial f^2}{\partial \theta}\frac{\partial g^2}{\partial \theta}\right\} |dz \wedge d\bar{z}|$$

with $g^2(z) = (z-a)^2(z+a)^2$.

Since the kinetic energy does not depend explicitly on t, we should have two integrals for the Euler–Lagrange equations for T. If we write z in polar coordinates $z = re^{i\varphi}$, then these integrals will correspond to the conservation laws for the energy and angular momentum:

$$T =: c_T = \text{const}, \qquad M = \rho^2 \dot{\theta} m_\perp =: c_M = \text{const}.$$

From these conservation laws, for the geodesic $\rho = \rho(\theta)$ we get an equation depending on two given constants c_T and c_M:

$$\theta = \int_\infty^{\rho(\theta)} \frac{\sqrt{m_\parallel/m_\perp}\, d\rho}{\rho\sqrt{2c_T m_\perp c_M^{-2}\rho^2 - 1}}$$

with the asymptotic condition $\rho(\theta) \to \infty$ as $\theta \to 0$.

From this equation we can determine, in particular, the main parameters characterizing the trajectory $\rho = \rho(\theta)$, namely, the *minimal distance to the origin* $\rho_{\min}$ and the *scattering angle* $\Delta\theta$. For the computation of $\rho_{\min}$ we have the equation

$$\frac{d\rho}{d\theta}(\rho_{\min}) = 0 \qquad \Leftrightarrow \qquad \frac{2c_T}{c_M^2} m_\perp(\rho_{\min}) = \frac{1}{\rho_{\min}^2}.$$

The scattering angle is given by the formula

$$\Delta\theta = 2\int_\infty^{\rho_{\min}} \frac{\sqrt{m_\parallel/m_\perp}\, d\rho}{\rho\sqrt{2c_T m_\perp c_M^{-2}\rho^2 - 1}}.$$

The most interesting limiting case corresponds to $\rho_{\min} \to 0$. In this limit the main contribution to the integral defining the scattering angle is made by its part near $\rho \sim 0$. For small ρ one can use the power series expansion of f^2 with respect to ρ^2:

$$f^2 = f_0^2(1 + \rho^2 f_1 + \rho^4 f_2 + \dots),$$

where f_0 is the radial function corresponding to the vortex solution $\Phi_0(z) = z^2 f_0$ (for $a = 0$). In this case

$$m_\perp = \mu\rho^2 + O(\rho^6), \qquad m_\parallel = m_\perp + O(\rho^6),$$

so for small ρ we have

$$\theta(\rho) \sim \int\limits_0^{\lambda(\theta)} \frac{d\lambda}{\sqrt{4c_T\mu c_M^{-2}\lambda^{-2} - \lambda^2}}\,\frac{1}{2}\arcsin\frac{c_M^2\lambda^2(\theta)}{\sqrt{2c_T\mu}},$$

where $\lambda(\theta) = 1/\rho(\theta)$. This implies the equation

$$\rho^2 \sin 2\theta = \frac{c_M}{\sqrt{2c_T\mu}} = \rho_{\min}^2,$$

where the second equality follows from the equation defining $\rho_{\min}$. Hence, the graph of $a = a(t)$ is the hyperbola defined by the equation

$$\operatorname{Re} a \cdot \operatorname{Im} a = \frac{\rho_{\min}^2}{2},$$

and the scattering angle coincides with $\Delta\theta = \pi/2$.

In a similar way we can study another limiting case corresponding to $\rho_{\min} \to \infty$ and show that in this limit $\Delta\theta \to \pi$. This means, in other words, that there is no long-distance action in our model.

Note that our conclusion that *two vortices after a head-on collision scatter at a right angle* also remains true for $\lambda \neq 1$. This is an experimental fact, although it is clear from general considerations that this phenomenon should persist for λ sufficiently close to the critical value $\lambda = 1$. This implies, in particular, that for $\lambda < 1$ sufficiently close to the critical value, there should exist an interesting dynamic solution called the *breather*. It has the following behavior. During the first movement two vortices collide with each other and after the collision scatter at a right angle. However, for $\lambda < 1$ the vortices are attracted to each other, so they cannot fly far away and after some time have to collide once again. After the collision they fly away again at a right angle, and so on. Such a solution is observed in real experiments where the breather can survive several movements until it exhausts its energy resource.

2.2.2. *The "vortex–antivortex" system.* As we have pointed out before, the adiabatic principle allows one to construct approximate dynamic solutions close to static ones (in other words, slowly moving systems of vortices). Now we formulate an important problem that cannot be solved in the framework of the adiabatic approach.

Recall that one of the corollaries of the Taubes theorem states that the static solutions of the Ginzburg–Landau equations are either vortices or antivortices. In other words, in the static case the system under study cannot contain vortices and antivortices simultaneously; such bound states must "annihilate" before the system transforms into the static state.

It is natural to ask whether a stable bound state of the *vortex–antivortex* type can be realized in the dynamic case. Such a solution exists, for example, in hydrodynamics. Indeed, one can remember the "smoke rings" well known to all smokers. If we cut such a "ring," which is a torus, by the plane along the meridian, then in the section we will have precisely a stable state of the vortex–antivortex type. However, it is still unknown whether such a state can be realized in the case of the Ginzburg–Landau system. Such a solution, if it exists, must move with velocity greater than a certain threshold value, so it cannot be constructed by using the adiabatic limit.

General considerations imply that the vortex and antivortex in such a system must move parallel to each other, so it seems that this solution cannot be realized on the plane (since due to the imposed asymptotic conditions the zeros of the function Φ cannot go to infinity). However, a solution of the vortex–antivortex type may exist on the torus.

2.2.3. *Centrally symmetric scattering of a system of d vortices.* Unfortunately, for the kinetic energy of a system of $d > 2$ vortices, we have no explicit formulas similar to those in the case $d = 2$. However, using only general properties of the kinetic metric, it is possible to obtain an extension of the result of Subsection 2.2.1 on the right angle scattering of two vortices under a head-on collision to the case of centrally symmetric scattering of a system of d vortices. This generalization, obtained by Palvelev [15], is presented in this subsection.

As we observed in Subsection 1.3.1, the moduli space of d-vortices $\mathcal{M}_d$ can be identified, according to the Taubes theorem, with the dth symmetric power $\operatorname{Sym}^d \mathbb{C}$ of the complex plane $\mathbb{C}$; namely, an arbitrary collection $\{z_1, \dots, z_d\}$ of d points on the complex plane (counted with multiplicities) can be assigned the monic polynomial $p(z)$ with zeros precisely at the points $z_1, \dots, z_d$:

$$p(z) = (z - z_1)(z - z_2) \dots (z - z_d) = z^d + s_1 z^{d-1} + \dots + s_{d-1} z + s_d.$$

With a given collection of points $\{z_1, \dots, z_d\} \in \mathcal{M}_d$ we associate a particular d-vortex solution (A_1, A_2, Φ) corresponding to the collection $\{z_1, \dots, z_d\}$ by virtue of the Taubes theorem: we fix the gauge by taking Φ in the form

$$\Phi(z) = (z - z_1)(z - z_2) \dots (z - z_d) f(z) \qquad \text{with} \quad f(z) > 0. \tag{2.11}$$

Now we choose the symmetric functions $s_1, \dots, s_d$ of the zeros of Φ (which coincide with the coefficients of the polynomial with zeros at the points $\{z_1, \dots, z_d\}$) as the coordinates on the space $\mathcal{M}_d$. Fixing the gauge by taking Φ in the form (2.11), we can consider the components of Φ, A_1, and A_2 of the vortex solution as functions of the complex parameters $s_1, \dots, s_d$. Differentiating them with respect to these parameters, we obtain solutions of the linearized vortex equations. Note that Φ, A_1, and A_2 *depend smoothly on* $s_1, \dots, s_d$ in the sense that for any fixed z^0 the function $\Phi(z^0; s_1, \dots, s_d)$ is a smooth function of $s_1, \dots, s_d$ and the same is true for A_1 and A_2.

We write the complex-valued function s_α, $\alpha = 1, \dots, d$, in the form $s_\alpha = s_{\alpha,1} + i s_{\alpha,2}$. We would like to write the *kinetic metric* on $\mathcal{M}_d$ in terms of the L^2-norms of the derivatives of the functions $(A_1, A_2, \Phi_1, \Phi_2)$ with respect to the variables $s_{\alpha,j}$, where $\alpha = 1, \dots, d$ and $j = 1, 2$. Unfortunately, these derivatives may turn out to be not square integrable; however, we can always replace the collection of functions $(A_1, A_2, \Phi_1, \Phi_2)$ by a gauge equivalent collection $(\widetilde{A}_1, \widetilde{A}_2, \widetilde{\Phi}_1, \widetilde{\Phi}_2)$ so that the obtained functions are L^2-integrable.

In more detail, we can find gauge factors $\chi_{\alpha,j}$ such that the functions

$$n_{\alpha,j} = \left(\frac{\partial \Phi_1}{\partial s_{\alpha,j}} - \chi_{\alpha,j} \Phi_2, \frac{\partial \Phi_2}{\partial s_{\alpha,j}} + \chi_{\alpha,j} \Phi_1, \frac{\partial A_1}{\partial s_{\alpha,j}} + \partial_1 \chi_{\alpha,j}, \frac{\partial A_2}{\partial s_{\alpha,j}} + \partial_2 \chi_{\alpha,j} \right) \tag{2.12}$$

belong to $(H^1)^4$ and satisfy the orthogonality condition. At the same time the map

$$(s_1, \dots, s_d) \mapsto n_{\alpha,j}(s_1, \dots, s_d),$$

acting from $\mathbb{C}^d$ to $(L^2)^4$, is smooth. The vectors $n_{\alpha,j}$, $\alpha = 1, \dots, n$, $j = 1, 2$, belong to the subspace $\operatorname{Ker} \mathcal{D}_{(A,\Phi)}$ and are linearly independent. So they form a basis of the subspace $\operatorname{Ker} \mathcal{D}_{(A,\Phi)}$. Using this basis, we can identify $T_{(A,\Phi)} \mathcal{M}_d$ with $\operatorname{Ker} \mathcal{D}_{(A,\Phi)}$.

Now we define the *kinetic metric* g on $\mathcal{M}_d$ by setting

$$g(v_1, v_2) := (v_1, v_2)_{(L^2)^4}$$

for arbitrary $v_1, v_2 \in T_{(A,\Phi)}\mathcal{M}_d = \operatorname{Ker} \mathcal{D}_{(A,\Phi)} \subset (H^1)^4$. Since the basis vectors $n_{\alpha,j}$ depend smoothly on $s_1, \dots, s_d$, the coefficients of the metric, which are equal to $g_{\alpha,j;\beta,k} = (n_{\alpha,j}, n_{\beta,k})_{(L^2)^4}$, are smooth functions of $s_1, \dots, s_d$.

Although we have no explicit formulas for this metric, we can describe the qualitative picture of vortex scattering by using its symmetry properties, namely, its invariance with respect to rotations and the complex conjugation. From these properties we can deduce the following assertion: *Under a centrally symmetric head-on collision the trajectories of the interacting vortices after collision rotate through the angle π/d.*

Let us describe the geodesic corresponding to this scattering process in the symmetric coordinates $s_1, \dots, s_d$. Consider the geodesic passing through the origin $s_1 = \ldots = s_d = 0$ and having the tangent vector $\vec{v} = (0, 0, \dots, 0, (-1)^{d+1})$ at this point. (The sign in this formula appears because of the relation $s_d = (-z_1)(-z_2)\dots(-z_d) = (-1)^d z_1 z_2 \dots z_d$.) This geodesic is written down in the form

$$s_1 = \ldots = s_{d-1} = 0, \qquad s_d(t) = \lambda(t),$$

where $\lambda(t)$ is a real-valued function with $\lambda(0) = 0$. Indeed, suppose that our geodesic is parameterized by $s_1 = s_1(t)$, …, $s_d = s_d(t)$. The rotation of the zeros of Φ through the angle φ, as defined by the map $z'_k = e^{i\varphi} z_k$, corresponds in the symmetric coordinates to the transformation

$$s'_1 = e^{i\varphi} s_1, \qquad s'_2 = e^{2i\varphi} s_2, \qquad \dots, \qquad s'_d = e^{id\varphi} s_d.$$

This implies that the rotation through the angle $2\pi/d$ does not influence the vector $\vec{v}$, because it is tangent to the trajectory $s_1 = \ldots = s_{d-1} = 0$, $(-1)^d s_d = -t$ invariant under this rotation. So the rotation through the angle $2\pi/d$ transforms our geodesic into a geodesic that also passes through the origin and has the same tangent vector $\vec{v}$ at this point. By the uniqueness property the two geodesics must coincide, which implies that our geodesic must be invariant under this transformation: $s_1(t) = e^{2\pi/d} s_1(t)$, …, $s_{d-1}(t) = e^{2\pi(d-1)/d} s_{d-1}(t)$. Hence $s_1(t) = \ldots = s_{d-1}(t) = 0$.

Now we use the complex conjugation. In the symmetric coordinates $s_1, \dots, s_d$ this operation corresponds to the transformation

$$s_1 \mapsto s'_1 = \bar{s}_1, \qquad s_2 \mapsto s'_2 = \bar{s}_2, \qquad \dots, \qquad s_d \mapsto s'_d = \bar{s}_d.$$

This transformation does not influence the vector $\vec{v}$ and so again leaves our geodesic invariant. It follows that the function $(-1)^d s_d(t) = \lambda(t)$ is real-valued. Moreover, $\lambda(0) = 0$, $\lambda(t)$ decreases when t increases, and $\lambda(t)$ changes its sign from the positive to negative one when t passes through zero.

In terms of the original coordinates $(z_1, \dots, z_d)$ the geodesic in question (which is smooth in the symmetric coordinates) describes the following motion of the system of d vortices. For $\lambda(t) > 0$ (i.e., for $t < 0$) we have d trajectories given by the formula

$$t \mapsto \bigl(\sqrt[d]{\lambda(t)},\, \sqrt[d]{\lambda(t)}\, e^{2\pi i/d},\, \dots,\, \sqrt[d]{\lambda(t)}\, e^{2\pi i(d-1)/d}\bigr),$$

while for $\lambda(t) < 0$ (i.e., for $t > 0$) we have d trajectories rotated with respect to the original ones through the angle π/d:

$$t \mapsto \bigl(\sqrt[d]{|\lambda(t)|}\, e^{\pi i/d},\, \sqrt[d]{|\lambda(t)|}\, e^{3\pi i/d},\, \dots,\, \sqrt[d]{|\lambda(t)|}\, e^{(2d-1)\pi i/d}\bigr).$$

In other words, this geodesic describes the scattering process of the following type. Up until the collision d vortices move towards the origin along the rays $\arg z = 0$, $\arg z = 2\pi/d$, …,

$\arg z = 2(d-1)\pi/d$. They collide at the origin and then run away along the rays $\arg z = \pi/d, \dots,$ $\arg z = (2(d-1)+1)\pi/d$.

In particular, for $d = 2$ we have a system of two vortices that move towards each other along the real axis, collide, and fly away along the imaginary axis. Hence, in this case we have the scattering at a right angle.

A detailed proof of these assertions is contained in [15]. (It is also worth comparing the considered problem on the scattering of vortices with a close problem on the scattering of monopoles that was studied in [1].)

2.2.4. *Periodic vortices.* As another application of the adiabatic principle, we will describe a periodic two-vortex solution on the Riemann sphere $S^2 = \mathbb{CP}^1$ (it was found by Stuart; see [23]).

Consider the *Abelian* $(2+1)$-*dimensional Higgs model* on the manifold

$$X = \mathbb{R}_t \times S^2$$

equipped with the Lorentz metric $ds^2 = dt^2 - g$, where g is the standard Riemannian metric on the sphere S^2_R of radius R in $\mathbb{R}^3$. The action in this model is given by the formula

$$S_{\lambda,\tau}(A,\Phi) = \int \{T(A,\Phi) - U_{\lambda,\tau}(A,\Phi)\}\, dt,$$

where

$$T(A,\Phi) = \frac{1}{2}\int_{S^2} \{|\dot{A} - dA_0|^2 + |\dot{\Phi} - A_0\Phi|^2\}\, d\mathrm{vol},$$

$$U_{\lambda,\tau}(A,\Phi) = \frac{1}{2}\int_{S^2} \left\{|dA|^2 + |d_A\Phi|^2 + \frac{\lambda}{4}(\tau - |\Phi|^2)^2\right\} d\mathrm{vol}.$$

Here, A is a U(1)-connection in a Hermitian line bundle $L \to S^2$ equipped with the Hermitian metric h, d_A is the associated covariant exterior derivative, and $d$vol is the volume form on the sphere S^2_R.

We suppose that L is extended to a Hermitian line bundle $\mathcal{L} \to X = \mathbb{R} \times S^2$ equipped with the U(1)-connection

$$\mathcal{A} = A_0\, dt + A = A_0\, dt + A_1\, dx_1 + A_2\, dx_2,$$

and Φ is a section of the bundle $\mathcal{L} \to X$. Assume also that the necessary solvability condition for the vortex equations on S^2 is satisfied, namely,

$$\tau > 4\pi \frac{d}{\mathrm{Vol}(S^2)}.$$

Consider dynamic solutions for τ close to the critical value

$$\tau_{\mathrm{cr}} = \frac{4\pi d}{\mathrm{Vol}(S^2)}.$$

Introduce the affine coordinates $x = (x_1, x_2)$ on $S^2_R \setminus \infty$ by using the stereographic projection $S^2_R \setminus \infty \to \mathbb{R}^2_{(x_1,x_2)}$ and identify $\mathbb{R}^2_{(x_1,x_2)}$ with the complex plane $\mathbb{C}_z$ equipped with the complex coordinate $z = x_1 + ix_2$. Suppose that the Hermitian metric h on L, transported to $\mathbb{C}_z$ with the help of the stereographic projection, is determined by a function $h(z)$ such that $|\Phi(z)|^2_h = h(z)|\Phi|^2$. The stereographic metric on $\mathbb{R}^2_{(x_1,x_2)}$ has the form

$$d\mathrm{vol} = \Lambda^2\, dx_1\, dx_2 \qquad \text{with} \quad \Lambda = \frac{4R^2}{(1+|x|^2)^2}.$$

The *dynamic Euler–Lagrange equations* for our action have the form

$$\begin{cases} \partial^2_{t,A_0}\Phi - \dfrac{1}{h\Lambda^2}\displaystyle\sum_{j=1}^{2}\partial_{j,A_j}(h\partial_{j,A_j}\Phi) - \dfrac{\lambda}{2}\,\Phi(\tau - |\Phi|^2) = 0, \\ \ddot{A}_j + \partial_j\dot{A}_0 + \varepsilon_{jk}\partial_k\left(\dfrac{F_{12}}{\Lambda^2}\right) = i\,\mathrm{Im}(\bar{\Phi}\partial_{j,A_j}\Phi), \quad j = 1,2, \\ \partial_j\dot{A}_j - \Delta A_0 = i\Lambda^2\,\mathrm{Im}(\bar{\Phi}\partial_{t,A_0}\Phi). \end{cases}$$

In the adiabatic limit these equations reduce to the Hamiltonian equations governed by the adiabatic Hamiltonian

$$H_{\mathrm{ad}} = T_{\mathrm{ad}} + U_{\mathrm{ad}}$$

on the moduli space of 2-vortices

$$\mathcal{M}_2 = \mathrm{Sym}^2 S^2 \cong \mathbb{C}\mathrm{P}^2.$$

In order to describe this Hamiltonian more explicitly, consider the affine part $\mathbb{C}^2$ of $\mathcal{M}_2$ with coordinates (z_1, z_2), assuming that the zeros of Φ are contained in $\mathbb{C}^2$, and introduce the coordinates of the center of mass

$$z_1 + z_2 = 0, \qquad z_1 z_2 = a^2,$$

where $a \in \mathbb{C}$ is written in the polar form as $-a^2 = \rho e^{i\theta}$. We use the small parameter $\delta > 0$ given by

$$\delta^2 = 4\pi(\tau R^2 - d),$$

where $d = 2$.

In these coordinates

$$T_{\mathrm{ad}} = \frac{1}{2}F(\rho)(\dot{\rho}^2 + \rho^2\dot{\theta}^2)$$

with

$$F(\rho) = 2\delta^2\frac{\rho^2 + 4\rho + 1}{(1+\rho)^2(1+\rho^2)^2} + O(\delta^4).$$

The *potential energy* is defined by the formula

$$U_{\mathrm{ad}} = \frac{|\lambda - 1|}{8}\int_{S^2}(\tau - |\Phi|^2)^2\,d\mathrm{vol}$$

and depends only on ρ (i.e., on the distance between the vortices). It admits the following power series expansion with respect to the small parameter δ:

$$U_{\mathrm{ad}} = \frac{|\lambda - 1|}{8}\left(4\pi\tau d - \tau\delta^2 + \frac{3\delta^4}{20\pi R^2} + \dots\right).$$

Denote by $r(\theta)$ the rotation of the complex plane $\mathbb{C}$ through the angle θ: $z \mapsto e^{i\theta}z$, and by $r_*(\theta)$ the induced action on pairs (A, Φ). (Note that the pull-back of $r(\theta)$ to L is defined up to gauge transformations, so we should also fix some pull-back of $r(\theta)$ to L.) By a *periodic trajectory* (of frequency ω) in the space $\mathcal{V}_d$ consisting of d-vortex solutions, we will mean any trajectory $t \mapsto (\widetilde{A}(t), \widetilde{\Phi}(t))$ in $\mathcal{V}_d$ of the form

$$\widetilde{A}(t) = r_*(\omega t)A + id\chi, \qquad \widetilde{\Phi}(t) = r_*(\omega t)\Phi e^{i\chi},$$

where $\chi = \chi(t,x)$ is obtained from the real-valued function $\chi_0(x)$ by averaging with respect to the circle action:

$$\chi(t,x) = \int\limits_0^{\omega t} \chi_0(e^{i\omega s}x)\, ds,$$

and the pair $(\widetilde{A}(t), \widetilde{\Phi}(t))$ satisfies the gauge-fixing condition

$$\delta^*_{(\widetilde{A},\widetilde{\Phi})}(\partial_t \widetilde{A}, \partial_t \widetilde{\Phi}) = 0.$$

Stuart showed in [23] that for sufficiently small $\tau - \tau_{\rm cr}$ and $|\lambda - 1|$ the adiabatic equations governed by the Hamiltonian $H_{\rm ad}$ have a periodic solution satisfying the conditions

$$\{\text{zeros of } \Phi\} = \pm\sqrt{\rho}, \qquad \{\text{zeros of } \widetilde{\Phi}(t)\} = \pm\sqrt{\rho}e^{i\omega_0 t}$$

for some ω_0. Moreover, he proved that for $\lambda = 1 - \varepsilon^2$ with sufficiently small ε the dynamic equations have a periodic solution close to an adiabatic one, with frequency of order ε and period of order $1/\varepsilon$. This proves the validity of the adiabatic principle for $\lambda \neq 1$ in this particular case.

2.2.5. *Abrikosov strings.* The method of the adiabatic limit can also be applied to the *Euclidean model* governed by the Ginzburg–Landau action functional in the space $\mathbb{R}^3$ with coordinates $x = (x_1, x_2, x_3)$, which describes the *Abrikosov strings* in $\mathbb{R}^3$. This functional has the form

$$E(A,\Phi) = \frac{1}{2}\int \left\{ |dA|^2 + |d_A\Phi|^2 + \frac{\lambda}{4}(1 - |\Phi|^2)^2 \right\} d^3x,$$

where A is a U(1)-connection on $\mathbb{R}^3$ given by the 1-form $A = \sum_{i=1}^3 A_i\, dx_i$ with smooth pure imaginary coefficients $A_i = A_i(x)$, and $\Phi = \Phi(x)$ is a smooth complex-valued function on $\mathbb{R}^3$. We will assume below that the gauge is chosen so that $A_3 = 0$.

The *Euler–Lagrange equations* for the functional $E(A,\Phi)$, which are otherwise called the *Euclidean Ginzburg–Landau equations*, have the same form as in the two-dimensional case:

$$\begin{cases} d^* F_A = i\,\mathrm{Im}(\bar{\Phi} d_A \Phi), \\ d_A^* d_A \Phi = \dfrac{\lambda}{2}\,\Phi(1 - |\Phi|^2). \end{cases}$$

A trajectory $\xi \mapsto [A(\xi), \Phi(\xi)]$ in $\mathcal{M}_d$ is called *adiabatic* if it is extremal for the functional $E(A,\Phi)$ restricted to the trajectories lying in $\mathcal{M}_d$. The gauge-fixing condition has the same form as in the dynamic $(2+1)$-dimensional case:

$$\delta^*_{(A,\Phi)}(\partial_3 A, \partial_3 \Phi) = 0.$$

As in the $(2+1)$-dimensional case, from the Euler–Lagrange equations for $E(A,\Phi)$ one can deduce the *adiabatic condition* of the form

$$\left(-\partial_3^2 A, -\partial_3^2 \Phi + \frac{1-\lambda}{2}\Phi(1 - |\Phi|^2) \right) \perp T_{(A,\Phi)}\mathcal{M}_d$$

(the only difference with the $(2+1)$-dimensional case is the change of the sign before the term $\Phi(1 - |\Phi|^2)$). It is equivalent (under the gauge-fixing condition) to the relation

$$\left(-\partial_3^2 A, -\partial_3^2 \Phi + \frac{1-\lambda}{2}\Phi(1 - |\Phi|^2) \right) \perp \operatorname{Ker}\mathcal{D}_{(A,\Phi)}.$$

It is a Hamiltonian equation on $T^*\mathcal{M}_d$ with Hamiltonian $H_{\rm ad}$ of the form

$$H_{\rm ad}(A,\Phi)=\frac{1}{2}\biggl\{\|\partial_3 A\|^2_{L^2}+\|\partial_3\Phi\|^2_{L^2}+\frac{|1-\lambda|}{4}\|1-\|\Phi|^2\|^2_{L^2}\biggr\}.$$

Using an L^2-basis $\{n_\mu\}$ of solutions of the linearized vortex equations

$$\mathcal{D}_{(A,\Phi)}n_\mu=0,\qquad \mu=1,\dots,2d,$$

we can rewrite the *adiabatic equation* in the form

$$\biggl\langle \partial^2_\xi(A,\Phi)+\frac{\lambda-1}{2}\Phi(1-|\Phi|^2),n_\mu\biggr\rangle=0,\qquad \mu=1,\dots,2d.$$

Solutions of this equation approximately describe the *Abrikosov strings close to the axis* (x_3).

3. CLIFFORD ALGEBRAS AND SPINOR GEOMETRY

This section is a digression in which we have collected basic notions from the theory of Clifford algebras and spinor geometry that will be used in the next section in the theory of Seiberg–Witten equations. A detailed exposition of spinor geometry can be found in book [12].

3.1. Clifford algebras and spinor groups.

3.1.1. *Clifford algebras.* Let V be an n-dimensional Euclidean vector space and $\{e_i\}_{i=1}^n$ be an orthonormal basis of V. The *Clifford algebra* $\mathrm{Cl}(V)$ is defined as an $\mathbb{R}$-algebra with unit 1 generated by the elements $1,e_1,e_2,\dots,e_n$ that satisfy the following relations:

$$e_i^2=-1,\qquad e_ie_j+e_je_i=0\quad\text{for}\quad i\neq j.$$

Note that $V\subset\mathrm{Cl}(V)$ and

$$uv+vu=-2(u,v)\qquad\text{for}\quad u,v\in V.$$

As a real vector space, $\mathrm{Cl}(V)$ is 2^n-dimensional and has a basis consisting of elements of the form 1 and $e_I:=e_{i_1}e_{i_2}\dots e_{i_k}$, where $I=\{i_1,i_2,\dots,i_k\}$ is a subset of $\{1,2,\dots,n\}$ composed of strictly increasing indices $i_1<i_2<\dots<i_k$ and $|I|:=k$.

Denote by $\mathrm{Cl}_k(V)$ the subset consisting of elements of degree k, and consider the subsets

$$\mathrm{Cl}_{\rm ev}(V):=\bigoplus_{k\ \text{even}}\mathrm{Cl}_k(V),\qquad \mathrm{Cl}_{\rm od}(V):=\bigoplus_{k\ \text{odd}}\mathrm{Cl}_k(V).$$

Then $\mathrm{Cl}_{\rm ev}(V)$ is a subalgebra of $\mathrm{Cl}(V)$ and

$$\mathrm{Cl}(V)=\mathrm{Cl}_{\rm ev}(V)\oplus\mathrm{Cl}_{\rm od}(V),$$

which equips $\mathrm{Cl}(V)$ with the structure of a *superalgebra.*

The Clifford algebra $\mathrm{Cl}(V)$ can be equipped with the *inner product* extending the inner product on V and with *conjugation* defined by the formula

$$x=\sum_{|I|=k}x_Ie_I\mapsto x^*=\sum_{|I|=k}\varepsilon_Ix_Ie_I,$$

where $\varepsilon_I=(-1)^{k(k+1)/2}$ on elements of degree k.

3.1.2. *Universal property.* The definition of the Clifford algebra $\mathrm{Cl}(V)$ does not in fact depend on the choice of the orthonormal basis because of the following *universal property*, which can be taken as the definition of the Clifford algebra. Namely, $\mathrm{Cl}(V)$ is a unique associative $\mathbb{R}$-algebra with unit and conjugation that contains V and has the following property: for any associative $\mathbb{R}$-algebra A with unit 1_A and conjugation $a \mapsto a^*$ and for an arbitrary linear map $f\colon V \to A$ satisfying the conditions

$$f^*(v) + f(v) = 0, \qquad f^*(v)f(v) = |v|^2\, 1_A,$$

there exists a unique extension of f to an algebra homomorphism $\widetilde{f}\colon \mathrm{Cl}(V) \to A$ preserving the conjugations:

$$\begin{array}{ccc} V & \xrightarrow{\ f\ } & A \\ {\scriptstyle i}\downarrow & \nearrow_{\widetilde{f}} & \\ \mathrm{Cl}(V) & & \end{array}$$

Example 1. Here are some standard examples of Clifford algebras:

(1) $\mathrm{Cl}(\mathbb{R}) = \mathbb{C}$ with $e_1 = i$;

(2) $\mathrm{Cl}(\mathbb{R}^2) = \mathbb{H}$ with $e_1 = i$, $e_2 = j$, and $e_1e_2 = k$;

(3) $\mathrm{Cl}(\mathbb{R}^4) = \mathbb{H}[2 \times 2]$ (2×2 matrices)

where $\mathbb{H}$ denotes the algebra of quaternions.

3.1.3. *Multiplicative group.* Let $\mathrm{Cl}^*(V)$ denote the group of invertible elements of the Clifford algebra $\mathrm{Cl}(V)$. Note that the set $V \setminus \{0\}$ is contained in $\mathrm{Cl}^*(V)$ because $v^{-1} := -v/|v|^2$ for $v \in V^*$. The group $\mathrm{Cl}^*(V)$ acts on $\mathrm{Cl}(V)$ by the *adjoint representation*

$$g \mapsto \mathrm{Ad}_g(x) := gxg^{-1},$$

where $g \in \mathrm{Cl}^*(V)$. For any $u \in V \setminus \{0\}$ and $v \in V$, the map

$$-\mathrm{Ad}_u(v) = v - \frac{2(u,v)}{|u|^2}u$$

is the reflection with respect to the hyperplane $u^\perp$ orthogonal to u.

In order to get rid of the minus sign on the left-hand side of the last formula, we introduce another action of the group $\mathrm{Cl}^*(V)$ on $\mathrm{Cl}(V)$ given by the *twisted adjoint representation*

$$g \mapsto \pi_g(x) := \alpha(g)xg^{-1},$$

where $g \in \mathrm{Cl}^*(V)$, $x \in \mathrm{Cl}(V)$, and $\alpha(g) := (-1)^{\deg g} g$ is the *grading map* ($\deg g$ is the degree of g). Then for $u \in V \setminus \{0\}$ the map $\pi_u\colon V \to V$ is the *reflection with respect to* $u^\perp$. Moreover, for $u \in V$ with $|u| = 1$ we have the equality

$$\pi_u(v) = uvu^*.$$

3.1.4. *The group* Pin. The *group* $\mathrm{Pin}(V)$ is defined as the subgroup of $\mathrm{Cl}^*(V)$ generated by the unit vectors $v \in V$, i.e., by the vectors v with $|v| = 1$. Since any such vector v generates the reflection π_v, i.e., an orthogonal transformation of the space V, we have a homomorphism

$$\pi\colon\ \mathrm{Pin}(V) \to \mathrm{O}(V).$$

On the other hand, since any orthogonal transformation is a composition of reflections, this map is an epimorphism. Moreover, there is an exact sequence

$$0 \to \mathbb{Z}_2 \to \mathrm{Pin}(V) \xrightarrow{\pi} \mathrm{O}(V) \to 0.$$

3.1.5. *The group* Spin. The *group* $\mathrm{Spin}(V)$ is by definition the identity connected component in the group $\mathrm{Pin}(V)$. It can be also defined as

$$\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap \mathrm{Cl}_{\mathrm{ev}}(V).$$

As in the case of the group $\mathrm{Pin}(V)$, there is an exact sequence

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}(V) \xrightarrow{\pi} \mathrm{SO}(V) \to 0.$$

The above definition of the group $\mathrm{Spin}(V)$ is equivalent to the following one:

$$\mathrm{Spin}(V) = \{x\colon\ x^*x = 1,\ xVx^* = V\}.$$

Example 2. Here are examples of Spin-groups:

(1) $\mathrm{Spin}(\mathbb{R}) = 1$;

(2) $\mathrm{Spin}(\mathbb{R}^2) = \mathrm{U}(1)$;

(3) $\mathrm{Spin}(\mathbb{R}^4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$.

3.1.6. *The group* $\mathrm{Spin}^{\mathrm{c}}$. Denote by $\mathrm{Cl}^{\mathrm{c}}(V) := \mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C}$ the *complexified Clifford algebra* and equip it with the Hermitian inner product and conjugation that extend the corresponding operations on $\mathrm{Cl}(V)$. Define the *group* $\mathrm{Spin}^{\mathrm{c}}(V)$ as

$$\mathrm{Spin}^{\mathrm{c}}(V) := \bigl\{z \in \mathrm{Cl}^{\mathrm{c}}_{\mathrm{ev}}(V)\colon\ z^*z = 1,\ zVz^* = V\bigr\}.$$

For this group the formula $\pi_z(v) = zvz^*$ for $v \in V$ defines a map

$$\pi\colon\ \mathrm{Spin}^{\mathrm{c}}(V) \to \mathrm{SO}(V),$$

and we have the exact sequence

$$0 \to \mathrm{U}(1) \to \mathrm{Spin}^{\mathrm{c}}(V) \xrightarrow{\pi} \mathrm{SO}(V) \to 0.$$

Note that the group $\mathrm{Spin}^{\mathrm{c}}(V)$ is the *circle extension* of the group $\mathrm{Spin}(V)$, i.e.,

$$\mathrm{Spin}^{\mathrm{c}}(V) = \bigl\{z = e^{i\theta}x\colon\ x \in \mathrm{Spin}(V),\ \theta \in \mathbb{R}\bigr\}.$$

Hence we have the exact sequence

$$0 \to \mathrm{Spin}(V) \to \mathrm{Spin}^{\mathrm{c}}(V) \xrightarrow{\delta} \mathrm{U}(1) \to 0,$$

where $\delta\colon xe^{i\theta} \mapsto e^{2i\theta}$. So

$$\mathrm{Spin}^{\mathrm{c}}(V) = \mathrm{Spin}(V) \times_{\mathbb{Z}_2} \mathrm{U}(1).$$

Combining the two exact sequences given above, we arrive at

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}^{\mathrm{c}}(V) \xrightarrow{(\pi,\delta)} \mathrm{SO}(V) \times \mathrm{U}(1) \to 0.$$

Note that the Lie algebra of the group $\mathrm{Spin}^{\mathrm{c}}(V)$ coincides with

$$\mathrm{spin}^{\mathrm{c}}(V) = \mathrm{cl}_2(V) \oplus i\mathbb{R},$$

where $\mathrm{cl}_2(V)$ is the component $\mathrm{Cl}_2(V)$ of the Clifford algebra equipped with the natural Lie algebra structure given by the commutator of elements.

Example 3. Here are examples of $\mathrm{Spin}^{\mathrm{c}}$-groups:

(1) $\mathrm{Cl}^{\mathrm{c}}(\mathbb{R}) = \mathbb{C} \oplus \mathbb{C}$, and $\mathrm{Spin}^{\mathrm{c}}(\mathbb{R})$ coincides with the group $\mathrm{U}(1)$ embedded in $\mathbb{C} \oplus \mathbb{C}$ by the diagonal map;

(2) $\mathrm{Cl}^{\mathrm{c}}(\mathbb{R}^2) = \mathbb{C}[2 \times 2]$, and $\mathrm{Spin}^{\mathrm{c}}(\mathbb{R}^2) = \mathrm{U}(1) \times \mathrm{U}(1)$, i.e., it consists of unitary diagonal matrices in $\mathbb{C}[2 \times 2]$.

3.1.7. *Spin representation.* The *spin representation* is defined as the linear map

$$\Gamma\colon\ V \to \operatorname{End} W$$

from the $2n$-dimensional Euclidean vector space V into the endomorphism group of a 2^n-dimensional Hermitian complex vector space W provided that the following conditions hold:

$$\Gamma^*(v) + \Gamma(v) = 0, \qquad \Gamma^*(v)\Gamma(v) = |v|^2 \operatorname{id}.$$

By the universal property, it extends to an algebra isomorphism

$$\Gamma\colon\ \mathrm{Cl^c}(V) \to \operatorname{End} W.$$

The action of $\mathrm{Cl^c}(V)$ on W is called the *Clifford multiplication*, and the elements of W are called *spinors*.

Define the *Clifford volume element* ω by setting

$$\omega := e_1 e_2 \dots e_{2n} \in \mathrm{Cl}_{2n}(V).$$

Then

$$\omega^2 = (-1)^{2n}, \qquad \omega v + v\omega = 0 \quad \text{for all} \quad v \in V.$$

Hence we can introduce the *semi-spinor spaces*

$$W^{\pm} := \{w \in W\colon\ \Gamma(\omega)w = \pm i^n w\}.$$

Then we obtain

$$W = W^+ \oplus W^-$$

and

$$\Gamma(v)\colon\ W^{\pm} \to W^{\mp} \qquad \text{for all} \quad v \in V.$$

Note that the subspaces $W^{\pm}$ are invariant under the Clifford multiplication by the elements of even degree.

Example 4. Here are examples of spin representations:

(1) $\Gamma\colon \mathrm{Cl^c}(\mathbb{R}^2) \to \mathbb{C}[2\times 2]$ is the *complexified Pauli map* γ^{c}, where

$$\gamma\colon\ \mathbb{H} \ni x = (x_0, x_1, x_2, x_3) \mapsto \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} \in \mathbb{C}[2\times 2];$$

(2) the map $\Gamma\colon \mathrm{Cl^c}(\mathbb{R}^4) = \mathrm{Cl^c}(\mathbb{H}) \to \mathbb{C}[4\times 4]$ is generated by the *complexified Dirac map* Γ^{c}, where

$$\Gamma\colon\ \mathbb{H} \ni x \mapsto \begin{pmatrix} 0 & \gamma(x) \\ -\gamma^*(x) & 0 \end{pmatrix}$$

and γ is the Pauli map. Under this map the group $\mathrm{Spin^c}(\mathbb{H}) = \mathrm{Spin^c}(\mathbb{R}^4)$ is realized as

$$\begin{aligned}\mathrm{Spin^c}(\mathbb{R}^4) &= \{(U,V) \in \mathrm{U}(W^+) \times \mathrm{U}(W^-)\colon\ \det U = \det V\} \\ &= \{(U,V) \in \mathrm{U}(2) \times \mathrm{U}(2)\colon\ \det U = \det V\},\end{aligned}$$

which implies that

$$\mathrm{Spin^c}(\mathbb{R}^4) = (\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_2 = \mathrm{Spin}(\mathbb{R}^4) \times_{\mathbb{Z}_2} \mathrm{U}(1).$$

3.1.8. *Exterior algebra.* Consider the exterior algebra $\Lambda^*(V)$ of the space V and introduce the map

$$\mathrm{Alt}_k\colon\ V\times\ldots\times V\to \mathrm{Cl}_k(V)$$

given by the formula

$$(v_1,\ldots,v_k)\mapsto \mathrm{Alt}_k(v_1,\ldots,v_k)=\frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)v_{\sigma(1)}\ldots v_{\sigma(k)}.$$

It generates the linear isomorphism

$$\mathrm{Alt}\colon\ \Lambda^*(V)\xrightarrow{\cong}\mathrm{Cl}(V).$$

By duality, we also have the isomorphism

$$\mathrm{Alt}^*\colon\ \Lambda^*(V^*)\to \mathrm{Cl}(V^*)\cong \mathrm{Cl}(V).$$

Using the spin representation $\Gamma\colon \mathrm{Cl}(V)\to \mathrm{End}\,W$, we can define

$$\rho:=\Gamma\circ\mathrm{Alt}^*\colon\ \Lambda^*(V^*)\to \mathrm{End}\,W.$$

The introduced map ρ determines the Clifford multiplication by forms from $\Lambda^*(V^*)$ in the space W. In particular, the Clifford multiplication by 2-forms leaves the subspaces $W_\pm$ invariant. More precisely, the map ρ sends real-valued 2-forms to skew-symmetric traceless endomorphisms of the subspaces $W^\pm$ and imaginary-valued 2-forms to Hermitian traceless endomorphisms of these subspaces.

If $\dim V=4$, then the subspace $\Lambda^2(V^*)$ decomposes into the direct sum

$$\Lambda^2(V^*)=\Lambda^2_+\oplus\Lambda^2_-$$

of the subspaces of self-dual and anti-self-dual forms with respect to Hodge $*$-operator. In this case the map ρ induces the isomorphisms: $\Lambda^2_\pm\to \mathrm{su}(W^\pm)$ and

$$\rho^\pm\colon\ \Lambda^2_\pm\otimes i\mathbb{R}\to \mathrm{Herm}_0(W^\pm).$$

The isomorphisms inverse to $\rho^\pm$ are denoted by

$$\sigma_\pm=(\rho_\pm)^{-1}\colon\ \mathrm{Herm}_0(W^\pm)\to\Lambda^2_\pm\otimes i\mathbb{R}.$$

3.1.9. *Kähler vector spaces.* Let V be an n-dimensional complex vector space equipped with a Hermitian metric. Then there exists a canonical spin representation $(W_{\mathrm{can}},\Gamma_{\mathrm{can}})$ with

$$W_{\mathrm{can}}=\Lambda^{0,*}(V^*):=\bigoplus_{q=0}^{n}\Lambda^{0,q}(V^*).$$

Note that in this case $V^*_{\mathbb{C}}=V^*\otimes_{\mathbb{R}}\mathbb{C}=V^{1,0}\oplus V^{0,1}$. So, given a $v\in V$, we also have the following representation for the dual covector $v^*\in V^*$:

$$v^*=v^{1,0}+v^{0,1}.$$

The *canonical spin representation* in these notations is given by the map

$$\Gamma_{\mathrm{can}}\colon\ V\to \mathrm{End}\,W_{\mathrm{can}},\qquad \Gamma_{\mathrm{can}}(v)w^{0,q}:=v^{0,1}\lrcorner\, w^{0,q}-v^{0,1}\wedge w^{0,q},$$

where $v\in V$ and $w^{0,q}\in\Lambda^{0,q}(V^*)$. Hence,

$$W^+_{\mathrm{can}}=\Lambda^{0,\mathrm{ev}}(V^*),\qquad W^-_{\mathrm{can}}=\Lambda^{0,\mathrm{od}}(V^*).$$

3.2. Spin$^{\mathbf{c}}$**-structures.**

3.2.1. Spin$^{\mathrm{c}}$*-structures on a principal bundle.* Let X be an oriented n-dimensional Riemannian manifold and $P_{\mathrm{SO}(n)} \to X$ be a principal $\mathrm{SO}(n)$-bundle of orthonormal bases on X. The Spin$^{\mathrm{c}}$-*structure* on $P_{\mathrm{SO}(n)}$ is an extension of this bundle to a principal $\mathrm{Spin}^{\mathrm{c}}(n)$-bundle $P_{\mathrm{Spin}^{\mathrm{c}}(n)} \to X$ together with a Spin$^{\mathrm{c}}$-invariant bundle epimorphism:

$$\begin{array}{ccc} P_{\mathrm{Spin}^{\mathrm{c}}(n)} & \longrightarrow & P_{\mathrm{SO}(n)} \\ \downarrow & & \downarrow \\ X & = & X \end{array}$$

where $\mathrm{Spin}^{\mathrm{c}}(n)$ acts on $P_{\mathrm{SO}(n)}$ by the homomorphism $\pi\colon \mathrm{Spin}^{\mathrm{c}}(n) \to \mathrm{SO}(n)$.

With the bundle $P_{\mathrm{Spin}^{\mathrm{c}}(n)}$ we can associate the principal $\mathrm{U}(1)$-bundle $P_{\mathrm{U}(1)} \to X$ so that the following diagram is commutative:

$$\begin{array}{ccc} P_{\mathrm{Spin}^{\mathrm{c}}(n)} & \xrightarrow{\ \delta\ } & P_{\mathrm{U}(1)} \\ \downarrow & & \downarrow \\ X & = & X \end{array}$$

where $\mathrm{Spin}^{\mathrm{c}}(n)$ acts on $P_{\mathrm{U}(1)}$ by the homomorphism $\delta\colon \mathrm{Spin}^{\mathrm{c}}(n) \to \mathrm{U}(1)$. The complex line bundle $L \to X$ associated with $P_{\mathrm{U}(1)} \to X$ is called the *characteristic bundle* of the given Spin$^{\mathrm{c}}$-structure, and its first Chern class $c_1(L)$ is called the *characteristic class* of the Spin$^{\mathrm{c}}$-structure.

3.2.2. Spin$^{\mathrm{c}}$*-structures on vector bundles.* In a similar way we can define the Spin$^{\mathrm{c}}$-structure on a rank n oriented Riemannian vector bundle $V \to X$ associated with a principal bundle $P_{\mathrm{SO}(n)} \to X$ and isomorphic to $V \cong P_{\mathrm{SO}(n)} \times_{\mathrm{SO}(n)} \mathbb{R}^n$. The Spin$^{\mathrm{c}}$-*structure* on $V \to X$ is the extension of its structure group from $\mathrm{SO}(n)$ to $\mathrm{Spin}^{\mathrm{c}}(n)$. In other words, the bundle $V \to X$ *admits* a Spin$^{\mathrm{c}}$-*structure* if it is associated with a principal $\mathrm{Spin}^{\mathrm{c}}(n)$-bundle $P_{\mathrm{Spin}^{\mathrm{c}}(n)} \to X$, i.e., if there exists a bundle isomorphism

$$P_{\mathrm{Spin}^{\mathrm{c}}(n)} \times_{\mathrm{Spin}^{\mathrm{c}}(n)} \mathbb{R}^n \to V,$$

where $\mathrm{Spin}^{\mathrm{c}}(n)$ acts on $\mathbb{R}^n$ by the homomorphism $\pi\colon \mathrm{Spin}^{\mathrm{c}}(n) \to \mathrm{SO}(n)$.

In particular, as V one can take the tangent bundle TX. In this case the Spin$^{\mathrm{c}}$-structure on TX is called the Spin$^{\mathrm{c}}$-*structure on the manifold* X.

In the case when the rank of V is equal to $2n$, we can give an equivalent definition of the Spin$^{\mathrm{c}}$-structure on V in terms of the spin representation. Namely, using this representation, in such a case we can construct a complex Hermitian vector bundle W of rank 2^n associated with the principal $\mathrm{Spin}^{\mathrm{c}}(2n)$-bundle $P_{\mathrm{Spin}^{\mathrm{c}}(2n)} \to X$:

$$W := P_{\mathrm{Spin}^{\mathrm{c}}(2n)} \times_{\mathrm{Spin}^{\mathrm{c}}(2n)} \mathbb{C}^{2^n} \to X,$$

where the action of the group $\mathrm{Spin}^{\mathrm{c}}(2n)$ on $\mathbb{C}^{2^n}$ is given by the spin representation $\Gamma\colon \mathrm{Spin}^{\mathrm{c}}(2n) \to \operatorname{End}\mathbb{C}^{2^n}$. This representation determines the linear homomorphism (denoted by the same letter)

$$\Gamma\colon\ V \to \operatorname{End} W,$$

which has the indicated characteristic property of spin representations. We call W the *spinor bundle.*

Hence, the definition of the Spin$^{\mathrm{c}}$-structure on V in this case is equivalent to the following one. A Spin$^{\mathrm{c}}$-*structure on the bundle* V of rank $2n$ is a pair (W, Γ) consisting of a complex Hermitian vector bundle $W \to X$ of rank 2^n and a bundle homomorphism $\Gamma\colon V \to \operatorname{End} W$ that has the property of spin representations

$$\Gamma^*(v) + \Gamma(v) = 0, \qquad \Gamma^*(v)\Gamma(v) = |v|^2 \,\mathrm{id}.$$

The bundle homomorphism $\Gamma\colon V \to \operatorname{End} W$ can be extended to a bundle homomorphism

$$\Gamma\colon \ \mathrm{Cl}^{\mathrm{c}}(V) \to \operatorname{End} W,$$

where $\mathrm{Cl}^{\mathrm{c}}(V)$ is the *bundle of complexified Clifford algebras* associated with the oriented Riemannian vector bundle V. In particular, W can be decomposed into the direct sum of $\Gamma(\omega)$-eigenbundles

$$W = W^+ \oplus W^-,$$

called the *semi-spinor bundles*. The *characteristic line bundle* of the $\mathrm{Spin}^{\mathrm{c}}$-*structure* (W,Γ) can be defined as

$$L_\Gamma := P_{\mathrm{Spin}^{\mathrm{c}}(2n)} \times_{\mathrm{Spin}^{\mathrm{c}}(2n)} \mathbb{C} \to X$$

where the action of the group $\mathrm{Spin}^{\mathrm{c}}(2n)$ on $\mathbb{C}$ is given by the homomorphism $\delta\colon \mathrm{Spin}^{\mathrm{c}}(2n) \to \mathrm{U}(1)$.

3.2.3. *Existence of* $\mathrm{Spin}^{\mathrm{c}}$-*structures. The space of* $\mathrm{Spin}^{\mathrm{c}}$-*structures.* It can be shown that a bundle $P_{\mathrm{SO}(n)}$ *admits the* $\mathrm{Spin}^{\mathrm{c}}$-*structure if and only if there exists a class* $c \in H^2(X,\mathbb{Z})$ *such that*

$$w_2(P_{\mathrm{SO}(n)}) \equiv c \mod 2,$$

where w_2 is the second Stiefel–Whitney class. This fact is proved using the following exact sequence:

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}^{\mathrm{c}}(n) \to \mathrm{SO}(n) \times \mathrm{U}(1) \to 0.$$

It is worthwhile to compare this criterion with the necessary and sufficient condition of existence of a Spin-structure on a principal bundle $P_{\mathrm{SO}(n)} \to X$. This condition has the form

$$w_2(P_{\mathrm{SO}(n)}) = 0.$$

It follows, in particular, that the $\mathrm{Spin}^{\mathrm{c}}$-structure exists on any spin manifold and any almost complex manifold X (in the latter case it is sufficient to put $c = c_1(X)$).

In fact it can be shown that the $\mathrm{Spin}^{\mathrm{c}}$-structure exists on any oriented compact 4-dimensional manifold X. (To prove this assertion, one can use the fact that in this case $w_2(X)\cdot\alpha \equiv \alpha\cdot\alpha \mod 2$ for all $\alpha \in H_2(X,\mathbb{Z})$.)

Assume that an oriented Riemannian vector bundle $V \to X$ of rank $2n$ has a $\mathrm{Spin}^{\mathrm{c}}$-structure (W,Γ). Then for any complex line bundle $E \to X$ we can introduce a new $\mathrm{Spin}^{\mathrm{c}}$-structure (W_E,Γ_E) by setting

$$W_E := W \otimes E, \qquad \Gamma_E := \Gamma \otimes \mathrm{id}.$$

This new $\mathrm{Spin}^{\mathrm{c}}$-structure (W_E,Γ_E) corresponds to the principal $\mathrm{Spin}^{\mathrm{c}}(2n)$-bundle

$$P_{\Gamma_E} = P_\Gamma \otimes_{\mathrm{U}(1)} P_E,$$

where P_Γ is the principal $\mathrm{Spin}^{\mathrm{c}}(2n)$-bundle associated with (W,Γ) and P_E is the principal $\mathrm{U}(1)$-bundle associated with E. The characteristic bundle of the $\mathrm{Spin}^{\mathrm{c}}$-structure (W_E,Γ_E) coincides with

$$L_{\Gamma_E} := L_\Gamma \otimes E^{\otimes 2}.$$

Thus the group $H^2(X,\mathbb{Z})$, which parameterizes the equivalence classes of complex linear bundles on X, acts on the space of $\mathrm{Spin}^{\mathrm{c}}$-structures on $V \to X$. If the latter space is not empty, then its quotient by the indicated action can be identified with $H^1(X,\mathbb{Z}_2)$, although not in the canonical way. However, in the almost complex case this identification becomes canonical due to the existence of the canonical $\mathrm{Spin}^{\mathrm{c}}$-structure $(W_{\mathrm{can}},\Gamma_{\mathrm{can}})$ on such manifolds.

3.2.4. Spin$^{\mathrm{c}}$-*structures on almost complex vector bundles.* Suppose that $V \to X$ is an almost complex vector bundle of (complex) rank n equipped with an almost complex structure J compatible with the Riemannian metric and orientation of V. Then V has a *canonical* Spin$^{\mathrm{c}}$-*structure* $(W_{\mathrm{can}}, \Gamma_{\mathrm{can}})$ that can be defined by

$$W_{\mathrm{can}} := \Lambda^{0,*}(V^*),$$

where V^* is equipped with the dual almost complex structure J^*. The Clifford multiplication map Γ_{can} is given by the same formula as in the case of Kähler vector spaces. The characteristic bundle L_{can} coincides with the anticanonical bundle K^* of V:

$$K^* = \Lambda^{0,n}(V^*).$$

We can construct new Spin$^{\mathrm{c}}$-structures on V by taking the tensor product of V with Hermitian line bundles $E \to X$ so that

$$W_E = W_{\mathrm{can}} \otimes E, \qquad L_{\Gamma_E} = K^* \otimes E^2.$$

3.3. Spin$^{\mathbf{c}}$-connections and Dirac operators.

3.3.1. Spin$^{\mathrm{c}}$-*connections in terms of principal bundles.* Let X be an oriented Riemannian manifold of dimension $2n$ equipped with a Spin$^{\mathrm{c}}$-structure (W, Γ). Denote by ∇ the Levi-Civita connection defined on the tangent bundle TX and generated by the Riemannian metric on X. Then the Spin$^{\mathrm{c}}$-*connection* on X is the extension of the connection ∇ to W. In more detail, it is the connection ∇ on W satisfying the condition

$$\nabla_u(\Gamma(v)\Phi) = \Gamma(v)\nabla_u\Phi + \Gamma(\nabla_u v)\Phi$$

for all $u, v \in \mathrm{Vect}(X)$ and $\Phi \in C^\infty(X, W)$. Such a connection preserves the semi-spinor bundles $W^\pm$, and any two connections of this form differ by a 1-form on X with pure imaginary values.

In terms of principal bundles, denote by $P_{\mathrm{SO}(2n)} \to X$ the bundle of orthonormal bases of the manifold X, and let $P_\Gamma := P_{\mathrm{Spin}^{\mathrm{c}}(2n)} \to X$ be its extension to a principal Spin$^{\mathrm{c}}(2n)$-bundle over X associated with the Spin$^{\mathrm{c}}$-structure (W, Γ). Then we have

$$W = P_{\mathrm{Spin}^{\mathrm{c}}(2n)} \times_{\mathrm{Spin}^{\mathrm{c}}(2n)} W_0,$$

where $W_0 = \mathbb{C}^{2^n}$ and the group Spin$^{\mathrm{c}}(2n)$ acts on W_0 by the standard spin representation Γ_0. Moreover,

$$TX = P_{\mathrm{Spin}^{\mathrm{c}}(2n)} \times_{\mathrm{Spin}^{\mathrm{c}}(2n)} V_0,$$

where $V_0 = \mathbb{R}^{2n} = \mathbb{C}^n$ and the group Spin$^{\mathrm{c}}(2n)$ acts on V_0 by the homomorphism $\pi\colon \mathrm{Spin}^{\mathrm{c}}(2n) \to \mathrm{SO}(2n)$, and

$$L_\Gamma = P_{\mathrm{Spin}^{\mathrm{c}}(2n)} \times_{\mathrm{Spin}^{\mathrm{c}}(2n)} \mathbb{C},$$

where the group Spin$^{\mathrm{c}}(2n)$ acts on $\mathbb{C}$ by the homomorphism $\delta\colon \mathrm{Spin}^{\mathrm{c}}(2n) \to \mathrm{U}(1)$.

Consider the standard spin representation

$$\Gamma_0\colon\ \mathrm{Cl}^{\mathrm{c}}(V_0) \to \mathrm{End}\, W_0$$

and denote by G the subgroup in $\mathrm{Aut}\, W_0$ that coincides with the image of the group Spin$^{\mathrm{c}}(2n)$ under the action of the map Γ_0: $G = \Gamma_0(\mathrm{Spin}^{\mathrm{c}}(2n))$. Its Lie algebra is equal to

$$\mathfrak{g} := \mathrm{Lie}\, G = \Gamma_0(\mathrm{cl}_2(V_0) \oplus i\mathbb{R}) = \Gamma_0(\mathrm{cl}_2(V_0)) \oplus i\mathbb{R} = \mathfrak{g}_0 \oplus i\mathbb{R},$$

where $\mathfrak{g}_0 := \Gamma_0(\mathrm{cl}_2(V_0))$.

Then the $\mathrm{Spin}^{\mathrm{c}}$-connection on W is generated by the 1-form of the connection $\mathcal{A} \in \Omega^1(P_\Gamma, \mathfrak{g})$. We can write

$$\mathcal{A} = \mathcal{A}_0 \oplus A,$$

where $\mathcal{A}_0 \in \Omega^1(P_\Gamma, \mathfrak{g}_0)$ is the traceless part of the form $\mathcal{A}$ and $A \in \Omega^1(P_\Gamma, i\mathbb{R})$ is its *trace part* equal to $A = \operatorname{Tr} A/2^n$.

The traceless part $\mathcal{A}_0$ generates a connection on TX, because $\mathfrak{g}_0 = \mathrm{so}(2n)$, and, by the definition of the $\mathrm{Spin}^{\mathrm{c}}$-connection, it must coincide with the Levi-Civita connection. Hence, $\mathcal{A}$ is completely determined by its trace part $A \in \Omega^1(P_\Gamma, i\mathbb{R})$. Since $\delta(e^{i\theta}\cdot 1) = e^{2i\theta}$, the trace part $A \in \Omega^1(P_\Gamma, i\mathbb{R})$ generates the connection $2A$ on the characteristic bundle $L_\Gamma (= L)$. If L has a square root $L^{1/2} \to X$ (this condition is satisfied, for example, in the case when X is a spin manifold), then A also generates a connection on $L^{1/2}$. In the general case A can be considered as a *virtual connection* on the *virtual line bundle* $L^{1/2}$. Denote by $\mathcal{A}(\Gamma)$ the space of all such virtual connections A in the virtual line bundle $L^{1/2}$.

3.3.2. *Dirac operator.* Denote by ∇_A the covariant derivative of sections of W that is generated by the connection $\mathcal{A} = \mathcal{A}_0 + A$. The *Dirac operator*

$$D_A\colon\ C^\infty(X, W^+) \to C^\infty(X, W^-)$$

associated with the virtual connection A is defined by the formula

$$D_A\Phi = \sum_{\nu=1}^{2n} \Gamma(e_\nu)\nabla_{A,e_\nu}\Phi,$$

where $\Phi \in C^\infty(X, W^+)$ and $\{e_\nu\}$ is a local orthonormal basis of the bundle TX. This definition is in fact independent of the choice of the local orthonormal basis $\{e_\nu\}$ (due to the existence of an equivalent invariant definition of this operator).

The *adjoint Dirac operator* is defined by duality as

$$D_A^*\colon\ C^\infty(X, W^-) \to C^\infty(X, W^+).$$

3.3.3. $\mathrm{Spin}^{\mathrm{c}}$*-connections and the Dirac operator on an almost complex manifold.* Let (X, J) be a $2n$-dimensional (over $\mathbb{R}$) almost complex manifold with an almost complex structure J compatible with the orientation and Riemannian metric g. Denote by $(W_{\mathrm{can}}, \Gamma_{\mathrm{can}})$ the canonical $\mathrm{Spin}^{\mathrm{c}}$-structure on X.

If the structure J is integrable and parallel with respect to g, i.e., X is *Kähler*, then the Levi-Civita connection $\nabla = \nabla_g$ preserves the spaces $\Omega^{0,q}(X)$ and can be extended to a canonical $\mathrm{Spin}^{\mathrm{c}}$-connection ∇_{can} on W_{can}. In particular, $2A_{\mathrm{can}}$ is the canonical connection on the canonical bundle

$$L_{\mathrm{can}} = K^*(X) = \Lambda^{0,n}(T^*X).$$

If the structure J is not integrable, then the Levi-Civita connection does not preserve the spaces $\Omega^{0,q}$; however, the canonical $\mathrm{Spin}^{\mathrm{c}}$-connection ∇_{can} on the bundle W_{can} can be defined by adding the term containing the *Nijenhuis tensor* of the structure J to the Levi-Civita connection.

Other $\mathrm{Spin}^{\mathrm{c}}$-structures on (X, J) can be constructed by setting

$$W_E = W_{\mathrm{can}} \otimes E,$$

where E is a Hermitian line bundle $E \to X$. The corresponding $\mathrm{Spin}^{\mathrm{c}}$-connection on W_E will have the form

$$A = A_{\mathrm{can}} \otimes \mathrm{id} + \mathrm{id} \otimes B,$$

where B is a Hermitian connection on $E \to X$. The corresponding Dirac operator

$$D_A\colon\ C^\infty(X, W_E^+) \to C^\infty(X, W_E^-),$$

where $W_E^+ = \Lambda^{0,\mathrm{ev}}(X,E)$ and $W_E^- = \Lambda^{0,\mathrm{od}}(X,E)$, coincides with

$$D_A = \bar\partial_B + \bar\partial_B^*.$$

3.3.4. *Weizenböck formulas.* Let X be an oriented $2n$-dimensional Riemannian manifold and

$$\nabla_A^*\colon\ C^\infty(X,W) \to C^\infty(X,W)$$

be the L^2-adjoint operator of ∇_A. Then the following *Weizenböck formulas* hold:

$$\begin{cases} D_A^* D_A\Phi = \nabla_A^*\nabla_A\Phi + \dfrac{1}{4}s\Phi + \rho^+(F_A)\Phi, \\ D_A D_A^*\Psi = \nabla_A\nabla_A^*\Psi + \dfrac{1}{4}s\Psi + \rho^-(F_A)\Psi, \end{cases}$$

where $\Phi \in C^\infty(X, W^+)$, $\Psi \in C^\infty(X, W^-)$, s is the scalar curvature of (X,g), and

$$\rho^\pm\colon\ \Lambda^2_\pm(T^*X)\otimes i\mathbb{R} \to \mathrm{Herm}_0(W^\pm)$$

are the maps introduced in Subsection 3.1.8. (Here, $\mathrm{Herm}_0(W^\pm)$ denotes the space of Hermitian traceless endomorphisms of the bundle $W^\pm$.)

4. DIMENSION FOUR: SEIBERG–WITTEN EQUATIONS

In this section we consider the Seiberg–Witten equations on compact oriented Riemannian manifolds of dimension 4 and study their solutions. In Subsection 4.1 we present some general properties of these equations and of the moduli spaces of their solutions. Subsection 4.2 is devoted to the special case of Kähler surfaces. Here we give a description of the moduli spaces in terms of complex curves that is analogous to the Bradlow theorem for vortex equations on compact Riemann surfaces. In the next subsections we study the Seiberg–Witten equations on symplectic 4-manifolds. (The topology of symplectic 4-manifolds and pseudoholomorphic curves on such manifolds were studied in [4, 6, 10, 32].) In Subsection 4.3 we discuss general properties of these equations and in Subsection 4.4 give a direct construction of the adiabatic (scale) limit that associates a pseudoholomorphic curve with a solution of the Seiberg–Witten equations. Subsection 4.5 is devoted to the derivation of the adiabatic equations that must hold for the sections of the vortex bundle over the limiting pseudoholomorphic curve. In the last Subsection 4.6 we consider the inverse construction that associates an approximate solution of the Seiberg–Witten equations with a section of the vortex bundle over a pseudoholomorphic curve. General references for this section are the original papers of Taubes [26–31] and review papers [14, 18].

4.1. Seiberg–Witten equations on Riemannian 4-manifolds.

4.1.1. *Seiberg–Witten equations.* Let X be a compact oriented Riemannian manifold of dimension 4. Suppose that it is equipped with a Spin^c-structure (W,Γ) and a Spin^c-connection ∇_A generated by a virtual connection $A \in \mathcal{A}(\Gamma)$ on L_Γ.

Consider the following *Seiberg–Witten equations*:

$$\begin{cases} D_A\Phi = 0, \\ F_A^+ = \sigma^+(\Phi\otimes\Phi^*)_0, \end{cases} \tag{4.1}$$

where

$$\sigma^+(\Phi\otimes\Phi^*)_0 := \sigma^+\biggl(\Phi\otimes\Phi^* - \frac{1}{2}|\Phi|^2\,\mathrm{id}\biggr).$$

Here, $\Phi\otimes\Phi^* - 1/2|\Phi|^2\,\mathrm{id}$ is the traceless Hermitian endomorphism of the bundle W^+ associated with Φ and

$$\sigma^+ := (\rho^+)^{-1}\colon\ \mathrm{Herm}_0(W^+)\to\Omega^2_+(X,i\mathbb{R}),$$

where $\Omega^2_\pm(X,i\mathbb{R})$ is the space of sections of the bundle $\Lambda^2_\pm(T^*X)\otimes i\mathbb{R}$ over X, i.e., the space of self-dual (respectively, anti-self-dual) 2-forms on X with pure imaginary values. Similarly, $F_A^+\in\Omega^2_+(X,i\mathbb{R})$ (respectively, $F_A^-\in\Omega^2_-(X,i\mathbb{R})$) denotes the self-dual (respectively, anti-self-dual) component of the form $F_A\in\Omega^2(X,i\mathbb{R})$.

4.1.2. *Seiberg–Witten functional.* Introduce the following *Seiberg–Witten functional*:

$$E(A,\Phi) = \frac{1}{2}\int\limits_X \biggl\{|F_A|^2 + |\nabla_A\Phi|^2 + \frac{|\Phi|^2}{4}(s+|\Phi|^2)\biggr\}\,d\mathrm{vol}, \tag{4.2}$$

where $s := s(g)$ denotes the scalar curvature of (X,g) and $d\mathrm{vol}$ is the volume element on (X,g). Note that $E(A,\Phi)$ can take negative values if the curvature s is negative.

Using the Weizenböck formula, one can deduce the *Bogomolny formula*

$$E(A,\Phi) = \frac{1}{2}\int\limits_X \bigl\{|D_A\Phi|^2 + 2|F_A^+ - \sigma^+(\Phi\otimes\Phi^*)_0|^2\bigr\}\,d\mathrm{vol} - \frac{\pi^2}{2}\langle c_1(L_\Gamma)^2,[X]\rangle.$$

To prove it, we use the following formula of the *Chern–Weil type*:

$$\pi^2\langle c_1(L_\Gamma)^2,[X]\rangle = -\int\limits_X F_A\wedge F_A = \|F_A^+\|^2 - \|F_A^-\|^2.$$

On the other hand,

$$\begin{aligned}|F_A^+ - \sigma^+(\Phi\otimes\Phi^*)_0|^2 &= |F_A^+|^2 + |\sigma^+(\Phi\otimes\Phi^*)_0|^2 - 2\langle F_A^+,\sigma^+(\Phi\otimes\Phi^*)_0\rangle\\ &= |F_A^+|^2 + \frac{1}{8}|\Phi|^4 - \frac{1}{2}\langle\rho^+(F_A)\Phi,\Phi\rangle\end{aligned}$$

and by the Weizenböck formula

$$\|D_A\Phi\|^2_{L^2} = \|\nabla_A\Phi\|^2_{L^2} + \frac{1}{4}\int\limits_X s|\Phi|^2\,d\mathrm{vol} + \langle\rho^+(F_A)\Phi,\Phi\rangle_{L^2}.$$

Now the Bogomolny formula follows from the last three relations.

The Bogomolny formula implies a lower estimate for the Seiberg–Witten functional:

$$E(A,\Phi)\ge -\frac{\pi^2}{2}\langle c_1(L_\Gamma)^2,[X]\rangle,$$

in which the equality is attained only on solutions of the Seiberg–Witten equations.

4.1.3. *Gauge transformations and perturbed Seiberg–Witten equations.* The Seiberg–Witten equations, as well as the Seiberg–Witten functional $E(A,\Phi)$, are invariant under the *gauge transformations* given by the formula

$$A \mapsto A + u^{-1}\,du, \qquad \Phi \mapsto u^{-1}\Phi,$$

where $u = e^{i\chi}$ and χ is a smooth real-valued function so that $u \in \mathcal{G} := C^\infty(X, \mathrm{U}(1))$. This action is always free except for the case $\Phi \equiv 0$.

In order to get rid of solutions of the form $(A,0)$, we consider the *perturbed Seiberg–Witten equations* defined in the following way:

$$\begin{cases} D_A\Phi = 0, \\ F_A^+ + \eta = \sigma^+(\Phi\otimes\Phi^*)_0, \end{cases} \tag{4.3}$$

where $\eta \in \Omega^2_+(X, i\mathbb{R})$. We will call them the SW_η-*equations* for brevity.

Note that if $b^2_+ := \dim H^2_+(X,\mathbb{R}) \ge 1$, then we can always choose η so that the SW_η-equations do not have solutions of the form $(A,0)$. (Recall that $H^2_+(X,\mathbb{R})$ denotes the positive definite subspace of cohomology with respect to the intersection form.) This fact holds by virtue of the following proposition, whose proof is left to the reader as an exercise.

Proposition 2. *Define the* Γ-*wall by setting*

$$\Omega^2_\Gamma(X, i\mathbb{R}) := \bigl\{\eta \in \Omega^2_+(X, i\mathbb{R})\colon \textit{ there exists } A \in \mathcal{A}(\Gamma) \textit{ with } F_A^+ + \eta = 0\bigr\}.$$

Then $\Omega^2_\Gamma(X, i\mathbb{R})$ *is an affine vector subspace in* $\Omega^2_+(X, i\mathbb{R})$ *of codimension* b_2^+.

4.1.4. *Moduli space of solutions.* The *moduli space of solutions of the Seiberg–Witten equations* is defined as

$$\mathcal{M}_\eta(X,\Gamma,g) := \{\mathrm{SW}_\eta\text{-solutions } (A,\Phi)\}/\mathcal{G}.$$

If $b^2_+ \ge 1$, then the manifold $\mathcal{M}_\eta(X,\Gamma,g)$ is smooth for an appropriate choice of η. Moreover, the following theorem is true.

Theorem 7. *If* $b^2_+ > 1$, *then for a generic form* $\eta \in \Omega^2_+(X, i\mathbb{R})$ *the moduli space* $\mathcal{M}_\eta(X,\Gamma,g)$ *is an oriented compact smooth manifold of dimension*

$$\dim \mathcal{M}_\eta(X,\Gamma,g) = \frac{\langle c_1(L_\Gamma)^2, [X]\rangle - 2\chi(X) - 3\sigma(X)}{4},$$

where $\chi(X)$ *is the Euler characteristic of* X *and* $\sigma(X)$ *is the signature of* $H^2(X,\mathbb{R})$.

According to this theorem, the homology class $[\mathcal{M}_\eta(X,\Gamma,g)]$ in the (infinite-dimensional) configuration space $\{(A,\Phi)\}/\mathcal{G}$ is well defined and does not depend on the choice of a generic form η and a metric g. It depends only on the Spin^c-structure Γ. (In the boundary case $b^2_+ = 1$, the moduli space $\mathcal{M}_\eta$ will, generally speaking, depend on the choice of η, because the Γ-wall has codimension 1 and divides the space $\Omega^2_+(X, i\mathbb{R})$ into two components.)

Assume, in particular, that $\dim \mathcal{M}_\eta(X,\Gamma,g) = 0$, i.e.,

$$\langle c_1(L_\Gamma)^2, [X]\rangle = 2\chi(X) + 3\sigma(X)$$

(note that this condition also arises in the *Wu theorem* on the existence of an almost complex structure on a given Riemannian manifold). Then the moduli space $\mathcal{M}_\eta(X,\Gamma,g)$ consists of a finite number of points with signs. In this case we can define the *Seiberg–Witten invariant* $\mathrm{SW}(X,\Gamma)$ by the formula

$$\mathrm{SW}(X,\Gamma) := \sum_{\text{points of } \mathcal{M}_\eta} (\text{sign of a point}) \in \mathbb{Z}.$$

The introduced quantity is *covariant* with respect to orientation-preserving diffeomorphisms f of the manifold X in the sense that

$$\mathrm{SW}(X,\Gamma)=\mathrm{SW}(f(X),f^*\Gamma).$$

4.1.5. *Scale transformations.* The SW_η-equations are not invariant under the change $g\mapsto\lambda^2 g$ of the scale of the base Riemannian metric. More precisely, there exists a bijective correspondence between the sets

$$\{\mathrm{SW}_\eta\text{-solutions } (A,\Phi) \text{ for the metric } g\} \quad\text{and}\quad \left\{\mathrm{SW}_\eta\text{-solutions } \left(A,\frac{1}{\lambda}\Phi\right) \text{ for the metric } \lambda^2 g\right\},$$

where $\lambda>0$ is a constant. Note that under a change of the scale the Seiberg–Witten functional transforms as

$$E_g(A,\Phi)=E_{\lambda^2 g}\left(A,\frac{1}{\lambda}\Phi\right).$$

4.2. Seiberg–Witten equations on a Kähler surface.

4.2.1. *Seiberg–Witten equations.* Let (X,ω,J) be a compact Kähler surface equipped with the canonical Spin^c-structure $(W_{\mathrm{can}},\Gamma_{\mathrm{can}})$ and canonical Spin^c-connection $\nabla_{\mathrm{can}}=\nabla_{A_{\mathrm{can}}}$, where $2A_{\mathrm{can}}$ is a connection on the anticanonical bundle K^*.

Let the Spin^c-structure on X be associated with some Hermitian line bundle $E\to X$ so that the semi-spinor bundles are given by

$$W_E^+=W_{\mathrm{can}}^+\otimes E=\Lambda^0(E)\oplus\Lambda^{0,2}(E),\qquad W_E^-=W_{\mathrm{can}}^-\otimes E=\Lambda^{0,1}(E).$$

The characteristic bundle coincides with

$$L_{\Gamma_E}=L_{\mathrm{can}}\otimes E^2=K^*\otimes E^2.$$

The Spin^c-connection ∇_A on W_E can be written in the form $\nabla_A=\nabla_{\mathrm{can}}+B$, where B is a Hermitian connection on the bundle $E\to X$. In this case the Dirac operator is written in the form

$$D_A=\bar\partial_B+\bar\partial_B^*$$

for $\Phi=(\varphi_0,\varphi_2)\in\Omega^0(X,E)\oplus\Omega^{0,2}(X,E)$.

The right-hand side of the second Seiberg–Witten equation (4.1) for the curvature can be rewritten as

$$\sigma^+(\Phi\otimes\Phi^*)=i\,\frac{|\varphi_0|^2-|\varphi_2|^2}{4}\,\omega+\frac{\overline{\varphi}_0\varphi_2-\varphi_0\overline{\varphi}_2}{2}.$$

Recall that on Kähler surfaces we have the decomposition of the complexified bundle $\Lambda^2_+\otimes\mathbb{C}$ of self-dual 2-forms into the components of the type

$$\Lambda^2_+\otimes\mathbb{C}=\Lambda^{2,0}\oplus\mathbb{C}[\omega]\oplus\Lambda^{0,2}.$$

Accordingly, the Seiberg–Witten equation for the curvature splits into the sum of the component parallel to ω, the $(0,2)$-component, and the $(2,0)$-component complex conjugate to the $(0,2)$-component.

So, the SW_η-*equations on a compact Kähler surface* can be rewritten in the form

$$\begin{cases}\bar\partial_B\varphi_0+\bar\partial_B^*\varphi_2=0,\\ F_B^{0,2}+\eta^{0,2}=\dfrac{\overline{\varphi}_0\varphi_2}{2},\\ F_{A_{\mathrm{can}}}^\omega+F_B^\omega=\dfrac{i}{4}(|\varphi_0|^2-|\varphi_2|^2)-\eta^\omega.\end{cases}\tag{4.4}$$

The first of these equations is the Dirac equation, the second one corresponds to the $(0,2)$-component of the equation for the curvature, and the third one corresponds to the curvature component parallel to ω.

4.2.2. *Solvability conditions.* Suppose next that η is a form of type $(1,1)$ and apply the $\bar{\partial}_B$-operator to the first Seiberg–Witten equation:

$$\begin{aligned}\bar{\partial}_B\bar{\partial}_B^*\varphi_2 &= -\bar{\partial}_B\bar{\partial}_B\varphi_0 && \text{(first equation (4.4))}\\ &= -F_B^{0,2}\varphi_0 && \text{(definition of } F_B^{0,2})\\ &= -\frac{|\varphi_0|^2\varphi_2}{2} && \text{(second equation (4.4))}.\end{aligned}$$

Taking the inner product of the obtained equation

$$\bar{\partial}_B\bar{\partial}_B^*\varphi_2 = -\frac{|\varphi_0|^2\varphi_2}{2}$$

with $\overline{\varphi}_2$, we integrate it over X. As a result we get

$$\|\bar{\partial}_B^*\varphi_2\|_{L^2} + \frac{\|\varphi_0\|_{L^2}^2\|\varphi_2\|_{L^2}^2}{2} = 0.$$

Hence,

$$\bar{\partial}_B^*\varphi_2 = \bar{\partial}_B\varphi_0 = \overline{\varphi}_0\varphi_2 \equiv 0,$$

which implies that either φ_0 or φ_2 should vanish identically.

In order to find out which of these quantities is identically equal to zero, we integrate the third equation (4.4). As a result we get

$$\int_X \frac{|\varphi_0|^2 - |\varphi_2|^2}{4}\,\omega\wedge\omega = i\int_X (F_{A_{\mathrm{can}}} + F_B + \eta)\wedge\omega = \pi(-c_1(K) + 2c_1(E))\cdot[\omega] + i\int_X \eta\wedge\omega.$$

Note that

$$\int_X \frac{|\varphi_0|^2 - |\varphi_2|^2}{4}\,\omega\wedge\omega = \frac{\|\varphi_0\|_{L^2}^2 - \|\varphi_2\|_{L^2}^2}{2}.$$

Consider first the case $\eta = 0$, which corresponds to the *unperturbed Seiberg–Witten equations.* Then

$$\|\varphi_2\|^2 - \|\varphi_0\|^2 = 2\pi\big(2c_1(E)\cdot[\omega] - c_1(K)\cdot[\omega]\big)$$

and we obtain the following *solvability conditions*:

- if $c_1(E)\cdot[\omega] > c_1(K)\cdot[\omega]/2$, then $\varphi_0 \equiv 0$ and $\varphi_2 \not\equiv 0$;
- if $c_1(E)\cdot[\omega] < c_1(K)\cdot[\omega]/2$, then $\varphi_0 \not\equiv 0$ and $\varphi_2 \equiv 0$.

Note that for a Kähler surface with $b_2^+ > 1$ the inequality $c_1(K)\cdot[\omega] \ge 0$ holds, because the canonical bundle K of such a surface admits a nontrivial holomorphic section. For the same reason,

- if $(E, \bar{\partial}_B)$ has a nontrivial holomorphic section φ_0, then $c_1(E)\cdot[\omega] \ge 0$;
- if $K\otimes E^*$ has a nontrivial holomorphic section φ_2, then $c_1(K)\cdot[\omega] \ge c_1(E)\cdot[\omega]$.

4.2.3. *The case of the trivial bundle E.* Consider next the SW_η-equations in the case of the trivial bundle E and set

$$\eta = -F^+_{A_{\mathrm{can}}} + i\lambda\omega$$

with $\lambda > 0$. From the third equation (4.4) we find that

$$4i(dB)^\omega = 4\lambda + |\varphi_2|^2 - |\varphi_0|^2.$$

Integrating this equation over X, we obtain

$$4\lambda \operatorname{Vol}(X) + \|\varphi_2\|^2 - \|\varphi_0\|^2 = 0.$$

Hence, $\varphi_2 \equiv 0$ and the SW_η-equations take the form

$$\begin{cases} \bar\partial_B\varphi_0 = 0, \\ F_B^{0,2} = 0, \\ 4i(dB)^\omega = 4\lambda - |\varphi_0|^2. \end{cases}$$

Since the bundle E is trivial, the equations admit the trivial solution

$$B \equiv 0, \qquad \varphi_0 \equiv 2\sqrt{\lambda}, \qquad \varphi_2 \equiv 0.$$

Using the fact that these equations are of Liouville type, one can show that the above solution is unique (up to gauge equivalence). So in this case

$$\mathrm{SW}(X, \Gamma_{\mathrm{can}}) = 1.$$

4.2.4. *Description of the moduli space in terms of effective divisors.* We will show that for the SW_η-equations on a Kähler surface an analog of the Bradlow theorem for vortex equations on a compact Riemann surface holds. Let $E \to X$ be a Hermitian line bundle over (X, ω, J). Suppose that for some $\lambda > 0$ its first Chern class satisfies the inequality

$$0 \le c_1(E)\cdot[\omega] < \frac{c_1(K)\cdot[\omega]}{2} + \lambda \operatorname{Vol}(X). \tag{4.5}$$

This inequality plays the same role as the stability condition $c_1(L) < (\tau/4\pi)\operatorname{Vol}_g(X)$ in the Bradlow theorem.

Under this condition the moduli space of SW_η-solutions with the form $\eta = \pi i\lambda\omega$ and $\mathrm{Spin}^{\mathrm{c}}$-structure (W_E, Γ_E) admits the following description: *there exists a bijective correspondence between the gauge classes of* SW_η*-solutions* (B, φ_0) *and effective divisors of degree* $c_1(E)$ *on* X. The latter space can be identified with the quotient of the space of holomorphic line bundles $(E, \bar\partial_E)$ that have a nontrivial holomorphic section φ_0 modulo the action of the complexified group of gauge transformations. Since $\bar\partial_E = \bar\partial_B$ for some Hermitian connection B, this space coincides with the space of solutions (B, φ_0) of the equations

$$\bar\partial_B\varphi_0 = 0, \qquad F_B^{0,2} = 0$$

modulo gauge transformations.

To prove this equivalence, we need to show that for any solution (B, φ_0) of the above equations there exists a unique $\mathcal{G}_{\mathbb{C}}$-equivalent solution (B_u, φ_u) satisfying the third Seiberg–Witten equation (4.4). Writing the gauge factor u in the form $u = e^\theta$ for a real-valued function $\theta \in \mathbb{R}$, we obtain the following Liouville type equation for θ:

$$8i(\partial\bar\partial\theta)^\omega + e^{-2\theta}|\varphi_0|^2 = 4\pi\lambda - 4i(F_B^\omega + F^\omega_{A_{\mathrm{can}}}).$$

According to the Kazdan–Warner theorem, this equation has a unique solution if condition (4.5) is satisfied.

4.3. Seiberg–Witten equations on a 4-dimensional symplectic manifold.

4.3.1. *Seiberg–Witten equations.* Let (X,ω,J) be a 4-dimensional compact symplectic manifold equipped with a compatible almost complex structure J. Let (W_E,Γ_E) be a Spin$^{\mathrm{c}}$-structure on X associated with a Hermitian line bundle $E\to X$ equipped with a Hermitian connection B.

The corresponding SW$_\eta$-equations have the form

$$\begin{cases} \bar\partial_B\varphi_0+\bar\partial_B^*\varphi_2=0,\\ F_{A_{\mathrm{can}}}^{0,2}+F_B^{0,2}+\eta^{0,2}=\dfrac{\overline{\varphi}_0\varphi_2}{2},\\ F_{A_{\mathrm{can}}}^{\omega}+F_B^{\omega}+\eta^{\omega}=\dfrac{|\varphi_2|^2-|\varphi_0|^2}{4}, \end{cases} \tag{4.6}$$

where $(\varphi_0,\varphi_2)\in\Omega^0(X,E)\oplus\Omega^{0,2}(X,E)$. Note that the form $F_{A_{\mathrm{can}}}$ is not necessarily of type $(1,1)$ for a general almost complex structure J.

As in the Kähler case, consider the perturbation η of the form

$$\eta=-F_{A_{\mathrm{can}}}^{+}+\pi i\lambda\omega,$$

where λ is a positive number, and introduce the normalized sections

$$\alpha:=\frac{\varphi_0}{\sqrt\lambda},\qquad \beta:=\frac{\varphi_2}{\sqrt\lambda}.$$

Then the SW$_\eta$-equations can be rewritten in the form

$$\begin{cases} \bar\partial_B\alpha+\bar\partial_B^*\beta=0,\\ \dfrac{2}{\lambda}F_B^{0,2}=\bar\alpha\beta,\\ \dfrac{4i}{\lambda}F_B^{\omega}=4\pi+|\beta|^2-|\alpha|^2. \end{cases} \tag{4.7}$$

We will call them the SW$_\lambda$-*equations* for brevity.

4.3.2. *Solvability conditions.* We first study the solvability conditions for the SW$_\lambda$-equations. They look similar to the Kähler case; however, in contrast to the latter, we now have

$$\bar\partial_B\bar\partial_B^*\beta=-\bar\partial_B\bar\partial_B\alpha=-F_B^{0,2}\alpha+\frac14(\partial_B\alpha)\circ N_J,$$

where N_J is the *Nijenhuis tensor* of the structure J. By some tedious though not complicated estimates based on the Weizenböck formula (which are presented, for example, in the paper by Kotschik [9]), one can prove that there exists a positive constant λ_0 depending only on N_J such that for all $\lambda\ge\lambda_0$ the following estimate holds:

$$\varepsilon\|d_B\alpha\|^2+\lambda\|\bar\alpha\beta\|^2+C\lambda\|\beta\|^2+\lambda\|4\pi-|\alpha|^2\|^2\le 16\pi^2c_1(E)\cdot[\omega], \tag{4.8}$$

where $\|\cdot\|:=\|\cdot\|_{L^2}$ and $\varepsilon,C>0$ are some constants. This inequality implies the following solvability condition:

$$c_1(E)\cdot[\omega]\ge 0. \tag{4.9}$$

Note that in the Kähler case this condition is a corollary to the existence of a $\bar\partial_B$-holomorphic section φ_0 of the bundle E.

Consider first the case when $c_1(E)\cdot[\omega]=0$. Then there exists a section α such that $|\alpha|\equiv 2\sqrt{\pi}$; hence the bundle E is necessarily trivial. In this case the SW$_\lambda$-equations admit the trivial solution

$$B\equiv 0,\qquad \alpha\equiv 2\sqrt{\pi},\qquad \beta\equiv 0,$$

and it can be shown that this solution is unique (up to gauge equivalence; see [26]). So in the case under consideration we have

$$\mathrm{SW}(X,\Gamma_{\mathrm{can}})=1.$$

Consider next the case when $c_1(E)\cdot[\omega]>0$. If we suppose that $\mathrm{SW}(X,\Gamma_E)\neq 0$, then, using (4.8), one can show that the validity of the inequality

$$0\le c_1(E)\cdot[\omega]\le c_1(K)\cdot[\omega]$$

is necessary and sufficient for the solvability of the SW$_\lambda$-equations. Note that the equality in the left sign $\le$ is attained only for the trivial E, while in the right sign $\le$ it is attained only for $E=K$.

4.4. From the Seiberg–Witten equations to pseudoholomorphic curves.

4.4.1. *Seiberg–Witten equations.* Let (X,ω) be a 4-dimensional compact symplectic manifold with a generic compatible almost complex structure J satisfying the condition $b_2^+>1$. Let E be a Hermitian line bundle over X with a Hermitian connection B. Assume that X is equipped with a Spin^c-structure (W_E,Γ_E) associated with E and with a Spin^c-connection generated by B. Consider the SW$_\eta$-equations corresponding to this Spin^c-structure with the form

$$\eta=-F^+_{A_{\mathrm{can}}}+\frac{i\lambda}{4}\omega,\qquad \lambda>0,$$

and normalized sections

$$\alpha=\frac{\varphi_0}{\sqrt{\lambda}}\in\Omega^0(X,E),\qquad \beta=\frac{\varphi_2}{\sqrt{\lambda}}\in\Omega^{0,2}(X,E).$$

The arising equations (still called the SW$_\lambda$-equations) have the form

$$\begin{cases}\bar\partial_B\alpha+\bar\partial_B^*\beta=0,\\ \dfrac{2}{\lambda}F_B^{0,2}=\bar\alpha\beta,\\ \dfrac{4i}{\lambda}F_B^\omega=1+|\beta|^2-|\alpha|^2.\end{cases}\tag{4.10}$$

In this subsection we will describe the *direct Taubes construction* that assigns a pseudoholomorphic curve C in X to a λ-dependent family of solutions of the SW$_\lambda$-equations for $\lambda\to\infty$, and the homology class $[C]$ of C is Poincaré dual to the Chern class $c_1(E)$. This construction is a nontrivial generalization to the case of 4-dimensional symplectic manifolds of the above description of the moduli space of SW$_\lambda$-solutions on compact Kähler surfaces in terms of effective divisors.

The following theorem is proved in [28].

Theorem 8 (Taubes)**.** *If* $\mathrm{SW}(X,\Gamma_E)\neq 0$ *and* $c_1(E)\cdot[\omega]>0$, *then there exists a compact pseudoholomorphic curve* C *embedded in* X *with the homology class* $[C]$ *Poincaré dual to the Chern class* $c_1(E)$.

Remark 7. The pseudoholomorphic curve C mentioned in the theorem may be disconnected, so that $C=\sum_{j=1}^k d_jC_j$, where C_j are pairwise disjoint connected pseudoholomorphic curves. Below we suppose for simplicity that $k=1$.

4.4.2. *A priori estimates.* In the case when $\mathrm{SW}(X,\Gamma_E)\neq 0$, the SW_λ-equations have solutions $(B_\lambda,(\alpha_\lambda,\beta_\lambda))$ for all $\lambda>0$. The following a priori estimates of these solutions can be obtained with the help of the Weizenböck formula and maximum principle:

$$|\alpha_\lambda|\le 1+\frac{C_1}{\lambda}, \tag{4.11}$$

$$|\beta_\lambda|^2\le \frac{C_2}{\lambda}(1-|\alpha|^2)+\frac{C_3}{\lambda^3}, \tag{4.12}$$

$$\|\bar\partial_{B_\lambda}\alpha_\lambda\|^2+\|d_{B_\lambda}\beta_\lambda\|^2\le\frac{C_4}{\lambda}, \tag{4.13}$$

$$2\pi c_1(E)\cdot[\omega]-\frac{C_5}{\lambda}\le\frac{\lambda}{2}\int\limits_X |1-|\alpha_\lambda|^2|\,d\mathrm{vol}\le 2\pi c_1(E)\cdot[\omega]+\frac{C_5}{\lambda}, \tag{4.14}$$

$$|F^{\pm}_{B_\lambda}|\le C_6\lambda(1-|\alpha_\lambda|^2)+C_7, \tag{4.15}$$

where $C_1,\dots,C_7$ are some constants depending only on $c_1(E)$ and the Riemannian metric.

From these estimates we deduce that $|\alpha_\lambda|\to 1$ as $\lambda\to\infty$ almost everywhere on X (outside the zeros of α_λ). Moreover, $\|\bar\partial_{B_\lambda}\alpha_\lambda\|\to 0$, i.e., α_λ tends to become a $\bar\partial_{B_\lambda}$-holomorphic section of the bundle E. At the same time, $\beta_\lambda\to 0$ everywhere (together with the first derivatives). So, as $\lambda\to\infty$, the situation becomes more and more similar to that in the Kähler case.

4.4.3. *Construction of the pseudoholomorphic curve.* Denote by $C_\lambda:=\alpha_\lambda^{-1}(0)$ the zero set of the section α_λ. The weak limit of this zero set is precisely the desired pseudoholomorphic curve C.

In more detail, with the SW_λ-solution $(B_\lambda,(\alpha_\lambda,\beta_\lambda))$ we associate the *current*

$$F_\lambda(\eta):=\frac{i}{2\pi}\int\limits_X F_{B_\lambda}\wedge\eta$$

on forms $\eta\in\Omega^2(X,\mathbb{R})$. The norms

$$\|F_\lambda\|=\sup_{0\neq\eta\in\Omega^2}\frac{|F_\lambda(\eta)|}{\sup_{x\in X}|\eta(x)|}$$

are bounded uniformly in λ, because relations (4.14) and (4.15) imply the estimate

$$\|F_\lambda\|\le\frac{1}{2\pi}\|F_{B_\lambda}\|_{L^1}<C,$$

where $C>0$ is a constant that does not depend on λ. So we can find a sequence $\lambda_n\to\infty$ such that the currents F_{λ_n} converge weakly to a functional $\mathcal{F}$ that is a closed positive integer $(1,1)$-current Poincaré dual to $c_1(E)$. The support of $\mathcal{F}$ is precisely the desired pseudoholomorphic curve C.

4.4.4. *Seiberg–Witten equations on* $\mathbb{R}^4$. Consider now the SW_1-equations (for $\lambda=1$) on the space $X=\mathbb{R}^4$ with the standard Euclidean metric g_0 and standard symplectic form ω_0. They will play the role of a local model for the SW_λ-equations on (X,ω,J) for $\lambda\to\infty$. We identify $(\mathbb{R}^4,\omega_0,J_0)$ with $\mathbb{C}^2$ and consider the trivial bundle E over $\mathbb{C}^2$. In this case the SW_1-equations are written in the form

$$\begin{cases}\bar\partial_B\alpha=0,\\ F_B^{0,2}=0,\\ 4iF_B^{\omega}=1-|\alpha|^2.\end{cases}$$

Solutions of these equations satisfy the following a priori estimates:

$$\begin{cases} |\alpha| \le 1, \quad |\nabla_B \alpha| \le C(1-|\alpha|^2), \\ |F_B^-| \le |F_B^+| = \dfrac{1}{4}(1-|\alpha|^2), \\ \displaystyle\int\limits_{B_R} (1-|\alpha|^2)\, d\mathrm{vol} \le CR^2 \end{cases} \tag{4.16}$$

in the ball $B_R = B_R(0)$ of radius R and

$$\int \{|F_B^+|^2 - |F_B^-|^2\}\, d\mathrm{vol} \le C < \infty. \tag{4.17}$$

These estimates imply the following *properties of solutions*:

1. Either $|\alpha| \equiv 1$ or $|\alpha| < 1$ everywhere on $\mathbb{C}^2$:

(a) if $|\alpha| \equiv 1$, then any solution is gauge equivalent to the trivial one, i.e., $B \equiv 0$ and $\alpha \equiv 1$;

(b) if $|\alpha| < 1$, then the zero set $\alpha^{-1}(0)$ coincides with the zero set of some complex polynomial on $\mathbb{C}^2$ of degree controlled by the constant C from estimates (4.16) and (4.17).

2. Either $|F_B^-| \equiv |F_B^+|$ or $|F_B^-| < |F_B^+|$ everywhere on $\mathbb{C}^2$. If $|F_B^-| \equiv |F_B^+|$, then there exists a $\mathbb{C}$-linear projection $\pi\colon \mathbb{C}^2 \to \mathbb{C}$ such that the pair (B,α) is gauge equivalent to a pair $\pi^*(B_1,\alpha_1)$ for a vortex solution (B_1,α_1) on $\mathbb{C}$ with finite energy.

3. The quantity

$$\frac{1}{4\pi}\int \{|F_B^+|^2 - |F_B^-|^2\}\, d\mathrm{vol}(X)$$

is a nonnegative integer.

4. The quantities $1-|\alpha|^2$ and $|d_B\alpha|^2$ decrease exponentially as the distance to $\alpha^{-1}(0)$ tends to zero.

4.4.5. *Reduction to the local model.* Let (X,ω,J) be a compact symplectic 4-dimensional manifold equipped with a compatible Riemannian metric g. For an arbitrary point $x_0 \in X$ we can define the *Gaussian coordinate chart* at the point x_0 by the embedding $h\colon \mathbb{R}^4 \hookrightarrow X$ sending the origin to x_0 so that

$$h^*g = g_0 + O(|y|^2), \qquad h^*\omega = \omega_0 + O(|y|).$$

Here, $(\mathbb{R}^4, g_0, \omega_0)$ is the standard 4-dimensional Euclidean space with coordinates $y = (y_1,y_2,y_3,y_4)$ and h^* is the tensor map induced by h.

Suppose that $(B,(\alpha,\beta)) := (B_\lambda,(\alpha_\lambda,\beta_\lambda))$ is an SW_λ-solution on (X,g,ω). Then $h^*(B,(\alpha,\beta))$ define the SW_λ-data on $\mathbb{R}^4$. Applying the dilatation

$$\delta_\lambda\colon\ y \mapsto \frac{y}{\sqrt{\lambda}}$$

(which is equivalent to introducing the slow time variable), we obtain the data

$$(\underline{B},(\underline{\alpha},\underline{\beta})) = \delta_\lambda^* h^*(B,(\alpha,\beta))$$

on $(\mathbb{R}^4,\underline{g},\underline{\omega})$ with

$$|\underline{g} - g_0| \le \frac{C}{\lambda}|y|^2, \qquad |\underline{\omega} - \omega_0| \le \frac{C}{\sqrt{\lambda}}|y|$$

on the "large" ball of radius $\sqrt{\lambda}$ (i.e., for $|y| \le \sqrt{\lambda}$).

The data $(\underline{B}, (\underline{\alpha}, \underline{\beta}))$ satisfy the SW_1-equations on $(\mathbb{R}^4, \underline{g}, \underline{\omega})$ and are estimated on the "small" ball of radius $1/\sqrt{\lambda}$ by the data $h^*(B, (\alpha, \beta))$ on $(\mathbb{R}^4, \underline{g}, \underline{\omega})$ with the help of the following inequalities:

$$\begin{cases} |\underline{\alpha}(y)| = \left|\alpha\left(\dfrac{y}{\sqrt{\lambda}}\right)\right| \le C, \quad |d_{\underline{B}}\,\underline{\alpha}(y)| \dfrac{1}{\sqrt{\lambda}}\left|d_B\alpha\left(\dfrac{y}{\sqrt{\lambda}}\right)\right| \le C, \\ |\underline{\beta}(y)| = \dfrac{1}{\lambda}\left|\beta\left(\dfrac{y}{\sqrt{\lambda}}\right)\right| \le \dfrac{C}{(\sqrt{\lambda})^3}, \quad |d_{\underline{B}}\,\underline{\beta}(y)| = \left(\dfrac{1}{\sqrt{\lambda}}\right)^3\left|d_B\beta\left(\dfrac{y}{\sqrt{\lambda}}\right)\right| \le \dfrac{C}{\lambda}, \\ |F_{\underline{B}}(y)| = \dfrac{1}{\lambda}\left|F_B\left(\dfrac{y}{\sqrt{\lambda}}\right)\right| \le C. \end{cases} \tag{4.18}$$

4.4.6. *Compactness lemma and existence of vortex-type solutions.* The following lemma is proved by Taubes in [28].

Lemma 1 (compactness lemma). *Suppose that the sequence $\lambda_n \to \infty$ as $n \to \infty$. Denote by $(B_n, (\alpha_n, \beta_n))$ the sequence of SW_{λ_n}-data on (X, g, ω). Let $\{x_n\}$ be an arbitrary sequence of points in X and $\{h_n\}$ be the corresponding sequence of Gaussian charts at the points x_n. Denote by $(\underline{B_n}, (\underline{\alpha_n}, \underline{\beta_n}))$ the sequence of SW_1-solutions on $(\mathbb{R}^4, \underline{g_n}, \underline{\omega_n})$ constructed from the data $(B_n, (\alpha_n, \beta_n))$ with the help of the Gaussian charts $\{h_n\}$ at the points $\{x_n\}$ in the same way as in Subsection 4.4.5. Then there exists a subsequence of $[\underline{B_n}, (\underline{\alpha_n}, \underline{\beta_n})]$ (where $[\underline{B_n}, (\underline{\alpha_n}, \underline{\beta_n})]$ denotes the gauge equivalence class of the data $(\underline{B_n}, (\underline{\alpha_n}, \underline{\beta_n}))$) converging in the C^∞-topology on compact subsets of $\mathbb{R}^4$ to some SW_1-solution $(B_0, (\alpha_0, 0)) =: (B_0, \alpha_0)$ on $(\mathbb{R}^4, g_0, \omega_0)$. This solution satisfies estimates (4.16) and (4.17) on $\mathbb{R}^4$.*

Let us try to apply this lemma to the study of the pseudoholomorphic curve C introduced above. Let $x_0 \in C \subset X$. We apply Lemma 1 to the sequences $\lambda_n \to \infty$ and $x_n \equiv x_0$. Then, according to the lemma, the corresponding sequence $[\underline{B_n}, (\underline{\alpha_n}, \underline{\beta_n})]$ contains a subsequence converging (on compact subsets of $\mathbb{R}^4$) to an SW_1-solution (B_0, α_0) on $\mathbb{R}^4$. By construction, this solution should not depend on the radius in spherical coordinates on $\mathbb{R}^4$. This means that $|F^+_{B_0}| \equiv |F^-_{B_0}|$ for such a solution, so, according to property 2 from Subsection 4.4.4, the pair (B_0, α_0) is a vortex-type solution.

Although this heuristic argument looks quite plausible, its justification is a hard problem. The proof of the existence of vortex-type solutions is based on the following localization lemma due to Taubes [28].

4.4.7. *Localization lemma.*

Lemma 2 (localization lemma). *Fix $\varepsilon > 0$, $\delta > 0$, $R \ge 1$, and $k \in \mathbb{N}$. Then there exists $\lambda_0 > 0$ such that for all $\lambda \ge \lambda_0$ and any SW_λ-solution $(B, (\alpha, \beta))$ on (X, g, ω) the following assertions are true:*

1. *For any fixed $x \in X$*

(a) *and any Gaussian chart h at the point x we can construct, as above, an SW_1-solution $(\underline{B}, (\underline{\alpha}, \underline{\beta}))$ on $(\mathbb{R}^4, \underline{g}, \underline{\omega})$. Then there exists an SW_1-solution (B_0, α_0) on $(\mathbb{R}^4, g_0, \omega_0)$ such that the distance between $[\underline{B}, (\underline{\alpha}, \underline{\beta})]$ and $[B_0, \alpha_0]$ measured on the ball $B_R \subset \mathbb{R}^4$ in the C^k-norm does not exceed ε;*

(b) *the pair (B_0, α_0) satisfies estimates (4.16) and (4.17) with a constant C depending only on g and $c_1(E)$ (but not on λ or $(B, (\alpha, \beta))$);*

(c) *the pair $(B_0, \alpha_0) \equiv (B_0^x, \alpha_0^x)$ depends only on x, and $\alpha_0 \not\equiv \mathrm{const}$ if $|\alpha(x)| < 1$.*

2. *For varying $x \in X$ there exists a constant C depending only on g and $c_1(E)$ such that the set*

$$\Omega_\delta := \left\{ x \in X \colon \int\limits_{B_R} \{ |F^+_{B_0^x}|^2 - |F^-_{B_0^x}|^2 \} \, d\mathrm{vol}(g_0) \geq \delta \right\}$$

can be covered by less than C/δ balls of radius $R/\sqrt{\lambda}$ (in other words, the set Ω_δ is thin).

Using this lemma, we can find a subsequence $x_n \to x_0 \in C$ and an associated subsequence $\delta_n \to 0$ such that

$$\int\limits_{B_R} \{ |F^+_{\underline{B_n}}|^2 - |F^-_{\underline{B_n}}|^2 \} \, d\mathrm{vol}(g_0) < \delta_n.$$

Then by the compactness lemma we can find a subsequence of $[\underline{B_n}, (\underline{\alpha_n}, \underline{\beta_n})]$ that converges on the ball B_R to an SW_1-solution (B_0, α_0) on $\mathbb{R}^4$ so that

$$\|F^+_{B_0}\|_{L^2(B_R)} = \|F^-_{B_0}\|_{L^2(B_R)}.$$

Since $|F^+_{B_0}| \geq |F^-_{B_0}|$, we have $|F^+_{B_0}| \equiv |F^-_{B_0}|$ on B_R. Hence (B_0, α_0) is a vortex-type solution.

This argument shows that for a given point $x_0 \in C$ one can find a subsequence of SW_{λ_n}-solutions $(B_n, (\alpha_n, \beta_n))$ converging to a vortex-type solution (B_0, α_0) with center at x_0. In this sense, the SW_λ-equations on X for $\lambda \to \infty$ *reduce to a family of vortex equations on the normal bundle $N \to C$.*

4.5. Adiabatic limit in the Seiberg–Witten equations. This subsection is devoted to the detailed study of the adiabatic limit in the Seiberg–Witten equations on a 4-dimensional compact symplectic manifold.

4.5.1. *Geometry in a neighborhood of a pseudoholomorphic curve.* Let C be a compact pseudoholomorphic curve in a compact symplectic 4-dimensional manifold (X, ω, J, g). We will suppose that the curve C is connected and smooth. Let

$$\pi \colon N \to C$$

be the normal bundle to the curve C with the fiber N_x for $x \in C$ identified with the orthogonal complement to the space T_xC in T_xX. Since the operator of the almost complex structure J preserves T_xC, it also preserves N_x. Hence, $\pi \colon N \to C$ is a complex line bundle.

Introduce a *fiberwise constant almost complex structure* J_0 on $\pi \colon N \to C$ by

$$J_0 := \pi^*(J|TC).$$

We also fix a J_0-holomorphic *fiber coordinate* s on N such that the curve C, identified with the zero section of the bundle $\pi \colon N \to C$, is given by the equation $s = 0$.

We equip N with the Riemannian metric induced by the Riemannian metric of the manifold X, and consider the disk subbundle $U \to C$ in $N \to C$ formed by the disks of some (constant) radius in fibers of the bundle $N \to C$. If the chosen radius is sufficiently small, then, using the exponential map $\exp \colon U \to X$, we can identify U with a tubular neighborhood u of the curve C in the manifold X. With the help of this identification we can transport the almost complex structure J from the neighborhood u to U by taking the inverse image of $J|u$ under the exponential map. Denote the *almost complex structure* on U obtained in this way by the same letter J. By definition, it coincides with the almost complex structure J_0 on the zero section of the bundle $\pi \colon U \to C$; however, in contrast to J_0, it is not fiberwise constant on U.

We define the *fiber linearization* $\overline{D}_J$ of the $\bar{\partial}_J$*-operator* with respect to the bundle $\pi\colon U \to C$. Note that

$$\bar{\partial}_J = \bar{\partial}_{J_0} + \nu s + \mu \bar{s} + O(|s|^2) \tag{4.19}$$

on U, where μ and ν are $(0,1)$-forms (with respect to the almost complex structure J_0) on C. Then by definition

$$\overline{D}_J := \bar{\partial}_{J_0} + \bar{\delta}_J = \bar{\partial}_{J_0} + \nu s + \mu \bar{s} \tag{4.20}$$

is the linear part of the operator $\bar{\partial}_J$ in s.

4.5.2. *Vortex bundle.* Recall that the moduli space of d-vortices $\mathcal{M}_d$ from Section 1 is identified with the vector space $\mathbb{C}^d$ by the map assigning the pair (A, Φ) with $A = i(\bar{\partial} - \partial)u$ and $\Phi = P_w e^{-u}$ to the collection $w = (w_1, w_2, \dots, w_d) \in \mathbb{C}^d$, where

$$P_w(z) = z^d + w_1 z^{d-1} + \ldots + w_d$$

and u is a unique solution of the Liouville-type equation

$$4i\partial\bar{\partial}u = *\frac{1}{2}(1 - |P_w|^2 e^{-2u})$$

that satisfies the asymptotic condition $u(z) \sim d\log|z|$ as $|z| \to \infty$.

In the moduli space $\mathcal{M}_d$ there is a U(1)-*action* generated by the natural U(1)-action on $\mathbb{C}^d$ given by the formula

$$(w_1, \dots, w_d) \mapsto (e^{i\theta} w_1, \dots, e^{id\theta} w_d).$$

The induced U(1)-action on $\mathcal{M}_d$ is given by

$$(A, \Phi) \mapsto (\delta_\theta^* A, e^{id\theta} \delta_\theta^* \Phi),$$

where $\delta_\theta\colon z \mapsto e^{i\theta} z$ is a rotation of the plane $\mathbb{C}$.

There is also a *scale transformation* generated by the dilatation

$$\rho_r\colon\ z \mapsto \sqrt{r} z$$

with $r > 0$. This dilatation induces the map

$$\rho_r^*\colon\ (A, \Phi) \mapsto (A_r, \Phi_r),$$

which sends a vortex solution (A, Φ) to a solution (A_r, Φ_r) of the *rescaled vortex equations*

$$\begin{cases} \bar{\partial}_{A_r} \Phi_r = 0, \\ 2i(dA_r) = r(1 - |\Phi_r|^2). \end{cases}$$

We return to the manifold (X, ω, J, g) and pseudoholomorphic curve C. Denote by $L \to C$ the circle subbundle in the bundle $\pi\colon N \to C$ with the natural U(1)-action and introduce the *d-vortex bundle* associated with the bundle $\pi\colon N \to C$:

$$\mathcal{L}_d := L \times_{\mathrm{U}(1)} \mathcal{M}_d \to C.$$

A section of the bundle $\mathcal{L}_d$, called otherwise the *d-vortex section*, is given by a family $\tau = \{\tau_x\}_{x \in C}$ of d-vortex solutions $\tau_x = [A_x, \Phi_x]$ in the normal planes N_x, $x \in C$. In particular, the zero section τ_0 corresponds to the family of radial d-vortex solutions. Note that the scale ρ_r^*-transformation extends naturally to the d-vortex sections.

Consider again the sequence $C_n \to C$. The curves C_n with sufficiently large n can be identified with sections of $\pi\colon U \to C$ of degree d or, in other words, with sections of the bundle $\pi\colon U^d \to C$. In another way, the curve C_n can be considered as a d-vortex section, i.e., as a section of the bundle $\mathcal{L}_d \to C$ given by some family

$$\gamma_n\colon\ x \in C_n \mapsto \gamma_n = [A_n, \Phi_n],$$

where the gauge class $[A_n, \Phi_n]$ is determined by the section C_n of the bundle $\pi\colon U^d \to C$ with the help of the identification $\mathcal{M}_d \cong \operatorname{Sym}^d \mathbb{C}$.

Denote by γ_0 the zero section of the bundle $U^d \to C$, which is identified with the pseudoholomorphic divisor $d[C]$. We are going to show that γ_0 is the adiabatic limit of γ_n for $n \to \infty$ in the same sense as in the Abelian Higgs model.

4.5.3. *From d-vortex sections to the Seiberg–Witten equations.* Our goal is to construct, starting from a d-vortex section $\gamma = [A, \Phi]$, the data for the SW_λ-equations on X that define the solution of these equations converging to the original d-vortex section in the adiabatic limit. These data are obtained by the smooth extension of the line bundle $\pi^* N^d \to U$, connection A, and section Φ from the set U identified with a tubular neighborhood u of the curve C in the manifold X to the whole X.

Let $U_\delta \to C$ be the disk subbundle (of radius $\delta < 1$) of the bundle $U \to C$. Let also $\tau = [A, \Phi]$ be a given d-vortex bundle. We will construct a bundle $E \to X$ by gluing together the trivial bundle over $X \setminus U_{1/2}$ and $\pi^* N^d \to U$ with the help of the *gluing map*

$$(U \setminus U_{1/2}) \times \mathbb{C} \to \pi^* N^d$$

sending (x, ζ) to $(x, \zeta\Phi_r/|\Phi_r|)$, where the scale r is chosen so that the zeros of the section Φ_r are contained in $U_{1/2}$. Then the class $c_1(E)$ will coincide with the Poincaré dual of the class $d[C]$.

To construct the data $(B, (\alpha, 0))$, we will use the cut-off function χ_δ on X satisfying the conditions $\chi_\delta \equiv 1$ on U_δ and $\chi_\delta \equiv 0$ outside $U_{1/2}$. We construct α by gluing together $\alpha_r \equiv 1$ on $X \setminus U_{1/2}$ and $\alpha_r = \Phi_r/(\chi_\delta + (1-\chi_\delta)|\Phi_r|)$. We also define B by gluing together the trivial connection on $X \setminus U_{1/2}$ and $B_r = \chi_\delta A_r + (1-\chi_\delta)\alpha_r^{-1}\nabla\alpha_r$.

Plugging these data into the Seiberg–Witten equations, we obtain the following inequalities:

$$\begin{cases} |D_{B_r}(\alpha_r, 0)| \le C e^{-c\sqrt{r}\cdot \mathrm{dist}}, \\[4pt] \left| F^+_{B_r} + \dfrac{ir}{4}(1 - |\alpha_r|^2)\omega \right| \le C\sqrt{r} e^{-c\sqrt{r}\cdot \mathrm{dist}}, \end{cases}$$

where dist denotes the distance from C. The second of these estimates is unsatisfactory for large r, because its right-hand side contains the factor $\sqrt{r}$, which increases with the growth of r. To obtain correct estimates, we should use the adiabatic equations, which we will deduce in the next subsection.

4.5.4. *Seiberg–Witten equations in a neighborhood of the zero section.* The Seiberg–Witten equations corresponding to the data $(E, (B, \alpha))$ in a small neighborhood U_δ of the zero section of the bundle $U \to C$ (U_δ is assumed to be formed by the disks of sufficiently small radius δ) can be written in the form

$$\begin{cases} \bar\partial_{J,A}\Phi = 0, \\[4pt] d_J^+ A + \dfrac{i}{4}(1 - |\Phi|^2)\omega = 0, \end{cases} \tag{4.21}$$

where the operator d_J^+ is given by the composition of the exterior derivative d with the projection to the self-dual 2-forms (note that the definition of these forms depends on the Riemannian metric g and, hence, on the almost complex structure J).

4.5.5. *Perturbations of vortex sections.* As in the $(2+1)$-dimensional case, consider perturbations $\gamma_\varepsilon = [A^\varepsilon, \Phi^\varepsilon]$ of the original d-vortex section $\gamma = [A, \Phi]$ of the form

$$A^\varepsilon = A + \varepsilon a, \qquad \Phi^\varepsilon = \Phi + \varepsilon\varphi, \tag{4.22}$$

where (a, φ) satisfy the orthogonality condition

$$(a, \varphi) \perp T_{(A,\Phi)}\mathcal{M}_d. \tag{4.23}$$

Using an H^2-basis $\{n_\mu\}$ of solutions of the linearized vortex equations $\mathcal{D}_{(A,\Phi)} n_\mu = 0$, we can rewrite this condition in the form

$$\langle (a, \varphi), n_\mu \rangle = 0 \qquad \text{for} \quad \mu = 1, \dots, 2d. \tag{4.24}$$

We suppose that, as in the $(2+1)$-dimensional case, n_μ and (a, φ) satisfy the gauge-fixing condition

$$\delta^*_{(A,\Phi)} n_\mu = 0, \qquad \delta^*_{(A,\Phi)}(a, \varphi) = 0. \tag{4.25}$$

Substituting the trajectories γ_ε given by formula (4.22) into the Seiberg–Witten equations (4.21), we get

$$\begin{cases} \bar\partial_{J,A}\Phi + \varepsilon\bar\partial_{J,A}\varphi + \varepsilon a_J^{(0,1)}\Phi \approx 0, \\ d_J^+ A + \dfrac{i}{4}(1 - |\Phi|^2)\omega + \varepsilon d_J^+ a - \dfrac{i\varepsilon}{2}\operatorname{Re}(\varphi\bar\Phi)\omega \approx 0; \end{cases} \tag{4.26}$$

here and below the sign "$\approx$" denotes an equality up to terms of order higher than 1 in ε, and $a_J^{(0,1)}$ is the $(0,1)$-component of the 1-form a with respect to the almost complex structure J.

4.5.6. *Vertical–horizontal decomposition.* At this stage we encounter a new effect that does not occur in the $(2+1)$-dimensional case. We decompose all differential operators d and their $\bar\partial$- and covariant analogs appearing in equations (4.26) into the *vertical*, i.e., normal to TC, and *horizontal*, i.e., tangent to TC, components: $d = d^N + d^C$, $\bar\partial = \bar\partial^N + \bar\partial^C$, and so on.

We study the vertical and horizontal components of equations (4.26) in different ways. While considering the normal derivations, we can suppose that the gauge class $[A_x, \Phi_x]$ at a point $x \in C$ is fixed (since the change of the gauge class means the change inside $\mathcal{M}_d$, i.e., along the base of the bundle $\mathcal{L}_d \to C$). However, the almost complex structure J_x is not constant in the normal direction (since it depends on the fiber parameter s on N_x). For the horizontal derivations (along C) we can, on the contrary, suppose that the almost complex structure J_x coincides with the (fiberwise constant) almost complex structure $J_{0,x}$ but remember that the gauge class $[A_x, \Phi_x]$ may change in the horizontal direction.

4.5.7. *Tangent component of the Seiberg–Witten equations.* Consider first the horizontal, or tangent, component of the Seiberg–Witten equations (4.26) at a point $x \in C$. Then we can suppose that all $\bar\partial$-derivatives are taken with respect to J_0 and omit the subscript J_0 in notations. The tangent component of equations (4.26) has the form

$$\begin{cases} \bar\partial_A^C\Phi + \varepsilon\big(\bar\partial_A^C\varphi + a^{(0,1)}\Phi\big) \approx 0, \\ d_+^C A + \varepsilon\Big(d_+^C a - \dfrac{i}{2}\operatorname{Re}(\varphi\bar\Phi)\omega\Big) \approx 0. \end{cases}$$

This equation can be rewritten in terms of the linearized vortex operator in the following way:

$$(\bar\partial_A^C\Phi, d_+^C A) + \varepsilon\mathcal{D}_{(A,\Phi)}(a, \varphi) \approx 0. \tag{4.27}$$

4.5.8. *Normal component of the Seiberg–Witten equations.* Consider next the vertical, or normal, component of the Seiberg–Witten equations (4.26). Here we should take into account the dependence of J on the fiber parameter s. According to formulas (4.19) and (4.20),

$$\bar\partial_J = \overline{D}_J + \ldots = \bar\partial_{J_0} + \bar\delta_J + \ldots$$

on C, where $\overline{D}_J$ is the linearized $\bar\partial_J$-operator and we denote by an ellipsis the terms of higher orders in s and $\bar s$. In a similar way,

$$\begin{cases} d_J^+ = d^+ + \delta_J^+, \\ \bar\partial_{J,A}^N\Phi = \bar\partial_A^N + \bar\delta_{J,A}^N\Phi + \ldots, \end{cases}$$

where we have again omitted the subscript J_0. Using these notations, we can write down the normal component of equations (4.26) in the form

$$\begin{cases} \bar\partial_A^N\Phi + \bar\delta_{J,A}^N\Phi + \varepsilon\bar\partial_{J,A}^N\varphi + \ldots \approx 0, \\ d_+^N A + \dfrac{i}{4}(1-|\Phi|^2)\omega + \delta_J^{+,N}A + \varepsilon d_J^{+,N}a + \ldots \approx 0. \end{cases}$$

Note that

$$\bar\partial_A^N\Phi = d_+^N A + \frac{i}{4}(1-|\Phi|^2)\omega = 0,$$

because the pair (A,Φ) satisfies the vortex equations on N_x, $x\in C$ (with respect to J_0). So we can rewrite the normal component of equations (4.26) in the following vector form:

$$(\bar\delta_{J,A}^N\Phi, \delta_J^{+,N}A) + \varepsilon(\bar\partial_{J,A}^N\varphi, d_J^{+,N}a) + \ldots \approx 0. \tag{4.28}$$

Comparing expression (4.27) with (4.28), we finally arrive at

$$(\bar\partial_A^C\Phi, d_+^C A) + (\bar\delta_{J,A}^N\Phi, \delta_J^{+,N}A) + \varepsilon\mathcal{D}_{(A,\Phi)}(a,\varphi) + \varepsilon(\bar\partial_{J,A}^N\varphi, d_J^{+,N}a) + \ldots \approx 0. \tag{4.29}$$

4.5.9. *Using the orthogonality condition.* As in the $(2+1)$-dimensional case, we should use the orthogonality condition (4.23) in order to get rid of the terms containing (a,φ) in (4.29).

Since $\langle(a,\varphi), n_\mu\rangle = 0$, there exists a unique H^2-solution (b,ψ) of the equation

$$\mathcal{D}_{(A,\Phi)}^*(b,\psi) = (a,\varphi), \tag{4.30}$$

where $\mathcal{D}_{(A,\Phi)}^*$ is the adjoint operator of $\mathcal{D}_{(A,\Phi)}$.

Introduce the *linearized vortex Laplacian*

$$L_{(A,\Phi)} := \mathcal{D}_{(A,\Phi)}\mathcal{D}_{(A,\Phi)}^* = \mathcal{D}_{(A,\Phi)}^*\mathcal{D}_{(A,\Phi)}$$

and note that $L_{(A,\Phi)}n_\mu = 0$. This implies that

$$\langle\mathcal{D}_{(A,\Phi)}(a,\varphi), n_\mu\rangle = \langle L_{(A,\Phi)}(b,\psi), n_\mu\rangle = \langle(b,\psi), L_{(A,\Phi)}n_\mu\rangle = 0.$$

Taking the inner product of both sides of equation (4.29) with n_μ and using the preceding equality, we obtain

$$\langle(\bar\partial_A^C\Phi, d_+^C A), n_\mu\rangle + \langle(\bar\delta_{J,A}^N\Phi, \delta_J^{+,N}A), n_\mu\rangle + \varepsilon\langle(\bar\partial_{J,A}^N\varphi, d_J^{+,N}a), n_\mu\rangle + \ldots \approx 0. \tag{4.31}$$

4.5.10. *Introducing the "slow" variable.* Following the same scheme as in the $(2+1)$-dimensional case, we introduce the "slow" variable $\xi = \varepsilon x$ and divide both sides by ε. Then we get

$$\langle(\bar\partial^C_A\Phi, d^C_+A), n_\mu\rangle + \langle(\bar\delta^N_{J,A}\Phi, \delta^{+,N}_J A), n_\mu\rangle + \ldots \approx 0,$$

where all derivatives are taken with respect to ξ. Setting here $\varepsilon = 0$ and $s = 0$, we obtain the *adiabatic equation* in the form

$$\langle(\bar\partial^C_A\Phi, d^C_+A), n_\mu\rangle + \langle(\bar\delta^N_{J,A}\Phi, \delta^{+,N}_J A), n_\mu\rangle = 0, \qquad \mu = 1,\ldots,2d. \tag{4.32}$$

This equation coincides with the equation obtained in [25] from other considerations (see the discussion of this point in Subsection 4.6.1).

4.5.11. *Particular cases.* The adiabatic equation (4.32) has the form of a nonlinear $\bar\partial$-equation for sections $\gamma = [A,\Phi]$ of the d-vortex bundle $\mathcal{L}_d \to C$. We call the d-vortex sections γ satisfying equation (4.32) *adiabatic*.

Equation (4.32) takes a simpler form for *constant sections*, i.e., for sections $\gamma\colon x \in C \mapsto (A_x, \Phi_x)$ with the gauge class of (A_x, Φ_x) independent of x, for example, for the *radial section*. For such sections the first term in (4.32) disappears and we obtain the equation

$$\langle(\bar\delta^N_{J,A}\Phi, \delta^{+,N}_J A), n_\mu\rangle = 0, \qquad \mu = 1,\ldots,2d.$$

Another particular case of equation (4.32) corresponds to $d = 1$. In this case formula (4.32) simplifies (in the notation from (4.19)) to

$$\bar\partial\sigma_\gamma + \nu\sigma_\gamma + \mu\bar\sigma_\gamma = 0,$$

where σ_γ is a section of the bundle $N \to C$ given by $\Phi^{-1}(0)$ for $\gamma = [A,\Phi]$. In particular, if σ_γ coincides with the zero section of the bundle $N \to C$, then the latter $\bar\partial$-equation reduces to the *pseudoholomorphicity condition* for the curve C.

4.5.12. *Description in terms of the action functional.* As in the $(2+1)$-dimensional case, we can introduce the *action functional on d-vortex solutions*. For a section $\gamma = [A,\Phi]$ of the bundle $\mathcal{L}_d \to C$ it is equal to

$$E(A,\Phi) = \frac{1}{2}\int\limits_C \Big\{\|\bar\partial_A\Phi\|^2 + 2\Big\|d^+A + \frac{i}{4}(|\Phi|^2 - 1)\Big\|^2\Big\}\, d\mathrm{vol}$$

and coincides (up to some topological term) with the restriction of the Seiberg–Witten action functional to the d-vortex sections γ of the bundle $\mathcal{L}_d \to C$ that are close to the zero section. It can be shown (using the same arguments as in the derivation of the adiabatic equation (4.32)) that the adiabatic sections are extremals of this action functional.

4.6. From pseudoholomorphic curves to the Seiberg–Witten equations.

4.6.1. *Construction of Seiberg–Witten data from d-vortex sections.* Now we consider the procedure for reconstructing a solution of the Seiberg–Witten equations from a pseudoholomorphic curve C and a family of solutions of the vortex equations in the planes normal to C, or, which is the same, from a section of the vortex bundle over C under the assumption that this section satisfies the adiabatic equation (4.32).

We write the adiabatic equation for the section $\tau = [A,\Phi] \in \mathcal{L}$ in the form

$$\langle p(A,\Phi), n_\mu\rangle = 0.$$

Then, as in (4.30), there exists a unique H^2-solution of the equation

$$\mathcal{D}^*_{(A,\Phi)}(b,\psi) = p(A,\Phi).$$

The desired Seiberg–Witten equation on X is reconstructed from the *Seiberg–Witten data* consisting of a Hermitian line bundle $E \to X$, a Hermitian connection B on this bundle, and its section (α,β). The bundle $E \to X$ over the manifold X was already defined in Subsection 4.5.3. The other data, denoted by $(\widetilde{B},(\widetilde{\alpha},\widetilde{\beta}))$, are constructed from the data $(B_r,(\alpha_r,0))$ introduced in Subsection 4.5.3. They are obtained by taking the data $(B_r,(\alpha_r,0))$ on $X\setminus U_{1/2}$ and the data $(B_r+\chi_\delta b_r,(\alpha_r,\chi_\delta\psi_r))$ on U and gluing them over $U\setminus U_{1/2}$. The new data satisfy the estimates

$$\begin{cases} |D_{\widetilde{B}_r}(\widetilde{\alpha}_r,\widetilde{\beta}_r)| \le \dfrac{C}{\sqrt{r}}\, e^{-c\sqrt{r}\cdot\mathrm{dist}}, \\ \Big|F^+_{\widetilde{B}_r} + \dfrac{ir}{4}(1-|\widetilde{\alpha}_r|^2+|\widetilde{\beta}_r|^2)\omega + \dfrac{r}{2}(\widetilde{\alpha}_r\overline{\widetilde{\beta}}_r - \overline{\widetilde{\alpha}}_r\widetilde{\beta}_r)\Big| \le Ce^{-c\sqrt{r}\cdot\mathrm{dist}}. \end{cases}$$

Note that the right-hand sides of these inequalities, in contrast to the analogous estimates given in Subsection 4.5.3, do not contain factors increasing with r. Taubes obtained equation (4.32) (in a different form) precisely from the condition of vanishing of the terms growing in r on the right-hand sides of the estimates from Subsection 4.5.3.

4.6.2. *The space of adiabatic sections.* Denote by $\mathcal{Z}$ the *space of adiabatic sections*, i.e., sections of the bundle $\mathcal{L}_d \to C$ satisfying the adiabatic equation.

This space is locally compact, and its smooth part $\mathcal{Z}_{\mathrm{reg}}$ consisting of the adiabatic sections for which the adiabatic equation is satisfied transversally is an oriented manifold of dimension

$$2d(1-g) + d(d+1)n,$$

where g is the genus of C and n is the degree of the map $\pi\colon N \to C$.

Using the above estimates and the implicit function theorem, Taubes proved the following result (see [30]).

Theorem 9 (Taubes)**.** *Let K be a relatively compact open subset in $\mathcal{Z}_{\mathrm{reg}}$. Then for $r \ge r_0$ there exists a continuous map from the set K to the moduli space of solutions of the Seiberg–Witten* SW_r*-equations on X that is given by the formula*

$$\tau = [A,\Phi] \mapsto (\vec{B}_r,(\vec{\alpha}_r,\vec{\beta}_r)),$$

where $\vec{B}_r = \widetilde{B}_r + \sqrt{r}B'$, $\vec{\alpha}_r = \widetilde{\alpha}_r + \alpha'$, $\vec{\beta}_r = \widetilde{\beta}_r + \beta'$, and the remainder section $\gamma' = (B',(\alpha',\beta'))$ satisfies the estimates

$$\|\nabla\gamma'\|_{L^2} + \sqrt{r}\|\gamma'\|_{L^2} \le \frac{C}{\sqrt{r}}, \qquad \sup_X|\nabla\gamma'| + \sqrt{r}\sup_X|\gamma'| \le C.$$

5. SUPPLEMENT: ADIABATIC LIMIT IN DIMENSION 1 + 1

In this supplement we consider the one-dimensional analogs of vortices that were introduced by Domrin [3]. They arise in the scale limit of a system governed by the action functional representing a one-dimensional analog of the static Ginzburg–Landau action functional.

Consider the *action functional* S_λ defined on the C^2-smooth trajectories in the complex plane of the form $\gamma\colon I := [0,1] \to \mathbb{C}$ by the formula

$$S_\lambda(\gamma) = \int_I \big\{|\gamma'(s)|^2 + \lambda(1-|\gamma(s)|^2)^2\big\}\,ds, \tag{5.1}$$

where $\lambda > 0$ is the scale parameter.

The *Euler–Lagrange equation* for this functional has the form

$$\gamma'' = 2\lambda\gamma(1 - |\gamma(s)|^2). \tag{5.2}$$

This equation can be considered as a *one-dimensional analog of the static Ginzburg–Landau equation.*

In order to formulate the Dirichlet problem for equation (5.2), we restrict the class of admissible trajectories γ. Namely, we will consider only those trajectories that do not pass through the origin and have endpoints lying on the unit circle. A trajectory γ of this type can be written in the form $\gamma(s) = \rho(s)e^{i\theta(s)}$, where $\rho(s) = |\gamma(s)|$ and θ is a continuous real-valued function on $I = [0, 1]$. The integer part $d := [\Delta\theta]$ of

$$\Delta\theta := \frac{1}{\pi}(\theta(1) - \theta(-1)),$$

equal to the number of full revolutions of the trajectory γ around the origin, plays the role of a *topological invariant* of the problem.

The function $\theta(s) - \theta(0)$ is odd and increases monotonically on the interval I, while the function $\rho(s)$ is even and strictly increases on the interval $(0, 1]$.

The Dirichlet problem in a given topological class can be stated in the following form. Fix a number $\phi > 0$ and consider the problem

$$\begin{cases} \gamma'' = 2\lambda\gamma(1 - |\gamma(s)|^2), \\ |\gamma(\pm 1)| = 1, \quad \theta(\pm 1) = \pm\phi. \end{cases} \tag{5.3}$$

We are interested in the behavior of solutions of the Dirichlet problem (5.3) for $\lambda \to +\infty$. As shown in [3], the limiting trajectories of solutions γ_λ, which can be considered as *one-dimensional analogs of the vortices*, may be of two kinds: regular and singular ones.

In the *regular case* the limiting trajectory is given by an arc of the unit circle of the form

$$\{\gamma(s) = e^{i\theta(s)} \colon -\phi \le \theta(s) \le \phi\},$$

which contains the whole circle passed around one or several times if $\phi \ge \pi$.

In the *singular case* (which may realize only for $\phi > \pi/2$) the trajectory $\gamma(s)$ behaves in the following way: in the beginning the point $\gamma(s)$ moves along the unit circle till the value $\theta = -\pi/2$, then "jumps" to the opposite point $\theta = \pi/2$, and continues the motion along the unit circle. (A rigorous formulation and a proof of this assertion are given in [3].)

The behavior of the singular trajectory of this type reminds the "*swinging*" of the rolling spin on the plane. When the speed of its rotation follows down to a critical one (determined by the physical parameters of the spin), it suddenly bends almost to the plane over which it is rotating, but then stands up again and continues to rotate. This process is continued until the spin finally falls down to the plane.

ACKNOWLEDGMENTS

This work is supported by the Russian Science Foundation under grant 14-50-00005.

REFERENCES

1. M. F. Atiyah and N. Hitchin, *The Geometry and Dynamics of Magnetic Monopoles* (Princeton Univ. Press, Princeton, NJ, 1988).
2. S. B. Bradlow, "Vortices in holomorphic line bundles over closed Kähler manifolds," Commun. Math. Phys. **135**, 1–17 (1990).
3. A. V. Domrin, "Ginzburg–Landau vortex analogues," Teor. Mat. Fiz. **124** (1), 18–35 (2000) [Theor. Math. Phys. **124**, 872–886 (2000)].

4. S. K. Donaldson, "An application of gauge theory to four dimensional topology," J. Diff. Geom. **18**, 279–315 (1983).
5. O. García-Prada, "A direct existence proof for the vortex equations over a compact Riemann surface," Bull. London Math. Soc. **26**, 88–96 (1994).
6. M. Gromov, "Pseudo holomorphic curves in symplectic manifolds," Invent. Math. **82**, 307–347 (1985).
7. A. Jaffe and C. Taubes, *Vortices and Monopoles: Structure of Static Gauge Theories* (Birkhäuser, Boston, 1980).
8. J. L. Kazdan and F. W. Warner, "Curvature functions for compact 2-manifolds," Ann. Math., Ser. 2, **99**, 14–47 (1974).
9. D. Kotschick, "The Seiberg–Witten invariants of symplectic four-manifolds (after C.H. Taubes)," in *Séminaire Burbaki, Volume 1995/96, Exposés 805–819* (Soc. Math. France, Paris, 1997), Exp. 812, Astérisque **241**, pp. 195–220.
10. P. B. Kronheimer and T. S. Mrowka, "The genus of embedded surfaces in the projective plane," Math. Res. Lett. **1**, 797–808 (1994).
11. E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics*, Part 2: *Theory of the Condensed State* (Pergamon Press, London, 1980).
12. H. B. Lawson, Jr. and M.-L. Michelsohn, *Spin Geometry* (Princeton Univ. Press, Princeton, NJ, 1989).
13. N. S. Manton, "A remark on the scattering of BPS monopoles," Phys. Lett. B **110**, 54–56 (1982).
14. J. W. Morgan, *The Seiberg–Witten Equations and Applications to the Topology of Smooth Four-Manifolds* (Princeton Univ. Press, Princeton, NJ, 1996).
15. R. V. Palvelev, "Scattering of vortices in the Abelian Higgs model," Teor. Mat. Fiz. **156** (1), 77–91 (2008) [Theor. Math. Phys. **156**, 1028–1040 (2008)].
16. R. V. Pal'velev, "Justification of the adiabatic principle in the Abelian Higgs model," Tr. Mosk. Mat. Obshch. **72** (2), 281–314 (2011) [Trans. Moscow Math. Soc. **2011**, 219–244 (2011)].
17. R. V. Palvelev and A. G. Sergeev, "Justification of the adiabatic principle for hyperbolic Ginzburg–Landau equations," Tr. Mat. Inst. im. V.A. Steklova, Ross. Akad. Nauk **277**, 199–214 (2012) [Proc. Steklov Inst. Math. **277**, 191–205 (2012)].
18. D. Salamon, "Spin geometry and Seiberg–Witten invariants," Preprint (Warwick Univ., 1996).
19. N. Seiberg and E. Witten, "Electric–magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang–Mills theory," Nucl. Phys. B **426**, 19–52 (1994).
20. N. Seiberg and E. Witten, "Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD," Nucl. Phys. B **431**, 484–550 (1994).
21. A. G. Sergeev, *Vortices and Seiberg–Witten Equations* (Nagoya Univ., Nagoya, 2002).
22. A. G. Sergeev and S. V. Chechin, "Scattering of slowly moving vortices in the Abelian $(2 + 1)$-dimensional Higgs model," Teor. Mat. Fiz. **85** (3), 397–411 (1990) [Theor. Math. Phys. **85**, 1289–1299 (1990)].
23. D. M. A. Stuart, "Periodic solutions of the Abelian Higgs model and rigid rotation of vortices," Geom. Funct. Anal. **9**, 568–595 (1999).
24. C. H. Taubes, "Arbitrary N-vortex solutions to the first order Ginzburg–Landau equations," Commun. Math. Phys. **72**, 277–292 (1980).
25. C. H. Taubes, "On the equivalence of the first and second order equations for gauge theories," Commun. Math. Phys. **75**, 207–227 (1980).
26. C. H. Taubes, "The Seiberg–Witten invariants and symplectic forms," Math. Res. Lett. **1**, 809–822 (1994).
27. C. H. Taubes, "The Seiberg–Witten and Gromov invariants," Math. Res. Lett. **2**, 221–238 (1995).
28. C. H. Taubes, "SW $\Rightarrow$ Gr: From the Seiberg–Witten equations to pseudo-holomorphic curves," J. Am. Math. Soc. **9**, 845–918 (1996).
29. C. H. Taubes, "Counting pseudo-holomorphic submanifolds in dimension 4," J. Diff. Geom. **44**, 818–893 (1996).
30. C. H. Taubes, "Gr $\Rightarrow$ SW: From pseudo-holomorphic curves to Seiberg–Witten solutions," J. Diff. Geom. **51**, 203–334 (1999).
31. C. H. Taubes, "Gr = SW: Counting curves and connections," J. Diff. Geom. **52**, 453–609 (1999).
32. E. Witten, "Monopoles and four-manifolds," Math. Res. Lett. **1**, 769–796 (1994).

Translated by the author