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V.A. Steklov's Problem of Estimating the Growth of Orthogonal Polynomials

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Abstract—The well-known problem of V.A. Steklov is closely related to the following extremal problem. For a fixed $n \in \mathbb{N}$, find $M_{n,\delta} = \sup_{\sigma \in S_\delta} \|\phi_n\|_{L^\infty(\mathbb{T})}$, where $\phi_n(z)$ is an orthonormal polynomial with respect to a measure $\sigma \in S_\delta$ and S_δ is the Steklov class of probability measures σ on the unit circle such that $\sigma'(\theta) \geq \delta/(2\pi) > 0$ at every Lebesgue point of σ. There is an elementary estimate $M_n \lesssim \sqrt{n}$. E.A. Rakhmanov proved in 1981 that $M_n \gtrsim \sqrt{n}/(\ln n)^{3/2}$. Our main result is that $M_n \gtrsim \sqrt{n}$, i.e., that the elementary estimate is sharp. The paper gives a survey of the results on the solution of this extremal problem and on the general problem of Steklov in the theory of orthogonal polynomials. The paper also analyzes the asymptotics of some trigonometric polynomials defined by Fejér convolutions. These polynomials can be used to construct asymptotic solutions to the extremal problem under consideration.

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1. INTRODUCTION

The theory of orthogonal polynomials occupied an important place in the work of V.A. Steklov. He published the total of 29 papers on this subject [11–39]. The first of them appeared in 1900, while the last ones are dated 1926. In the 1940 survey on orthogonal polynomials [9], all these publications were summarized and it was precisely indicated what new properties of orthogonal polynomials had been obtained there. In the Russian literature, a detailed analysis of Steklov's work was carried out in 1977 by Suetin in his monographic survey [40]. The main attention in [40] was paid to Steklov's conjecture, which represented an open question at the time and attracted increased interest to this field and related problems. First of all, interest was aroused by the problem of estimating orthogonal polynomials on the support of the orthogonality weight, which was called *Steklov's problem* in [40]: find an estimate on $(-1,1)$ for a polynomial sequence $\{P_n(x)\}_{n=0}^{\infty}$ that is orthonormal

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, \rho(x)\, dx = \delta_{n,m}, \qquad n,m = 0,1,2,\dots, \tag{1.1}$$

with respect to a strictly positive weight ρ:

$$\rho(x) \geq \delta > 0, \qquad x \in [-1,1]. \tag{1.2}$$

In 1921, Steklov conjectured that the sequence $\{P_n(x)\}$ is bounded at the points $x \in (-1,1)$, i.e.,

$$\limsup_{n\to\infty} |P_n(x)| < \infty, \tag{1.3}$$

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if the weight ρ does not vanish on $[-1,1]$ (see [35, p. 321] or a citation in [40]). In the present study, we mark the progress made after 1977 concerning the conjecture and problem of Steklov.

We will use the more convenient terminology of polynomials orthogonal on a circle. Let $\{\phi_n\}$ be a sequence of polynomials in $z = e^{i\theta}$ that are orthonormal on the unit circle,

$$\int_0^{2\pi} \phi_n \overline{\phi}_m \, d\sigma(\theta) = \delta_{n,m}, \qquad n,m = 0,1,2,\dots, \tag{1.4}$$

with respect to a measure σ. The *Steklov class* S_δ is a class of probability measures σ on the unit circle that satisfy the condition

$$\sigma' \geq \frac{\delta}{2\pi} \tag{1.5}$$

at every Lebesgue point. In these terms, Steklov's conjecture states that the polynomials ϕ_n generated by a measure in the Steklov class should be uniformly bounded in n on the support of the orthogonality measure.

Steklov's conjecture was disproved in 1979 by Rakhmanov [7]. He constructed polynomials with conditions (1.4) and (1.5) such that

$$\limsup_{n\to\infty} \|\phi_n(z;\sigma)\|_{L^\infty(\mathbb{T})} = \infty. \tag{1.6}$$

More precisely, in Rakhmanov's construction (below we will dwell on it in more detail), polynomials displayed logarithmic growth (along a subsequence) and their orthogonality measure (from the Steklov class) contained a discrete component. The Rakhmanov counterexample was extended to continuous measures (weights) in [2].

The following extremal problem played an important role in Rakhmanov's construction and in the subsequent research on Steklov's problem: for a fixed n, find

$$M_{n,\delta} = \sup_{\sigma\in S_\delta} \|\phi_n(z;\sigma)\|_{L^\infty(\mathbb{T})}. \tag{1.7}$$

There is a trivial upper bound (see [41]):

$$M_{n,\delta} \leq \sqrt{\frac{n+1}{\delta}}, \qquad n \in \mathbb{N}. \tag{1.8}$$

Indeed, (1.8) follows from the normalization condition (1.4) and the Cauchy–Schwarz inequality:

$$1 \geq \frac{\delta}{2\pi} \int_{\mathbb{T}} |\phi_n|^2 \, d\theta = \delta \sum_{j=0}^{n} |c_j|^2 \geq \delta \frac{\|\phi_n(z;\sigma)\|^2_{L^\infty(\mathbb{T})}}{n+1}, \qquad \phi_n(z;\sigma) =: \sum_{j=0}^{n} c_j z^j.$$

In 1981, Rakhmanov proved [8] the inequality

$$C\sqrt{\frac{n+1}{\delta \ln^3 n}} \leq M_{n,\delta}, \qquad C > 0, \quad \delta \ll 1, \tag{1.9}$$

which allowed him to significantly improve his previous result (1.6) from [7]. Namely, he proved that for any sequence $\{\beta_n\}$: $\beta_n \to 0$, there exists a $\sigma \in S_\delta$, $\delta \ll 1$, such that

$$\|\phi_{k_n}(z;\sigma)\|_{L^\infty(\mathbb{T})} > \beta_{k_n} \sqrt{\frac{k_n}{\ln^3 k_n}} \tag{1.10}$$

for some sequence $\{k_n\} \subset \mathbb{N}$. This estimate is almost sharp in view of the following result by Geronimus [5, Theorem 3.5] and its refinement due to Nevai [6]. For $\sigma \in S_\delta$, we have

$$\|\phi_n(z;\sigma)\|_{L^\infty(\mathbb{T})} = \bar{o}(\sqrt{n}). \tag{1.11}$$

Thus, Rakhmanov's result left a very narrow interval to which the quantity $M_{n,\delta}$ and the growth rate of the norms of subsequences of orthonormal polynomials can belong.

Recently, we have succeeded in proving that the well-known upper bounds (1.8) and (1.11) are asymptotically sharp with respect to the growth order. The main result of our study [3] is

Theorem 1.1. *Let $\delta \in (0,1)$. Then*

(1) *for sufficiently large $n > n_0 > 0$, there exists a constant $C(\delta) > 0$ such that*

$$M_{n,\delta} > C(\delta)\sqrt{n}; \tag{1.12}$$

(2) *for an arbitrary sequence $\{\beta_n\}\colon \beta_n \to 0$, there exists an absolutely continuous probability measure $\sigma^*\colon d\sigma^* = \sigma^{*\prime}\,d\theta$, $\sigma^* \in S_\delta$, such that*

$$\|\phi_{k_n}(z;\sigma^*)\|_{L^\infty(\mathbb{T})} > \beta_{k_n}\sqrt{k_n}, \qquad \beta_{k_n}(\delta) > 0, \tag{1.13}$$

for some sequence $\{k_n\} \subset \mathbb{N}$.

Notice that δ in the theorem is not necessarily small and may be arbitrarily close to unity. Of course, this is possible at the cost of the value of the constant in (1.12), which in this case tends to zero.

The structure of the paper. In Section 2, we present preliminary results concerning extremal problems of the form (1.7). These results allow us to describe the structure of the extremal measure in problem (1.7) and solve problem (1.7) for δ small compared with $1/n$. Then we discuss approaches to the design of orthonormal polynomials with large norm as well as the constructions of such polynomials (Section 3). In the concluding Section 4, for one of such constructions (that was not used in [3]) we present estimates for a polynomial close to the extremal one and for its derivatives.

2. EXTREMAL MEASURE AND SMALL δ

In this section, we consider extremal problems of the form (1.7), but on other classes of orthogonality measures. Namely, we consider the problem in the class of measures with derivative bounded below (by a parameter δ) and above (by a parameter Δ):

$$\sup_{\sigma \in S_\delta^\Delta} \|\phi_n(z;\sigma)\|_{L^\infty(\mathbb{T})} =: M_{n,\delta}^\Delta, \tag{2.1}$$

where S_δ^Δ is the class of probability measures σ such that $\Delta \ge \sigma'(\theta) \ge \delta > 0$. We also consider the problem in the class of nonnormalized measures:

$$\sup_{\sigma'(\theta) \ge \delta > 0} \|\phi_n(z;\sigma)\| =: \widetilde{M}_{n,\delta}. \tag{2.2}$$

2.1. Structure of an extremal measure. We begin with characterizing the extremal measures in (2.1) and (1.7).

Theorem 2.1. *The following assertions are valid:*

(1) *there exists an extremal measure σ_δ^Δ for the extremal problem* (2.1); *its density $d\sigma_\delta^\Delta(\theta)/d\theta$ takes only two values Δ and δ and has at most $2n$ switches;*

(2) *there exists an extremal measure σ_δ^* for the extremal problem* (1.7); *it can be expressed as*

$$d\sigma_\delta^* = \delta\,d\theta + \sum_{k=1}^{n} m_k\,\delta(\theta - \theta_k). \tag{2.3}$$

Proof. A detailed proof of the theorem is given in [3]. Here we demonstrate the main idea of the proof of assertion (1) of Theorem 2.1.

Our extremal problems (1.7) and (2.1) are variational problems for a functional $F(s_0, s_1, \dots, s_n)$ defined on a finite number of moments $\{s_0, s_1, \dots, s_n\}$ of a measure in the class S_δ or S_δ^Δ, respectively. We can set

$$F(s_0, s_1, \dots, s_n) = |\phi_n(1)|.$$

The functional F is differentiable with respect to the moments. The set S_δ is the weak closure of the sets S_δ^Δ:

$$S_\delta = \overline{\bigcup_{\Delta > \delta} S_\delta^\Delta}.$$

Thus, we can consider the extremal problems (1.7) and (2.1) as optimal control problems (see [1])

$$F(s_0, s_1, \dots, s_n) \to \sup \tag{2.4}$$

with constraints

$$\int_0^{2\pi} e^{ik\theta}\, d\sigma(\theta) = s_k, \qquad k = 0, \dots, n, \quad \sigma \in S_\delta. \tag{2.5}$$

Since F is continuous and the moments are continuous in the weak topology, it follows that

$$\sup_{S_\delta} F = \lim_{\Delta \to \infty} \sup_{S_\delta^\Delta} F.$$

In turn, the problem

$$F \to \sup, \qquad \int_0^{2\pi} e^{ik\theta}\, d\sigma(\theta) = s_k, \quad k = 0, \dots, n, \quad \sigma \in S_\delta^\Delta, \tag{2.6}$$

always has a solution because it is considered on the compact set

$$\begin{cases} \displaystyle\int_0^{2\pi} \cos k\theta\, d\sigma(\theta) = \operatorname{Re} s_k, \quad k = 0, \dots, n, \\ \displaystyle\int_0^{2\pi} \sin k\theta\, d\sigma(\theta) = \operatorname{Im} s_k, \quad k = 1, \dots, n, \end{cases} \qquad \sigma \in S_\delta^\Delta, \quad s_0 \le C.$$

Let us write the Lagrangian for this case:

$$\begin{aligned} \lambda_0 F(s_0, s_1, \dots, s_n) &+ \sum_{k=0}^{n} \left(\lambda_{2k+1} \int_0^{2\pi} \cos k\theta\, d\sigma(\theta) - \operatorname{Re} s_k \right) \\ &+ \sum_{k=1}^{n} \left(\lambda_{2k} \int_0^{2\pi} \sin k\theta\, d\sigma(\theta) - \operatorname{Im} s_k \right) + L(s_0 - C) \\ &= \lambda_0 F(s_0, s_1, \dots, s_n) - \sum_{k=0}^{n} (\operatorname{Re} s_k + \operatorname{Im} s_k) + L(s_0 - C) \\ &+ \int_0^{2\pi} \left(\sum_{k=0}^{n} \lambda_{2k+1} \cos k\theta + \sum_{k=1}^{n} \lambda_{2k} \sin k\theta \right) d\sigma(\theta). \end{aligned}$$

The optimality condition for $\sigma \in S_\delta^\Delta$ in the variational problem (2.6) yields

$$\sigma'(\theta) = \frac{\Delta + \delta}{2} + \frac{\Delta - \delta}{2} \operatorname{sign}\left(\sum_{k=0}^{n} \lambda_{2k+1} \cos k\theta + \sum_{k=1}^{n} \lambda_{2k} \sin k\theta\right); \tag{2.7}$$

i.e., $\sigma'(\theta)$ takes only two values Δ and δ and has at most $2n$ switches (since the degree of the control trigonometric polynomial is at most n). $\square$

2.2. Exact solution to the extremal problem with unconstrained mass of the measure. Now we pass to the extremal problem (2.2).

Theorem 2.2. *The following assertions are valid:*

(1) *it holds that*

$$\widetilde{M}_{n,\delta} = \sqrt{\frac{n+1}{\delta}}; \tag{2.8}$$

(2) *a maximizing sequence* $\{\sigma_l\}$ *for the extremal problem* (2.2) *is given by*

$$d\sigma_l = \delta\, d\theta + \sum_{k=1}^{n} m_k^{(l)}\, \delta(\theta - \theta_k), \qquad \theta_k = \frac{k}{n+1} 2\pi, \quad k = 1, \dots, n, \tag{2.9}$$

and

$$\{m_k^{(l)}\}\colon \quad \lim_{l\to\infty} \min_k m_k^{(l)} = \infty. \tag{2.10}$$

Remark. We stress that in the class of measures $S_{\delta,M}$, $M > 0$ (i.e., without constraint on the total mass M), the elementary upper bound (1.8) is sharp and a maximizing sequence is given by (2.3).

Proof of Theorem 2.2. Introduce the notation

$$\Pi_n(z) = \prod_{k=1}^{n} (z - \varepsilon_k), \qquad \varepsilon_k := e^{i\theta_k}, \tag{2.11}$$

and let $\Phi_n(z) = z^n + \dots$ be a monic orthogonal polynomial on $\mathbb{T}$,

$$\frac{1}{2\pi} \int_0^{2\pi} \Phi_n(e^{i\theta}) e^{-i\nu\theta}\, d\sigma(\theta) = 0, \qquad \nu = 0, \dots, n-1, \tag{2.12}$$

where the orthogonality measure has the form

$$d\sigma_l = \delta\, d\theta + \sum_{k=1}^{n} m_k\, \delta(\theta - \theta_k).$$

The polynomial $\Phi_n(z)$ can be represented as

$$\Phi_n(z) = \Pi_n(z) \left(1 + \sum_{j=1}^{n} \frac{C_j}{z - \varepsilon_j}\right). \tag{2.13}$$

Notice that the orthogonality relations are equivalent to the following:

$$0 = \left\langle \Phi_n(z), \frac{\Pi_n(z)}{z - \varepsilon_k} \right\rangle = \frac{1}{2\pi} \int_0^{2\pi} \Phi_n(z) \overline{\left(\frac{\Pi_n(z)}{z - \varepsilon_k}\right)}\, d\sigma(\theta) = 0, \qquad k = 1, \dots, n, \quad z = e^{i\theta}.$$

Using (2.13) and the equality

$$\frac{\overline{\Pi}_n(z)}{\overline{z}-\overline{\varepsilon}_k} = \frac{\Pi_n^*(z)}{z^{n-1}(1-z\overline{\varepsilon}_k)},$$

we further have

$$0 = \frac{\delta}{2\pi}\int\limits_0^{2\pi} \frac{\Pi_n(z)\,\Pi_n^*(z)\,d\theta}{z^{n-1}(1-z\overline{\varepsilon}_k)} + \sum_{j=1}^{n}\frac{C_j\delta}{2\pi}\int\limits_0^{2\pi}\frac{\Pi_n(z)\,\Pi_n^*(z)\,d\theta}{z^{n-1}(z-\varepsilon_j)(1-z\overline{\varepsilon}_k)} + m_kC_k|\Pi_k'(\varepsilon_k)|^2 \tag{2.14}$$

for $k=1,\dots,n$. Now, suppose that (2.10) is valid. In view of (2.14), this implies

$$\|\overrightarrow{C}\| \lesssim \frac{1}{\min_k m_k}, \qquad \overrightarrow{C} = (C_1,\dots,C_n). \tag{2.15}$$

Then we calculate the norm

$$\|\Phi_n\|_\sigma^2 = \frac{1}{2\pi}\int\limits_0^{2\pi}\left(1+\sum_{j=1}^{n}\frac{C_j}{z-\varepsilon_j}\right)\left(1+\sum_{j=1}^{n}\frac{\overline{C_j}}{\overline{z}-\overline{\varepsilon}_j}\right)\Pi_n(z)\,\overline{\Pi_n(z)}\,d\sigma(z)$$

$$= \|\Pi_n\|_{\delta\,d\theta}^2 + \sum_{j=1}^{n}C_j\int\limits_0^{2\pi}(\dots)\,d\theta + \sum_{j=1}^{n}\overline{C_j}\int\limits_0^{2\pi}(\dots)\,d\theta + \sum_{k=1}^{m}m_k|C_k|^2|\Pi_n'(\varepsilon_k)|^2.$$

Hence, taking into account (2.14) and (2.15), we obtain

$$\min_k m_k \to \infty \qquad \Rightarrow \qquad \begin{cases} \|\Phi_n\|_\sigma^2 \to \|\Pi_n\|_{\delta\,d\theta}^2, \\ \Phi_n(1) \to \Pi_n(1). \end{cases}$$

Finally, set

$$\Pi_n(z) = \frac{z^{n+1}-1}{z-1} = \sum_{j=0}^{n} z^j,$$

which yields

$$\|\Pi_n\|_{\delta\,d\theta}^2 = \delta(n+1), \qquad \Pi_n(1) = n+1.$$

Thus,

$$\frac{\Phi_n(1)}{\|\Phi_n\|_\sigma} \to \sqrt{\frac{n+1}{\delta}} \qquad \text{if} \quad \min_k m_k \to \infty.$$

The theorem is proved. □

Remark. This theorem has the following corollary for our original problem (1.7). Consider the class S_δ for δ small compared with $1/n$. Then, choosing the scale $\phi_n(z;\alpha\mu) = \alpha^{-1/2}\phi_n(z;\mu)$ for any $\alpha > 0$, we obtain

$$M_{n,\delta_n} = \sqrt{\frac{n+1}{\delta_n}}(1+\overline{o}(1)),$$

where

$$\delta_n = \frac{C}{nm_n}, \qquad m_n \to +\infty \quad \text{as} \quad n\to\infty.$$

Thus, for small δ, the elementary upper bound (1.8) for $M_{n,\delta}$ is sharp. If we take $m_n = 1/n$ and make the total mass in the proof finite, then the above-constructed polynomials ϕ_n will be bounded with respect to n for $\delta \sim 1$.

3. DESIGN OF ORTHONORMAL POLYNOMIALS WITH LARGE NORM

3.1. Orthogonal polynomials on a circle: The main properties. To describe approaches and constructions related to the extremal problem (1.7), we need some concepts from the theory of orthogonal polynomials on a circle (see [5, 10]).

For an arbitrary polynomial $P_n(z) = p_n z^n + \ldots + p_1 z + p_0$, its nth inverse (or its $*$-transform) is defined as

$$P_n^*(z) = z^n \overline{P_n\left(\frac{1}{\overline{z}}\right)} = \overline{p}_0 z^n + \overline{p}_1 z^{n-1} + \ldots + \overline{p}_n.$$

Notice that if $z^* \neq 0$ is a root of $P_n(z)$, then $(\overline{z^*})^{-1}$ will be a root of $P_n^*(z)$. We also note that the definition of the nth inverse polynomial does not exclude the vanishing of the leading coefficients of $P_n(z)$; in this case the polynomial $P_n^*(z)$ has a zero (of the corresponding order) at the origin. It is well known [5] that all zeros of ϕ_n are located inside $\mathbb{D}$; thus, ϕ_n^* has no zeros in $\overline{\mathbb{D}}$.

For monic orthogonal polynomials, one uses the notation Φ_n:

$$\Phi_n(z;\mu) = z^n + \ldots : \qquad \phi_n(z;\mu) = \frac{\Phi_n(z;\mu)}{\|\Phi_n\|_{2,\mu}}.$$

Using these polynomials, one can define circular parameters γ_n such that

$$\Phi_n(0;\mu) = -\overline{\gamma}_{n-1}.$$

Then (see [10])

$$\Phi_n(z;\mu) = \phi_n(z;\mu)\bigl(\rho_0 \ldots \rho_{n-1}\bigr), \qquad \rho_n = \sqrt{1-|\gamma_n|^2}.$$

The circular parameters allow one to write recurrence relations for polynomials orthonormal with respect to a probability measure and their nth inverses (see [5]):

$$\begin{cases} \phi_{n+1} = \rho_n^{-1}(z\phi_n - \overline{\gamma}_n \phi_n^*), & \phi_0 = \sqrt{\dfrac{1}{|\mu|}} = 1, \\[2ex] \phi_{n+1}^* = \rho_n^{-1}(\phi_n^* - \gamma_n z \phi_n), & \phi_0^* = \sqrt{\dfrac{1}{|\mu|}} = 1. \end{cases}$$

Along with the polynomials ϕ_n and ϕ_n^*, we will need polynomials of the second kind ψ_n and ψ_n^* that are defined by recurrence relations of the same form but with circular parameters $-\gamma_n$, i.e. (see [10, p. 57]),

$$\begin{cases} \psi_{n+1} = \rho_n^{-1}(z\psi_n + \overline{\gamma}_n \psi_n^*), & \psi_0 = \sqrt{|\mu|} = 1, \\ \psi_{n+1}^* = \rho_n^{-1}(\psi_n^* + \gamma_n z \psi_n), & \psi_0^* = \sqrt{|\mu|} = 1. \end{cases} \tag{3.1}$$

An important role in the theory of orthogonal polynomials on a circle is played by two analytic functions in the disk whose boundary values are related to the orthogonality measure $d\mu$. We mean the Carathéodory function

$$F\colon \quad \operatorname{Re} F(z) > 0, \quad z \in \mathbb{D}, \qquad F(z) = \int_{\mathbb{T}} C(z, e^{i\theta})\, d\mu(\theta), \quad C(z,\xi) = \frac{\xi + z}{\xi - z}, \quad \xi \in \mathbb{T},$$

and the Szegő function

$$\Pi\colon \quad \Pi(z) \neq 0, \quad z \in \mathbb{D}, \qquad \Pi(z) = \exp\left(\frac{1}{4\pi}\int_0^{2\pi} C(z, e^{i\theta}) \log \mu'(e^{i\theta})\, d\theta\right).$$

The Szegő function is considered under Szegő's condition on the absolutely continuous part of the orthogonality measure:

$$\exp\left(\frac{1}{4\pi}\int\limits_{-\pi}^{\pi}\ln(2\pi\mu'(\theta))\,d\theta\right)=\rho>0 \qquad\Leftrightarrow\qquad \{\gamma_n\}\in\ell^2,$$

since $\rho=\prod_{j\ge 0}\rho_j$ by the Szegő theorem (see [10]). The Carathéodory function is considered without constraints on the measure; its Taylor coefficients are equal to the moments of the measure $d\mu$. The relationship between the boundary values and the absolutely continuous part of the measure is given by the formulas

$$2\pi\,\mathrm{Re}\,F=\mu' \qquad\text{and}\qquad |\Pi|^{-2}=2\pi\mu' \qquad\text{on }\ \mathbb{T}.$$

A key place in the theory of orthogonal polynomials on a circle is occupied by the Bernstein–Szegő approximations for the Carathéodory function and the orthogonality measure:

$$F_n(z)=\frac{\psi_n^*(z)}{\phi_n^*(z)}=\int\limits_{\mathbb{T}}C(z,e^{i\theta})\,d\mu_n(\theta), \qquad d\mu_n(\theta)=\frac{d\theta}{2\pi|\phi_n(e^{i\theta})|^2}=\frac{d\theta}{2\pi|\phi_n^*(e^{i\theta})|^2}.$$

The first n Taylor coefficients of F_n and the first $2n$ moments of $d\mu_n$ coincide with the corresponding coefficients of F and moments of the measure $d\mu$. Finally, note three remarkable limit relations of the theory of orthogonal polynomials on a circle

$$d\mu_n\to d\mu, \qquad F_n(z)\to F(z), \qquad\text{and}\qquad \phi_n^*(z)\to\Pi(z), \qquad z\in\overline{\mathbb{D}},$$

where the first relation is satisfied in the sense of weak convergence of measures, and the second and third hold uniformly in $z\in\overline{\mathbb{D}}$ under additional conditions on $d\mu$ (for example, for μ in S_δ with smooth μ').

3.2. Construction of extremal polynomials: Rakhmanov's approach. Consider the Christoffel–Darboux kernel

$$K_n(\xi,z,\mu)=\sum_{j=0}^{n}\overline{\phi_j(\xi;\mu)}\,\phi_j(z;\mu).$$

The following important result is due to Geronimus [5]:

Lemma 3.1. *Consider a measure $\mu(t)=(1-t)\mu+t\delta_0$ with $t\in(0,1)$. Then one has the identity*

$$\Phi_n(z;\mu(t))=\Phi_n(z;\mu)-t\frac{\Phi_n(1;\mu)K_{n-1}(1,z,\mu)}{1-t+tK_{n-1}(1,1,\mu)}. \tag{3.2}$$

In [7], Rakhmanov used the following generalization, in which point masses are added at several specially chosen points rather than at a single point.

Lemma 3.2. *Let μ be a measure on the circle $\mathbb{T}$, $\Phi_n(z;\mu)$ be monic orthogonal polynomials, and $K_n(\xi,z,\mu)=\sum_{l=0}^{n}\overline{\phi_j(\xi;\mu)}\,\phi_j(z;\mu)$ be the Christoffel–Darboux kernel. If points $\xi_j\in\mathbb{T}$, $j=1,\dots,m$, $m\le n$, are chosen so that*

$$K_{n-1}(\xi_j,\xi_l,\mu)=0, \qquad j\ne l, \tag{3.3}$$

then

$$\Phi_n(z;\eta)=\Phi_n(z;\mu)-\sum_{k=1}^{m}\frac{m_k\Phi_n(\xi_k;\mu)}{1+m_kK_{n-1}(\xi_k,\xi_k,\mu)}K_{n-1}(\xi_k,z_k,\mu), \tag{3.4}$$

where

$$\eta = \mu + \sum_{k=1}^{m} m_k \delta_{\theta_k}, \qquad z_k = e^{i\theta_k}, \qquad m_k \ge 0.$$

The application of this formula to $d\mu = d\theta$ immediately yields the estimate

$$M_{n,\delta} \ge C(\delta) \ln n$$

for an appropriate choice of the weights $\{m_j\}$. Moreover, it is easy to show that one cannot improve the logarithmic growth by changing $\{m_k\}$. When the measure $d\mu$ is different from $d\theta$, the calculation of the polynomials, the Christoffel–Darboux kernel, and its zeros appears to be problematic.

In the subsequent paper [8], Rakhmanov again used the idea of modifying the weight by a point mass, but this time he defined the orthogonality measure implicitly.

3.3. Construction of extremal polynomials: An alternative approach. In [3], to estimate from below the solution of the extremal problem (2.5), i.e., to construct (for a fixed n) an orthogonal polynomial $\phi_n^*(z;\sigma)$ for σ in the Steklov class S_δ,

$$\phi_n^*(1;\sigma) > C(\delta)\sqrt{n}, \qquad \sigma \in S_\delta, \tag{3.5}$$

we proposed a fundamentally different approach.

First, we rewrote the Steklov condition in the form of several relations that generally involve an arbitrary polynomial and an arbitrary Carathéodory function. Then we presented all necessary parameters in an explicit form, and the greater part of the proof consists in verifying that these parameters satisfy the desired conditions. In this approach, the absolute value of the orthogonal polynomial on the circle is given explicitly, and the orthogonality measure can easily be found from available formulas. In particular, the measure can be thoroughly analyzed.

The following lemma from [3] provides a reformulation of problem (3.5).

Lemma 3.3. *To prove* (3.5), *it suffices to find a polynomial* ϕ_n^* *and a Carathéodory function* $\widetilde{F}$ *with the following properties*:

(1) $\phi_n^*(z)$ *has no roots in* $\mathbb{D}$;

(2) $\phi_n^*(z)$ *is normalized as*

$$\int_{\mathbb{T}} |\phi_n^*(z)|^{-2}\, d\theta = 2\pi, \qquad \phi_n^*(0) > 0; \tag{3.6}$$

(3) ϕ_n^* *has a large uniform norm, i.e.,*

$$|\phi_n^*(1)| \sim \sqrt{n};$$

(4) $\widetilde{F} \in C^\infty(\mathbb{T})$, $\operatorname{Re} \widetilde{F} > 0$ *on* $\mathbb{T}$, *and*

$$\frac{1}{2\pi} \int_{\mathbb{T}} \operatorname{Re} \widetilde{F}(e^{i\theta})\, d\theta = 1; \tag{3.7}$$

(5) *in addition,*

$$|\phi_n^*(z)| + \bigl|\widetilde{F}(z)\bigl(\phi_n(z) - \phi_n^*(z)\bigr)\bigr| < C_1(\delta)\bigl(\operatorname{Re} \widetilde{F}(z)\bigr)^{1/2} \tag{3.8}$$

uniformly in $z \in \mathbb{T}$.

Proof. For a detailed proof of the lemma, see [3]; here we only outline this proof.

It is well known (we have already pointed this out) that polynomials orthonormal with respect to a probability measure possess properties (1) and (2) indicated in the lemma. One can easily show (and we do this in [3]) that the converse is also valid; i.e., any polynomial $\phi_n(z)$ possessing properties (1) and (2) is orthonormal with respect to some probability measure and also defines the first n circular parameters $\gamma_0, \dots, \gamma_{n-1}$. Property (3) gives the necessary growth of the norm.

Now, let us show that properties (4) and (5) guarantee the existence of a measure σ in the Steklov class S_δ for which ϕ_n is the nth orthonormal polynomial.

Note that $\widetilde{F}$ defines (according to property (4) in the lemma) the corresponding probability measure $\widetilde{\sigma}$, which is absolutely continuous and has a positive smooth density $\widetilde{\sigma}'$ defined by the equality

$$\widetilde{\sigma}'(\theta) = \frac{\operatorname{Re} \widetilde{F}(e^{i\theta})}{2\pi}. \tag{3.9}$$

Denote its circular parameters by $\{\widetilde{\gamma}_j\}$, $j = 0, 1, \dots$, and the corresponding orthonormal polynomials of the first and second kind by $\{\widetilde{\phi}_j\}$ and $\{\widetilde{\psi}_j\}$, $j = 0, 1, \dots$, respectively. Notice that the normalization of the measure $\widetilde{\sigma}$ implies $\widetilde{\phi}_0 = \widetilde{\psi}_0 = 1$. Baxter's theorem (see [10]) implies that $\widetilde{\gamma}_j \in \ell^1$ (in fact, the decrease is much faster, but ℓ^1 is sufficient for our purposes). Then we form a probability measure σ with the following circular parameters:

$$\gamma_0, \dots, \gamma_{n-1}, \widetilde{\gamma}_0, \widetilde{\gamma}_1, \dots .$$

We will show that this measure is the sought measure in the Steklov class for which ϕ_n is the nth orthonormal polynomial.

Let us prove that $\sigma \in S_\delta$. Set

$$\gamma_n = \widetilde{\gamma}_0, \qquad \gamma_{n+1} = \widetilde{\gamma}_1, \qquad \dots . \tag{3.10}$$

Baxter's theorem states that σ is absolutely continuous, σ' belongs to the Wiener class $W(\mathbb{T})$, and σ' is positive on $\mathbb{T}$. The first n orthonormal polynomials corresponding to the measure σ are $\{\phi_j\}$, $j = 0, \dots, n-1$.

In [3], we obtained a fundamental identity that allows one to calculate the polynomials ϕ_j and ψ_j (orthonormal with respect to σ) for indices $j > n$:

$$2\phi^*_{n+m} = \phi_n(\widetilde{\phi}^*_m - \widetilde{\psi}^*_m) + \phi^*_n(\widetilde{\phi}^*_m + \widetilde{\psi}^*_m) = \widetilde{\phi}^*_m\big(\phi_n + \phi^*_n + \widetilde{F}_m(\phi^*_n - \phi_n)\big), \tag{3.11}$$

where

$$\widetilde{F}_m(z) = \frac{\widetilde{\psi}^*_m(z)}{\widetilde{\phi}^*_m(z)}.$$

Since $\{\widetilde{\gamma}_n\} \in \ell^1$ and $\{\gamma_n\} \in \ell^1$, we have [10, p. 225]

$$\widetilde{F}_m \to \widetilde{F}, \quad m \to \infty, \qquad \phi^*_n \to \Pi, \quad \widetilde{\phi}^*_n \to \widetilde{\Pi}, \quad n \to \infty,$$

uniformly on $\overline{\mathbb{D}}$. The functions Π and $\widetilde{\Pi}$ are the Szegő functions for σ and $\widetilde{\sigma}$, respectively; i.e., they are external functions in $\mathbb{D}$, which yields the factorization

$$|\Pi|^{-2} = 2\pi\sigma', \qquad |\widetilde{\Pi}|^{-2} = 2\pi\widetilde{\sigma}'. \tag{3.12}$$

Now in (3.11) we let $m \to \infty$ and obtain

$$2\Pi = \widetilde{\Pi}\big(\phi_n + \phi^*_n + \widetilde{F}(\phi^*_n - \phi_n)\big). \tag{3.13}$$

Thus, the first formula in (3.12) shows that in the class of sufficiently regular measures, the Steklov condition $\sigma' > \delta/(2\pi)$ is equivalent to

$$\bigl|\widetilde{\Pi}\bigl(\phi_n + \phi_n^* + \widetilde{F}(\phi_n^* - \phi_n)\bigr)\bigr| \le \frac{2}{\sqrt{\delta}}, \qquad z = e^{i\theta} \in \mathbb{T}. \tag{3.14}$$

Since $|\phi_n| = |\phi_n^*|$ on $\mathbb{T}$, we have

$$\bigl|\widetilde{\Pi}\bigl(\phi_n + \phi_n^* + \widetilde{F}(\phi_n^* - \phi_n)\bigr)\bigr| \le 2|\widetilde{\Pi}|\bigl(|\phi_n^*| + |\widetilde{F}(\phi_n^* - \phi_n)|\bigr) < 2C_1(\delta)|\widetilde{\Pi}|(\operatorname{Re}\widetilde{F})^{1/2} = 2C_1(\delta)$$

owing to (3.8) and (3.9) and the second formula in (3.12). Thus, for (3.14) to hold, we must set $C_1(\delta) := \delta^{-1/2}$ in (3.8). Since we assume that δ is fixed, explicit formulas for $C(\delta)$ and $C_1(\delta)$ do not matter. □

3.4. Constructions of an extremal polynomial. As a polynomial asymptotically close to an extremal one, in [3] we proposed the following construction. The polynomial ϕ_n^* was taken in the form

$$\phi_n^*(z) = C_n f_n(z), \qquad f_n(z) = P_m(z) + Q_m(z) + Q_m^*(z), \tag{3.15}$$

where P_m and Q_m are some polynomials of degree $2m-1$ and $m-1$, respectively, $m = [\delta_1 n]$ with a sufficiently small $\delta_1 > 0$, and the zeros of the polynomial Q_m lie outside the disk $\mathbb{D}$. Note that here Q_m^* is defined by the application of the operation $*$ of order n. The constant C_n is chosen so that

$$\int_{-\pi}^{\pi} |\phi_n^*|^{-2}\, d\theta = 2\pi$$

(i.e., the orthogonality measure of the polynomial ϕ_n is a probability measure, see (3.6)). Note that one of the main technical difficulties in applying Lemma 3.3 in [3] was the verification of the fact that

$$C_n = \left(\int_{-\pi}^{\pi} |f_n|^{-2}\, d\theta\right)^{1/2} \sim 1 \tag{3.16}$$

uniformly in n. Then, to prove the desired lower bound (3.5), it suffices to show that f_n possesses the other properties indicated in the lemma.

Let us explain the purpose of the terms constituting the polynomial ϕ_n^* in (3.15). Since Q_m is chosen with zeros outside $\mathbb{D}$, all n zeros of $Q_m + Q_m^*$ are located on the unit circle $\mathbb{T}$. Since the polynomial ϕ_n^* defined in (3.15) is expected to be orthogonal on $\mathbb{T}$, the zeros of $Q_m + Q_m^*$ in (3.15) must be "pushed out" of $\mathbb{D}$ by an appropriately chosen polynomial P_m. This "pushing" polynomial P_m serves no other purpose and has a small absolute value. Thus, the main contribution to the polynomial ϕ_n^* on the unit circle is made by $Q_m + Q_m^*$ (the corresponding proposition is proved in [3]). Let us give an appropriate representation for this term. We have

$$\begin{aligned} Q_m + Q_m^* &= |Q_m| \exp(i \operatorname{Arg}(Q_m)) + |Q_m| \exp(in\theta - i \operatorname{Arg}(Q_m)) \\ &= 2|Q_m| \exp\left(\frac{in\theta}{2}\right) \cos\left(\frac{n\theta}{2} - \operatorname{Arg}(Q_m)\right) \\ &= 2 \exp\left(\frac{in\theta}{2}\right) \sqrt{\mathcal{A}(\theta)} \cos\Bigl(\Bigl(\frac{n}{2} - m + 1\Bigr)\theta + \Phi(\theta)\Bigr). \end{aligned}$$

Here we have used the notation

$$\mathcal{A}(\theta) := |Q(e^{i\theta})|^2, \qquad \Phi(\theta) := \operatorname{Arg}(Q_m^*(e^{i\theta})), \qquad \theta \in (0, 2\pi),$$

where the operation $*$ is of degree $m-1$. It is clear that when verifying condition (3.16), one must control the argument of Q_m and its derivative. We have

$$\Phi(\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathcal{A}'(\varphi)}{\mathcal{A}(\varphi)} \ln\left|\sin\frac{\varphi-\theta}{2}\right| d\varphi, \tag{3.17}$$

$$\Phi'(\theta) = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\mathcal{A}'(\varphi)}{\mathcal{A}(\varphi)}\right)' \ln\left|\sin\frac{\varphi-\theta}{2}\right| d\varphi. \tag{3.18}$$

These representations show that we need the asymptotics for $|Q_m|^2$ and estimates for its two derivatives in order to control $Q_m + Q_m^*$ on the unit circle.

In [3], we defined the polynomial $Q_m(z)$ as an algebraic polynomial (in powers of z) without zeros in $\mathbb{D}$ that is obtained by the Riesz–Fejér factorization of a positive trigonometric polynomial:

$$|Q_m(z)|^2 := \mathcal{G}_m(\theta) + \left|R_{(m,\alpha/2)}(e^{i\theta})\right|^2, \tag{3.19}$$

which consists of

$$\mathcal{G}_m(\theta) = \mathcal{F}_m(\theta) + \frac{1}{2}\mathcal{F}_m\Big(\theta - \frac{\pi}{m}\Big) + \frac{1}{2}\mathcal{F}_m\Big(\theta + \frac{\pi}{m}\Big), \tag{3.20}$$

where $\mathcal{F}_m$ is the Fejér kernel,

$$\mathcal{F}_m(\varphi) = \frac{\sin^2\frac{m\varphi}{2}}{m\sin^2\frac{\varphi}{2}}, \qquad \mathcal{F}_m(0) = m, \qquad \int_0^{2\pi} \mathcal{F}_m(\varphi)\,d\varphi = 2\pi, \tag{3.21}$$

and of the Taylor approximation $R_{(k,\alpha)}$ of the function $(1-z)^{-\alpha}$,

$$R_{(k,\alpha)}(z) = c_0 + \sum_{j=1}^{k} c_j z^j, \tag{3.22}$$

with a parameter $\alpha \in (1/2, 1)$.

In [3], the polynomial P_m (from the definition of the polynomial ϕ_n^* in (3.15)), which pushes the zeros out of the unit circle, is defined as

$$P_m(z) = Q_m(z)(1-z)\big(1 - 0.1R_{(m,-(1-\alpha))}(z)\big). \tag{3.23}$$

Its degree $\deg P_m = 2m+1$ is less than n in view of the choice of a small δ_1.

Finally, as the Carathéodory function $\widetilde{F}$ (from property (4) in Lemma 3.3) that guarantees the Steklov condition (by property (5) in Lemma 3.3) for the orthogonality measure of the polynomial ϕ_n, in [3] we proposed the following function:

$$\widetilde{F}(z) = \widetilde{C}_n\big(\rho(1+\epsilon_n - z)^{-1} + (1+\epsilon_n - z)^{-\alpha}\big), \tag{3.24}$$

where $\epsilon_n = n^{-1}$, $\rho \in (0, \rho_0)$, ρ_0 is sufficiently small, and $\widetilde{C}_n$ is a positive constant that we specify when verifying the properties indicated in Lemma 3.3. It is clear that $\widetilde{F}$ is a smooth function with positive real part in $\overline{\mathbb{D}}$.

Thus, as proved in [3], formulas (3.15), (3.19), (3.20), and (3.22)–(3.24) give an explicit expression for the orthogonal polynomial ϕ_n and the Carathéodory function $\widetilde{F}$ that satisfy Lemma 3.3 and thus solve problem (3.5). Note that the explicit expressions given here solve problem (3.5) for small δ. To prove assertions (1) and (2) of Theorem 1.1 for $\delta \in (0,1)$ (i.e., to solve problem (1.12) and construct a subsequence of maximum possible growth), we successively complicated these explicit constructions of ϕ_n and $\widetilde{F}$. Nevertheless, the main idea of the new method proposed in [3] is given by the explicit constructions presented here for small δ.

Now, let us explain the purpose of both terms constituting the positive trigonometric polynomial $|Q_m(z)|^2$ in (3.19). We can see that the first term $\mathcal{G}_m$ in (3.19) ("hat") guarantees the desired growth of the orthogonal polynomial:

$$|\phi_n(1)| \sim \sqrt{n}.$$

In our construction, the trigonometric polynomial Q_m must preserve a large absolute value on an interval of length $\sim 1/m$. We need this in order to keep the bounded derivatives of the polynomial and, hence, to have sufficiently smooth Szegő functions that guarantee the Steklov condition for the orthogonality measure. Since the Fejér kernel decays very rapidly, we take the sum of three kernels (see (3.20)) to meet these requirements. We also require that outside this interval of order $1/m$, the polynomial should have a controlled decay, which should not be so fast as that of the Fejér kernel. In (3.19), the polynomial $R_{(m,\alpha/2)}$ ("wings") serves this purpose.

The asymptotics for the norm of the trigonometric polynomial Q_m (defined in (3.19)–(3.22)) and upper bounds for its first and second derivatives were obtained in [3, Appendices A, B] (see [3, Lemmas 5.2, 5.3 and the proof of Lemma 6.1]). These rather delicate technical estimates lead to the key lemma [3, Lemma 6.1] (needed to prove (3.16)), which controls the phase of $Q_m(e^{i\theta})$ (denote it by Φ) for $|\theta| < \upsilon$, where υ is a small fixed number:

$$|\Phi'(\theta)| \lesssim m. \tag{3.25}$$

In the present paper, to diversify approaches and constructions for verifying technically difficult places in the proof of Theorem 1.1, we represent the polynomials Q_m from (3.15) in another explicit form, different from that used in [3] (see (3.20) and (3.22)), and prove estimates for $|Q_m|$ and its derivatives that are necessary to verify (3.16) (see Section 4). Preserving the above-described general requirements for the polynomial Q_m, we build Q_m of terms different from those in (3.20) and (3.22). Set (see [4])

$$\mathcal{A}(\theta) := |Q(e^{i\theta})|^2 := (q \otimes \mathcal{F}_m)(\theta), \tag{3.26}$$

where $q \otimes \mathcal{F}_m$ is the convolution with the Fejér kernel $\mathcal{F}_m$ (3.21) of the function

$$q(\theta) = m e^{-m^2 \sin^2(\theta/2)} + \left(m^{-2} + \sin^2 \frac{\theta}{2}\right)^{-\alpha/2} =: q_1 + q_2, \tag{3.27}$$

which is split into two terms (in accordance with (3.19)), i.e., into the hat q_1 and wings q_2.

4. ESTIMATES FOR FEJÉR CONVOLUTIONS IN THE CONSTRUCTION OF THE POLYNOMIAL Q_m

The aim of this section is to obtain estimates matching the results from [3, Appendices], which are needed to prove (3.25); this finally leads to the verification of (3.16) for the new input data (3.26) and (3.27).

4.1. Formulation of the results. In this subsection, we formulate the estimates to be proved (see [4]). We begin with the hat

$$\mathcal{A}_1(x) := (q_1 \otimes \mathcal{F}_m)(x) := \int_{-\pi}^{\pi} q_1(t)\mathcal{F}_m(x-t)\,dt, \qquad q_1(t) = me^{-m^2\sin^2(t/2)}.$$

Prior to formulating the result on the asymptotics of $\mathcal{A}_1(x)$ and its derivatives $\mathcal{A}_1'(x)$ and $\mathcal{A}_1''(x)$ as $m \to \infty$, we introduce a sequence of entire functions $\{E_l\}$:

$$E_0(r) := (r-1)e^{-r^2}, \qquad E_1(r) := r(r-1)e^{-r^2}, \qquad E_2(r) := (r-1)(2r^2-1)e^{-r^2}.$$

Denote by $\{c_j^{(l)}\}$ the coefficients of the power series expansion of $E_l(r)$ about $r=1$:

$$E_l(r) = \sum_{j=1}^{\infty} c_j^{(l)}(1-r)^j.$$

This allows us to form two other sets of entire functions

$$\widetilde{C}_l(t) := \sum_{\nu=0}^{\infty}(-1)^\nu \widetilde{c}_\nu^{(l)} t^{2\nu}, \qquad \widetilde{S}_l(t) := \sum_{\nu=0}^{\infty}(-1)^\nu \widetilde{s}_\nu^{(l)} t^{2\nu+1}$$

with coefficients

$$\widetilde{c}_\nu^{(l)} := \sum \frac{c_j^{(l)} j!}{(j+2\nu+1)!}, \qquad \widetilde{s}_\nu^{(l)} := \sum \frac{c_j^{(l)} j!}{(j+2\nu+2)!}.$$

Lemma 4.1. *For any $\varepsilon > 0$, as $m \to \infty$, we have*

$$\mathcal{A}_1(x) = \begin{cases} m\dfrac{\sqrt{\pi}x^2}{\sin^2\frac{x}{2}}\widetilde{C}_0(mx) + O(mx), & |x|m \le \dfrac{1}{\varepsilon}, \\ \dfrac{1}{m}\,\dfrac{\sqrt{\pi}\big(1-\frac{\cos mx}{e}\big)}{\sin^2\frac{x}{2}} + O\Big(\dfrac{1}{m^2x^2}\Big), & |x|m \ge \dfrac{1}{\varepsilon}, \end{cases}$$

and, for the derivatives,

$$\mathcal{A}_1'(x) = \begin{cases} m^2\dfrac{-\sqrt{\pi}x^2}{\sin^2\frac{x}{2}}\widetilde{S}_1(mx) + O(m), & |x|m \le \dfrac{1}{\varepsilon}, \\ \dfrac{\sqrt{\pi}}{e}\,\dfrac{\sin mx}{\sin^2\frac{x}{2}} + O\Big(\dfrac{1}{mx^2}\Big), & |x|m \ge \dfrac{1}{\varepsilon}, \end{cases}$$

$$\mathcal{A}_1''(x) = \begin{cases} m^3\dfrac{-\sqrt{\pi}x^2}{2\sin^2\frac{x}{2}}\big(\widetilde{C}_0(mx) + \widetilde{C}_2(mx)\big) + O(m^{7/3}), & |x|m \le \dfrac{1}{\varepsilon}, \\ m\dfrac{\sqrt{\pi}}{e}\,\dfrac{\cos mx}{\sin^2\frac{x}{2}} + O\Big(\dfrac{1}{mx^3}\Big), & |x|m \ge \dfrac{1}{\varepsilon}. \end{cases}$$

Here the function $\widetilde{C}_0(\xi)$ is even and

$$\widetilde{C}_0(\xi) > 0 \qquad \forall \xi \in \mathbb{R}. \tag{4.1}$$

Now we pass to the wings

$$\mathcal{A}_2(\theta) := (q_2 \otimes \mathcal{F}_m)(\theta) := \int_{-\pi}^{\pi} q_2(t)\mathcal{F}_m(\theta - t)\,dt.$$

Lemma 4.2.

$$\mathcal{A}_2(\theta) \asymp \max\left(\frac{1}{m}, |\theta|\right)^{-\alpha}, \qquad \theta \in [-\pi, \pi]. \tag{4.2}$$

For the derivatives $\mathcal{A}_2' = q_2' \otimes \mathcal{F}_m$ and $\mathcal{A}_2'' = q_2'' \otimes \mathcal{F}_m$, we obtained the following upper bound (cf. [3, Lemma 5.2]).

Lemma 4.3.

$$|\mathcal{A}_2'(\theta)| \lesssim m^{\alpha+1} \min\left(1, \frac{1}{m\theta}\right)^{\alpha+1}, \qquad |\mathcal{A}_2''(\theta)| \lesssim m^{\alpha+2} \min\left(1, \frac{1}{m\theta}\right)^{2}.$$

4.2. Asymptotics of the hat and its derivatives. Here we prove Lemma 4.1 on the asymptotics of

$$\mathcal{A}_1^{(p)}(x) := \int\limits_{-\pi}^{\pi} q_1^{(p)}(t)\mathcal{F}_m(x-t)\,dt, \qquad p = 0, 1, 2, \tag{4.3}$$

where, recall,

$$q_1(t) = me^{-m^2\sin^2(t/2)}, \qquad \mathcal{F}_m(\varphi) = \frac{\sin^2\frac{m\varphi}{2}}{m\sin^2\frac{\varphi}{2}}.$$

Proof of Lemma 4.1. We begin with a general approach to the estimate (4.3) for an arbitrary p. Then we specify the general result for $p = 0, 1, 2$. The general approach consists of the following steps:

1. We split the integral (4.3) into two parts

$$\mathcal{A}_1^{(p)} = \int\limits_{-m^{-2/3}}^{m^{-2/3}} \ldots + \int\limits_{[-\pi,\pi]\setminus[-m^{-2/3}, m^{-2/3}]} \ldots =: \widetilde{\mathcal{A}}_{1p} + \widetilde{\widetilde{\mathcal{A}}}_{1p},$$

where the second integral is estimated as

$$\widetilde{\widetilde{\mathcal{A}}}_{1p} = O\left(m^{2+2p}e^{-\frac{m^{2/3}}{4}}\right). \tag{4.4}$$

2. Introduce the notation

$$S_m(x,t) := e^{-\frac{m^2t^2}{4}}\frac{\sin^2\frac{m}{2}(x-t)}{\left(\frac{x-t}{2}\right)^2}, \qquad f_p(x,t) := \frac{q_1^{(p)}(t)\left(\frac{x-t}{2}\right)^2}{me^{-\frac{m^2t^2}{4}}\sin^2\frac{x-t}{2}}.$$

Thus,

$$\widetilde{\mathcal{A}}_{1p} = \int\limits_{-m^{-2/3}}^{m^{-2/3}} f_p(x,t)\,S_m(x,t)\,dt. \tag{4.5}$$

Consider the expansion

$$f_p(x,t) = \sum_{j=0}^{\infty} \widetilde{F}_{p,j}(x,m)t^j \tag{4.6}$$

and notice that the coefficients $\widetilde{F}_{p,j}$ are bounded for $x \in (-\pi, \pi)$. Now we leave the first N terms in the power series expansion and estimate the remainder of the series using the inequality $|t| < m^{-2/3}$:

$$f_p(x,t) = \sum_{j\le N} \widetilde{F}_{p,j}(x,m)t^j + \sum_{j>N} O(m^{k_j})t^{l_j}. \tag{4.7}$$

The sharpness of the asymptotics will depend on N.

3. Next, we substitute (4.7) into (4.5) and, using estimates similar to (4.4), extend the integration interval in (4.5) from $[-m^{-2/3}, m^{-2/3}]$ to $[-\infty, \infty]$:

$$\mathcal{A}_1^{(p)} = \sum_{j\le N} \widetilde{F}_{p,j}(x,m)\, J_j(x,m) + \sum_{j>N} O(m^{k_j})\, \widetilde{J}_{l_j}(x,m), \tag{4.8}$$

where

$$J_j(x,m) := \int_{-\infty}^{\infty} t^j S_m(x,t)\, dt, \qquad \widetilde{J}_j(x,m) := \int_{-\infty}^{\infty} |t|^j S_m(x,t)\, dt.$$

This representation implies

$$J_{2k}(x,m) = \widetilde{J}_{2k}(x,m) > 0, \qquad x\in\mathbb{R}, \quad k\in\mathbb{N}. \tag{4.9}$$

4. Then we consider the integrals J_j. We use the identity

$$\frac{\sin^2\frac{m(x-t)}{2}}{\left(\frac{x-t}{2}\right)^2} = 2m^2\int_0^1\int_0^s \cos(rm(x-t))\,dr\,ds = 2m^2\int_0^1\int_r^1 \cos(rm(x-t))\,ds\,dr.$$

It yields

$$J_j(x,m) = 2m^2\int_0^1\int_{-\infty}^{\infty} t^j e^{-\frac{m^2t^2}{4}}\int_r^1 \cos(rm(x-t))\,ds\,dt\,dr.$$

We can explicitly integrate with respect to s and t:

$$\begin{aligned} J_{2k} &= 2^{k+2}\frac{(-1)^{k+1}\sqrt{\pi}}{m^{2k-1}}\int_0^1 \cos(rmx)\,E_{2k}(r)\,dr,\\ J_{2k+1} &= 2^{k+3}\frac{(-1)^{k+1}\sqrt{\pi}}{m^{2k}}\int_0^1 \sin(rmx)\,E_{2k+1}(r)\,dr, \end{aligned} \tag{4.10}$$

where $E_l(r)$ are some entire functions:

$$E_0 := (r-1)e^{-r^2}, \qquad E_1 = r(r-1)e^{-r^2}, \qquad E_2 := (r-1)(2r^2-1)e^{-r^2}, \qquad \dots .$$

Thus, to complete the description of the general approach, we should explain how to obtain the asymptotics of $J_l(x,m)$ as $m\to\infty$. We will find the asymptotics separately in two domains

$$|x|m \le \frac{1}{\varepsilon} \qquad \text{and} \qquad |x|m \ge \frac{1}{\varepsilon} \qquad \forall\varepsilon>0.$$

5. For bounded $|x|m$, we take the power series expansions of the entire functions $E_l(r)$ at the point $r=1$,

$$E_l(r) = \sum_{j=1}^{\infty} c_j^{(l)}(1-r)^j,$$

and substitute them into (4.10). Expanding the resulting integrals in mx,

$$\int_0^1 \cos(rmx)\,(1-r)^j\,dr = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu j!\,(mx)^{2\nu}}{(j+2\nu+1)!}, \qquad \int_0^1 \sin(rmx)\,(1-r)^j\,dr = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu j!\,(mx)^{2\nu+1}}{(j+2\nu+2)!},$$

we form other entire functions

$$\widetilde{C}_l(t) := \sum_{\nu=0}^{\infty} (-1)^\nu \widetilde{c}_\nu^{(l)} t^{2\nu}, \qquad \widetilde{S}_l(t) := \sum_{\nu=0}^{\infty} (-1)^\nu \widetilde{s}_\nu^{(l)} t^{2\nu+1}$$

with coefficients

$$\widetilde{c}_\nu^{(l)} := \sum_{j=1}^{\infty} \frac{c_j^{(l)} j!}{(j+2\nu+1)!}, \qquad \widetilde{s}_\nu^{(l)} := \sum_{j=1}^{\infty} \frac{c_j^{(l)} j!}{(j+2\nu+2)!}.$$

Thus, for finite $|x|m$, we obtain the following representation for the integrals (4.10):

$$J_{2k}(x,m) = \frac{(-1)^{k+1}\sqrt{\pi}\cdot 2^{k+2}}{m^{2k-1}} \widetilde{C}_{2k}(mx), \qquad J_{2k+1}(x,m) = \frac{(-1)^{k+1}\sqrt{\pi}\cdot 2^{k+3}}{m^{2k}} \widetilde{S}_{2k+1}(mx). \quad (4.11)$$

For $k=0$, the first of these formulas together with (4.9) yields (4.1).

6. For growing $|x|m$, we integrate by parts in (4.10). We have

$$J_0(x,m) = -4\sqrt{\pi}m\Biggl\{\Bigl(\frac{\cos mx}{e} - 1\Bigr)(mx)^{-2} + \frac{4}{e}\sin mx\,(mx)^{-3} + \Biggl[\frac{6}{e}(1-\cos mx) + 4\int_0^1 (\cos rmx - 1)\mathcal{P}_5(r)e^{-r^2}\,dr\Biggr](mx)^{-4}\Biggr\},$$

where $\mathcal{P}_5(r)$ is a polynomial with $\deg \mathcal{P}_5 = 5$. Thus,

$$J_0(x,m) = 4\sqrt{\pi}m\Bigl(1 - \frac{\cos mx}{e}\Bigr)\frac{1}{(mx)^2} + O\Bigl(\frac{1}{m^2x^3}\Bigr). \quad (4.12)$$

Similarly,

$$J_1(x,m) = -\frac{8\sqrt{\pi}}{e}\sin mx \frac{1}{(mx)^2} + O\Bigl(\frac{1}{m^3x^3}\Bigr), \quad (4.13)$$

$$J_2(x,m) = \frac{8\sqrt{\pi}}{m}\Bigl(\frac{\cos mx}{e} + 1\Bigr)\frac{1}{(mx)^2} + O\Bigl(\frac{1}{m^4x^3}\Bigr),$$

$$J_3(x,m) = \frac{16\sqrt{\pi}}{m^2}\Bigl(\frac{\sin mx - mx}{e}\Bigr)\frac{1}{(mx)^2} + O\Bigl(\frac{1}{m^5x^3}\Bigr),$$

and so on. Now we can analyze the special cases of $\mathcal{A}_1^{(p)}$, $p=0,1,2$. We will use the notation

$$a(x) := \Biggl(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\Biggr)^2, \qquad b(x) := \frac{x^2\cos\frac{x}{2} - 2x\sin\frac{x}{2}}{4\sin^3\frac{x}{2}}.$$

Case $p = 0$. We have

$$f_0(x,t) = \exp\left\{\frac{m^2t^2}{4} - m^2\sin^2\frac{t}{2}\right\}a(x-t)$$
$$= a(x) + b(x)t + O(1)t^2 + O(1)t^3 + \left(\frac{1}{48}a(x)m^2 + O(1)\right)t^4 + \dots.$$

Notice that the exponents in this series have the following periodic structure:

$$O(m^{2k})t^{4k}, \quad O(m^{2k})t^{4k+1}, \quad O(m^{2k})t^{4k+2}, \quad O(m^{2k})t^{4k+3}, \qquad k \in \mathbb{N}.$$

Thus,

$$f_0(x,t) = a(x) + O(1)t, \qquad t \in \big[-m^{-2/3}, m^{-2/3}\big].$$

This yields

$$\mathcal{A}_1 = a(x)J_0(x,m) + O(1)\widetilde{J}_1(x,m),$$

where $O(1)$ is bounded as a function of x and m. Hence, using (4.11) and (4.12), we arrive at the assertion of Lemma 4.1 for $p = 0$.

Case $p = 1$. We have

$$f_1(x,t) = -\frac{m^2}{2}\exp\left\{\frac{m^2t^2}{4} - m^2\sin^2\frac{t}{2}\right\}\sin t\cdot a(x-t)$$
$$= -m^2\frac{a(x)}{2}t - m^2\frac{b(x)}{2}t^2 + O(1)m^2t^3 + O(1)m^2t^4 - \left(\frac{m^4a(x)}{96} + O(1)m^2\right)t^5 + \dots.$$

Thus,

$$f_1(x,t) = -m^2\frac{a(x)}{2}t + O(m^2)t^2, \qquad t \in \big[-m^{-2/3}, m^{-2/3}\big].$$

This yields

$$\mathcal{A}_1^{(1)} = -m^2\frac{a(x)}{2}J_1(x,m) + O(m^2)J_2(x,m).$$

Now, using (4.11) and (4.13), we obtain the assertion of Lemma 4.1 for $p = 1$.

Case $p = 2$. We have

$$f_2(x,t) = -\frac{m^2}{4}\exp\left\{\frac{m^2t^2}{4} - m^2\sin^2\frac{t}{2}\right\}(2\cos t - m^2\sin^2 t)a(x-t).$$

Expanding in t^ν implies

$$f_2(x,t) = -m^2\frac{a(x)}{2}\left(1 - \frac{m^2t^2}{2}\right) - m^2\frac{b(x)}{2}\left(t - \frac{m^2t^3}{2}\right) + \frac{m^6a(x)}{192}t^6 + O(m^{4/3})$$

for $t \in [-m^{-2/3}, m^{-2/3}]$. This yields

$$\mathcal{A}_1^{(2)} = -m^2\frac{a(x)}{2}\left(J_0 - \frac{m^2}{2}J_2\right) - m^2\frac{b(x)}{2}\left(J_1 - \frac{m^2}{2}J_3\right) + O(m^{4/3})J_0.$$

Integrating by parts, we get

$$J_0 - \frac{m^2}{2} J_2 = -8\sqrt{\pi} m \int_0^1 \cos(rmx)\, r^2(r-1)e^{-r^2}\, dr = -\frac{8\sqrt{\pi}}{e} m \cos mx \frac{1}{(mx)^2} + O\Big(\frac{1}{m^3 x^4}\Big),$$

$$J_1 - \frac{m^2}{2} J_3 = -16\sqrt{\pi} m \int_0^1 \sin(rmx)\, r(r+1)(r-1)^2 e^{-r^2}\, dr$$
$$= \frac{32\sqrt{\pi}}{e}(2\cos mx + e)\frac{1}{(mx)^3} + O\Big(\frac{1}{m^4 x^4}\Big).$$

Now, using (4.11), we obtain the assertion of Lemma 4.1 for $p = 2$.

Lemma 4.1 is proved. □

4.3. Asymptotics of the wings. Here we prove Lemma 4.2 on the asymptotics of the second term in representation (3.26), (3.27) for $\mathcal{A}$:

$$\mathcal{A}_2(\theta) := (q_2 \otimes \mathcal{F}_m)(\theta) := \int_{-\pi}^{\pi} q_2(t)\mathcal{F}_m(\theta - t)\, dt.$$

Proof of Lemma 4.2. We begin with the lower bound. For the Fejér kernel, we have

$$\mathcal{F}_n(t) \ge \begin{cases} \dfrac{4n}{\pi^2}, & |t| < \dfrac{\pi}{n}, \\ 0, & |t| > \dfrac{\pi}{n}. \end{cases}$$

Hence (since q_2 is positive),

$$(q_2 \otimes \mathcal{F}_n)(\theta) \ge \int_{\theta-\frac{\pi}{n}}^{\theta+\frac{\pi}{n}} \frac{4n}{\pi^2} q_2(t)\, dt \ge \int_{\theta-\frac{\pi}{n}}^{\theta+\frac{\pi}{n}} \frac{4n}{\pi^2}\Big(\frac{1}{n} + \frac{|t|}{2}\Big)^{-\alpha} dt,$$

where we used the fact that $\alpha > 0$. There are two possibilities: either

$$|\theta| < \frac{\pi}{n} \qquad \Rightarrow \qquad |t| \lesssim \frac{1}{n} \qquad \Rightarrow \qquad \Big(\frac{1}{n} + \frac{|t|}{2}\Big)^{-\alpha} \asymp n^\alpha$$

or

$$|\theta| \ge \frac{\pi}{n} \qquad \Rightarrow \qquad \frac{1}{n} + \frac{|t|}{2} \asymp \frac{1}{n} + \frac{|\theta|}{2} \asymp |\theta|.$$

The lower bound in (4.2) is proved.

Now, let us proceed to the upper bound. For the Fejér kernel, we have

$$\mathcal{F}_n(t) \le \begin{cases} n, & |t| < \dfrac{\pi}{n}, \\ \dfrac{\pi^2}{nt^2}, & |t| > \dfrac{\pi}{n}. \end{cases} \tag{4.14}$$

Hence,

$$(q_2 \otimes \mathcal{F}_n)(\theta) \le \int\limits_{\theta-\frac{\pi}{n}}^{\theta+\frac{\pi}{n}} n\left(\frac{|t|}{\pi}\right)^{-\alpha} dt + \int\limits_{\theta+\frac{\pi}{n}}^{\pi} \frac{\pi^2}{n}\left(\frac{|t|}{\pi}\right)^{-\alpha} \frac{dt}{(\theta-t)^2} + \int\limits_{-\pi}^{\theta-\frac{\pi}{n}} \frac{\pi^2}{n}\left(\frac{|t|}{\pi}\right)^{-\alpha} \frac{dt}{(\theta-t)^2}.$$

Denote the integrals on the right-hand side by I_1, I_2, and I_3. Using the symmetry, we can assume that $\theta \in (0, \pi/2)$. The case of $\theta \in (\pi/2, \pi)$ is similar. For I_1, we have

$$|\theta| \le \frac{\pi}{n} \qquad \Rightarrow \qquad h := \frac{n}{\pi}|\theta| \le 1 \qquad \Rightarrow \qquad I_1 = \frac{\pi n^\alpha}{1-\alpha}\big((1+h)^{1-\alpha} + (1-h)^{1-\alpha}\big) \asymp n^\alpha,$$

where we used the fact that $\alpha < 1$, and

$$|\theta| > \frac{\pi}{n} \qquad \Rightarrow \qquad h > 1 \qquad \Rightarrow \qquad I_1 = \frac{\pi n^\alpha}{1-\alpha}\big((h+1)^{1-\alpha} - (h-1)^{1-\alpha}\big) \asymp n^\alpha h^{-\alpha} = |\theta|^{-\alpha}.$$

To consider I_2, we make a change of variables:

$$u\colon \qquad \frac{1}{\theta - t} = \frac{-u}{\theta}.$$

Then

$$I_2 = \int\limits_{\frac{\theta}{\pi-\theta}}^{\frac{n\theta}{\pi}} \left(\frac{\pi^2}{n\theta}\right)\left(\frac{\theta}{\pi}\,\frac{u+1}{u}\right)^{-\alpha} du = \frac{\pi^2}{n}\pi^{-\alpha}\theta^{-\alpha-1} \int\limits_{\frac{\theta}{\pi-\theta}}^{\frac{n\theta}{\pi}} \left(\frac{u+1}{u}\right)^{-\alpha} du.$$

Note that in the integrand $\big(\frac{u+1}{u}\big)^{-\alpha} \in (0,1)$; hence, either

$$|\theta| > \frac{\pi}{n} \qquad \Rightarrow \qquad I_2 = O(n^{-1}\theta^{-\alpha-1}n\theta) = O(\theta^{-\alpha})$$

or

$$|\theta| \le \frac{\pi}{n} \qquad \Rightarrow \qquad I_2 < \frac{\pi^2}{n}\pi^{-\alpha}\theta^{-\alpha-1} \int\limits_0^{\frac{n\theta}{\pi}} u^\alpha\, du = O(n^\alpha).$$

To consider I_3, we make another change of variables:

$$u\colon \qquad \frac{1}{\theta - t} = \frac{u}{\theta}.$$

We have

$$I_3 = \int\limits_{\frac{\theta}{\pi+\theta}}^{\frac{n\theta}{\pi}} \frac{\pi^2}{n\theta}\left(\frac{\theta}{\pi}\left|\frac{1-u}{u}\right|\right)^{-\alpha} du = \frac{\pi^{2+\alpha}}{n\theta^{1+\alpha}} \int\limits_{\frac{\theta}{\pi+\theta}}^{\frac{n\theta}{\pi}} u^\alpha |1-u|^{-\alpha}\, du.$$

We estimate this integral for three domains of values of the parameter θ. In the first case,

$$\frac{n\theta}{\pi} < \frac{1}{2} \qquad \Rightarrow \qquad |1-u|^{-\alpha} < 2^\alpha \qquad \Rightarrow \qquad I_3 < \frac{2^\alpha \pi^{2+\alpha}}{n\theta^{1+\alpha}} \int\limits_0^{\frac{n\theta}{\pi}} u^\alpha\, du \sim n^\alpha.$$

In the second case, we have

$$\frac{1}{2} \le \frac{n\theta}{\pi} < 1 \qquad \Rightarrow \qquad I_3 < \pi n^\alpha \left(\frac{n\theta}{\pi}\right)^{-\alpha-1} \int\limits_0^1 u^\alpha (1-u)^{-\alpha}\, du = 2^{\alpha+1}\pi n^\alpha \frac{\pi\alpha}{\sin \pi\alpha} \sim n^\alpha.$$

Finally, in the third case $1 \le n\theta/\pi$, we split the integral into three parts:

$$I_3 < \frac{\pi^{2+\alpha}}{n\theta^{1+\alpha}} \left(\int_0^1 u^\alpha (1-u)^{-\alpha}\, du + \int_1^\infty \Big[\Big(\frac{u}{u-1} \Big)^\alpha - 1 - \frac{\alpha}{u} \Big]\, du + \int_1^{\frac{n\theta}{\pi}} \Big(1 - \frac{\alpha}{u} \Big)\, du \right)$$

$$= \frac{\pi^{2+\alpha}}{n\theta^{1+\alpha}} \left(\mathrm{const}(\alpha) + \frac{n\theta}{\pi} + \alpha \ln \frac{n\theta}{\pi} \right) \sim \theta^{-\alpha}.$$

Lemma 4.2 is proved. □

4.4. Wings: Estimates for the derivatives. Let us prove Lemma 4.3 on upper bounds for the derivatives

$$\mathcal{A}_2' = q_2' \otimes \mathcal{F}_m, \qquad \mathcal{A}_2'' = q_2'' \otimes \mathcal{F}_m, \qquad \text{where} \qquad q_2 = \left(\frac{1}{m^2} + \sin^2 \frac{t}{2} \right)^{-\frac{\alpha}{2}}.$$

Proof of Lemma 4.3. We have

$$q_2' = -\frac{\alpha}{4} \sin t \left(\frac{1}{m^2} + \sin^2 \frac{t}{2} \right)^{-\frac{\alpha}{2}-1},$$

$$q_2'' = \frac{\alpha(\alpha+2)}{16} \sin^2 t \left(\frac{1}{m^2} + \sin^2 \frac{t}{2} \right)^{-\frac{\alpha}{2}-2} - \frac{\alpha}{4} \cos t \left(\frac{1}{m^2} + \sin^2 \frac{t}{2} \right)^{-\frac{\alpha}{2}-1}.$$

The inequality between the geometric and arithmetic means yields

$$a \left(\frac{1}{m^2} + a^2 \right)^{\frac{-\beta-1}{2}} = \frac{a}{\sqrt{\frac{1}{m^2} + a^2}} \left(\frac{\frac{1}{m^2\beta}}{\frac{1}{m^2} + a^2} \right)^{\frac{\beta}{2}} (\beta m^2)^{\frac{\beta}{2}} \le (\beta m^2)^{\frac{\beta}{2}} \left(\frac{\frac{a^2}{\frac{1}{m^2}+a^2} + \beta \frac{\frac{1}{m^2\beta}}{\frac{1}{m^2}+a^2}}{1+\beta} \right)^{\frac{\beta+1}{2}}$$

$$= (\beta m^2)^{\frac{\beta}{2}} (\beta+1)^{-\frac{\beta+1}{2}};$$

this allows us to estimate $q_2'(t)$ and $q_2''(t)$ from above on $t \in [-\pi, \pi]$:

$$|q_2'| \le \frac{\alpha}{2} \sin \frac{t}{2} \left(\frac{1}{m^2} + \sin^2 \frac{t}{2} \right)^{-\frac{\alpha}{2}-1} \le \frac{\alpha}{2} \sqrt{\frac{(\alpha+1)^{\alpha+1}}{(\alpha+2)^{\alpha+2}}}\, m^{\alpha+1},$$

$$|q_2''| \le \frac{\alpha(\alpha+2)}{4} \sin^2 \frac{t}{2} \left(\frac{1}{m^2} + \sin^2 \frac{t}{2} \right)^{-\frac{\alpha}{2}-2} + \frac{\alpha}{4} \left(\frac{1}{m^2} + \sin^2 \frac{t}{2} \right)^{-\frac{\alpha}{2}-1}$$

$$\le \frac{\alpha}{4} \left(2 + \left(\frac{\alpha+2}{\alpha+4} \right)^{\frac{\alpha+4}{2}} \right) m^{\alpha+2}.$$

We can also obtain an upper bound using a power series expansion in t:

$$|q_2'(t)| < \frac{\alpha}{4} |t| \left(\left(\frac{|t|}{\pi} \right)^2 \right)^{-\frac{\alpha}{2}-1} = \frac{\alpha \pi^{\alpha+2}}{4} |t|^{-\alpha-1},$$

$$|q_2''(t)| < \frac{\alpha(\alpha+2)}{16} t^2 \left(\left(\frac{|t|}{\pi} \right)^2 \right)^{-\frac{\alpha}{2}-2} + \frac{\alpha}{4} \left(\left(\frac{|t|}{\pi} \right)^2 \right)^{-\frac{\alpha}{2}-1} = \frac{\alpha \pi^{\alpha+2}}{4} \left(1 + (\alpha+2) \frac{\pi^2}{4} \right) |t|^{-\alpha-2}.$$

Thus,

$$|q_2'(t)| \lesssim \min\left(m, \frac{\pi}{|t|} \right)^{\alpha+1}, \qquad |q_2''(t)| \lesssim \min\left(m, \frac{\pi}{|t|} \right)^{\alpha+2}.$$

Now we can consider $\mathcal{A}_2'$ and $\mathcal{A}_2''$ on $[-\pi,\pi]$. We split the integral into two parts

$$\mathcal{A}_2'(\theta)=\int\limits_{\mathbb{T}} q_2'(t)\mathcal{F}_m(\theta-t)\,dt=\int\limits_{\frac{\theta}{2}-\pi}^{\frac{\theta}{2}} q_2'(t)\mathcal{F}_m(\theta-t)\,dt+\int\limits_{\frac{\theta}{2}}^{\frac{\theta}{2}+\pi} q_2'(t)\mathcal{F}_m(\theta-t)\,dt$$

and obtain

$$|\mathcal{A}_2'(\theta)|\le\int\limits_{\frac{\theta}{2}-\pi}^{\frac{\theta}{2}}|q_2'(t)|\,dt\max_{t\in[\frac{\theta}{2},\frac{\theta}{2}+\pi]}|\mathcal{F}_m(t)|+\int\limits_{\frac{\theta}{2}-\pi}^{\frac{\theta}{2}}|\mathcal{F}_m(t)|\,dt\max_{t\in[\frac{\theta}{2},\frac{\theta}{2}+\pi]}|q_2'(t)|.$$

We have used the periodicity of the functions $\mathcal{F}$, $q_2^{(j)}$, and $\mathcal{A}_2^{(j)}$ and their symmetry about zero. Now we recall (4.14):

$$|\mathcal{F}_m(t)|\le\frac{1}{m}\min\left(m,\frac{\pi}{|t|}\right)^2,\qquad t\in[-\pi,\pi].$$

We fix $\theta>0$, fix a period so that $\theta/2$ is in the middle of the period, and continue

$$|\mathcal{A}_2'(\theta)|\lesssim\frac{1}{m}\min\left(m,\frac{2\pi}{|\theta|}\right)^2\int\limits_{-\pi}^{\pi}\min\left(m,\frac{\pi}{|t|}\right)^{\alpha+1}dt+\min\left(m,\frac{2\pi}{|\theta|}\right)^{\alpha+1}\int\limits_{-\pi}^{\pi}\frac{1}{m}\min\left(m,\frac{\pi}{|t|}\right)^2dt.$$

As a result,

$$|\mathcal{A}_2'(\theta)|\lesssim m^{\alpha+1}\min\left(1,\frac{1}{m\theta}\right)^2+m^{\alpha+1}\min\left(1,\frac{1}{m\theta}\right)^{\alpha+1}\lesssim m^{\alpha+1}\min\left(1,\frac{1}{m\theta}\right)^{\alpha+1}.$$

For $\mathcal{A}_2''$, we perform the same procedure with the exponent $\alpha+1$ replaced by $\alpha+2$:

$$|\mathcal{A}_2''(\theta)|\lesssim m^{\alpha+2}\min\left(1,\frac{1}{m\theta}\right)^2+m^{\alpha+2}\min\left(1,\frac{1}{m\theta}\right)^{\alpha+2}\lesssim m^{\alpha+2}\min\left(1,\frac{1}{m\theta}\right)^2;$$

however, now the dominant term is the first one.

Lemma 4.3 is proved. □

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