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One Method for Solving Systems of Nonlinear Partial Differential Equations

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Abstract—A method for reducing systems of partial differential equations to corresponding systems of ordinary differential equations is proposed. A system of equations describing two-dimensional, cylindrical, and spherical flows of a polytropic gas; a system of dimensionless Stokes equations for the dynamics of a viscous incompressible fluid; a system of Maxwell's equations for vacuum; and a system of gas dynamics equations in cylindrical coordinates are studied. It is shown how this approach can be used for solving certain problems (shockless compression, turbulence, etc.).

Keywords: systems of nonlinear partial differential equations, investigation method for nonlinear partial differential equations, exact solutions.

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INTRODUCTION

In a number of earlier papers, the authors studied the direction of the most intensive development of a process described by a nonlinear partial differential equation (PDE) and developed a geometric method, which made it possible to obtain some classes of solutions of such equations by reducing them to an ordinary differential equation (ODE) or to a system of ODEs and to study the character of the process [1–3].

In the present paper, this method is used to obtain some solutions of systems of nonlinear differential equations: a system of equations describing two-dimensional, cylindrical, and spherical flows of a polytropic gas; a system of dimensionless Stokes equations for the dynamics of a viscous incompressible fluid; a system of Maxwell's equations for vacuum; and a system of gas dynamics equations in cylindrical coordinates. Solution of these systems are obtained after reducing them to systems of ODEs.

1. Consider a system describing spherical, cylindrical, and two-dimensional waves with constant entropy [4, 5]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{2}{k-1}c\frac{\partial c}{\partial r} = 0, \quad \frac{\partial c}{\partial t} + u\frac{\partial c}{\partial r} + \frac{k-1}{2}\Big(c\frac{\partial u}{\partial r} + \frac{Nuc}{r}\Big) = 0. \tag{0.1}$$

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Here, r is the space coordinate, u is the speed of the gas, c is the sound speed in the gas, $c^2 = dp/d\rho$, p is pressure, ρ is density, $p = p(\rho)$ is the gas equation, $k = \text{const}$ is the adiabatic exponent, and $N = \text{const}$. The wave is two-dimensional for $N = 0$, cylindrical for $N = 1$, and spherical for $N = 2$.

2. Some solutions of the system of Stokes equations describing the dynamics a viscous incompressible fluid are obtained in dimensionless form [6]:

$$
\begin{aligned}
&S\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + E\frac{\partial p}{\partial x} - \frac{1}{R}\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\Big) = 0,\\
&S\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + E\frac{\partial p}{\partial y} - \frac{1}{R}\Big(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\Big) = 0,\\
&S\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + E\frac{\partial p}{\partial z} - \frac{1}{R}\Big(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\Big) = \frac{1}{F},\\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.
\end{aligned}
\tag{0.2}
$$

Here, S is the Strouhal number, E is the Euler number, R is the Reynolds number, F is the Froude number, $\{u, v, w\}$ are the components of the velocity vector, and p is pressure.

3. The above approach is applied for obtaining some solutions of the system of Maxwell's equations in vacuum [7]:

$$
\frac{\partial \mathbf{E}}{\partial t} - \operatorname{curl}\mathbf{H} = 0, \qquad \frac{\partial \mathbf{H}}{\partial t} + \operatorname{curl}\mathbf{E} = 0. \tag{0.3}
$$

Here, $\mathbf{E} = (u_1, u_2, u_3)$ is the electric field vector and $\mathbf{H} = (u_4, u_5, u_6)$ is the magnetic field vector.

4. The following system of ODEs for gas dynamics equations written in cylindrical coordinates is obtained by the geometric method [8]:

$$
\begin{aligned}
&\frac{d\varrho}{dt} + \varrho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial w}{\partial \varphi}\right) = -\frac{1}{r}\varrho v, \quad \frac{du}{dt} + \frac{1}{\varrho}\frac{\partial p}{\partial x} = 0, \quad \frac{dv}{dt} + \frac{1}{\varrho}\frac{\partial p}{\partial r} = \frac{w^2}{r},\\
&\frac{dw}{dt} + \frac{1}{r\varrho}\frac{\partial p}{\partial \varphi} = -\frac{vw}{r}, \quad \frac{dp}{dt} + c^2\varrho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial w}{\partial \varphi}\right) = -c^2\varrho\frac{v}{r},\\
&\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial r} + \frac{w}{r}\frac{\partial}{\partial \varphi} \ \text{ is the material derivative,} \quad c^2 = \kappa\frac{p}{\varrho}.
\end{aligned}
\tag{0.4}
$$

Here, $\{x, r, \varphi\}$ are cylindrical coordinates, $\{u, v, w\}$ are the components of the velocity vector, p is pressure, ϱ is the gas density, and κ is the adiabatic exponent.

For a particular case of system (0.1), it is shown that the approach described in the present paper produces singular solutions that are well known in gas dynamics [4] — Riemann simple waves [9].

1. THE SYSTEM OF ONE-DIMENSIONAL UNSTEADY MOTION OF A POLYTROPIC GAS

Consider system (0.1).

Assertion 1. *Under certain conditions, system* (0.1) *can be reduced by the geometric method to a system of ODEs.*

Proof. Let there exist a coordinate system in which $u = u(\psi(r,t))$ and $c = c(\psi(r,t))$. Then, the level surface $\psi = \psi(r,t)$ of these functions and system (0.1) can be represented in the form

$$u'\psi_t + \Big[uu' + \frac{(c^2)'}{k-1}\Big]\psi_r = 0, \quad (c^2)'r\psi_t + [u(c^2)' + (k-1)c^2u']r\psi_r + (k-1)Nuc^2 = 0. \tag{1.1}$$

Here, the prime ($'$) denotes differentiation with respect to ψ. Lower indices at the function $\psi(r,t)$ denote differentiation with respect to the corresponding variables.

Suppose that, in the first equation of (1.1), $\psi_t \neq 0$ and $\psi_r/\psi_t = f(\psi)$, where $f(\psi)$ is some function. Then $\psi_r = f(\psi)\psi_t$. Substituting this expression into the second equation of the system, we find that $r\psi_t = (1-k)Nuc^2/\{(c^2)' + f[u(c^2)' + (k-1)c^2u']\}$. The right-hand side of the obtained expression depends on ψ only. Denote it by $g_1(\psi)$; then, $r\psi_t = g_1(\psi)$ and $r\psi_r = r\psi_t f = g_1(\psi)f(\psi) = g_2(\psi)$ (by definition, $g_2(\psi) = f(\psi)g_1(\psi)$). Computing the derivative ψ_{tr}, we get $\psi_{tr} = (g_2g_1' - g_1)/r^2$. Computing ψ_{rt}, we get $\psi_{rt} = g_2'g_1/r^2$. Equating the mixed derivatives, we obtain the dependence $g_2g_1' - g_1g_2' = g_1$. Hence, under the assumption $g_1 \neq 0$, we have

$$g_2 = g_1 f(\psi), \quad f = C - w, \quad w = \int \frac{d\psi}{g_1}. \tag{1.2}$$

Since $\psi_r = \psi_t f(\psi)$, the solution of this quasi-linear first-order equation can be represented in the form [7] $\psi = \psi(t + fr)$ or $t = -f(\psi)r + G(\psi)$. Differentiating the latter relation with respect to t, we get $1 = G'\psi_t - rf'\psi_t$. Hence, $\psi_t = 1/(G' - f'r)$. On the other hand, $\psi_t = g_1/r$. Equating these relations and substituting the value $f' = -1/g_1$ from (1.2) into the resulting expression, we find that the equality is possible only when $G(\psi) = \text{const}$. Define $G = t_0$. From the relation $t = G - fr$, we get $f(\psi) = -(t - t_0)/r$. Hence, $\psi = \psi(y)$ and $y = (t - t_0)/r$. Then, however, $u = u(y)$ and $c = c(y)$. Substituting such functions $u(y)$ and $c(y)$ into system (0.1), we come to the system of ODEs

$$\frac{du}{dy} = \frac{Nc^2uy}{c^2y^2 - (1-uy)^2}, \qquad \frac{dc^2}{dy} = \frac{(k-1)(1-uy)Nc^2u}{c^2y^2 - (1-uy)^2};$$

passing to the independent variable $z = 1/y$, we obtain the system

$$\frac{du}{dz} = -\frac{Nc^2u}{z[c^2 - (u-z)^2]}, \qquad \frac{dc^2}{dz} = \frac{(k-1)(u-z)Nc^2u}{z[c^2 - (u-z)^2]}, \tag{1.3}$$

which was to be proved. □

Thus, we have found a class of solutions to system (0.1) of PDEs given by system (1.3) of ODEs. It is easy to see that system (1.3) has a solution of the form $u = az$, $c = bz$, where $a = \text{const}$ and $b = \text{const}$. Substituting into the system a solution of this form, we obtain

$$u = \frac{2z}{(k+1) + N(k-1)}, \qquad c^2 = \frac{(N+1)(k-1)^2z^2}{[(k+1) + N(k-1)]^2}.$$

For system (1.3), the following problem is solved numerically: the initial values $u = u_0 = 25$ and $c = c_0 = 1$ are given for $z = z_0 = 4.5$, and the behavior of $c(z)$ is studied for $z < z_0$. Since the system describes processes in a polytropic gas, we have $\rho \longmapsto \infty$ as $c \longmapsto \infty$, where ρ is density. In this problem, shockless gas compression is observed for some values $z^* < z < z_0$; for $z = z^*$, the sound speed is discontinuous at the point where the denominators $[c^2 - (u-z)^2]$ of system (1.3) change their signs.

Assertion 2. *In the special case $\psi = c$, system* (0.1) *reduces to the ODE*

$$\frac{2}{k-1}Ncuu'' - (N+1)uu'^3 - \frac{2}{k-1}cu'^2 + \frac{2}{k-1}uu'\Big(\frac{2}{k-1} - N\Big) + \frac{8}{(k-1)^3} = 0. \tag{1.4}$$

Proof. For system (0.1), consider a special case producing a simple wave. Let the independent variable be $\psi = c$; hence, $u = u(c)$. In this case, system (0.1) reduces to the system

$$u'\frac{\partial c}{\partial t} + \Big(uu' + \frac{2}{k-1}c\Big)\frac{\partial c}{\partial r} = 0, \qquad \frac{\partial c}{\partial t} + \Big(u + \frac{k-1}{2}cu'\Big)\frac{\partial c}{\partial r} = -\frac{k-1}{2}\frac{Nuc}{r}.$$

Here, the prime ($'$) denotes differentiation with respect to c. Finding from this system the derivatives of c with respect to t and r and equating the mixed derivatives, we come to the following equation for $u = u(c)$:

$$\frac{2}{k-1}Ncuu'' - (N+1)uu'^3 - \frac{2}{k-1}cu'^2 + \frac{2}{k-1}uu'\Big(\frac{2}{k-1} - N\Big) + \frac{8}{(k-1)^3} = 0,$$

which was to be proved. □

Let us write a specific solution of equation (1.4)

$$u = \pm\frac{1}{k-1}\sqrt{\frac{2[(3-k) - N(k-1)]}{N+1}}\Big[c - \frac{2(N+1)}{3-k-N(k-1)}\Big].$$

We see that u is a real-valued function in the case $N = 0$ if $k < 3$, in the case $N = 1$ if $k < 2$, and in the case $N = 2$ if $k < 5/3$.

2. THE SYSTEM OF STOKES EQUATIONS

Consider system (0.2).

Assertion 3. *Under certain conditions, system* (0.2) *can be reduced by the geometric method to a system of ODEs.*

Proof. Suppose that $u = u(\psi(x,y,z,t))$, $v = v(\psi(x,y,z,t))$, $w = w(\psi(x,y,z,t))$, and $p = p(\psi(x,y,z,t))$. Then, $\psi(x,y,z,t) = \text{const}$ is a level surface for u, v, w, and p. Under this assumption, system (0.2) can be written in the form

$$\begin{aligned}
&Su'\psi_t + uu'\psi_x + vu'\psi_y + wu'\psi_z - \frac{1}{R}[u''(\psi_x^2+\psi_y^2+\psi_z^2) + u'(\psi_{xx}+\psi_{yy}+\psi_{zz})] = 0,\\
&Sv'\psi_t + uv'\psi_x + vv'\psi_y + wv'\psi_z - \frac{1}{R}[v''(\psi_x^2+\psi_y^2+\psi_z^2) + v'(\psi_{xx}+\psi_{yy}+\psi_{zz})] = 0,\\
&Sw'\psi_t + uw'\psi_x + vw'\psi_y + ww'\psi_z - \frac{1}{R}[w''(\psi_x^2+\psi_y^2+\psi_z^2) + w'(\psi_{xx}+\psi_{yy}+\psi_{zz})] = \frac{1}{F},\\
&u'\psi_x + v'\psi_y + w'\psi_z = 0.
\end{aligned} \tag{2.1}$$

In system (2.1), the prime ($'$) denotes differentiation with respect to ψ and lower indices denote differentiation of ψ with respect to the corresponding variables. Let $\psi_t \neq 0$, $\psi_x/\psi_t = f_1(\psi)$, $\psi_y/\psi_t = f_2(\psi)$, $\psi_z/\psi_t = f_3(\psi)$, and $(\psi_x^2 + \psi_y^2 + \psi_z^2)/\psi_t = f_4(\psi)$, where $f_i(\psi)$ are arbitrary functions ($i = 1,2,3,4$). Then, $\psi_t = f_4/(f_1^2 + f_2^2 + f_3^2) = g(\psi)$. From the equality of the mixed derivatives, we find that $\psi_x = ag(\psi)$, $\psi_y = bg(\psi)$, $\psi_z = cg(\psi)$, $a = \text{const}$, $b = \text{const}$, and $c = \text{const}$. Therefore, $\psi = \psi(l)$ and $l = t + ax + by + cz$. Then, $u = u(l)$, $v = v(l)$, $w = w(l)$, $p = p(l)$, and system (0.2) reduces to the system of ODEs

$$\begin{aligned}
&(S + au + bv + cw)u_l + Eap_l - \frac{1}{R}u_{ll}(a^2 + b^2 + c^2) = 0,\\
&(S + au + bv + cw)v_l + Ebp_l - \frac{1}{R}v_{ll}(a^2 + b^2 + c^2) = 0,\\
&(S + au + bv + cw)w_l + Ecp_l - \frac{1}{R}w_{ll}(a^2 + b^2 + c^2) = \frac{1}{F},\\
&au_l + bv_l + cw_l = 0,
\end{aligned} \tag{2.2}$$

which was to be proved. □

Differential consequences of the last equation of (2.2) yield the relations $au_{ll} + bv_{ll} + cw_{ll} = 0$ and $au + bv + cw = A$, $A = \text{const}$. If we multiply the first equation of system (2.2) by a, the second equation by b, and the third equation by c and add the obtained relations, we obtain $p_l = c/[FE(a^2 + b^2 + c^2)]$. Then, the first three equations of system (2.2) take the form

$$\begin{aligned}
&\frac{a^2 + b^2 + c^2}{R}u_{ll} - (S + A)u_l - \frac{ca}{F(a^2 + b^2 + c^2)} = 0,\\
&\frac{a^2 + b^2 + c^2}{R}v_{ll} - (S + A)v_l - \frac{cb}{F(a^2 + b^2 + c^2)} = 0,\\
&\frac{a^2 + b^2 + c^2}{R}w_{ll} - (S + A)w_l + \frac{a^2 + b^2}{F(a^2 + b^2 + c^2)} = 0.
\end{aligned} \tag{2.3}$$

A general solution of linear system (2.3) with constant coefficients is easily written. For $A \neq (-S)$,

$$\begin{aligned}
&u = \frac{U_0}{\alpha}\exp(\alpha l) - \frac{\beta}{\alpha}l + U_1, \quad \alpha = \frac{R(S + A)}{a^2 + b^2 + c^2}, \quad \beta = -\frac{caR}{F(a^2 + b^2 + c^2)^2},\\
&U_0 = \text{const}, \quad U_1 = \text{const};\\
&v = \frac{V_0}{\alpha}\exp(\alpha l) - \frac{b\beta}{a\alpha}l + V_1, \quad V_0 = \text{const}, \quad V_1 = \text{const};\\
&w = -\frac{aU_0 + bV_0}{c\alpha}\exp(\alpha l) + \frac{\beta(a^2 + b^2)}{ac\alpha}l + \frac{A - aU_1 - bV_1}{c};\\
&p = \frac{c}{FE(a^2 + b^2 + c^2)}l + p_0, \quad p_0 = \text{const}, \quad l = t + ax + by + cz.
\end{aligned} \tag{2.4}$$

If $A = -S$, then

$$\begin{aligned}
&u = -0.5\beta l^2 + U_2 l + U_3, \quad U_2 = \text{const}, \quad U_3 = \text{const};\\
&v = -0.5\beta l^2 + V_2 l + V_3, \quad V_2 = \text{const}, \quad V_3 = \text{const};\\
&w = \frac{0.5\beta(a + b)}{c}l^2 - \frac{aU_2 + bV_2}{c}l - \frac{aU_3 + bV_3 + S}{c}.
\end{aligned} \tag{2.5}$$

As follows from formulas (2.4), these laws describe blow-up regimes, which results in turbulence and supports the conjecture proposed by Jean Leray in 1933. The greater the Reynolds number, the faster the speed goes to infinity. If the Reynolds number is small, the blow-up also occurs but later. In the case $A = -S$ (see formulas (2.5)), the situation is different for the same Reynolds numbers: the growth of the sound speed is shifted in time and is insignificant in comparison with the its growth in the case $A \neq -S$.

3. THE SYSTEM OF MAXWELL'S EQUATIONS

Consider system (0.3).

Assertion 4. *Under certain conditions, system* (0.3) *can be reduced by the geometric method to a system of ODEs.*

Proof. Let $u_i = u_i(\psi(t,x,y,z))$ for $i = 1,2,3,4,5,6$. Then, (0.3) can be written in the form

$$\begin{aligned} &u_1'\psi_t + u_5'\psi_z - u_6'\psi_y = 0, \quad u_2'\psi_t + u_6'\psi_x - u_4'\psi_z = 0, \quad u_3'\psi_t + u_4'\psi_y - u_5'\psi_x = 0, \\ &u_4'\psi_t - u_2'\psi_z + u_3'\psi_y = 0, \quad u_5'\psi_t - u_3'\psi_x + u_1'\psi_z = 0, \quad u_6'\psi_t - u_1'\psi_y + u_2'\psi_x = 0. \end{aligned} \tag{3.1}$$

Here, the prime ($'$) denotes differentiation with respect to ψ and lower indices at the function $\psi(t,x,y,z)$ denote the derivatives with respect to the corresponding variables. Consider the case $\psi_t \neq 0$. Assume that

$$\frac{\psi_z}{\psi_t} = f_1(\psi), \quad \frac{\psi_y}{\psi_t} = f_2(\psi), \quad \frac{\psi_x}{\psi_t} = f_3(\psi), \tag{3.2}$$

where $f_j(\psi)$ for $j = 1,2,3$ are arbitrary functions. It is easy to verify that the solution of system (3.2) has the form

$$\psi = \psi(t + zf_1(\psi) + yf_2(\psi) + xf_3(\psi)). \tag{3.3}$$

System (3.1) has a nontrivial solution if the corresponding determinant is zero. This is so if $f_1^2 + f_2^2 + f_3^2 = 1$. Hence, setting $f_3 = \pm\sqrt{1 - f_1^2 - f_2^2}$, we obtain the system of relations

$$\begin{aligned} u_1' &= \mp\left(\frac{f_2}{\sqrt{1-f_1^2-f_2^2}}u_2' + \frac{f_1}{\sqrt{1-f_1^2-f_2^2}}u_3'\right), \\ u_4' &= f_1u_2' - f_2u_3', \\ u_5' &= \pm\left(\frac{f_1f_2}{\sqrt{1-f_1^2-f_2^2}}u_2' + \frac{1-f_2^2}{\sqrt{1-f_1^2-f_2^2}}u_3'\right), \\ u_6' &= \mp\left(\frac{1-f_1^2}{\sqrt{1-f_1^2-f_2^2}}u_2' + \frac{f_1f_2}{\sqrt{1-f_1^2-f_2^2}}u_3'\right), \end{aligned} \tag{3.4}$$

which was to be proved. □

The functions $f_1(\psi)$, $f_2(\psi)$, $u_2(\psi)$, and $u_3(\psi)$ in relations (3.4) are arbitrary. Consider the case when $f_1 = f_2 = \psi$ and $\psi = t + zf_1 + yf_2 + x\sqrt{1 - f_1^2 - f_2^2}$ in relation (3.3). Assume also that $u_2 = a\psi + c_2$, $u_3 = b\psi + c_3$, $a = \text{const}$, $b = \text{const}$, $c_2 = \text{const}$, and $c_3 = \text{const}$. Then, the solution of system (3.4) has the form

$$\begin{aligned} &u_1 = 0.5(a+b)\sqrt{1-2\psi^2} + c_1, \quad u_4 = 0.5(a-b)\psi^2 + c_2, \\ &u_5 = 0.25(b-a)\psi\sqrt{1-2\psi^2} + \frac{a+3b}{4\sqrt{2}}\arcsin(\psi\sqrt{2}) + c_5, \end{aligned}$$

$$u_6 = 0.25(a-b)\psi\sqrt{1-2\psi^2} + \frac{3a+b}{4\sqrt{2}}\arcsin(\psi\sqrt{2}) + c_6,$$

$$c_1 = \text{const}, \quad c_4 = \text{const}, \quad c_5 = \text{const}, \quad c_6 = \text{const},$$

where

$$\psi = \frac{t(1-y-z) \pm \sqrt{t^2(1-y-z)^2 - (t^2-x^2)[(1-y-z)^2+2x^2]}}{(1-y-z)^2+2x^2}.$$

4. FINDING THE STRENGTHS OF THE MAGNETIC AND ELECTRIC FIELDS THAT PRODUCE THE MOTION OF CHARGED PARTICLES WITH A GIVEN CONSTANT SPEED

Lorentz established [10] that the drift direction of the center of a charged particle coincides with the vector product of the vectors $\mathbf{E}$ and $\mathbf{H}$ of the electric and magnetic fields.

Let us solve for system (0.3) the following problem. Assume that the vector $\mathbf{U} = (u, v, w)$ of velocity drift is given for the center of a charge; let $u = \text{const}$, $v = \text{const}$, and $w = \text{const}$. Then [10], the following dependences hold: $u = u_2u_6 - u_3u_5$, $v = u_3u_4 - u_1u_6$, and $w = u_1u_5 - u_2u_4$. Hence, $uu_1 + vu_2 + wu_3 = 0$ and $uu_4 + vu_5 + wu_6 = 0$. From system (3.4), we find that

$$\begin{gathered}\Big(f_1 \pm \frac{vf_1f_2}{u\sqrt{1-f_1^2-f_2^2}} \mp \frac{w(1-f_1^2)}{u\sqrt{1-f_1^2-f_2^2}}\Big)u_2' \\ + \Big(-f_2 \pm \frac{v(1-f_2^2)}{u\sqrt{1-f_1^2-f_2^2}} \mp \frac{wf_1f_2}{u\sqrt{1-f_1^2-f_2^2}}\Big)u_3' = 0, \\ \Big(\pm\frac{f_2}{\sqrt{1-f_1^2-f_2^2}} - \frac{v}{u}\Big)u_2' + \Big(\pm\frac{f_1}{\sqrt{1-f_1^2-f_2^2}} - \frac{w}{u}\Big)u_3' = 0.\end{gathered} \tag{4.1}$$

System (4.1) has a nontrivial solution if $(vf_2 + wf_1 \pm u\sqrt{1-f_1^2-f_2^2})^2 = u^2+v^2+w^2$. Finding f_2 from this relation, we conclude that the function f_2 is real-valued if the discriminant of the obtained quadratic equation is zero. Consequently, we have

$$f_1 = \pm\frac{w}{\sqrt{u^2+v^2+w^2}}, \qquad f_2 = \pm\frac{v}{\sqrt{u^2+v^2+w^2}}.$$

Substituting the obtained values f_1 and f_2 into system (3.4), we get

$$\begin{aligned} u_1 &= \mp\Big(\frac{v}{u}u_2 + \frac{w}{u}u_3\Big) + a_1, \quad u_4 = \pm\Big(\frac{w}{\sqrt{u^2+v^2+w^2}}u_2 - \frac{v}{\sqrt{u^2+v^2+w^2}}u_3\Big) + a_2, \\ u_5 &= \pm\Big(\frac{vw}{u\sqrt{u^2+v^2+w^2}}u_2 + \frac{u^2+w^2}{u\sqrt{u^2+v^2+w^2}}u_3\Big) + a_3, \\ u_6 &= \mp\Big(\frac{u^2+v^2}{u\sqrt{u^2+v^2+w^2}}u_2 + \frac{vw}{u\sqrt{u^2+v^2+w^2}}u_3\Big) + a_4, \quad a_i = \text{const} \quad (\mathrm{i} = 1,2,3,4). \end{aligned} \tag{4.2}$$

Note once again that $\mathbf{U} = \mathbf{E}\times\mathbf{H}$; consequently, $u = u_2u_6 - u_3u_5$, $v = u_3u_4 - u_1u_6$, and $w = u_1u_5 - u_2u_4$. Substituting into these expression the values of the components of the vectors $\mathbf{E}$ and $\mathbf{H}$ from (4.2), we find that $u_2 = u_2(u, v, w, a_1, a_2, a_3) = \text{const}$, $u_3 = u_3(u, v, w, a_1, a_2, a_3) = \text{const}$, and $a_4 = a_4(u, v, w, a_1, a_2, a_3) = \text{const}$. Thus, we find all constant components of the vectors of the electric and magnetic fields that produce the drift of the charged particle in a given direction with a given constant speed. If the given drift is variable, we can similarly obtain electric and magnetic fields producing this drift.

5. THE SYSTEM OF GAS DYNAMICS EQUATIONS IN CYLINDRICAL COORDINATES

Consider system (0.4).

Assertion 5. *Under certain conditions, system* (0.4) *can be reduced by the geometric method to a system of ODEs.*

Proof. As in the systems considered earlier, we assume that $u = u(\psi)$, $v = v(\psi)$, $w = w(\psi)$, $p = p(\psi)$, and $\varrho = \varrho(\psi)$. Then, system (0.4) reduces to the system of ODEs

$$\begin{gathered} r\varrho'(\psi_t + u\psi_x + v\psi_r) + \varrho' w\psi_\varphi + \varrho(u'r\psi_x + v'r\psi_r + w'\psi_\varphi) = -\varrho v, \\ r\varrho u'(\psi_t + u\psi_x + v\psi_r) + w\varrho u'\psi_\varphi + rp'\psi_x = 0, \\ r\varrho v'(\psi_t + u\psi_x + v\psi_r) + w\varrho v'\psi_\varphi + rp'\psi_r = w^2\varrho, \\ r\varrho w'(\psi_t + u\psi_x + v\psi_r) + w\varrho w'\psi_\varphi + p'\psi_\varphi = -wv\varrho, \\ rp'(\psi_t + u\psi_x + v\psi_r) + p'w\psi_\varphi + \kappa p(u'r\psi_x + v'r\psi_r + w'\psi_\varphi) = -\kappa pv. \end{gathered} \tag{5.1}$$

As above, the prime (′) denotes differentiation with respect to ψ, and lower indices denote the derivatives of the function ψ with respect to the corresponding variables.

Let, in system (5.1), $r\psi_t = f_1(\psi)$, $r\psi_x = f_2(\psi)$, $\psi_\varphi = f_3(\psi)$, and $r\psi_r = f_4(\psi)$. Then, equating the mixed derivatives, we obtain the dependences $f_2 = c_2 f_1$, $f_3 = c_3 f_1$,

$$f_4 = f_1 g(\psi), \quad g(\psi) = c_4 - \int \frac{d\psi}{f_1}, \quad \psi_r = g(\psi)\psi_t, \quad c_2 = \text{const} \neq 0, \quad c_3 = \text{const} \neq 0, \quad c_4 = \text{const}.$$

In view of all the dependences between the first derivatives of ψ and equating all mixed derivatives, we find that $f_3 = 0$ and $(t + c_2 x + g(\psi)r) = \text{const}$ (see the complete proof in Section 1). Therefore, $g(\psi) = (t_0 - t - c_2 x)/r$. Let $g(\psi) = -\psi$; then, $\psi = (t - t_0 + c_2 x)/r$. Since $\mathbf{U} = \mathbf{U}(\psi)$ and $\mathbf{U} = \{p, \varrho, u, v, w\}$, the substitution of such functions into system (0.4) yields the system of ODEs

$$\begin{gathered} (q/\varrho)\varrho' + c_2 u' - \psi v' = -v, \quad qu' + (c_2/\varrho)p' = 0, \quad qv' - (\psi/\varrho)p' = w^2, \quad qw' = -vw, \\ [q/(\kappa p)]p' + c_2 u' - \psi v' = -v, \quad \text{where} \quad q = 1 + c_2 u - \psi v, \end{gathered} \tag{5.2}$$

which was to be proved. □

Comparing the first and the last equations of system (5.2), we obtain the dependence $p = a\varrho^\kappa$, where $a = \text{const} > 0$. From the first equation of system (5.2), we have $c_2 u' - \psi v' = -v - q(\varrho'/\varrho) = q(w'/w - \varrho'/\varrho)$; then, $q' = c_2 u' - \psi v' - v = -2v - q(\varrho'/\varrho) = q(2w'/w - \varrho'/\varrho)$. Hence, $q'/q = 2w'/w - \varrho'/\varrho$ and $q\varrho/w^2 = c_0$, where $c_0 = \text{const} > 0$ if $w \neq 0$, $\varrho \neq 0$, and $q > 0$.

Using the obtained first integrals and assuming that $[a\kappa\varrho^{\kappa-1}(c_2^2 + \psi^2) - q^2] \neq 0$, write system (5.2) in a form resolved with respect to the derivatives:

$$\begin{gathered} u' = \frac{\kappa a c_2 \varrho^{\kappa-1}(\psi\varrho - vc_0)}{c_0[a\kappa\varrho^{\kappa-1}(c_2^2 + \psi^2) - q^2]}, \qquad v' = \frac{\kappa a \varrho^{\kappa-1}(c_2^2\varrho + \psi v c_0) - q^2\varrho}{c_0[a\kappa\varrho^{\kappa-1}(c_2^2 + \psi^2) - q^2]}, \\ \varrho' = \frac{q\varrho(vc_0 - \psi\varrho)}{c_0[a\kappa\varrho^{\kappa-1}(c_2^2 + \psi^2) - q^2]}. \end{gathered} \tag{5.3}$$

System of equations (5.3) contains three arbitrary constants: $a > 0$, $c_2 > 0$, and $c_0 > 0$. If, for example, $c_0 = c_0(c_2, a)$, then the solution of the system depends on an arbitrary function.

Let us write exact solutions of system (0.4):

$$\varrho = \varrho_0 = \text{const}, \quad p = a\varrho_0^\kappa = \text{const}, \quad u = u_0 = \text{const}, \quad v = \varrho_0\psi/c_0,$$
$$w = \sqrt{(1 + c_2u_0 - \varrho_0\psi^2/c_0)\varrho_0/c_0}.$$

$$\varrho = \varrho_0 = \text{const}, \quad p = a\varrho_0^\kappa = \text{const}, \quad u = \psi^2\varrho_0/c_0 - 1 + \sqrt{a\kappa\varrho_0^{(\kappa-1)}(c_2^2 + \psi^2)}, \quad v = \psi\varrho_0/c_0,$$
$$w = \pm(\varrho_0/c_0)^{1/2}[a\kappa\varrho_0^{(\kappa-1)}(c_2^2 + \psi^2)]^{1/4}.$$

6. CONCLUSIONS

The geometric method, which was earlier used for studying and solving nonlinear PDEs, can be applied for nonlinear and linear systems of PDEs. In this case, systems of PDEs are reduced to systems of ODEs; the solution of the latter systems makes it possible to find in a number of cases exact solutions of original systems of equations (see systems (0.2) and (0.4)) and solve some other problems (systems (0.1)–(0.3)).

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