ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2015, Vol. 288, Suppl. 1, pp. S162–S172.

published in Trudy Instituta Matematiki i Mekhaniki UrO RAN, 2014, Vol. 20, No. 1.

Construction of Orthogonal Multiwavelet Bases

E. A. Pleshcheva[1] **and N. I. Chernykh**[1,2]

Received April 5, 2013

Abstract—We propose a method for constructing orthogonal multiwavelet bases of the space $\mathbf{L}^2(\mathbb{R})$ for any known multiscaling functions that generate a multiresolution analysis of dimension greater than 1.
Keywords: multiwavelet, multiresolution analysis, multiscaling function, mask, matrix mask.

DOI: 10.1134/S0081543815020169

INTRODUCTION

Consider, as in [3, 5–7], instead of one scaling function a system of k functions $\{\varphi^l(x)\colon l=\overline{1,k}\}$ whose shifts and compressions generate by Mallat's classical scheme the corresponding multiresolution analysis of dimension $k>1$. Keinert noted [2, Ch. 10] that there exist methods of wavelet construction from known multiscaling functions satisfying "basic regularity conditions." In [4], a universal method was proposed for constructing biorthogonal bases of multiwavelets in the case when multiscaling functions are compactly supported. In our paper, the method of multiwavelet construction from multiscaling functions does not involve any additional constraints except for condition (e) of the following known definition of an MRA$_k$.

Definition. A sequence of nested closed subspaces

$$\ldots\subset V_j\subset V_{j+1}\subset\ldots\qquad (j\in\mathbb{Z})\tag{0.1}$$

of the space $\mathbf{L}^2(\mathbb{R})$ is called its multiresolution analysis of dimension k (MRA$_k$) if it satisfies the following conditions:

(a) $\overline{\bigcup_j V_j}=\mathbf{L}^2(\mathbb{R})$;

(b) $\bigcap_j V_j=\{0\}$;

(c) $f(x)\in V_j\Leftrightarrow \forall l\in\mathbb{Z}\ f(x-l/2^j)\in V_j$;

(d) $f(x)\in V_0\Leftrightarrow \forall j\in\mathbb{Z}\ f(2^jx)\in V_j$;

(e) there exist functions $\varphi^s(x)$, $s=\overline{1,k}$, from $V_0\subset\mathbf{L}^2(\mathbb{R})$ such that the set of their integer shifts $\varphi^s(x-n)$, $s=\overline{1,k}$, $n\in\mathbb{Z}$, forms a basis of the space V_0 orthonormal in $\mathbf{L}^2(\mathbb{R})$.

The functions $\varphi^1(x),\varphi^2(x),\ldots,\varphi^k(x)$ are called *multiscaling.*

[1]Institute of Mathematics and Mechanics, Ural Branch of the Russian Academy of Sciences, ul. S. Kovalevskoi 16, Yekaterinburg, 620990 Russia
emails: eplescheva@gmail.com, chernykh@imm.uran.ru
[2]Institute of Mathematics and Computer Science, Ural Federal University, pr. Lenina 51, Yekaterinburg, 620000 Russia

The main result of the present paper is an algorithm for constructing orthonormal bases of multiwavelet spaces from known orthonormal bases of multiresolution analysis spaces.

Similarly to the case of one scaling function, wavelet subspaces W_j corresponding to an MRA_k are defined by the conditions

$$W_j \dotplus V_j = V_{j+1}, \quad V_j \in \mathrm{MRA}_k, \quad j \in \mathbb{Z}, \tag{0.2}$$

where $\dotplus$ means the orthogonal sum of subspaces.

1. NECESSARY ORTHOGONALITY CONDITIONS IN TERMS OF MASKS OF SCALING FUNCTIONS

The material presented in this section is well known (see, for example, [2, Ch. 7]), similarly to the case $k = 1$, and is given here with the aim of introducing necessary notions and notation. Let us write scaling relations for functions that form a basis of spaces of an MRA_k. For this, we introduce a scaling vector function (a column)

$$\Phi(x) = \Big(\varphi^1(x), \varphi^2(x), \dots, \varphi^k(x)\Big)^T.$$

As follows from the definition of an MRA_k, the components of the vector function $\Phi_{j,n}(x) = 2^{j/2}\Phi(2^j x - n)$ form an orthonormal basis of the space V_j; hence, condition (0.1) is equivalent to the equality

$$\Phi(x) \overset{\mathbf{L}^2(\mathbb{R})}{=} \sum_{n\in\mathbb{Z}} H_n \Phi_{1,n}(x) \tag{1.3}$$

with matrix coefficients

$$H_n = \begin{pmatrix} h_n^{1,1} & h_n^{1,2} & \dots & h_n^{1,k} \\ h_n^{2,1} & h_n^{2,2} & \dots & h_n^{2,k} \\ \dots & & & \\ h_n^{k,1} & h_n^{k,2} & \dots & h_n^{k,k} \end{pmatrix}$$

and componentwise convergence in $\mathbf{L}^2(\mathbb{R})$ of the series in (1.3). This is equivalent to the fact that the sequences of complex numbers $\{h_n^{r,s}\}_{n\in\mathbb{Z}}$ $(r, s = \overline{1,k})$ belong to $l^2(\mathbb{Z})$.

After the Fourier transform, equality (1.3) takes the form

$$\widehat{\Phi}(\omega) = M\Big(\frac{\omega}{2}\Big)\widehat{\Phi}\Big(\frac{\omega}{2}\Big), \tag{1.4}$$

where

$$\widehat{\Phi}(\omega) = \Big(\widehat{\varphi^1}(\omega), \widehat{\varphi^2}(\omega), \dots, \widehat{\varphi^k}(\omega)\Big)^T,$$

$$M(\omega) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} H_n e^{-2\pi i n\omega} = \begin{pmatrix} m^{1,1}(\omega) & m^{1,2}(\omega) & \dots & m^{1,k}(\omega) \\ m^{2,1}(\omega) & m^{2,2}(\omega) & \dots & m^{2,k}(\omega) \\ \dots & & & \\ m^{k,1}(\omega) & m^{k,2}(\omega) & \dots & m^{k,k}(\omega) \end{pmatrix}.$$

The matrix $M(\omega)$ is called a *mask of the system of scaling functions*; its elements

$$m^{r,s}(\omega) \overset{\mathbf{L}^2}{=} \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} h_n^{r,s} e^{-2\pi i n\omega}$$

are 1-periodic functions square integrable on $[0,1)$, i.e., functions from $\mathbf{L}^2[0,1)$. Let us write $\big\langle \Phi(x), \Phi(x-n)\big\rangle = \int_{\mathbb{R}} \Phi(x)[\Phi(x-n)]^* dx$, where $[\Phi(x)]^* = [\overline{\varphi^1}(x), \overline{\varphi^2}(x), \ldots, \overline{\varphi^k}(x)]$. Here and elsewhere, $A^* = (\overline{A})^T$ is the complex conjugate transpose of A. It is clear that the condition $\int_{\mathbb{R}} \varphi^r(x)\overline{\varphi^s(x-n)}dx = \delta_{r,s}\delta_{0,n}$ $(r, s = \overline{1,k}, n \in \mathbb{Z})$ of the orthonormality of the system $\{\varphi^s(x-n) : s = \overline{1,k}, n \in \mathbb{Z}\}$ is equivalent to the matrix equality (with the unit matrix I of dimension k)

$$\big\langle \Phi(x), \Phi(x-n)\big\rangle = \delta_{0,n} I.$$

As in the classical case, passing to Fourier transforms, we get

$$\big\langle \Phi(x), \Phi(x-n)\big\rangle = \big\langle \widehat{\Phi}(\omega), \widehat{\Phi}(\omega)e^{-2\pi i n\omega}\big\rangle = \int_{\mathbb{R}} \widehat{\Phi}(\omega)[\widehat{\Phi}(\omega)]^* e^{2\pi i n\omega} d\omega = \delta_{0,n} I.$$

Representing the integral from this relation as the sum of integrals over the intervals $[l, l+1]$ and replacing in these integrals ω by $\omega - l$, as in the classical case (see, for example, [1, Ch. 1]), we obtain

$$\int_{\mathbb{R}} \widehat{\Phi}(\omega)[\widehat{\Phi}(\omega)]^* e^{2\pi i n\omega} d\omega = \sum_{l\in\mathbb{Z}} \int_l^{l+1} \widehat{\Phi}(\omega)[\widehat{\Phi}(\omega)]^* e^{2\pi i n\omega} d\omega$$

$$= \sum_{l\in\mathbb{Z}} \int_0^1 \widehat{\Phi}(\omega-l)[\widehat{\Phi}(\omega-l)]^* e^{2\pi i n(\omega-l)} d\omega = \int_0^1 \Big[\sum_{l\in\mathbb{Z}} \widehat{\Phi}(\omega-l)[\widehat{\Phi}(\omega-l)]^*\Big] e^{2\pi i n\omega} d\omega = \delta_{0,n} I,$$

which yields a necessary and sufficient condition of orthogonality in $\mathbf{L}^2(\mathbb{R})$ of the system $\{\varphi^s(x-n)\colon s = \overline{1,k},\ n \in \mathbb{Z}\}$:

$$\sum_{l\in\mathbb{Z}} \widehat{\Phi}(\omega-l)\big[\widehat{\Phi}(\omega-l)\big]^* \overset{\text{a.a.}}{=} I. \tag{1.5}$$

This condition implies the known necessary condition for the masks $M(\omega)$, which will be given here in a new form reflected in the following statement.

Statement. *Let*

$$\mathfrak{M}(\omega) = \Big[M(\omega); M\Big(\omega+\frac{1}{2}\Big)\Big] = \begin{pmatrix} m^{1,1}(\omega) & \ldots & m^{1,k}(\omega) & m^{1,1}\Big(\omega+\frac{1}{2}\Big) & \ldots & m^{1,k}\Big(\omega+\frac{1}{2}\Big) \\ m^{2,1}(\omega) & \ldots & m^{2,k}(\omega) & m^{2,1}\Big(\omega+\frac{1}{2}\Big) & \ldots & m^{2,k}\Big(\omega+\frac{1}{2}\Big) \\ \ldots & & & & & \\ m^{k,1}(\omega) & \ldots & m^{k,k}(\omega) & m^{k,1}\Big(\omega+\frac{1}{2}\Big) & \ldots & m^{k,k}\Big(\omega+\frac{1}{2}\Big) \end{pmatrix}.$$

If the system $\{\varphi^s(x-n)\colon s = \overline{1,k},\ n \in \mathbb{Z}\}$ *is orthonormal, then*

$$\mathfrak{M}(\omega)(\mathfrak{M}(\omega))^* \overset{\text{a.a.}}{=} I.$$

Proof. As follows from equalities (1.4) and $(AB)^* = B^*A^*$,

$$\sum_{l\in\mathbb{Z}} \widehat{\Phi}(\omega-l)\big[\widehat{\Phi}(\omega-l)\big]^* = \sum_{l\in\mathbb{Z}} M\Big(\frac{\omega-l}{2}\Big)\widehat{\Phi}\Big(\frac{\omega-l}{2}\Big)\Big[\widehat{\Phi}\Big(\frac{\omega-l}{2}\Big)\Big]^*\Big[M\Big(\frac{\omega-l}{2}\Big)\Big]^* \overset{\text{a.a.}}{=} I. \tag{1.6}$$

Applying a known scheme, we decompose the latter sum into two sums, with even and odd l. Then, using condition (1.5), we get

$$I \overset{\text{a.a.}}{=} M\Big(\frac{\omega}{2}\Big)\Big[M\Big(\frac{\omega}{2}\Big)\Big]^* + M\Big(\frac{\omega}{2}+\frac{1}{2}\Big)\Big[M\Big(\frac{\omega}{2}+\frac{1}{2}\Big)\Big]^*. \tag{1.7}$$

Changing in this equality $\omega/2$ for ω, we can easily verify that it can be written in terms of the introduced matrix $\mathfrak{M}(\omega)$ in the following equivalent form:

$$\mathfrak{M}(\omega)(\mathfrak{M}(\omega))^* \overset{\text{a.a.}}{=} I. \tag{1.8}$$

Indeed, for example, the first element in the first row of this matrix is

$$\sum_{s=1}^{k}\big|m^{1,s}(\omega)\big|^2 + \sum_{s=1}^{k}\Big|m^{1,s}\Big(\omega+\frac{1}{2}\Big)\Big|^2,$$

where the first and second sums are the first elements of the first row of the matrices $M(\omega)[M(\omega)]^*$ and $M(\omega+1/2)[M(\omega+1/2)]^*$, respectively. The remaining equalities are verified similarly. □

2. BASES OF MULTIWAVELET SPACES

Let us construct wavelet spaces W_j $(j \in \mathbb{Z})$ using the matrices $\mathfrak{M}(\omega)$. As follows from (0.2), it is sufficient to construct W_0 such that $W_0 \dotplus V_0 = V_1$, since it is clear that $W_j = d_2^j W_0$ for remaining $j \in \mathbb{Z}$, where d_2 is the operator of binary compression: $(d_2 f)(x) = f(2x)$. Further, we construct a vector function

$$\Psi(x) = \Big(\psi^1(x), \psi^2(x), \dots, \psi^k(x)\Big)^T, \quad \psi^s(x) \in \mathbf{L}^2(\mathbb{R}), \quad s = \overline{1,k}, \tag{2.1}$$

such that $\{\psi^s(x-n)\}$, $s = \overline{1,k}$, $n \in \mathbb{Z}$, is an orthonormal system in the space $W_0 \subset V_1$.

For the function $\psi^s(x)$ to lie in $W_0 \subset V_1$, there must exist matrices H_n^ψ composed of elements $\{h_n^{r,s,\psi}\}_{n\in\mathbb{Z}} \in l_2$ such that

$$\Psi(x) \overset{\mathbf{L}^2(\mathbb{R})}{=} \sum_{n\in\mathbb{Z}} H_n^\psi \sqrt{2}\,\Phi(2x-n). \tag{2.2}$$

Here, the series of vector functions in the right-hand side of the equality converges componentwise in $\mathbf{L}^2(\mathbb{R})$. In terms of Fourier transforms, equality (2.2) looks as follows:

$$\widehat{\Psi}(\omega) = M^\psi\Big(\frac{\omega}{2}\Big)\widehat{\Phi}\Big(\frac{\omega}{2}\Big), \tag{2.3}$$

where the mask of the multiwavelet system $M^\psi(\omega)$ is given by the expression

$$M^\psi(\omega) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} H_n^\psi e^{-2\pi i n\omega} = \begin{pmatrix} m_\psi^{1,1}(\omega) & m_\psi^{1,2}(\omega) & \dots & m_\psi^{1,k}(\omega) \\ m_\psi^{2,1}(\omega) & m_\psi^{2,2}(\omega) & \dots & m_\psi^{2,k}(\omega) \\ \dots & & & \\ m_\psi^{k,1}(\omega) & m_\psi^{k,2}(\omega) & \dots & m_\psi^{k,k}(\omega) \end{pmatrix}, \tag{2.4}$$

$$m_\psi^{r,s}(\omega) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} h_n^{r,s,\psi} e^{-2\pi i n\omega} \in \mathbf{L}^2[0,1) \quad (r,s = \overline{1,k}).$$

Introduce the matrix

$$\mathfrak{M}^{\psi}(\omega)=\begin{pmatrix} m_{\psi}^{1,1}(\omega) & \dots & m_{\psi}^{1,k}(\omega) & m_{\psi}^{1,1}\Big(\omega+\dfrac{1}{2}\Big) & \dots & m_{\psi}^{1,k}\Big(\omega+\dfrac{1}{2}\Big) \\ m_{\psi}^{2,1}(\omega) & \dots & m_{\psi}^{2,k}(\omega) & m_{\psi}^{2,1}\Big(\omega+\dfrac{1}{2}\Big) & \dots & m_{\psi}^{2,k}\Big(\omega+\dfrac{1}{2}\Big) \\ \dots & & & & & \\ m_{\psi}^{k,1}(\omega) & \dots & m_{\psi}^{k,k}(\omega) & m_{\psi}^{k,1}\Big(\omega+\dfrac{1}{2}\Big) & \dots & m_{\psi}^{k,k}(\omega+\dfrac{1}{2}) \end{pmatrix} \tag{2.5}$$

and express in terms of the matrices $\mathfrak{M}(\omega)$ and $\mathfrak{M}^{\psi}(\omega)$ conditions of orthonormality of the system $\{\psi^1(x-n),\psi^2(x-n),\dots,\psi^k(x-n)\}_{n\in\mathbb{Z}}$ and orthogonality of the spaces V_0 and $\widetilde{W}_0=\overline{\mathrm{span}}\{\psi^r(x-n)\}$.

Theorem 1. *Let $\{\varphi^s(x-n),\ s=\overline{1,k},\ n\in\mathbb{Z}\}$ be an orthonormal system. Then, a system $\{\psi^s(x-n),\ s=\overline{1,k},\ n\in\mathbb{Z}\}$ of form* (2.2) *is orthonormal if only if*

$$\mathfrak{M}^{\psi}(\omega)(\mathfrak{M}^{\psi}(\omega))^{*}\overset{a.a.}{=}I, \tag{2.6}$$

and the spaces $\widetilde{W}_0$ and V_0 are orthogonal if and only if

$$\mathfrak{M}^{\psi}(\omega)(\mathfrak{M}(\omega))^{*}\overset{a.a.}{=}\mathbf{0}, \tag{2.7}$$

where $\mathbf{0}$ is the zero matrix of dimension $k\times k$.

Proof. The proof is similar the case of an MRA_1. Let a system $\{\psi^s(x-n),\ s=\overline{1,k},\ n\in\mathbb{Z}\}$ be orthonormal. Then, the validity of equality (2.6) is obtained by a modification of the proof of the statement from Section 1: one should use (2.3) instead of (1.4), replace $\widehat{\Phi}(\omega-l)$ by $\widehat{\Psi}(\omega-l)$, preserve $\widehat{\Phi}$ in the right-hand side of (1.6), and replace M by M^{ψ} applying notation (2.5). As a result, we obtain (2.6) instead of (1.8). It is easily seen that, in contrast to the statement, this argument is invertible if we assume the orthogonality of the system $\{\varphi^s(x-n),\ s=\overline{1,k},\ n\in\mathbb{Z}\}$.

Let us prove the equality

$$\sum_{l\in\mathbb{Z}}\widehat{\Phi}(\omega-l)[\widehat{\Psi}(\omega-l)]^{*}=\mathbf{0}$$

by the usual scheme. As always, the condition $\langle\psi^r(x-n),\varphi^s(x-l)\rangle=0$, $r,s=\overline{1,k}$, $n,l\in\mathbb{Z}$, of orthogonality of the spaces W_0 and V_0 is equivalent to the conditions

$$\langle\Phi(x),\Psi(x-n)\rangle=\mathbf{0},\quad n\in\mathbb{Z}, \tag{2.8}$$

or, in terms of Fourier transforms, to the condition

$$\langle\widehat{\Phi}(\omega),\widehat{\Psi}(\omega)e^{-2\pi in\omega}\rangle=\int_{\mathbb{R}}\widehat{\Phi}(\omega)[\widehat{\Psi}(\omega)]^{*}e^{2\pi in\omega}d\omega=\mathbf{0}.$$

Applying standard transformations (see the proof of formula (1.5)), we find that (2.8) is equivalent to the equalities

$$\int_0^1\sum_{l\in\mathbb{Z}}\widehat{\Phi}(\omega-l)[\widehat{\Psi}(\omega-l)]^{*}e^{2\pi in\omega}d\omega\overset{\text{a.a.}}{=}\mathbf{0},\quad n\in\mathbb{Z},$$

which imply

$$\sum_{l\in\mathbb{Z}} \widehat{\Phi}(\omega-l)[\widehat{\Psi}(\omega-l)]^* = \mathbf{0}.$$

Substituting representations (2.3) and (1.4) into this equality, we get

$$\sum_{l\in\mathbb{Z}} \widehat{\Phi}(\omega-l)[\widehat{\Psi}(\omega-l)]^* = \sum_{l\in\mathbb{Z}} M\Big(\frac{\omega-l}{2}\Big)\widehat{\Phi}\Big(\frac{\omega-l}{2}\Big)\Big[M^{\psi}\Big(\frac{\omega-l}{2}\Big)\widehat{\Phi}\Big(\frac{\omega-l}{2}\Big)\Big]^*$$

$$= \sum_{l\in\mathbb{Z}} M\Big(\frac{\omega-l}{2}\Big)\widehat{\Phi}\Big(\frac{\omega-l}{2}\Big)\Big[\widehat{\Phi}\Big(\frac{\omega-l}{2}\Big)\Big]^*\Big[M^{\psi}\Big(\frac{\omega-l}{2}\Big)\Big]^* \overset{\text{a.a.}}{=} \mathbf{0}.$$

As in the classical case, using the transformations from the derivation of (1.7), we find that (2.8) implies the equality

$$M\Big(\frac{\omega}{2}\Big)\Big[M^{\psi}\Big(\frac{\omega}{2}\Big)\Big]^* + M\Big(\frac{\omega}{2}+\frac{1}{2}\Big)\Big[M^{\psi}\Big(\frac{\omega}{2}+\frac{1}{2}\Big)\Big]^* \overset{\text{a.a.}}{=} \mathbf{0}. \tag{2.9}$$

Inserting in (2.9) between the factors M and M^{ψ} the matrix I in the form

$$\sum \widehat{\Phi}\Big(\frac{\omega-m}{2}\Big)\Big[\widehat{\Phi}\Big(\frac{\omega-m}{2}\Big)\Big]^*$$

for $m=2l$ and $m=2l+1$, we see that these transformations are also invertible: (2.9) implies (2.8).

As in the proof of the statement, it is easy to verify that equality (2.9) after changing $\omega/2$ for ω is written in terms of the matrices $\mathfrak{M}(\omega)$ and $\mathfrak{M}^{\psi}(\omega)$ in the form

$$\mathfrak{M}(\omega)(\mathfrak{M}^{\psi}(\omega))^* \overset{\text{a.a.}}{=} \mathbf{0}. \tag{2.10}$$

Indeed, for example, the first element in the first row of this matrix is $\sum_{s=1}^{k} m^{1,s}(\omega)\overline{m_{\psi}^{1,s}}(\omega) + \sum_{s=1}^{k} m^{1,s}(\omega+1/2)\overline{m_{\psi}^{1,s}}(\omega+1/2)$, and the first and second sums are the first elements in the first row of the matrices $M(\omega/2)[M^{\psi}(\omega/2)]^*$ and $M(\omega/2+1/2)[M^{\psi}(\omega/2+1/2)]^*$, respectively. Similar arguments are carried out for the remaining elements. $\square$

If we find a matrix $\mathfrak{M}^{\psi}(\omega)$ satisfying Theorem 1, then the functions $\psi^s(x)$, $s=\overline{1,k}$, are uniquely found from the part $M^{\psi}(\omega)$ of this matrix and formulas (2.2) or (2.3). By Theorem 1, integer shifts of these functions generate an orthonormal system in the space W_0.

Let us describe an algorithm for constructing masks of multiwavelets from known masks of multiscaling functions.

Consider the case $k=2$ and then extend the method of constructing $\mathfrak{M}^{\psi}(\omega)$ to all even k. Compose the determinant

$$\overrightarrow{b_1(\omega)} = \det\begin{pmatrix} \overrightarrow{i_1} & \overrightarrow{i_2} & \overrightarrow{i_3} & \overrightarrow{i_4} \\ \overline{m^{1,1}}(\omega) & \overline{m^{1,2}}(\omega) & \overline{m^{1,1}}\Big(\omega+\frac{1}{2}\Big) & \overline{m^{1,2}}\Big(\omega+\frac{1}{2}\Big) \\ \overline{m^{2,1}}(\omega) & \overline{m^{2,2}}(\omega) & \overline{m^{2,1}}\Big(\omega+\frac{1}{2}\Big) & \overline{m^{2,2}}\Big(\omega+\frac{1}{2}\Big) \\ a^1(\omega) & a^2(\omega) & a^1\Big(\omega+\frac{1}{2}\Big) & a^2\Big(\omega+\frac{1}{2}\Big) \end{pmatrix}, \tag{2.11}$$

where $\overrightarrow{i_s}$ are unit vectors: $\overrightarrow{i_1} = (1,0,0,0)$, …, $\overrightarrow{i_4} = (0,0,0,1)$; $m^{r,s}(\omega)$ are elements of the mask $M(\omega)$; the vector $\overrightarrow{a(\omega)} = \big(a^1(\omega), a^2(\omega), a^1(\omega+1/2), a^2(\omega+1/2)\big)$ is linearly independent with the vectors

$$\overrightarrow{m_1(\omega)} = \Big(\overline{m^{1,1}}(\omega), \overline{m^{1,2}}(\omega), \overline{m^{1,1}}\Big(\omega+\frac{1}{2}\Big), \overline{m^{1,2}}\Big(\omega+\frac{1}{2}\Big)\Big),$$

$$\overrightarrow{m_2(\omega)} = \Big(\overline{m^{2,1}}(\omega), \overline{m^{2,2}}(\omega), \overline{m^{2,1}}\Big(\omega+\frac{1}{2}\Big), \overline{m^{2,2}}\Big(\omega+\frac{1}{2}\Big)\Big);$$

and $a^1(\omega)$ and $a^2(\omega)$ are arbitrary 1-periodic functions from $\mathbf{L}^2[0,1)$.

It is easy to see that the scalar product of a vector of type $\overrightarrow{b_1(\omega)}$ and the vectors $\sum \overrightarrow{i_s} c^s$ in the space l_4^2 coincides with the determinant obtained by replacing the first row in determinant (2.11) by $(\overline{c_1}, \overline{c_2}, \overline{c_3}, \overline{c_4})$. Therefore, the vector $\overrightarrow{b_1(\omega)} := \big(b_1^1(\omega), b_1^2(\omega), b_1^3(\omega), b_1^4(\omega)\big)$ constructed in this way is orthogonal in the space l_4^2 to the vectors

$$\overrightarrow{m^1(\omega)} = \Big(m^{1,1}(\omega), m^{1,2}(\omega), m^{1,1}\Big(\omega+\frac{1}{2}\Big), m^{1,2}\Big(\omega+\frac{1}{2}\Big)\Big),$$

$$\overrightarrow{m^2(\omega)} = \Big(m^{2,1}(\omega), m^{2,2}(\omega), m^{2,1}\Big(\omega+\frac{1}{2}\Big), m^{2,2}\Big(\omega+\frac{1}{2}\Big)\Big);$$

it is also 1-periodic since the elements and the corresponding algebraic complements of the vectors in the first row of determinant (2.11) are 1-periodic. It is seen from the formulas

$$b_1^1(\omega) = \det \begin{pmatrix} \overline{m^{1,2}}(\omega) & \overline{m^{1,1}}\Big(\omega+\frac{1}{2}\Big) & \overline{m^{1,2}}\Big(\omega+\frac{1}{2}\Big) \\ \overline{m^{2,2}}(\omega) & \overline{m^{2,1}}\Big(\omega+\frac{1}{2}\Big) & \overline{m^{2,2}}\Big(\omega+\frac{1}{2}\Big) \\ a^2(\omega) & a^1\Big(\omega+\frac{1}{2}\Big) & a^2\Big(\omega+\frac{1}{2}\Big) \end{pmatrix},$$

$$b_1^3(\omega) = \det \begin{pmatrix} \overline{m^{1,1}}(\omega) & \overline{m^{1,2}}(\omega) & \overline{m^{1,2}}\Big(\omega+\frac{1}{2}\Big) \\ \overline{m^{2,1}}(\omega) & \overline{m^{2,2}}(\omega) & \overline{m^{2,2}}\Big(\omega+\frac{1}{2}\Big) \\ a^1(\omega) & a^2(\omega) & a^2\Big(\omega+\frac{1}{2}\Big) \end{pmatrix}$$

that $b_1^3(\omega) = b_1^1(\omega+1/2)$, since the determinant $b_1^1(\omega+1/2)$ is obtained from the determinant $b_1^3(\omega)$ by an even permutation of columns. It is verified similarly that $b_1^4(\omega) = b_1^2(\omega+1/2)$. Thus, the vector $\overrightarrow{\overline{b_1}(\omega)}$ has the form

$$\overrightarrow{\overline{b_1}(\omega)} = \big(b_1^1(\omega), b_1^2(\omega), b_1^1(\omega+1/2), b_1^2(\omega+1/2)\big).$$

Substituting now into the fourth row of determinant (2.11) the vector $\overrightarrow{\overline{b_1}(\omega)}$ instead of $\overrightarrow{a(\omega)}$, we obtain the new vector

$$\overrightarrow{b_2(\omega)} = \det \begin{pmatrix} \overrightarrow{i_1} & \overrightarrow{i_2} & \overrightarrow{i_3} & \overrightarrow{i_4} \\ \overline{m^{1,1}}(\omega) & \overline{m^{1,2}}(\omega) & \overline{m^{1,1}}\Big(\omega+\frac{1}{2}\Big) & \overline{m^{1,2}}\Big(\omega+\frac{1}{2}\Big) \\ \overline{m^{2,1}}(\omega) & \overline{m^{2,2}}(\omega) & \overline{m^{2,1}}\Big(\omega+\frac{1}{2}\Big) & \overline{m^{2,2}}\Big(\omega+\frac{1}{2}\Big) \\ \overline{b_1^1}(\omega) & \overline{b_1^2}(\omega) & \overline{b_1^1}\Big(\omega+\frac{1}{2}\Big) & \overline{b_1^2}\Big(\omega+\frac{1}{2}\Big) \end{pmatrix}. \tag{2.12}$$

It is clear that the vector $\overrightarrow{b_2(\omega)}$ also has the form

$$\overrightarrow{b_2(\omega)} := \Big(b_2^1(\omega), b_2^2(\omega), b_2^1\Big(\omega+\frac{1}{2}\Big), b_2^2\Big(\omega+\frac{1}{2}\Big)\Big)$$

and is orthogonal to the vectors $\overrightarrow{m^1(\omega)}$ and $\overrightarrow{m^2(\omega)}$. Replacing in determinant (2.12) the first row by $\overrightarrow{\overline{b_1}(\omega)}$, we see that $\overrightarrow{b_1(\omega)} \perp \overrightarrow{b_2(\omega)}$ in the space l_4^2.

Using these vectors, we can construct

$$\overrightarrow{m_\psi^s(\omega)} = \Big(m_\psi^{s,1}(\omega), m_\psi^{s,2}(\omega), m_\psi^{s,1}\Big(\omega+\frac{1}{2}\Big), m_\psi^{s,2}\Big(\omega+\frac{1}{2}\Big)\Big), \quad s = 1, 2,$$

setting $\overrightarrow{m_\psi^s(\omega)} = \overrightarrow{b_s(\omega)}/\|\overrightarrow{b_s(\omega)}\|_{l_4^2}$. Thus, we have constructed the masks and the matrix $\mathfrak{M}_\psi$ of multiwavelets from the system of scaling functions of an MRA$_2$, more exactly, from the elements of the mask matrix of this system. By construction, the mask $M^\psi(\omega)$ satisfies Theorem 1, and its elements look as follows:

$$m^{r,s}(\omega) = \frac{b_s^r(\omega)}{\|\overrightarrow{b_s(\omega)}\|_{l_4^2}}, \quad r, s = 1, 2.$$

Similarly, we construct wavelet bases for other even k:

$$\overrightarrow{b_1(\omega)} = \det\begin{pmatrix} \overrightarrow{i_1} & \dots & \overrightarrow{i_k} & \overrightarrow{i_{k+1}} & \dots & \overrightarrow{i_{2k}} \\ \overline{m^{1,1}}(\omega) & \dots & \overline{m^{1,k}}(\omega) & \overline{m^{1,1}}\Big(\omega+\frac{1}{2}\Big) & \dots & \overline{m^{1,k}}\Big(\omega+\frac{1}{2}\Big) \\ \dots \\ \overline{m^{k,1}}(\omega) & \dots & \overline{m^{k,k}}(\omega) & \overline{m^{k,1}}\Big(\omega+\frac{1}{2}\Big) & \dots & \overline{m^{k,k}}\Big(\omega+\frac{1}{2}\Big) \\ a_1^1(\omega) & \dots & a_1^k(\omega) & a_1^1\Big(\omega+\frac{1}{2}\Big) & \dots & a_1^k\Big(\omega+\frac{1}{2}\Big) \\ \dots \\ a_{k-1}^1(\omega) & \dots & a_{k-1}^k(\omega) & a_{k-1}^1\Big(\omega+\frac{1}{2}\Big) & \dots & a_{k-1}^k\Big(\omega+\frac{1}{2}\Big) \end{pmatrix}, \tag{2.13}$$

choosing arbitrary 1-periodic functions $\overrightarrow{a_j^s(\omega)}$ from $\mathbf{L}^2[0,1)$ so that the last $2k-1$ rows of the determinant are linearly independent.

Further, we write $\overrightarrow{\overline{b_1}(\omega)}$ instead of $\overrightarrow{a_1(\omega)}$ in (2.13) and obtain $\overrightarrow{b_2(\omega)}$, and so on. As a result, we get the system of vectors $\{\overrightarrow{b_1(\omega)}, \dots, \overrightarrow{b_k(\omega)}\}$, which, after a unit normalization in l_{2k}^2, defines the matrix $\mathfrak{M}_\psi(\omega)$. Obviously, this matrix has form (2.5) and satisfies properties (2.6) and (2.7).

Now, let k be odd. Consider the vector

$$\overrightarrow{\widetilde{b_1}(\omega)} = \det\begin{pmatrix} \overrightarrow{i_1} & \dots & \overrightarrow{i_k} & \overrightarrow{i_{k+1}} & \dots & \overrightarrow{i_{2k}} \\ \overline{m^{1,1}}(\omega) & \dots & \overline{m^{1,k}}(\omega) & \overline{m^{1,1}}\Big(\omega+\frac{1}{2}\Big) & \dots & \overline{m^{1,k}}\Big(\omega+\frac{1}{2}\Big) \\ \dots \\ \overline{m^{k,1}}(\omega) & \dots & \overline{m^{k,k}}(\omega) & \overline{m^{k,1}}\Big(\omega+\frac{1}{2}\Big) & \dots & \overline{m^{k,k}}\Big(\omega+\frac{1}{2}\Big) \\ a_1^1(\omega) & \dots & a_1^k(\omega) & a_1^1\Big(\omega+\frac{1}{2}\Big) & \dots & a_1^k\Big(\omega+\frac{1}{2}\Big) \\ \dots \\ a_{k-1}^1(\omega) & \dots & a_{k-1}^k(\omega) & a_{k-1}^1\Big(\omega+\frac{1}{2}\Big) & \dots & a_{k-1}^k\Big(\omega+\frac{1}{2}\Big) \end{pmatrix}. \tag{2.14}$$

The constructed vector $\overrightarrow{\widetilde{b_1}(\omega)} = (\widetilde{b_1^1}(\omega), \widetilde{b_1^2}(\omega), \ldots, \widetilde{b_1^{2k}}(\omega))$ is 1-periodic in ω; it is not equal to $(\widetilde{b_1^1}(\omega), \ldots \widetilde{b_1^k}(\omega), \widetilde{b_1^1}(\omega+1/2) \ldots, \widetilde{b_1^k}(\omega+1/2))$ as in the case of even k, because, since the number of permutations of columns of algebraic complements to elements from the first row of matrix (2.14) is odd, we have the equalities $\widetilde{b_1^s}(\omega+1/2) = -\widetilde{b_1^{k+s}}(\omega)$ for $s = \overline{1,k}$. Transform the vector $\overrightarrow{\widetilde{b_1}(\omega)}$ so that it satisfies the required condition. For this, we multiply it by the function $\lambda_1(\omega)$. The resulting vector $\lambda_1(\omega)\overrightarrow{\widetilde{b_1}(\omega)}$ is orthogonal in l^2_{2k} to all $\overrightarrow{m^s(\omega)}$, $s = 1, \ldots, k$, and is written as follows:

$$\overrightarrow{b_1(\omega)} := \lambda_1(\omega)\overrightarrow{\widetilde{b_1}(\omega)} = \big(\lambda_1(\omega)\widetilde{b_1^1}(\omega), \ldots, \lambda_1(\omega)\widetilde{b_1^k}(\omega), \lambda_1(\omega)\widetilde{b_1^{k+1}}(\omega), \ldots, \lambda_1(\omega)\widetilde{b_1^{2k}}(\omega)\big).$$

Let us find conditions on $\lambda_1(\omega)$ that are necessary and sufficient for the vector $\overrightarrow{b_1(\omega)}$ to have the same structure as in the case of even k, i.e., conditions for the coincidence of the component $b_1^s(\omega+1/2) = \lambda_1(\omega+1/2)\widetilde{b_1^s}(\omega+1/2)$ for $s = \overline{1,k}$ with the component $b_1^{k+s}(\omega) = \lambda_1(\omega)\widetilde{b_1^{k+s}}(\omega)$. Obviously, by the above equalities for $\widetilde{b_1^s}(\omega+1/2)$, it is sufficient to impose on the function $\lambda_1(\omega)$ the condition

$$\lambda_1(\omega) = -\lambda_1\Big(\omega + \frac{1}{2}\Big) \tag{2.15}$$

additionally to the condition of 1-periodicity. We now can substitute the obtained row function $\overrightarrow{b_1(\omega)} = \lambda_1(\omega)\overrightarrow{\widetilde{b_1}(\omega)}$ into determinant (2.14), replacing the row $\overrightarrow{a_1(\omega)}$ by it. Further, proceeding similarly, we find $\overrightarrow{\widetilde{b_2}(\omega)}$ and construct $\overrightarrow{b_2(\omega)} = \lambda_2(\omega)\overrightarrow{\widetilde{b_2}(\omega)}$ with $\lambda_2(\omega) = -\lambda_2(\omega+1/2)$, and so on. As a result, after the normalization in l^2_{2k} of the row vectors $\overrightarrow{b_r}$ for $r = \overline{1,k}$, we obtain matrix functions $\mathfrak{M}^\psi$ and M^ψ with the same properties as for even k.

Having obtained the masks $M^\psi(\omega)$ of multiwavelets, we find, as in the classical case, an expression for Fourier transforms of multiwavelets in terms of Fourier transforms of multiscaling functions by formulas (2.3). The inverse Fourier transform yields vector function (2.1) and, hence, the family of multiwavelets $\psi^s_{j,l}$, $s = \overline{1,k}$, $j, l \in \mathbb{Z}$. The same result can be obtained by expanding $M^\psi(\omega)$ into a trigonometric series with matrix coefficients H^ψ_n and then applying formula (2.2) for the construction of vector function (2.1).

In the following theorem, we assume that the masks $M^\psi(\omega)$ are constructed according to the presented scheme as the corresponding submatrices of the matrix $\mathfrak{M}^\psi(\omega)$.

Theorem 2. *The system of functions $\psi^s(x-n)$, $s = \overline{1,k}$, $n \in \mathbb{Z}$, recovered from Fourier transform* (2.3) *of the corresponding vector function $\Psi(x)$, where the mask $M^\psi(\omega)$ is defined above, forms a basis of the space W_0.*

Proof. Let $f(x) \in W_0$, i.e., (1) $f(x) \in V_1$ and (2) $f(x) \perp V_0$. By condition (1), we have $f(x) = \sum_{n\in\mathbb{Z}} C^f_n \Phi_{1,n}(x)$, where elements of the row vector C^f_n lie in $l^2(\mathbb{Z})$, or, equivalently,

$$\widehat{f}(\omega) = \overrightarrow{m^f\Big(\frac{\omega}{2}\Big)}\widehat{\Phi}\Big(\frac{\omega}{2}\Big),$$

where $\overrightarrow{m^f(\omega)} = \sum_{n\in\mathbb{Z}} C^f_n e^{2\pi i n\omega}$ is a 1-periodic vector function of dimension $1 \times k$.

By condition (2), $\langle f(x), \varphi^l(x-n)\rangle_{\mathbf{L}^2(\mathbb{R})} = 0$ for $l = \overline{1,k}$. Since the components of the vector functions $\widehat{f}(\omega)\widehat{\Phi}(\omega)$ are integrable, this condition can be written in terms of Fourier transforms in the form

$$\int_{\mathbb{R}} \overrightarrow{m^f\Big(\frac{\omega}{2}\Big)}\widehat{\Phi}\Big(\frac{\omega}{2}\Big)\Big[e^{-2\pi i n\omega}\overrightarrow{m\Big(\frac{\omega}{2}\Big)}\widehat{\Phi}\Big(\frac{\omega}{2}\Big)\Big]^* d\omega = 0 \quad \forall n \in \mathbb{Z}.$$

Divide the integral into the sum of integrals over the intervals $[\nu, \nu+1]$ and then pass to the integral over the interval $[0,1]$. Considering separately the sums over even and odd ν and using the properties of a matrix product and equality (1.5), we obtain the equivalent equality

$$\int\limits_{\mathbb{R}} \left(\overrightarrow{m^f\Big(\frac{\omega}{2}\Big)} \Big[\overrightarrow{m\Big(\frac{\omega}{2}\Big)}\Big]^* + \overrightarrow{m^f\Big(\frac{\omega}{2}+\frac{1}{2}\Big)} \Big[\overrightarrow{m\Big(\frac{\omega}{2}+\frac{1}{2}\Big)}\Big]^* \right) e^{2\pi i n\omega} d\omega = 0 \quad \forall n \in \mathbb{Z}.$$

Therefore,

$$\overrightarrow{m^f(\omega)}\big[M(\omega)\big]^* + \overrightarrow{m^f\Big(\omega+\frac{1}{2}\Big)}\Big[M\Big(\omega+\frac{1}{2}\Big)\Big]^* \overset{\text{a.a.}}{=} \overrightarrow{0}. \tag{2.16}$$

By (2.6), we have

$$\overrightarrow{m^f(\omega)} = \overrightarrow{\alpha(\omega)} M^\psi(\omega),$$

where $\overrightarrow{\alpha(\omega)}$ is a 1-periodic row vector of dimension k. Using equalities (2.16) and (2.10), we find conditions on $\overrightarrow{\alpha(\omega)}$. It is easy to see that

$$\overrightarrow{\alpha(\omega)} M^\psi(\omega)[M(\omega)]^* + \overrightarrow{\alpha\Big(\omega+\frac{1}{2}\Big)} M^\psi\Big(\omega+\frac{1}{2}\Big)\Big[M\Big(\omega+\frac{1}{2}\Big)\Big]^*$$

$$= \left(\overrightarrow{\alpha\Big(\omega+\frac{1}{2}\Big)} - \overrightarrow{\alpha(\omega)}\right) M^\psi\Big(\omega+\frac{1}{2}\Big)\Big[M\Big(\omega+\frac{1}{2}\Big)\Big]^* \overset{\text{a.a.}}{=} \overrightarrow{0},$$

$$\overrightarrow{\alpha\Big(\omega+\frac{1}{2}\Big)} = \overrightarrow{\alpha(\omega)}, \;\; \widehat{f}(\omega) = \overrightarrow{\alpha\Big(\frac{\omega}{2}\Big)} M^\psi\Big(\frac{\omega}{2}\Big)\widehat{\Phi}\Big(\frac{\omega}{2}\Big) = \overrightarrow{\alpha\Big(\frac{\omega}{2}\Big)}\widehat{\Psi}(\omega). \tag{2.17}$$

Here, as seen from (2.17), $\overrightarrow{\alpha(\omega)}$ is a 1/2-periodic row vector of dimension k; consequently, $\widehat{f}(\omega)$ is the product of a 1-periodic vector and $\widehat{\Psi}(\omega)$. Applying the inverse Fourier transform, we find that

$$f(x) = \sum_{n\in\mathbb{Z}} D_n \Psi(x-n),$$

where D_n are the matrix coefficients from the expansion

$$\overrightarrow{\alpha\Big(\frac{\omega}{2}\Big)} = \sum_{n\in\mathbb{Z}} D_n e^{-2\pi i n\omega}. \qquad \square$$

These bases are defined nonuniquely, up to the chosen vectors $\overrightarrow{a_s(\omega)}$, $s = 1, \dots, k-1$, and functions $\lambda_s(\omega)$, $s = 1, \dots, k$, with the required properties. For even k, we can multiply the constructed vectors $\overrightarrow{b_r(\omega)}$ by the functions $\lambda_r(\omega)$ with period 1/2. However, in the normalization of the vectors $\overrightarrow{b_r(\omega)}$, $r = \overline{1,k}$, for any $k \in \mathbb{N}$, the arbitrariness in $\lambda_s(\omega)$ remains only in the form of the factors $\mu_r(\omega)$ for even k and $e^{i\pi\omega}\mu_r(\omega)$ for odd k with 1/2-periodic functions $\mu_r(\omega)$ such that $|\mu_r(\omega)| = 1$. Note that the method makes it possible to construct for a given family of multiscaling functions a family of wavelet bases that is richer as compared to the classical case due to the mentioned arbitrariness in the choice of the vectors $\overrightarrow{a_s(\omega)}$, $s = 1, \dots, k-1$.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research (project no. 12-01-00004), by the Program for State Support of Leading Universities of the Russian Federation (agreement no. 02.A03.21.0006 of August 27, 2013), and by the Russian President's Grant for State Support of Leading Scientific Schools (project no. NSh-4538.2014.1).

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Translated by E. Vasil'eva