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The Structure of Finite Monoids Satisfying the Relation $\mathscr{R} = \mathscr{H}$

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Abstract—It is shown that any finite monoid S on which Green's relations $\mathscr{R}$ and $\mathscr{H}$ coincide divides the monoid of all upper triangular row-monomial matrices over a finite group. The proof is constructive; given the monoid S, the corresponding group and the order of matrices can be effectively found. The obtained result is used to identify the pseudovariety generated by all finite monoids satisfying $\mathscr{R} = \mathscr{H}$ with the semidirect product of the pseudovariety of all finite groups and the pseudovariety of all finite $\mathscr{R}$-trivial monoids.
Keywords: finite monoids, Green's relations, monoid representation, monoid pseudovariety, upper triangular matrices.

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INTRODUCTION

The study of monoids satisfying some constraints on Green's relations is important both for the general theory of semigroups and for the theory of formal languages. The classical result of the first type is Straubing's theorem [8], which states that a finite monoid is $\mathscr{J}$-trivial if and only if it divides the monoid of all reflexive binary relations on a set of n elements for some natural n or, equivalently, some monoid of monotone extensive transformations of a partially ordered set (moreover, a chain of n elements, as was specified by Pin [3]). Another classical result in this direction (see [3]) is the assertion that any finite $\mathscr{R}$-trivial monoid can be isomorphically embedded in the monoid $\mathscr{E}_n$ of all extensive transformations on the set $\{1, \dots, n\}$ for some n. The connection with formal languages is illustrated by well-known theorems: Schützenberger's theorem [5] on the correspondence between aperiodic languages and $\mathscr{H}$-trivial monoids and Simon's theorem [6] on $\mathscr{J}$-trivial monoids and piecewise testable languages.

Equality-type constraints and their applications were also studied quite intensively. For example, Volkov and Pastijn [2] characterized varieties whose semigroups satisfy one of the relations $\mathscr{D} = \mathscr{H}$, $\mathscr{D} = \mathscr{L}$, $\mathscr{D} = \mathscr{R}$, or $\mathscr{D} = \mathscr{J}$. The correspondence between a certain class of finite monoids satisfying the relation $\mathscr{D} = \mathscr{R}$ and finite prefix codes was shown in [1].

In what follows, we assume the finiteness of all objects. We study monoids on which $\mathscr{R} = \mathscr{H}$.

A finite group G with adjoined zero 0 is denoted by G^0. Let us call a matrix of order n over G^0 *row-monomial* if any of its rows contains exactly one nonzero element. For all elements $g \in G^0$, we set additionally $g + 0 = 0$. Denote by $TM_n(G)$ the monoid of all upper triangular row-monomial

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matrices of order n over G^0. In view of the additional condition, two monomial matrices from $TM_n(G)$ are multiplied by the usual rule of matrix multiplication. Recall that a monoid S *divides* a monoid T if S is a homomorphic image of some submonoid of T. The main result of this paper is the following theorem.

Theorem. *Any finite monoid satisfying the relation $\mathscr{R} = \mathscr{H}$ divides the monoid $TM_n(G)$ for an appropriate group G and an appropriate positive integer n.*

The proof of the theorem is constructive; given a finite monoid, the group G and the number n are effectively calculated.

We use the theorem to describe the pseudovariety **RH** generated by all possible finite monoids on which the relations $\mathscr{R}$ and $\mathscr{H}$ coincide in the form of the semidirect product of the pseudovariety of all finite groups **G** and the pseudovariety of all finite $\mathscr{R}$-trivial monoids **R**.

1. PRELIMINARIES

We adopt definitions and notation from [11]. A semigroup S with adjoined identity is denoted by S^1. Green's relations $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$, $\mathscr{D}$, and $\mathscr{H}$ on a semigroup S are defined by the formulas

$$a\mathscr{R}b \Leftrightarrow aS^1 = bS^1, \quad a\mathscr{L}b \Leftrightarrow S^1a = S^1b, \quad a\mathscr{J}b \Leftrightarrow S^1aS^1 = S^1bS^1, \quad \mathscr{D} = \mathscr{R} \vee \mathscr{L}, \quad \mathscr{H} = \mathscr{R} \cap \mathscr{L}.$$

In finite semigroups, the relations $\mathscr{D}$ and $\mathscr{J}$ coincide. It follows immediately from the definition of Green's relations that $a\mathscr{R}b$ if and only if there exist elements $c, d \in S^1$ such that $ac = b$ and $bd = a$, $a\mathscr{L}b$ if and only if there exist elements $c, d \in S^1$ such that $ca = b$ and $db = a$, and so on. We also define the following relation on S:

$$a \leq_{\mathscr{R}} b \Leftrightarrow aS^1 \subseteq bS^1.$$

A semigroup is called *$\mathscr{R}$-trivial* if $a\mathscr{R}b$ implies $a = b$. In this case, $\leq_{\mathscr{R}}$ is a partial order on S.

Let H be an arbitrary $\mathscr{H}$-class of a semigroup S. The set $St_r(H) = \{x \in S^1 \mid Hx = H\}$ is called the *right stabilizer of the class* H and forms a submonoid in S^1. On $St_r(H)$, we define a relation $\sim$, setting $x \sim y$ if and only if $hx = hy$ for some (and, hence, for any) $h \in H$. This relation is a congruence on $St_r(H)$. Denote by δ the canonical monoid homomorphism of $St_r(H)$ to $\Gamma_r(H) = St_r(H)/\sim$, and call $\Gamma_r(H)$ the transition monoid of the class H. The transition monoid $\Gamma_r(H)$ is a simply transitive permutation group on H. If $\mathscr{H}$-classes H_1 and H_2 are contained in the same $\mathscr{D}$-class, then the permutation groups $\Gamma_r(H_1)$ and $\Gamma_r(H_2)$ are isomorphic.

The abstract group $\Gamma_r(H)$ is called the *group of the $\mathscr{D}$-class D containing* H, or the *Schützenberger group of the class D*.

The right stabilizer $St_r(L)$ of an $\mathscr{L}$-class L is defined as $St_r(L) = \{x \in S^1 \mid Lx \subseteq L\}$ or, equivalently, $St_r(L) = \{x \in S^1 \mid Lx \cap L \neq \varnothing\}$. Then, $St_r(H)$ is a submonoid of the monoid $St_r(L)$. Similarly, we denote by $St_r(K) = \{x\colon Kx \subseteq K\}$ the right stabilizer of an arbitrary subset $K \subseteq S$.

The relation $\sim$ on $St_r(H)$ used to define $\Gamma_r(H)$ can be naturally continued to $St_r(L)$. More exactly, for $x, y \in St_r(L)$, we set $x \sim y$ if and only if $lx = ly$ for some (and, hence, for any) $l \in L$. As above, this relation is a congruence on $St_r(L)$. The canonical homomorphism from $St_r(L)$ to $St_r(L)/\sim$ is again denoted by δ, and the quotient $\Sigma_r(L) = St_r(L)/\sim$ is called the *transition monoid of the class* L. The Schützenberger group $\Gamma_r(H)$ of an arbitrary $\mathscr{H}$-class $H \subseteq L$ is the group of invertible elements in $\Sigma_r(L)$. According to [11, Proposition 3.7], the equality $\Sigma_r(L) = \Gamma_r(H)$ holds in finite semigroups.

2. AUXILIARY STATEMENTS

Proposition 1. *Any monoid $TM_n(G)$ satisfies the relation $\mathscr{R} = \mathscr{H}$.*

Proof. For an arbitrary matrix $a \in TM_n(G)$, we denote its elements by a_{ij}, $1 \le i \le n$, $1 \le j \le n$. The columns of the matrix a will be denoted by a_j, $1 \le j \le n$. Let us associate with the matrix a the following family of numbers $l(a,j)$, $1 \le j \le n$:

$$l(a,j) = \begin{cases} \max\{i \,|\, a_{ij} \neq 0\} & \text{if the column } a_j \text{ is nonzero,} \\ 0 & \text{if the column } a_j \text{ is zero.} \end{cases}$$

In other words, $l(a,j)$ is the index of the row that contains the lowest nonzero element in the column a_j, and $l(a,j)$ is 0 if the column a_j is zero.

Let us describe the relation $\mathscr{L}$ on $TM_n(G)$. We show that, for arbitrary matrices $a,b \in TM_n(G)$, the relation $a\mathscr{L}b$ is equivalent to the fact that $l(a,j) = l(b,j)$ for all $1 \le j \le n$.

Assume that $a\mathscr{L}b$. Then, there exist matrices $x, y \in TM_n(G)$ such that $xa = b$ and $yb = a$. Hence, if a column a_j is zero for some index j, then the column b_j is also zero, and vice versa. Therefore, $l(a,j) = 0$ if and only if $l(b,j) = 0$, and it remains to consider the case when columns a_j and b_j are nonzero. Arguing by contradiction, assume that there exists an index j, $i \le j \le n$, for which $l(a,j) < l(b,j)$. Define $p = l(a,j)$ and $q = l(b,j)$; then, $p < q$. Since $xa = b$, there exists a number k, $1 \le k \le n$, such that $x_{qk}a_{kj} = b_{qj}$, where $x_{qk}, a_{kj}, b_{qj} \in G$. Then, $q \le k$ since the matrices are upper triangular. Since $p < q$, we have $l(a,j) < k$, which contradicts the definition of $l(a,j)$. Thus, $l(a,j) \ge l(b,j)$. Similar arguments for the equality $yb = a$ yield the inequality $l(b,j) \ge l(a,j)$, which implies $l(a,j) = l(b,j)$, $1 \le j \le n$.

Let us show the converse implication. We have $l(a,j) = l(b,j)$, $1 \le j \le n$. Let us construct a matrix $x \in TM_n(G)$ with property $xa = b$; a matrix y with property $ya = b$ is constructed similarly. By the above conclusions for zero columns, assume that $b_{ij} \neq 0$. Then, $i \le l(b,j) = l(a,j)$; consequently, there exists an element $a_{kj} \neq 0$ and, moreover, $k > j$. Then, we assume $x_{ik} = b_{ij}a_{kj}^{-1}$, which is correct since $i \le k$ and b is a monomial matrix. As a result, we construct an upper triangular matrix x with property $xa = b$.

Now, we describe the relation $\mathscr{R}$. For $g \in G$, denote by $a_k g$ the column in which each element of a_k is multiplied by g on the right. Let us show that, for any matrices $a, b \in TM_n(G)$, the relation $a\mathscr{R}b$ is equivalent to the fact that, for any $1 \le k \le n$, there exists $g_k \in G$ such that $a_k g_k = b_k$ (here, elements g_k of the group G are indexed by the indices of the corresponding columns a_k).

Assume that $a\mathscr{R}b$. Then, there exist matrices $x, y \in TM_n(G)$ with properties $ax = b$ and $by = a$. Let us take an arbitrary element $a_{ik} \neq 0$; then, there exists an element $x_{kj} \neq 0$ such that $a_{ik}x_{kj} = b_{ij}$, $i \le k \le j$. Let us show that $k = j$. Assume by contradiction that $k < j$. Since $b_{ij} \neq 0$, we have $b_{ij}y_{js} = a_{is}$ for the corresponding nonzero element y_{js}. We also have $i \le k < j \le s$; hence, a_{is} is different from a_{ik}, which contradicts the monomiality of a. Consequently, $k = j$; i.e., $a_{ik} \neq 0$ implies $b_{ik} \neq 0$. The converse is also true. Thus, nonzero elements of the matrices a and b are arranged identically. In addition, the equality $a_{ik}x_{kk} = b_{ik}$ holds for all $1 \le i \le n$; i.e., $a_k x_{kk} = b_k$.

Conversely, let matrices $a, b \in TM_n(G)$ satisfy the necessary condition: for any $1 \le k \le n$, there exists $g_k \in G$ such that $a_k g_k = b_k$. Let us set $x_{kk} = g_k$ and thus define a matrix $x \in TM_n(G)$ with property $ax = b$. A matrix y with property $by = a$ is constructed similarly, and, as a result, we have $a\mathscr{R}b$.

It follows immediately from the description of the relation $\mathscr{R}$ that if $a\mathscr{R}b$ for matrices $a, b \in TM_n(G)$, then $l(a,k) = l(b,k)$ for all $1 \le k \le n$. Consequently, $a\mathscr{R}b$ implies $a\mathscr{L}b$, and $TM_n(G)$ satisfies the relation $\mathscr{R} = \mathscr{H}$. The proposition is proved. □

Here and elsewhere, S is a finite monoid on which the equality $\mathscr{R}=\mathscr{H}$ holds.

Proposition 2. *The smallest congruence on S containing the relation $\mathscr{R}$ is contained in $\mathscr{L}$.*

Proof. The congruence $\mathscr{R}^\sharp$ generated by the relation $\mathscr{R}$ can be obtained as follows: $a\mathscr{R}^\sharp b$ if and only if there exist $a_0, a_1, \dots a_n \in S$ such that $a=a_0$, $b=a_n$, and the equalities $a_i = s_i x_i t_i$ and $a_{i+1}=s_i y_i t_i$ hold for any $0\le i\le n-1$, where $x_i\mathscr{R}y_i$ and $s_i,t_i\in S$. In view of the relation $x_i\mathscr{R}y_i$, we obtain $s_ix_i\mathscr{R}s_iy_i$, since $\mathscr{R}$ is a left congruence. Then, $s_ix_i\mathscr{L}s_iy_i$, since $\mathscr{R}\subseteq\mathscr{L}$. Hence, $s_ix_it_i\mathscr{L}s_iy_it_i$, since $\mathscr{L}$ is a right congruence. Therefore, $a_i\mathscr{L}a_{i+1}$ for all $0\le i\le n-1$, which implies $a\mathscr{L}b$. As a result, we have $\mathscr{R}\subseteq\mathscr{R}^\sharp\subseteq\mathscr{L}$. The proposition is proved. □

In what follows, we preserve the notation $\mathscr{R}^\sharp$ for the congruence generated by $\mathscr{R}$. Let us now describe the relation $\mathscr{R}$ on the quotient $S/\mathscr{R}^\sharp$.

Proposition 3. *The quotient $S/\mathscr{R}^\sharp$ is $\mathscr{R}$-trivial.*

Proof. Denote Green's right relation on the monoid $S/\mathscr{R}^\sharp$ by $\bar{\mathscr{R}}$, and denote the $\mathscr{R}^\sharp$-class of an element $a\in S$ by $\bar a$. Let $\bar a\bar{\mathscr{R}}\bar b$ for some $a,b\in S$. Then, there exist elements $x,y\in S$ such that $\bar a=\bar b\bar x$ and $\bar b=\bar a\bar y$. In the monoid S, we then have $a\mathscr{R}^\sharp bx$ and $b\mathscr{R}^\sharp ay$. Hence, since $\mathscr{R}^\sharp$ is a congruence, we conclude that $a\mathscr{R}^\sharp ayx$. By Proposition 2, $\mathscr{R}^\sharp\subseteq\mathscr{L}$, which implies $a\mathscr{L}ayx$. Since, clearly, $ayx\le_{\mathscr{R}} a$, we can use Corollary 2.3.11 from [11] and obtain $a\mathscr{R}ayx$. Then, we have $a\mathscr{R}ay\mathscr{R}^\sharp b$, which implies $a\mathscr{R}^\sharp b$; i.e., $\bar a=\bar b$. The proposition is proved. □

By Proposition 3, the relation $\le_{\bar{\mathscr{R}}}$ is a partial order on $S/\mathscr{R}^\sharp$. In view of [3, Proposition 0.1], there exists a linear order $\le$ on $S/\mathscr{R}^\sharp$ such that, for $u,v\in S/\mathscr{R}^\sharp$, the relation $u\le_{\bar{\mathscr{R}}} v$ implies $v\le u$. Let us enumerate $\mathscr{R}^\sharp$-classes of the monoid S in ascending order in accordance with the order $\le$ and fix this enumeration. Obviously, the class containing the identity of the monoid S can be assigned index 1. Since $ux\le_{\bar{\mathscr{R}}} u$ for any elements $x,u\in S/\mathscr{R}^\sharp$, we have $u\le ux$; i.e., any element of $S/\mathscr{R}^\sharp$ corresponds to an extensive transformation on $S/\mathscr{R}^\sharp$ with respect to the order $\le$.

Let us note an important property of elements that stabilize a fixed $\mathscr{L}$-class of the monoid S. As mentioned above [11, Proposition 3.7], for finite semigroups, the transition monoid $\Sigma_r(L)$ of an arbitrary $\mathscr{L}$-class L coincides with the Schützenberger group $\Gamma_r(H)$ of an arbitrary $\mathscr{H}$-class $H\subseteq L$. In this case, if $x\in St_r(L)$, then $x\in St_r(H)$ for an arbitrary $\mathscr{H}$-class $H\subseteq L$. Indeed, assume that $H_1,H_2\subseteq L$ are some distinct $\mathscr{H}$-classes, and suppose that elements $h_1\in H_1$ and $h_2\in H_2$ are such that $h_1x=h_2$. Since $\Gamma_r(H)$ is a group, the action of the element x is invertible; i.e., there exists an element $y\in St_r(L)$ such that $h_2y=h_1$. Then, we have $h_1\mathscr{R}h_2$, which contradicts the condition $H_1\ne H_2$. Hence, $x\in St_r(H)$ for any $\mathscr{H}$-class $H\subseteq L$. Consequently, if $x\in St_r(L)$, then $x\in St_r(K)$ for any $\mathscr{R}^\sharp$-class K since any such class is a union of $\mathscr{H}$-classes.

Definition 1. Let us call classes K_i and K_j *neighboring with respect to an order $\le$ and the action of a class K* if $K_i\le K_j$; $K_iK\subseteq K_j$; and, for any $\mathscr{R}^\sharp$-classes P and Q such that $PQ\subseteq K$, exactly one of the two conditions holds:

(1) $K_iP\subseteq K_i$ and $K_iQ\subseteq K_j$;

(2) $K_iP\subseteq K_j$ and $K_jQ\subseteq K_j$.

Consider now $\mathscr{R}^\sharp$-classes K_i and K_j that are neighboring with respect to the order $\le$ and the action of a class K. We have $K_iK\subseteq K_j$. In the following proposition, we describe the action of any element of the class K from K_i to K_j. For this, let us fix an arbitrary element $a\in K$.

Proposition 4. *For any element $b\in K$, there exist $x\in St_r(K_i)$ and $y\in St_r(K_j)$ such that the action of b on the class K_i coincides with the action of xay on the class K_i.*

Proof. Fix an $\mathscr{H}$-class $H\subseteq K_i$ and consider a minimal by inclusion union of $\mathscr{H}$-classes $T_a=\cup_k H_k$ covering the set $Ha\subseteq K_j$. Let $T_b=\cup_l H_l$ be a similar union for the set Hb.

Let us prove the equality $T_a = T_b$. Since $a, b \in K$, there exists, as noted above, a sequence of elements $a_0, a_1 \ldots a_{n-1}, a_n \in S$ such that $a_0 = a$, $a_n = b$, $a_i = p_i x_i$, and $a_{i+1} = q_i x_i$, where $p_i \mathscr{R} q_i$ and $p_i, q_i \in S$.

Consider Ha_0 and $Ha_1 \subseteq K_j$. We have $Ha_0 = Hp_0x_0$ and $Ha_1 = Hq_0x_0$. Since the relation $p_0 \mathscr{R} q_0$ holds and K_i and K_j are neighboring classes with respect to the order $\le$ and the action of the class K, we have two possibilities:

(1) $K_ip_0, K_iq_0 \subseteq K_i$; then, $p_0, q_0 \in St_r(K)$, which implies $p_0, q_0 \in St_r(H)$. Consequently, $Ha_0 = Hp_0x_0 = Hx_0 = Hq_0x_0 = Ha_1$, and we obtain $T_{a_0} = T_{a_1}$.

(2) $K_ip_0, K_iq_0 \subseteq K_j$. Since $p_0 \mathscr{R} q_0$, we get $hp_0 \mathscr{R} hq_0$ for any $h \in H$; hence, $T_{p_0} = T_{q_0}$. Further, $x_0 \in St_r(K_j)$, which implies $T_{p_0x_0} = T_{p_0}x_0 = T_{q_0}x_0 = T_{q_0x_0}$. Again, we obtain $T_{a_0} = T_{a_1}$.

Now, arguing by induction, we obtain $T_a = T_b$, as required.

Take now some $h \in H$, and let $ha = s$ and $hb = t$. Since $T_a = T_b$, there exists an element $h_1 \in H$ with property $h_1 a \mathscr{H} t$ and, consequently, $h_1 a \mathscr{R} t$ since $\mathscr{R} = \mathscr{H}$. Then, by the transitivity of the corresponding Schützenberger groups, there exist an element $x \in St_r(K_i)$ such that $hx = h_1$ and an element $y \in St_r(K_j)$ such that $h_1ay = t$. Hence, $hb = hxay$. Let us show that the same property holds for arbitrary $h \in K_i$. Indeed, let $h_2 \in K_i$; then, there exists an element $l \in S$ such that $lh = h_2$ since $h \mathscr{L} h_2$. Then, $h_2b = lhb = lhxay = h_2xay$. Consequently, for the element $b \in K$, the elements $x \in St_r(K_i)$ and $y \in St_r(K_j)$ are constructed such that the equality $hb = hxay$ holds for arbitrary $h \in K_i$, as required. The proposition is proved. □

Let us formulate the result of Proposition 4 in terms of Schützenberger groups. For a fixed mapping between K_i and K_j implemented by an element $a \in K$, an element $b \in K$ is assigned a pair $(\delta(x), \delta(y)) \in \Gamma_r(H_i) \times \Gamma_r(H_j)$, where H_i and H_j are arbitrary $\mathscr{H}$-classes from K_i and K_j, respectively.

Note that, in general, a pair $(\delta(x), \delta(y))$ is defined ambiguously since it depends on the choice of an element $h_1 \in H$. Indeed, if $h_2 \in H$, $x_2 \in St_r(K_i)$, and $y_2 \in St_r(K_j)$ are other elements with properties $hx_2 = h_2$, $h_2 a \mathscr{R} t$, and $hx_2ay_2 = hb$, then the elements $\delta(x_1)$ and $\delta(x_2)$ of the group $\Gamma_r(H)$ are different, since they act differently on h. Similarly, we cannot guarantee the coincidence of $\delta(y_1)$ and $\delta(y_2)$. In the case $h_1a \ne h_2a$ and $h_1ay_1 = h_2ay_2$, the elements $\delta(y_1)$ and $\delta(y_2)$ are different since the Schützenberger group of an arbitrary $\mathscr{H}$-class is simply transitive.

3. PROOF OF THE THEOREM

Consider a direct product of groups $G_1 \times \ldots \times G_n$, where $G_i = \Gamma_r(H_i)$ for an arbitrary $\mathscr{H}$-class $H_i \subseteq K_i$ and $n = \mathrm{card}(S/\mathscr{R}^\sharp)$. The enumeration of $\mathscr{R}^\sharp$-classes $K_1 \ldots K_n$ was introduced above.

In Proposition 4, we described the action of an arbitrary element b from an $\mathscr{R}^\sharp$-class K in the case when b acts from K_i to K_j and the classes K_i and K_j are neighboring with respect to the action of the class K. In this case, for a fixed element $a \in K$, there exist elements $x \in St_r(K_i)$ and $y \in St_r(K_j)$ such that, for any element $h \in K_i$, we have $hb = hxay$. Now, we describe the action of an element $b \in K$ in the general case, when $\mathscr{R}^\sharp$-classes K_i and K_j are not necessarily neighboring with respect to the action of the class K.

Definition 2. Let an element b take K_i to K_j; i.e., let $K_ib = K_j$. Suppose that $b = b_1 \ldots b_p$ (where $b_1 \ldots b_p \in S$) is a decomposition of the element b and $K_{i_1}, \ldots, K_{i_{p+1}}$ (where $K_{i_1} = K_i$ and $K_{i_{p+1}} = K_j$) are $\mathscr{R}^\sharp$-classes such that:

(1) $K_{i_l} \ne K_{i_{l+1}}$ and $K_{i_1}b_1 \subseteq K_{i_2}, \ldots, K_{i_p}b_p \subseteq K_{i_{p+1}}$; the classes K_{i_l} and $K_{i_{l+1}}$ are neighboring with respect to b_l;

(2) for any other decomposition $b = c_1 \dots c_s$ of the element b (where $c_1 \dots c_s \in S$), the following condition holds: if $\{K_{i_1} \dots K_{i_p}\} \subseteq \{K_{j_1} \dots K_{j_s}\}$, then $\{K_{i_1} \dots K_{i_p}\} = \{K_{j_1} \dots K_{j_s}\}$.
Then, the decomposition $b = b_1 \dots b_p$ is called *dense*. Its existence is obvious since the partially ordered set of $\mathscr{R}^\sharp$-classes is finite in view of the finiteness of S.

Hence, by Proposition 4, each element b_l, $1 \le l \le p$, is assigned a pair $(g_{i_l}, g^*_{i_{l+1}}) \in G_{i_l} \times G_{i_{l+1}}$ specifying the action of the element b_l on the $\mathscr{R}^\sharp$-class K_{i_l}.

Let us assign to an element $b \in K$ an element $(f_1, \dots, f_n)$ of the group $G_1 \times \dots \times G_n$ as follows:

(1) $f_i = e_i$ (the identity of the group G_i) if $K_i \notin \{K_{i_1} \dots K_{i_p}\}$;

(2) $f_{i_1} = g_{i_1}$, $f_{i_{p+1}} = g^*_{i_{p+1}}$;

(3) $f_{i_l} = g^*_{i_{l-1}} g_{i_l}$ for $l = 2, \dots, p$.

As a result, the action of b from K_i to K_j corresponds to a family $f_{ij}(b)$ of elements of the group $G_1 \times \dots \times G_n$ constructed by all possible dense decompositions of b. Elements of this family constructed by the above scheme depend on the choice of $(g_{i_l}, g^*_{i_{l+1}}) \in (G_{i_l}, G_{i_{l+1}})$ for any element b_l from a specific decomposition and on the choice of a decomposition. If, for a pair (i, j), the class K does not take K_i to K_j, then we assume $f_{ij}(b) = \varnothing$.

Now, we assign to an element $b \in K$ a family $f(b)$ of upper triangular row-monomial matrices over the group $G_1 \times \dots \times G_n$. The arrangement of nonzero elements in each matrix is specified by the type of the matrix that corresponds to the $\mathscr{R}^\sharp$-class K as to an element of the $\mathscr{R}$-trivial monoid $S/\mathscr{R}^\sharp$ under its embedding into the monoid of partial extensive transformations. Hence, this arrangement is the same for all elements $b \in K$.

Now, we can completely specify a matrix $B \in f(b)$ by taking the actions of element b on any class K_i, $1 \le i \le n$, and fixing one dense decomposition for any such action. We define

$$B_{ij} = \begin{cases} (f_1, \dots, f_n) \in f_{ij}(b, & \text{if } f_{ij}(b) \text{ is nonempty,} \\ 0 & \text{if } f_{ij}(b) \text{ is empty;} \end{cases}$$

i.e., all matrices in $f(b)$ are obtained when nonzero elements B_{ij} of each matrix $B \in f(b)$ run independently over elements from the respective families $f_{ij}(b)$.

Consider the set $f(S) = \{f(z) \mid z \in S\}$. Define a mapping $\varphi\colon f(S) \to S$ as follows: $\varphi(Z) = z$ for any matrix $Z \in f(z)$.

Let us show that the definition is correct. Let $a, b \in S$ be arbitrary nonequal elements. Then, let us prove that $f(a) \cap f(b) = \varnothing$.

Consider the action of a and b on the $\mathscr{R}^\sharp$-class K_1 containing the identity e of the monoid S. Assume by contradiction that there exists a matrix $X \in f(a) \cap f(b)$. Then, we have two cases:

(1) Suppose that $a \in K_i$, $b \in K_j$, and $K_i \ne K_j$. Then, since $K_1 a \subseteq K_i$ and $K_1 b \subseteq K_j$, we have $X_{1i} \ne 0$ and $X_{1j} \ne 0$ for $i \ne j$, which contradicts the monomiality of X.

(2) Let $a, b \in K_i$, and let $X_{1i} = (f_1, \dots, f_n)$. In this case, the element X_{1i} corresponds to some dense decompositions $a = a_1 \dots a_p$ and $b = b_1 \dots b_p$ and to the chain of $\mathscr{R}^\sharp$-classes $K_1 = K_{i_1}, \dots, K_{i_{p+1}} = K_i$. Since the family of elements $\{s_{i_l}\}$ defining mappings from K_{i_l} in $K_{i_{l+1}}$, $1 \le l \le p$, was fixed, we have $a = ea = e f_{i_1} s_{i_1} f_{i_2} s_{i_2} \dots s_{i_p} f_{i_{p+1}} = eb = b$, which implies $a = b$. This contradicts the assumption (here, multiplication by f_{i_l} is understood as the action of an element of the corresponding Schützenberger group).

Thus, $f(a) \cap f(b) = \varnothing$, and the correctness is proved. It remains to show that $f(S)$ is a monoid and φ is a homomorphism. Obviously, φ is surjective by construction. It is sufficient to show that, if $A \in f(a)$ and $B \in f(b)$, then $AB \in f(ab)$.

Let $AB = C$. If $C_{ij} \neq 0$, then, in accordance with multiplication of monomial matrices, we have $A_{il}, B_{lj} \neq 0$ and $A_{il}B_{lj} = C_{ij}$ for some l. The element $A_{ik} \in f_{ik}(a)$ of the matrix A corresponds to some dense decomposition $a_1 \dots a_p$ and the family of classes $\{K_{i_1}, \dots, K_{i_p}\}$ under the action $K_i a$. Define $A_{ik} = (f_1(a), \dots, f_n(a))$. Similarly, the element $B_{kj} \in f_{kj}(b)$ of the matrix B corresponds to some dense decomposition $b = b_1 \dots b_s$ and the family of classes $\{K_{i_p} \dots K_{i_{p+s-1}}\}$.

Let $B_{kj} = (f_1(b) \dots f_n(b))$. We have $C_{ij} = (f_1(a)f_1(b), \dots, f_n(a)f_n(b))$. Then,

$$f_i(a)f_i(b) = \begin{cases} e_i e_i = e_i & \text{for } i \notin \{i_1 \dots i_{p+s-1}\}, \\ f_i(a)e_i = f_i(a) & \text{for } i \in \{i_1 \dots i_{p-1}\}, \\ f_{i_p}(a)f_{i_p}(b) & \text{for } i = i_p, \\ e_i f_i(b) = f_i(b) & \text{for } i \notin \{i_{p+1} \dots i_{p+s-1}\}. \end{cases}$$

On the other hand, we have $K_i a \subseteq K_l$ and $K_l b \subseteq K_j$. Hence, $K_i c = K_i ab \subseteq K_j$. Now, note that, for $c = ab$, the decomposition $c = a_1 \dots a_p b_1 \dots b_s$ is dense since, otherwise, $a = a_1 \dots a_p$ or $b = b_1 \dots b_s$ could be densed, which is impossible. Under the action $K_i c \subseteq K_j$, this decomposition corresponds exactly to the element C_{ij}.

Since the argument was given for arbitrary $C_{ij} \neq 0$, we conclude that $C \in f(c) = f(ab)$. Consequently, the set $f(S) = \{f(z) \mid z \in S\}$ forms a submonoid in $TM_n(G)$, since the identity of the monoid S corresponds to the identity matrix E_n. The mapping $\varphi\colon f(S) \to S$ defined by the rule $\varphi(Z) = z$ for any matrix $Z \in f(z)$ is a homomorphism of $f(S)$ on S. Thus, S divides $TM_n(G)$. The theorem is proved. □

4. EXAMPLE

In [10], a series of semigroups S_r satisfying the relation $\mathscr{L} = \mathscr{H}$ was constructed. Let us present a similar construction for semigroups satisfying $\mathscr{R} = \mathscr{H}$ and apply to them our results. These semigroup will also be denoted by S_r.

Let p be a fixed prime. The generating elements $g, f_1, \dots, f_r$ of the semigroup S_r satisfy the relations

$$g^{p+1} = g; \quad f_1 g = f_1 g^p; \quad f_1 g f_r = 0;$$

$$f_i{}^2 = f_i \ \text{ for all } i = 1, \dots, r;$$

$$f_j f_i = 0 \ \text{ for all } i \neq j, j-1;$$

$$f_i g = 0 \ \text{ for } i = 2, \dots, r; \quad g f_i = 0 \ \text{ for } i = 1, \dots, r-1.$$

It is easy to calculate [10] that S_r consists of the following $\dfrac{r(r+3)}{2} + (r+2)p + 1$ elements:

$$0, g, g^2, \dots, g^p = e;$$

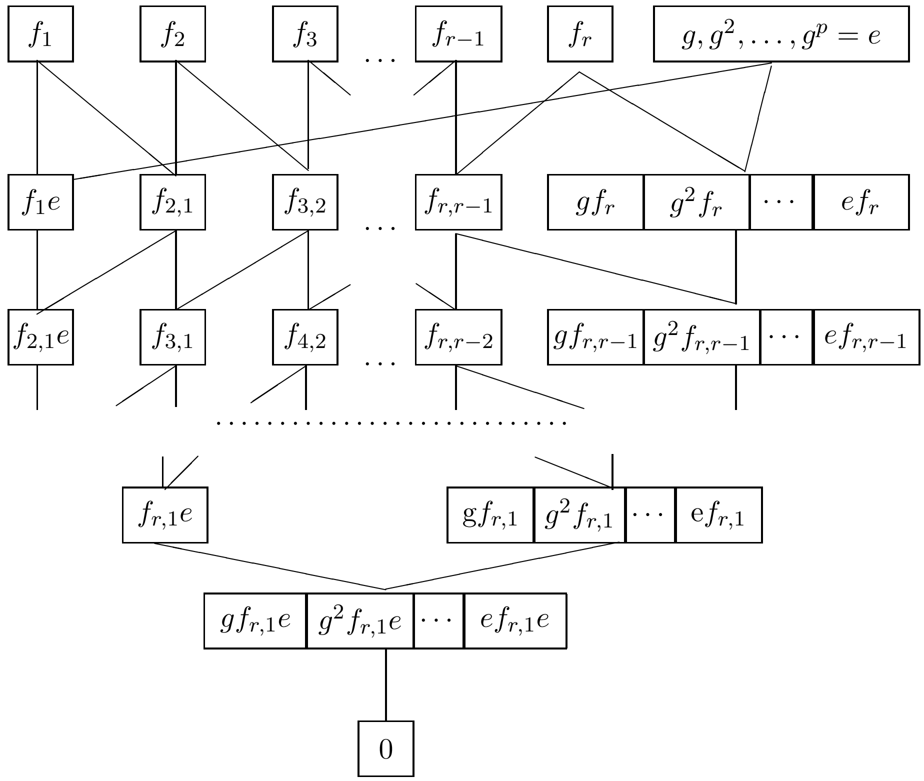
$$f_j f_{j-1} \cdots f_i = f_{j,i}, \quad \text{where } 1 \leq i \leq j \leq r;$$

$$f_{j,1}e, \ g^m f_{r,i}, \quad \text{where } 1 \leq i, j \leq r, \quad 1 \leq m \leq p;$$

$$g^m f_{r,1} e, \quad \text{where } 1 \leq m \leq p,$$

and nonzero products in S_r are exhausted by the following elements:

$$g^m \cdot g^n = g^{m+n \,(\mathrm{mod}\ p)};$$



The $\mathscr{J}$-structure of the semigroup S_r.

$$f_{k,j} \cdot f_{j,i} = f_{k,j+1} f_{j,i} = f_{k,i};$$

$$f_{j,1} \cdot g^m = f_{j,1}e \cdot g^m = f_{j,1}e;$$

$$g^m \cdot f_{r,i} = g^m f_{r,i}, \quad g^n \cdot g^m f_{r,i} = g^{m+n(\text{mod } p)} f_{i,r};$$

$$g^n f_{r,1} \cdot g^m = g^n f_{r,1}e \cdot g^m = g^n f_{r,1}e;$$

$$g^m \cdot f_{r,1}e = g^m f_{r,1}e, \quad g^n \cdot g^m f_{r,1}e = g^{m+n(\text{mod } p) f_{r,1}e}.$$

The $\mathscr{J}$-structure of the semigroup S_r is shown in figure. The cells denote $\mathscr{H}$-classes. Horizontally adjacent cells standardly denote $\mathscr{H}$-classes of the same $\mathscr{R}$-class. However, for the compactness of the figure, we use this notation for $\mathscr{H}$-classes of the same $\mathscr{L}$-class (or, equivalently, the same $\mathscr{J}$-class).

The $\mathscr{H}$-class of the element g is denoted by H_g. Let us construct the quotient $S_r/\mathscr{R}^\sharp$. Obviously, one-element $\mathscr{L}$-classes and H_g are $\mathscr{R}^\sharp$-classes. Since $\mathscr{L}$-classes of the form $\{g^i f_r\}_{i=1}^p$ and $\{g^i f_{r,r-1}\}_{i=1}^p \dots \{g^i f_{r,1}\}_{i=1}^p$ are the products of $\mathscr{R}$-classes $H_g\{f_r\}, \dots, H_g\{f_{r,1}\}$, respectively, all these $\mathscr{L}$-classes are also $\mathscr{R}^\sharp$-classes. The $\mathscr{L}$-class $\{g^i f_{r,1}e\}_{i=1}^p$ is the product of $\mathscr{R}$-classes $H_g\{f_{r,1}e\}$; hence, it is also an $\mathscr{R}^\sharp$-class.

As a result, any $\mathscr{L}$-class is an $\mathscr{R}^\sharp$-class. The class H_g, without loss of generality, is assigned index 1. Obviously, $\Gamma_r(H_g) \cong H_g$, and the Schützenberger groups of the remaining $\mathscr{H}$-classes are identity groups. Therefore, any element acting on an $\mathscr{R}^\sharp$-class different from H_g corresponds to the element $(e_1, \dots, e_n)$ independently of a decomposition, where $e_1 \dots e_n$ are the identities of respective Schützenberger groups and $n = \text{card}(S/\mathscr{R}^\sharp)$.

In the action on H_g, only the $\mathscr{R}^\sharp$-classes $\{g^i f_r\}_{i=1}^p, \dots, \{g^i f_{r,1}\}_{i=1}^p$ and H_g do not take it to 0. Let us take multiplication by f_r (i.e., $g^i \to g^i f_r$) as a fixed mapping from H_g to $\{g^i f_r\}$. Obviously, a unique dense decomposition of any element of the form $g^i x$ for $x \in \{f_{r,r-1}, \dots, f_{r,1}\}$ is $g^i f_r f_{r-1} \dots f_k$ for corresponding k. Each of the elements $g^i f_r$ is already represented in the form of its unique dense decomposition. Each of the elements $g^i f_{r,1} e$ has a dense decomposition of the form $\{g^i f_r f_{r-1} \dots f_1 g^l\}_{l=1}^p$, but the actions of all g^l on $\{g^i f_r f_{r-1} \dots f_1\}$ coincide.

Thus, elements of the form $g^i x$, where $x \in \{f_r, f_{r,r-1}, \dots, f_{r,1}, f_{r,1}e\}$, always correspond to the element $(g^i, e_2 \dots e_n)$, and we find that any $g^i x$ is assigned a unique matrix. The remaining elements form one-element $\mathscr{R}^\sharp$-classes, and, as shown above, each of them is also assigned a unique matrix with nonzero elements equal to $(e_1, \dots, e_n)$.

We obtain an embedding of S_r to $TM_n(G)$, where $G \cong H_g \times \{e_2\} \times \dots \times \{e_n\} \cong H_g$. Hence, we can identify $(e_1, \dots, e_n)$ with the identity $e_1 = e$ of H_g, identify the elements $(g^i, e_2 \dots, e_n)$ with g^i, and write e and g^i in the corresponding cells of matrices. All nonzero elements of matrices of the obtained subsemigroup, thus, are equal to e, except for the elements A_{1k} of matrices corresponding to the elements of the form $g^i x$, $x \in \{f_r, f_{r,r-1}, \dots, f_{r,1}\}$ of the semigroup S_r.

5. THE CONSTRUCTION OF THE PSEUDOVARIETY **RH**

Let us present several necessary definitions (they can be found, for example, in [9]).

Suppose that V and W are finite monoids, $v, v_1, v_2 \in V$, and $w, w_1, w_2 \in W$. The *semidirect product* $V * W$ is the set $V \times W$, where W acts on V on the left according to the rules $w(v_1 v_2) = w(v_1)w(v_2)$ and $w_1(w_2(v)) = (w_1 w_2)(v)$, and the product of pairs is defined as follows:

$$(v_1, w_1)(v_1, w_2) = (v_1 w_1(v_2), w_1 w_2).$$

The *wreath product* $V \circ W$ is the set $V^W \times W$ equipped with the product

$$(v_1, w_1)(v_2, w_2) = (s, w_1 w_2), \qquad s\colon W \to V, \quad s(w) = v_1(w) v_2(w w_1).$$

A *monoid pseudovariety* is a class of monoids closed under taking submonoids, homomorphic images, and finite direct products.

Let $\mathbf{V}$ and $\mathbf{W}$ be monoid pseudovarieties. Denote by $\mathbf{V} * \mathbf{W}$ the *semidirect product* of $\mathbf{V}$ and $\mathbf{W}$. A monoid S belongs to $\mathbf{V} * \mathbf{W}$ if S divides some semidirect product $V * W$ for some $V \in \mathbf{V}$ and $W \in \mathbf{W}$. Let us define also the *wreath product* $\mathbf{V} \circ \mathbf{W}$ of pseudovarieties $\mathbf{V}$ and $\mathbf{W}$. A monoid S belongs to $\mathbf{V} \circ \mathbf{W}$ if S divides the wreath product $V \circ W$ for some $V \in \mathbf{V}$ and $W \in \mathbf{W}$.

It is easy to show that the class of all monoids satisfying $\mathscr{R} = \mathscr{H}$ does not form a pseudovariety; i.e., the statement converse to the theorem does not hold. There exist monoids dividing $TM_n(G)$ or even submonoids of $TM_n(G)$ that do not satisfy the relation $\mathscr{R} = \mathscr{H}$. Consider the following example. Let H be a group consisting of the identity 1 and an involution q. Let T be the submonoid of the monoid $TM_3(H)$ generated by the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = BA = \begin{pmatrix} 0 & q & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

For these matrices, we also have $B = CA$, which implies $B \mathscr{R} C$ in T. However, the equation $XB = C$ is not solvable in T since, otherwise, the element X_{11} should necessary be equal to q, which, obviously, does not hold for any matrix in T. Thus, $B\mathscr{R}C$ does not imply $B\mathscr{L}C$; consequently, the relations $\mathscr{R}$ and $\mathscr{H}$ do not coincide in T.

Let us describe the pseudovariety $\mathbf{RH}$ generated by all possible finite monoids satisfying $\mathscr{R} = \mathscr{H}$ in terms of the semidirect product of the pseudovariety of all finite groups $\mathbf{G}$ and the pseudovariety of all finite $\mathscr{R}$-trivial monoids $\mathbf{R}$.

Proposition 5. $\mathbf{RH} = \mathbf{G} * \mathbf{R}$.

Proof. It is well known that the monoid $TM_n(G)$ is a special semidirect product of the direct degree G^n and the monoid $\mathscr{E}_n$ of all extensive transformations on the set $\{1, \dots, n\}$. For the convenience of further reasoning, let us recall this construction in detail. Any matrix z from $TM_n(G)$ can be represented as a pair (g, r), where $g = (g_1, \dots, g_n) \in G^n$ with elements g_i indexed by rows of the matrix and $r \in \mathscr{E}_n$ is a transformation of the set $\{1, \dots, n\}$ represented by the matrix over the set $\{0, 1\}$ with the same arrangement of nonzero elements as in the matrix z. According to matrix multiplication, the product of pairs is defined as follows:

$$(g, r_1)(h, r_2) = (s, r_1 r_2), \qquad s_k = g_k h_{r_1(k)}, \quad 1 \le k \le n,$$

which corresponds to the operation of semidirect product. Thus, since any monoid satisfying the equality $\mathscr{R} = \mathscr{H}$ divides $TM_n(G) = G^n * \mathscr{E}_n$, we have $\mathbf{RH} \subseteq \mathbf{G} * \mathbf{R}$.

Let us consider now the wreath product of pseudovarieties $\mathbf{G} \circ \mathbf{R}$. Suppose that the monoid S belongs to $\mathbf{G} \circ \mathbf{R}$; i.e., S divides the wreath product $G \circ T$ for some $G \in \mathbf{G}$ and $T \in \mathbf{R}$. Recall that the wreath product $G \circ T$ is the set $G^T \times T$ equipped with the product

$$(g, t_1)(h, t_2) = (s, t_1 t_2), \qquad s\colon T \to G, \quad s(t) = g(t)h(tt_1).$$

If we now consider the monoid T as an index set ordered by a linear order $\le$ such that $t_1 \le_{\mathscr{R}} t_2$ implies $t_2 \le t_1$ in T, then multiplication by t_1 corresponds to an extensive transformation on this set since $tt_1 \le_{\mathscr{R}} t$. Hence, the monoid $G \circ T$ is isomorphic to the monoid $G^{|T|} * T$ with the operation of semidirect product defined above. Thus, $G \circ T$ is a submonoid in $G^{|T|} * \mathscr{E}_{|T|}$. Let us fix the chosen indexation of elements of the monoid T.

Let us show that the relations $\mathscr{R}$ and $\mathscr{H}$ coincide in any monoid $G \circ T$. Let $a, b \in G \circ T$. If $a\mathscr{R}b$ in $G \circ T$, then $a\mathscr{R}b$ in $TM_{|T|}(G) = G^{|T|} * \mathscr{E}_{|T|}$ as well; i.e., a and b, at least, have the same arrangements of nonzero elements by Proposition 1.

Since the identity e of the monoid T induces the identical transformation on it, the submonoid $G^T \times e$ of the monoid $G \circ T$ consists of diagonal matrices with all possible arrangements of elements of G on the diagonal. Based on this, we define diagonal matrices x and y from $G^T \times e$ by the rule

$$x_{ii} = b_{ij} a_{ij}^{-1}, \quad 1 \le j \le |T|; \qquad y_{ii} = a_{ij} b_{ij}^{-1}, \quad 1 \le j \le |T|.$$

Then, $xa = b$ and $yb = a$, which implies $a\mathscr{L}b$. Thus, $a\mathscr{R}b$ implies $a\mathscr{L}b$ in $G \circ T$, as required. Therefore, $\mathbf{G} \circ \mathbf{R} \subseteq \mathbf{RH}$.

As Tilson showed in [9, Appendix A] for pseudovarieties of monoids, the equality $\mathbf{V} * \mathbf{W} = \mathbf{V} \circ \mathbf{W}$ holds if $\mathbf{W}$ is not an extended pseudovariety of groups. Since $\mathbf{R}$ is not such and $\mathbf{G}$ is a monoid pseudovariety, we conclude that $\mathbf{RH} \subseteq \mathbf{G} * \mathbf{R} = \mathbf{G} \circ \mathbf{R} \subseteq \mathbf{RH}$, i.e., $\mathbf{RH} = \mathbf{G} * \mathbf{R}$. The proposition is proved. □

Proposition 5 and the proof of the theorem imply the following statement.

Corollary. *If all Schützenberger groups of a monoid S satisfying the equality $\mathscr{R} = \mathscr{H}$ belong to a group pseudovariety $\mathbf{H}$, then $S \in \mathbf{H} * \mathbf{R}$.*

Remark. Note that the obtained representation of the pseudovariety **RH** correlates with Stiffler's results [7]. According to them, $\mathbf{G} * \mathbf{R} \subseteq \mathbf{R} * \mathbf{G} = \mathbf{ER}$, where **ER** denotes the pseudovariety consisting of monoids whose idempotents generate an $\mathscr{R}$-trivial submonoid. It is easy to see that each of the monoids $TM_n(G)$ satisfies this property. As follows from the results of Section 2, idempotents in $TM_n(G)$ are exactly matrices whose nonzero columns contain the identity of the group on the main diagonal. Multiplying idempotents, we still find that nonzero diagonal elements are the identity. Let M be the submonoid of $TM_n(G)$ generated by all its idempotents, and let $a, b \in M$. If $a\mathscr{R}b$ in M, then there exists an element $x \in M$ such that $ax = b$. In addition to this, the relation $a\mathscr{R}b$ also holds in $TM_n(G)$, and, according to the proof of Proposition 1, for any index k, $1 \le k \le n$, we obtain the equality $a_k x_{kk} = b_k$ for the corresponding columns a_k and b_k of the matrices a and b. Since all elements x_{kk} of the matrix x are equal either to the identity or to zero, we have $a = b$, and the submonoid M is $\mathscr{R}$-trivial.

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