ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2014, Vol. 286, pp. 253–267. © Pleiades Publishing, Ltd., 2014.

On the Holomorphic Torus-Bott Tower of Aspherical Manifolds

Yoshinobu Kamishima[a] **and Mayumi Nakayama**[a]

Received February 2013

Abstract—We introduce a notion of *holomorphic torus-Bott tower* which is an iterated holomorphic Seifert fiber space with fiber a complex torus. This is thought of as a holomorphic version of a *real Bott tower*. The top space of the holomorphic torus-Bott tower is called a holomorphic torus-Bott manifold. We discuss the structure of holomorphic torus-Bott manifolds and particularly the *holomorphic rigidity* of holomorphic torus-Bott manifolds.

DOI: 10.1134/S0081543814060157

1. INTRODUCTION

A holomorphic torus-Bott tower is a sequence of holomorphic Seifert fiber bundles by a complex torus fiber $T^1_{\mathbb{C}}$:

$$M = M_n \to M_{n-1} \to \ldots \to M_1 \to \{\mathrm{pt}\}. \tag{1.1}$$

The top space M of the tower (1.1) is said to be a *holomorphic torus-Bott manifold* of dimension $2n$ (see Definition 2.1 below for more details). Inductively from (1.1), M turns out to be a closed aspherical manifold. Then it is shown that the fundamental group Γ of M is virtually nilpotent. Let $\mathrm{E}(N) = N \rtimes K$ be the semidirect product of a simply connected nilpotent Lie group N with a compact group K in which K is a maximal compact group of the automorphism group $\mathrm{Aut}(N)$. When we forget a complex structure on M, it is proved that M is *diffeomorphic* to an infranil-manifold $N/\rho(\Gamma)$ where $\rho\colon \Gamma \to \mathrm{E}(N)$ is a discrete faithful representation. In particular, *Seifert rigidity* implies that two holomorphic torus-Bott manifolds with isomorphic fundamental groups are *diffeomorphic*.

In this paper we are interested in a *holomorphic version* of structure and rigidity for holomorphic torus-Bott manifolds.

By a *holomorphic nilmanifold* we shall mean a complex nilmanifold with left invariant complex structure. Refer to [17] for the recent results of deformation of left invariant nilpotent Lie algebras. On the other hand, denote by $T^k_{\mathbb{C}}$ a complex k-dimensional torus. Recall the structure theorem from S. Murakami's classical result [15].

Theorem. *Let $T^1_{\mathbb{C}} \to Y \to T^k_{\mathbb{C}}$ be a principal holomorphic torus bundle. Then Y is biholomorphic to a holomorphic nilmanifold N/Δ where N is a two-step nilpotent Lie group with left invariant complex structure containing a discrete uniform subgroup Δ.*

To study the holomorphic rigidity of our holomorphic torus-Bott manifolds, we need to generalize this result to the case of holomorphic torus bundles (orbibundles) over holomorphic infranil-manifolds (infranil-orbifolds).

We refer to [14, 3] for a *holomorphic Seifert fibration*. We shall prove the following Theorem 6.1:

Let M be a $2n$-dimensional holomorphic torus-Bott manifold which is a holomorphic fiber bundle over $\widehat{M}$ with fiber $T^1_{\mathbb{C}}$. Then M is biholomorphic to a holomorphic infranil-manifold N/Γ in which

[a] Department of Mathematics and Information Sciences, Tokyo Metropolitan University, Minami-Osawa 1-1, Hachioji, Tokyo 192-0397, Japan.

E-mail addresses: kami@tmu.ac.jp (Y. Kamishima), nakayama-mayumi@ed.tmu.ac.jp (M. Nakayama).

N/Γ has a holomorphic Seifert fibration $T^1_{\mathbb{C}} \to N/\Gamma \to \widehat{N}/\widehat{\Gamma}$ such that $\widehat{M}$ is biholomorphic to a holomorphic infranil-manifold $\widehat{N}/\widehat{\Gamma}$.

The proof of this theorem is organized as follows: As the fundamental group of M is virtually nilpotent, the smooth classification implies that M is diffeomorphic to an infranil-manifold N/Γ. Even if N/Γ supports a complex structure, it does not follow that M is *biholomorphic* to N/Γ. However, N has a central extension $1 \to \mathbb{C} \to N \to \widehat{N} \to 1$ in this case. Assume inductively that $\widehat{M}$ is *biholomorphic* to a holomorphic infranil-manifold $\widehat{N}/\widehat{\Gamma}$. Then we can find a nilpotent Lie group N' isomorphic to N. The group N' admits an $\mathrm{E}(N')$-invariant complex structure J for which the central extension $1 \to \mathbb{C} \to N' \to \widehat{N} \to 1$ becomes a principal holomorphic bundle. Moreover, N' is biholomorphic to the complex space $\mathbb{C}^n$; indeed, this fact is due to Oka's principle that the universal covering (N', J) is biholomorphic as a principal holomorphic bundle to the product $(\mathbb{C} \times \widehat{N}, J_0 \times \widehat{J})$ inductively. Speculating on the cohomology exact sequence induced from a short exact sequence $1 \to \mathbb{Z}^2 \xrightarrow{i} \mathbb{C} \xrightarrow{j} T^1_{\mathbb{C}} \to 1$,

$$\ldots \to H^1_\phi(\widehat{\Gamma}; \mathrm{hol}(\widehat{N}, \mathbb{C})) \xrightarrow{j} H^1_\phi(\widehat{\Gamma}; \mathrm{hol}(\widehat{N}, T^1_{\mathbb{C}})) \xrightarrow{\delta} H^2_\phi(\widehat{\Gamma}; \mathbb{Z}^2) \to \ldots,$$

we can show that M is biholomorphic to a holomorphic infranil-manifold N'/Γ' where $\Gamma' \leq \mathrm{E}_J(N')$ which is the semidirect product $N' \rtimes K'$ invariant under the complex structure J. There we construct a deformation N'/Γ' of N/Γ (see Theorem 5.1 below). Of course, N'/Γ' is nothing but N/Γ topologically.

The paper consists of the following sections. In Section 2 we introduce a notion of holomorphic torus-Bott tower and prove some topological results. We construct complex structures on holomorphic infranil-manifolds in Section 3. We study holomorphic infranil actions and holomorphic Seifert actions in Section 4. In Section 5, we prove the following Theorem 5.1, which is a key tool to prove Theorem 6.1. Suppose that there is an equivariant holomorphic Seifert action $(\mathbb{Z}^2, \mathbb{C}) \to (\Gamma, N) \to (\widehat{\Gamma}, \widehat{N})$ such that $\widehat{N}/\widehat{\Gamma}$ is an infranil-manifold.

Let (N, Γ) be a holomorphic Seifert action. Then there exists a nilpotent Lie group N' and a discrete subgroup $\Gamma' \leq \mathrm{E}_J(N')$ for which the quotient N/Γ is biholomorphic to the holomorphic infranil-manifold N'/Γ'.

In Section 6 we prove the above Theorem 6.1. As an application, each holomorphic fiber bundle $T^1_{\mathbb{C}} \to M_i \to M_{i-1}$ of (1.1) gives rise to a group extension of the fundamental groups: $1 \to \mathbb{Z}^2 \to \pi_i \to \pi_{i-1} \to 1$. This group extension represents a cocycle in $H^2_\phi(\pi_{i-1}; \mathbb{Z}^2)$. A holomorphic torus-Bott manifold is said to be of *finite type* if each cocycle has *finite order*; otherwise it is said to be of *infinite type* (cf. Definition 7.1). In Section 7, we apply Theorem 6.1 to show the following Theorem 7.2:

A holomorphic torus-Bott manifold M of finite type is biholomorphic to a complex euclidean space form $\mathbb{C}^n/\Gamma$ with holonomy group $L(\Gamma)$ lying in $\prod_{i=1}^n H_i$ where H_i is either $\{1\}$, $\mathbb{Z}_2$, $\mathbb{Z}_4$ or $\mathbb{Z}_6$.

An example of finite type is a *Kähler Bott tower*, i.e. each M_i is a Kähler manifold such that $T^1_{\mathbb{C}} \to M_i \to M_{i-1}$ is a Kähler submersion (see Subsection 7.2). It is shown in Theorem 7.5 that every Kähler Bott manifold M is biholomorphic to a complex euclidean space form $\mathbb{C}^n/\Gamma$ of Theorem 7.2. In Section 8 we study holomorphic torus-Bott manifolds of infinite type. As the fundamental group of such a manifold is virtually nilpotent (but not virtually abelian), it is a non-Kähler manifold. It would be difficult to obtain a *holomorphic classification* of holomorphic torus-Bott manifolds of *infinite type*. We shall consider what non-Kähler geometric structures exist on holomorphic torus-Bott manifolds of infinite type. In Theorem 8.4, we provide two classes of geometric structures:

(i) a $(2n+2)$-dimensional *locally homogeneous locally conformal Kähler* manifold $M = \mathbb{R} \times \mathcal{N}/\Gamma$ where $\mathcal{N}$ is the Heisenberg nilpotent Lie group and $\Gamma \leq \mathbb{R} \times (\mathcal{N} \rtimes \mathrm{U}(n))$ is a discrete uniform subgroup;

(ii) a complex $(2n+1)$-dimensional locally homogeneous *complex contact* manifold $\mathcal{L}/\Gamma$ where $\mathcal{L} = \mathcal{L}_{2n+1}$ is a complex $(2n+1)$-dimensional complex nilpotent Lie group and Γ is a discrete uniform subgroup of $\mathcal{L} \rtimes (\mathrm{Sp}(n) \cdot S^1)$.

In particular, $\mathcal{L}_3$ is the Iwasawa nilpotent Lie group.

2. HOLOMORPHIC TORUS-BOTT TOWER

Suppose that there is a tower of fiber bundles (1.1),

$$M = M_n \to M_{n-1} \to \ldots \to M_1 \to \{\mathrm{pt}\}.$$

Each (M_m, J_m) is a complex manifold such that

$$T^1_{\mathbb{C}} \to M_m \to M_{m-1} \tag{2.1}$$

is a holomorphic fiber bundle $(m = 1, \ldots, n)$ which induces a group extension

$$1 \to \mathbb{Z}^2 \to \pi_m \to \pi_{m-1} \to 1. \tag{2.2}$$

For $m = 1$, $M_1 = T^1_{\mathbb{C}}$ with $\pi_1 = \mathbb{Z}^2$. Let (X_m, J_m) be the universal covering space of M_m $(m = 1, \ldots, n)$ such that $X_1 = \mathbb{C}$.

Definition 2.1. The *holomorphic torus-Bott tower* is a tower (1.1) which satisfies the following conditions:

(1) There is an equivariant holomorphic principal bundle

$$(\mathbb{Z}^2, \mathbb{C}) \to (\pi_m, X_m, J_m) \xrightarrow{p_m} (\pi_{m-1}, X_{m-1}, J_{m-1}) \tag{2.3}$$

associated with the group extension (2.2).

(2) Each π_m normalizes the holomorphic action of $\mathbb{C}$.

We call the top space $M\,(= M_n)$ a *holomorphic torus-Bott manifold* (*of depth* n).

There are several remarks. Condition (2) for m is equivalent to say that $T^1_{\mathbb{C}} \to M_m \to M_{m-1}$ is a Seifert fiber space in the smooth case. It is not necessarily true that the universal covering X_m is biholomorphic to the product $\mathbb{C} \times X_{m-1}$. So, contrary to the smooth case, holomorphic Seifert actions are not described explicitly on the product $\mathbb{C} \times X_{m-1}$ in general. However, our holomorphic Seifert actions on the universal covering of a holomorphic torus-Bott manifold can be described. In fact, let $(X, J)\,(= (X_n, J_n))$ be the universal covering of a holomorphic torus-Bott manifold $M = M_n$. Put $(X_{n-1}, J_{n-1}) = (\widehat{X}, \widehat{J})$.

Proposition 2.2. *(X, J) is biholomorphic as a holomorphic principal bundle to the product $(\mathbb{C} \times \widehat{X}, J_0 \times \widehat{J})$.*

Proof. By Definition 2.1, $X_1 = \mathbb{C}$. We assume inductively that $\widehat{X} = X_{n-1}$ is biholomorphic to $\mathbb{C}^{n-1}$. By condition (2) of Definition 2.1, $\mathbb{C} \to X \to \widehat{X}$ is a holomorphic principal bundle. When A_{h} is the sheaf of germs of (local) holomorphic functions on $\widehat{X}$, Oka's principle says that $H^1(\widehat{X}, A_{\mathrm{h}}) = 0$ (see [9, pp. 167–168]). Thus (X, J) is holomorphically *bundle isomorphic* to the product $(\mathbb{C} \times \widehat{X}, J_0 \times \widehat{J})$. □

2.1. Holomorphic Seifert action. As a consequence of Proposition 2.2, the holomorphic action of π on (X, J) is a holomorphic action of π on $(\mathbb{C} \times \widehat{X}, J_0 \times \widehat{J})$. Assume that $(\widehat{\pi}, \widehat{X}, \widehat{J})$ is a holomorphic action. Let $(\mathbb{Z}^2, \mathbb{C}) \to (\pi, \mathbb{C} \times \widehat{X}, J) \xrightarrow{p} (\widehat{\pi}, \widehat{X}, \widehat{J})$ be an equivariant holomorphic principal bundle as in condition (1) of Definition 2.1.

- The group extension $1 \to \mathbb{Z}^2 \to \pi \to \widehat{\pi} \to 1$ represents a cocycle $f\colon \widehat{\pi} \times \widehat{\pi} \to \mathbb{Z}^2$ such that each element $\gamma \in \pi$ is viewed as $(n, \alpha) \in \mathbb{Z}^2 \times \widehat{\pi}$ with the group law

$$(n,\alpha)(m,\beta) = \big(n + \phi(\alpha)(m) + f(\alpha,\beta),\ \alpha\beta\big).$$

Here $\phi\colon \widehat{\pi} \to \mathrm{Aut}(\mathbb{Z}^2)$ is the homomorphism induced by the conjugation of π.

Since π normalizes the left translations $\mathbb{C}$ on $\mathbb{C} \times \widehat{X}$ by condition (2) of Definition 2.1, we can describe the action of π explicitly:

- There is a holomorphic map $\chi(\alpha)\colon (\widehat{X}, \widehat{J}) \to (\mathbb{C}, J_0)$ for each $\alpha \in \widehat{\pi}$ such that the action $(\pi, \mathbb{C} \times \widehat{X})$ is described as

$$(n,\alpha)(x,w) = \big(n + \bar{\phi}(\alpha)(x) + \chi(\alpha)(\alpha w),\ \alpha w\big) \tag{2.4}$$

for all $(n,\alpha) \in \pi$ and $(x,w) \in \mathbb{C} \times \widehat{X}$. Here $\bar{\phi}\colon \widehat{\pi} \to \mathrm{Aut}(\mathbb{C})$ is a unique extension of ϕ.

By the definition, (π, X) is a holomorphic Seifert action (cf. [6, 14, 3]).

2.2. Topology of a holomorphic torus-Bott manifold. From (2.2) there is a homomorphism induced by conjugation, $\phi\colon \pi_{m-1} \to \mathrm{Aut}(\mathbb{Z}^2)$. Since each element of π_m is almost complex and normalizes $\mathbb{C}$, there exists a matrix $P \in \mathrm{GL}(2,\mathbb{R})$ such that

$$P^{-1} \cdot \phi(\pi_{m-1}) \cdot P \le \mathrm{U}(1).$$

If we let $P^{-1} \cdot \phi(\alpha) \cdot P = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ for $\alpha \in \pi_{m-1}$, then the trace condition shows that $\cos\theta = 0, \pm 1/2, \pm 1$. It follows that respectively

$$\phi(\alpha) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\pm 1}, \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{\pm 1}, \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}. \tag{2.5}$$

So ϕ extends uniquely to an automorphism $\bar{\phi}\colon \pi_{m-1} \to \mathrm{Aut}_J(\mathbb{C}) = \mathbb{C}^*$ such that

$$\bar{\phi}(\alpha) = \pm \mathbf{i},\ e^{\pm \mathbf{i}\pi/3} \ \text{ or } \ \pm 1 \qquad \forall\, \alpha \in \pi_{m-1}, \tag{2.6}$$

respectively. In particular, $\bar{\phi}(\pi_{m-1})$ is a cyclic group of order 1, 2, 4 or 6.

Lemma 2.3. *Each π_m is virtually nilpotent.*

Proof. As $\mathbb{Z}^2 = \pi_1$, we suppose inductively that π_{m-1} is virtually nilpotent. Since $\phi(\pi_{m-1}) \le \mathrm{Aut}(\mathbb{Z}^2)$ is a finite cyclic group, we choose a finite index normal nilpotent subgroup Δ_{m-1} of π_{m-1} such that $\phi(\Delta_{m-1}) = \{1\}$. Then the group extension of (2.2) induces a central extension:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \pi_m & \longrightarrow & \pi_{m-1} & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \Delta_m & \longrightarrow & \Delta_{m-1} & \longrightarrow & 1 \end{array} \tag{2.7}$$

And hence Δ_m is nilpotent, which proves the induction step. □

For a holomorphic torus-Bott manifold M, there is a holomorphic fiber bundle $T^1_{\mathbb{C}} \to M \to M_{n-1}$. As the fundamental group π of M is virtually nilpotent, there exists a simply connected nilpotent Lie group N and a discrete faithful homomorphism $\rho\colon \pi \to \Gamma \le \mathrm{E}(N)$ such that the quotient N/Γ is an infranil-manifold (cf. [1] for instance). *Seifert rigidity* for nil-fiber [11] (see also [10, 14]) implies the following

Proposition 2.4. *Any holomorphic torus-Bott manifold M is diffeomorphic to an infranil-manifold N/Γ.*

Moreover, the diffeomorphism h between them preserves the fiber, i.e. there is a commutative diagram of equivariant diffeomorphisms:

$$\begin{array}{ccccc} (\mathbb{Z}^2, \mathbb{C}) & \longrightarrow & (\pi, X) & \xrightarrow{p} & (\widehat{\pi}, \widehat{X}) \\ {\scriptstyle \mathrm{id}}\big\downarrow & & {\scriptstyle \widetilde{h}}\big\downarrow & & {\scriptstyle \hat{h}}\big\downarrow \\ (\mathbb{Z}^2, \mathbb{C}) & \longrightarrow & (\Gamma, N) & \xrightarrow{p} & (\widehat{\Gamma}, \widehat{N}) \end{array} \tag{2.8}$$

3. INVARIANT METRIC ON A NILPOTENT LIE GROUP

3.1. Holomorphic infranil-manifolds. Let N be a simply connected nilpotent Lie group with left invariant complex structure J. Denote by $\mathrm{Aut}_J(N)$ the group of automorphisms of N which preserve J, i.e. $\alpha_* \circ J = J \circ \alpha_*$ on T_1N. Choose a maximal compact subgroup K from $\mathrm{Aut}_J(N)$ and put $\mathrm{E}_J(N) = N \rtimes K$. Each element $h = (a, \alpha) \in \mathrm{E}_J(N)$ acts on N as $h(x) = a \cdot \alpha(x)$ for all $x \in N$. Then $\mathrm{E}_J(N) = N \rtimes K$ acts holomorphically on N. If Γ is a discrete (torsion-free) uniform subgroup of $\mathrm{E}_J(N)$, the quotient N/Γ is said to be a *holomorphic infranil-orbifold* (*infranil-manifold*). It is well known that a finite cover of N/Γ is a nilmanifold.

3.2. Construction of an $\mathrm{E}(N)$-invariant complex structure. Let N be a simply connected nilpotent Lie group which has a central group extension $1 \to \mathbb{C} \to N \xrightarrow{\pi} \widehat{N} \to 1$. Let $\mathrm{E}(N) = N \rtimes K$ be the semidirect product. As $\mathbb{C}$ is normal in $\mathrm{E}(N)$, π induces an equivariant (continuous) homomorphism

$$\pi\colon\ (\mathrm{E}(N), N) \to (\mathrm{E}(\widehat{N}), \widehat{N}). \tag{3.1}$$

As $K \leq \mathrm{Aut}(N)$ normalizes $\mathbb{C}$, there is a homomorphism $\rho\colon K \to \mathrm{GL}(2, \mathbb{R})$. In order to be holomorphic on $\mathbb{C}$, we require that $\rho(K) \leq \mathrm{U}(1) \leq \mathrm{GL}(1, \mathbb{C}) = \mathrm{Aut}(\mathbb{C})$. Equivalently, for all $k \in K$,

$$k_* \circ J_0 = J_0 \circ k_* \qquad \text{on } T\mathbb{C}. \tag{3.2}$$

Suppose that $\widehat{J}$ is a left invariant complex structure on the $(2n-2)$-dimensional nilpotent Lie group $\widehat{N}$. As before, $\mathrm{E}_{\widehat{J}}(\widehat{N})$ denotes the holomorphic semidirect product $\widehat{N} \rtimes \widehat{K}$ of $\widehat{N}$ with a compact group $\widehat{K} \leq \mathrm{Aut}_{\widehat{J}}(\widehat{N})$.

Proposition 3.1. *There exists an* $\mathrm{E}(N)$*-invariant complex structure on* N *under the requirement* (3.2). *Moreover,*

$$(\mathbb{C}, J_0) \to (N, J) \xrightarrow{\pi} (\widehat{N}, \widehat{J})$$

is a principal holomorphic bundle.

Proof. Choose an N-invariant Riemannian metric on N and average it by the compact group K. Since K normalizes N, this gives an $\mathrm{E}(N)$-invariant Riemannian metric g on N. Let $T\mathbb{C}^\perp = \{X \in TN \mid g(X, A) = 0\ \forall\, A \in T\mathbb{C}\}$. As g is $\mathrm{E}(N)$-invariant and $\mathbb{C}$ is normal in $\mathrm{E}(N)$, it is easy to see that $T\mathbb{C}^\perp$ is $\mathrm{E}(N)$-invariant. Then the projection $\pi\colon N \to \widehat{N}$ induces an isomorphism $\pi_*\colon T\mathbb{C}^\perp \to T\widehat{N}$ at each point of N. Define an almost complex structure J on $T\mathbb{C}^\perp$ by the following correspondence at each point of N:

$$\pi_* J X = \widehat{J} \pi_* X. \tag{3.3}$$

Let J_0 be the standard complex structure on $\mathbb{C}^k$ $(k \geq 1)$. If we note that $TN = T\mathbb{C} \oplus T\mathbb{C}^\perp$, then we define

$$J(A + X) = J_0 A + JX, \qquad A \in T\mathbb{C}, \quad X \in T\mathbb{C}^\perp. \tag{3.4}$$

It follows that J is an *almost complex* structure on N. Since $\mathrm{E}(N)$ leaves $T\mathbb{C}^\perp$ invariant and normalizes $\mathbb{C}$, the decomposition is preserved by any element $h \in \mathrm{E}(N)$; $h_*A + h_*X \in T\mathbb{C} \oplus T\mathbb{C}^\perp$. In view of (3.1), the hypothesis that $\widehat{J}$ is $\mathrm{E}(\widehat{N})$-invariant shows that

$$\pi_*(h_*JX) = \pi(h)_*\pi_*(JX) = \pi(h)_*\widehat{J}\pi_*(X) = \widehat{J}\pi(h)_*\pi_*(X) = \widehat{J}\pi_*(h_*X) = \pi_*(Jh_*X),$$

and so $h_*JX = Jh_*X$ for all $X \in T\mathbb{C}^\perp$. As $\mathbb{C}$ is the center of N, $x_*J_0 = J_0x_*$ on $T\mathbb{C}$ for all $x \in N$. Each $\alpha \in K$ satisfies $\alpha_*J_0 = J_0\alpha_*$ on $T\mathbb{C}$ by our requirement (3.2). In particular, if $h = (x, \alpha) \in \mathrm{E}(N)$, then $h_*J_0 = J_0h_*$ on $T\mathbb{C}$. Taking into account these equalities, we have

$$Jh_*(A + X) = J_0h_*A + Jh_*X = h_*J_0A + h_*JX = h_*J(A + X),$$

and hence J is $\mathrm{E}(N)$-invariant. Obviously $(\mathbb{C}, J_0) \to (N, J) \xrightarrow{\pi} (\widehat{N}, \widehat{J})$ is an almost complex principal fiber bundle with respect to J. Let $\varphi\colon (\pi^{-1}(U), J) \to (U \times \mathbb{C}, J_0 \times \widehat{J})$ be a local trivialization isomorphism for this bundle. As $\widehat{J}$ is a complex structure by the hypothesis, so is J on N. □

3.3. Trivialization. Let $(\mathbb{C}, J_0) \to (N, J) \xrightarrow{\pi} (\widehat{N}, \widehat{J})$ be a principal holomorphic bundle from Proposition 3.1. We assume that $(\widehat{N}, \widehat{J})$ is biholomorphic to $(\mathbb{C}^{n-1}, J_0)$. By Proposition 2.2 we have

Corollary 3.2. *(N, J) is biholomorphic as a holomorphic principal bundle to the product $(\mathbb{C} \times \widehat{N}, J_0 \times \widehat{J})$.*

Let $\mathrm{E}_J(N) = N \rtimes K$ be the holomorphic semidirect product. Choose a torsion-free discrete cocompact subgroup Γ from $\mathrm{E}_J(N)$ so that N/Γ is a holomorphic infranil-manifold.

4. HOLOMORPHIC INFRANIL ACTION

4.1. Seifert infranil-manifold. We observe that a holomorphic infranil-manifold N/Γ will be a holomorphic Seifert manifold.

The central group extension $1 \to \mathbb{C} \to N \xrightarrow{\pi} \widehat{N} \to 1$ is viewed as a holomorphic principal bundle by Proposition 3.1. Under the hypothesis in Subsection 3.3, Corollary 3.2 shows that $N = \mathbb{C} \times \widehat{N}$ biholomorphically with the group law

$$(x, z) \cdot (y, w) = (x + y + \mathsf{f}(z, w), z \cdot w). \tag{4.1}$$

Here $\mathsf{f}\colon \widehat{N} \times \widehat{N} \to \mathbb{C}$ is a 2-cocycle. Put $\mathrm{E}(N) = \mathrm{E}_J(N)$ for brevity. Since $\mathrm{E}(N)$ normalizes $\mathbb{C}$, there is a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{C} & \longrightarrow & N & \xrightarrow{\pi} & \widehat{N} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathrm{E}(N) & \xrightarrow{\pi} & \widehat{N} \circ K & \longrightarrow & 1 \end{array} \tag{4.2}$$

where we put $\mathrm{E}(N)/\mathbb{C} = \widehat{N} \circ K$. As $\mathrm{E}(N)$ is the semidirect product $N \rtimes K$, $\widehat{N} \circ K$ has the group law; for $\alpha = a \circ k, \beta = b \circ h \in \widehat{N} \circ K$,

$$\alpha \cdot \beta = ak(b) \circ kh.$$

As $K \leq \mathrm{Aut}(N)$, there is a homomorphism $\widehat{\rho}\colon K \to \mathrm{Aut}(\widehat{N})$. If we recall that $\widehat{K}$ is the maximal compact subgroup of $\mathrm{Aut}(\widehat{N})$, $\widehat{\rho}(K) \leq \widehat{K}$ up to conjugation. It follows that

$$\widehat{N} \circ K = \widehat{N} \rtimes \widehat{\rho}(K) \leq \mathrm{E}(\widehat{N}). \tag{4.3}$$

Let $\phi\colon \widehat{N}\circ K \to \mathrm{Aut}(\mathbb{C})$ be a homomorphism induced by the conjugation from (4.2). Then $\mathrm{E}(N)$ is viewed as the set $\mathbb{C}\times(\widehat{N}\circ K)$ with the group law

$$(x,\alpha)\cdot(y,\beta)=\bigl(x+\phi(\alpha)(y)+\bar{\mathsf{f}}(\alpha,\beta),\ \alpha\cdot\beta\bigr) \tag{4.4}$$

in which $\bar{\mathsf{f}}\colon \widehat{N}\circ K\times\widehat{N}\circ K\to\mathbb{C}$ is a 2-cocycle extending f on $\widehat{N}\times\widehat{N}$ of (4.1). The action of $\mathrm{E}(N)$ on N is interpreted in terms of group law (4.4): $\mathrm{E}(N)\times\mathrm{E}(N)\to\mathrm{E}(N)\to N$; let $\alpha=a\circ k\in\widehat{N}\circ K$ with $(x,\alpha)\in\mathrm{E}(N)$ and $b\in\widehat{N}$ with $(y,b)\in N$. Then

$$(x,\alpha)\cdot(y,b)=(x+\phi(\alpha)(y)+\bar{\mathsf{f}}(\alpha,b),ak(b)\circ k)\mapsto(x+\phi(\alpha)(y)+\bar{\mathsf{f}}(\alpha,b),ak(b))\in N. \tag{4.5}$$

As in Subsection 3.1, $\mathrm{E}(N)$ normalizes $\mathbb{C}$, so the holomorphic action of $\mathrm{E}(N)$ on N induces a holomorphic action of $\widehat{N}\circ K$ on $\widehat{N}$ by $\alpha b=ak(b)$ for all $\alpha=a\circ k\in\widehat{N}\circ K$ and $b\in\widehat{N}$. By the definition of Subsection 2.1, we obtain a *holomorphic Seifert fibration* associated with the group extensions of (4.2):

$$(\mathbb{C},\mathbb{C})\to(\mathrm{E}(N),N)\xrightarrow{\pi}(\widehat{N}\circ K,\widehat{N})$$

where $N=\mathbb{C}\times\widehat{N}$ biholomorphically. Let $(y,w)\in N$. If $h=(x,\alpha)\in\mathrm{E}(N)$ with $\alpha\,(=a\cdot k)\in\widehat{N}\circ K$, then as in (2.4) the *holomorphic Seifert action* implies that there is a holomorphic map $\mu(\alpha)\colon(\widehat{N},\widehat{J})\to(\mathbb{C},J_0)$ such that

$$h(y,w)=\bigl(x+\phi(\alpha)(y)+\mu(\alpha)(\alpha w),\alpha w\bigr). \tag{4.6}$$

Using μ (cf. [14]), one can describe $\bar{\mathsf{f}}\colon\widehat{N}\circ K\times\widehat{N}\circ K\to\mathbb{C}$ as $\bar{\mathsf{f}}(\alpha,\beta)=\delta^1\mu(\alpha,\beta)(w)$ for all $w\in\widehat{N}$, i.e.

$$\bar{\mathsf{f}}(\alpha,\beta)=\phi(\alpha)(\mu(\beta)(\alpha^{-1}\cdot w))+\mu(\alpha)(w)-\mu(\alpha\beta)(w)\qquad\forall\,\alpha,\beta\in\widehat{N}\circ K,\quad\forall\,w\in\widehat{N}. \tag{4.7}$$

Here the set $\mathrm{hol}(\widehat{N},\mathbb{C})$ is an $(\widehat{N}\circ\widehat{K})$-module defined by

$$(\alpha\cdot g)(w)=\phi(\alpha)(g(\alpha^{-1}\cdot w))\qquad\forall\,g\in\mathrm{hol}(\widehat{N},\mathbb{C}),\quad\forall\,\alpha\in\widehat{N}\circ K. \tag{4.8}$$

4.2. Holomorphic Seifert manifold. Consider a torsion-free discrete uniform subgroup Γ lying in $\mathrm{E}(N)=\mathrm{E}_J(N)$:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \Gamma & \xrightarrow{\pi} & \widehat{\Gamma} & \longrightarrow & 1\\
 & & \cap & & \cap & & \cap & & \\
1 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathrm{E}(N) & \xrightarrow{\pi} & \widehat{N}\circ K & \longrightarrow & 1
\end{array} \tag{4.9}$$

Here $\mathbb{Z}^2=\mathbb{C}\cap\Gamma$ and $\widehat{\Gamma}=\pi(\Gamma)$. Then the group extension of Γ is represented by a 2-cocycle $[f]\in H^2_\phi(\widehat{\Gamma};\mathbb{Z}^2)$ where $\phi=\phi_{|\widehat{\Gamma}}\colon\widehat{\Gamma}\to\mathrm{Aut}(\mathbb{Z}^2)$ is a homomorphism restricted to $\widehat{\Gamma}$. Note that $\mathbb{Z}^2$ is a $\widehat{\Gamma}$-module through ϕ. In view of (4.6), we have shown that

Proposition 4.1. *Given a holomorphic infranil action of $\widehat{\Gamma}$ (i.e. $\widehat{\Gamma}\le\mathrm{E}_{\widehat{J}}(\widehat{N})$), a holomorphic infranil action of Γ on (N,J) is a holomorphic Seifert action of Γ on $(\mathbb{C}\times\widehat{N},J_0\times\widehat{J})$ which can be determined by a holomorphic map $\mu(\alpha)\colon\widehat{N}\to\mathbb{C}$ for each $\alpha\in\widehat{\Gamma}$ such that*

$$(n,\alpha)(x,w)=(n+\phi(\alpha)(x)+\mu(\alpha)(\alpha w),\ \alpha w)\qquad\forall\,(n,\alpha)\in\Gamma,\quad\forall\,(x,w)\in N. \tag{4.10}$$

Moreover, the cocycle f representing the group extension of Γ in (4.9) satisfies $\delta^1\mu=f$ as in (4.7).

Comparing (4.5) with (4.10) implies that

$$\bar{\mathsf{f}}(\alpha,w)=\mu(\alpha)(\alpha w)\qquad\forall\,\alpha\in\widehat{\Gamma},\quad\forall\,w\in\widehat{N}. \tag{4.11}$$

5. DEFORMATION OF NILPOTENT LIE GROUPS

Let $\mathrm{hol}(\widehat{N},\mathbb{C})$ be the set of all holomorphic maps from $(\widehat{N},\widehat{J})$ to $\mathbb{C}$. It is endowed with a $\widehat{\Gamma}$-module as in (4.8), and similarly for $\mathrm{hol}(\widehat{N},T^1_{\mathbb{C}})$ and $\mathbb{Z}^2$ (cf. Subsection 4.2).

Recall that a short exact sequence $1 \to \mathbb{Z}^2 \xrightarrow{i} \mathbb{C} \xrightarrow{j} T^1_{\mathbb{C}} \to 1$ induces a long cohomology exact sequence (cf. [14, 3])

$$\ldots \to H^1_\phi(\widehat{\Gamma};\mathbb{Z}^2) \xrightarrow{i} H^1_\phi(\widehat{\Gamma};\mathrm{hol}(\widehat{N},\mathbb{C})) \xrightarrow{j} H^1_\phi(\widehat{\Gamma};\mathrm{hol}(\widehat{N},T^1_{\mathbb{C}})) \xrightarrow{\delta} H^2_\phi(\widehat{\Gamma};\mathbb{Z}^2) \to \ldots. \qquad (5.1)$$

Put $\widehat{\mu} = j\circ\mu\colon \widehat{N} \to T^1_{\mathbb{C}}$ for a holomorphic function μ of Proposition 4.1. Then (4.10) implies that $\delta[\widehat{\mu}] = [f]$ by the definition. For any element $[\nu] \in H^1(\widehat{\Gamma};\mathrm{hol}(\widehat{N},\mathbb{C}))$, we have an element $j[\nu]\cdot[\widehat{\mu}]$ such that $\delta(j[\nu]\cdot[\widehat{\mu}]) = [f]$. Note that j maps $\mu+\nu$ to $j\nu\cdot\widehat{\mu}$. From Proposition 4.1, $\delta^1\mu = f$ and so it follows that $\delta^1(\mu+\nu) = f$, which still defines the same group extension $1 \to \mathbb{Z}^2 \to \Gamma \to \widehat{\Gamma} \to 1$.

We study a holomorphic Seifert action of Γ by this replacement $\mu+\nu$ which is given by

$$(n,\alpha)(x,w) = \bigl(n + \phi(\alpha)(x) + \mu(\alpha)(\alpha w) + \nu(\alpha)(\alpha w), \alpha w\bigr), \qquad n\in\mathbb{Z}^2,\quad \alpha\in\widehat{\Gamma},\quad (x,w)\in N. \qquad (5.2)$$

Theorem 5.1. *There exists a nilpotent Lie group N' isomorphic to N such that the complex structure J is invariant under $\mathrm{E}(N')$. The above action (Γ,N) is equivariantly biholomorphic to an infranil action of Γ' on N' (i.e. $\Gamma' \le \mathrm{E}_J(N')$). Here Γ' is a discrete uniform subgroup isomorphic to Γ. Specifically the quotient N/Γ is biholomorphic to the holomorphic infranil-manifold N'/Γ'. (In particular, $\Delta' = \Gamma'\cap N'$ is a finite index subgroup of Γ' such that N'/Δ' is a holomorphic nilmanifold.)*

Proof. First, when we take a $\widehat{\Gamma}$-module $\mathrm{Map}(\widehat{N},\mathbb{C})$ consisting of smooth maps from $\widehat{N}$ to $\mathbb{C}$ instead of $\mathrm{hol}(\widehat{N},\mathbb{C})$, we note that

$$H^q_\phi\bigl(\widehat{\Gamma};\mathrm{Map}(\widehat{N},\mathbb{C})\bigr) = 0, \qquad q\ge 1$$

(see [4, 14]).

If $[\nu] \in H^1_\phi(\widehat{\Gamma};\mathrm{hol}(\widehat{N},\mathbb{C}))$ is relaxed to be in $H^1_\phi(\widehat{\Gamma};\mathrm{Map}(\widehat{N},\mathbb{C}))$, then there is an element $\lambda \in \mathrm{Map}(\widehat{N},\mathbb{C})$ such that $\delta^1\lambda = \nu$, i.e. $\nu(\alpha)(\alpha w) = \delta^1\lambda(\alpha)(\alpha w) = \alpha\circ\lambda(\alpha w) - \lambda(\alpha w)$; hence (cf. (4.8))

$$\nu(\alpha)(\alpha w) = \phi(\alpha)(\lambda(w)) - \lambda(\alpha w) \qquad \forall\,\alpha\in\widehat{\Gamma},\quad \forall\, w\in\widehat{N}. \qquad (5.3)$$

A function $\mathsf{f}'\colon \widehat{N}\times\widehat{N} \to \mathbb{C}$ is defined to be

$$\mathsf{f}'(z,w) = \mathsf{f}(z,w) + \delta^1\lambda(z,w), \qquad z,w\in\widehat{N}. \qquad (5.4)$$

As $1 \to \mathbb{C} \to N \to \widehat{N} \to 1$ is a central extension, $\delta^1\lambda(z,w) = z\cdot\lambda(w) - \lambda(z\cdot w) + \lambda(z) = \lambda(z) + \lambda(w) - \lambda(z\cdot w)$. It is easy to see that $\delta^1\mathsf{f}' = 0$, so f' is a 2-cocycle in $H^2(\widehat{N};\mathbb{C})$. Let $N' = \mathbb{C}\times\widehat{N}$ be the product with the group law

$$(x,z)\circ(y,w) = (x+y+\mathsf{f}'(z,w),\ z\cdot w).$$

N' becomes a Lie group. Moreover, if $\varphi\colon N\to N'$ is a map defined by

$$\varphi(x,z) = (x-\lambda(z), z), \qquad (5.5)$$

then

$$\begin{aligned}\varphi((x,z)\cdot(y,w)) &= \varphi(x+y+\mathsf{f}(z,w),\ z\cdot w) = (x+y+\mathsf{f}(z,w) - \lambda(z\cdot w),\ z\cdot w)\\ &= (x+y+\mathsf{f}(z,w)+\delta^1\lambda(z,w) - \lambda(z) - \lambda(w),\ z\cdot w)\\ &= (x+y+\mathsf{f}'(z,w) - \lambda(z) - \lambda(w),\ z\cdot w)\\ &= (x-\lambda(z), z)\circ(y-\lambda(w), w) = \varphi(x,z)\circ\varphi(y,w). \end{aligned} \qquad (5.6)$$

Thus $\varphi\colon N\to N'$ is a Lie group isomorphism.

Let $\lambda\colon \widehat{N} \to \mathbb{C}$ be the map as above. We extend λ to $\widehat{N}\circ K$. Let $\alpha = a\cdot k \in \widehat{N}\circ K$. Since $K \le \mathrm{Aut}(N)$, evaluated at $1\in N$, we simply put

$$\bar{\lambda}(\alpha) = \lambda(a). \tag{5.7}$$

We can define a 2-cocycle $\bar{\mathsf{f}}'\colon (\widehat{N}\circ K)\times(\widehat{N}\circ K)\to\mathbb{C}$ to be

$$\bar{\mathsf{f}}'(\alpha,\beta) = \bar{\mathsf{f}}(\alpha,\beta) + \delta^1\bar{\lambda}(\alpha,\beta), \qquad \alpha,\beta\in \widehat{N}\circ K, \tag{5.8}$$

where

$$\delta^1\bar{\lambda}(\alpha,\beta) = \phi(\alpha)(\bar{\lambda}(\beta)) - \bar{\lambda}(\alpha\beta) + \bar{\lambda}(\alpha). \tag{5.9}$$

Then we have a group G as the set $\mathbb{C}\times(\widehat{N}\circ K)$ with the group law

$$(x,\alpha)\circ(y,\beta) = \bigl(x+\phi(\alpha)(y) + \bar{\mathsf{f}}'(\alpha,\beta),\ \alpha\beta\bigr). \tag{5.10}$$

By construction, there is an exact sequence $1\to N'\to G \xrightarrow{\pi} K \to 1$. As N' is a simply connected nilpotent Lie group, it follows that $G = N'\rtimes K'$ for which π maps K' isomorphically onto K. In particular, $G = \mathrm{E}(N')$. As in (5.6), if we define $\overline{\varphi}\colon \mathrm{E}(N)\to\mathrm{E}(N') = G$ to be

$$\overline{\varphi}(x,\alpha) = (x-\bar{\lambda}(\alpha),\alpha), \tag{5.11}$$

then

$$\begin{aligned}
\overline{\varphi}((x,\alpha)\cdot(y,\beta)) &= \bigl(x+\phi(\alpha)(y)+\bar{\mathsf{f}}(\alpha,\beta)-\bar{\lambda}(\alpha\beta),\ \alpha\beta\bigr)\\
&= \bigl(x+\phi(\alpha)(y)+\bar{\mathsf{f}}(\alpha,\beta)+\delta^1\bar{\lambda}(\alpha,\beta)-\phi(\alpha)(\bar{\lambda}(\beta))-\bar{\lambda}(\alpha),\ \alpha\beta\bigr)\\
&= \bigl(x+\phi(\alpha)(y)+\bar{\mathsf{f}}'(\alpha,\beta)-\phi(\alpha)(\bar{\lambda}(\beta))-\bar{\lambda}(\alpha),\ \alpha\beta\bigr)\\
&= (x-\bar{\lambda}(\alpha),\alpha)\circ(y-\bar{\lambda}(\beta),\beta) = \overline{\varphi}(x,\alpha)\circ\overline{\varphi}(y,\beta).
\end{aligned} \tag{5.12}$$

Hence $\overline{\varphi}\colon \mathrm{E}(N)\to\mathrm{E}(N')$ is an isomorphism. By formula (5.11), $\overline{\varphi}_{|\mathbb{C}} = \mathrm{id}$ and the induced homomorphism $\widehat{\varphi}\colon \widehat{N}\to\widehat{N}$ of $\overline{\varphi}$ is id on $\widehat{N}\circ K$. This induces the following exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathrm{E}(N) & \xrightarrow{\pi} & \widehat{N}\circ K & \longrightarrow & 1\\
 & & \mathrm{id}\big\downarrow & & \overline{\varphi}\big\downarrow & & \mathrm{id}\big\downarrow & & \\
1 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathrm{E}(N') & \xrightarrow{\pi} & \widehat{N}\circ K & \longrightarrow & 1
\end{array} \tag{5.13}$$

We recall the infranil action of $\mathrm{E}(N')$ on N'. As in (4.5), for $\alpha = a\circ k\in\widehat{N}\circ K$ with $(x,\alpha)\in\mathrm{E}(N')$ and $w\in\widehat{N}$ with $(y,w)\in N'$, it follows that

$$(x,\alpha)\circ(y,w) = \bigl(x+\phi(\alpha)(y)+\bar{\mathsf{f}}'(\alpha,w), ak(w)\circ k\bigr) \mapsto \bigl(x+\phi(\alpha)(y)+\bar{\mathsf{f}}'(\alpha,w),\alpha w\bigr)\in N' \tag{5.14}$$

where $\alpha w = ak(w)$. So we put this infranil action $(\mathrm{E}(N'),N')$ to be

$$(x,\alpha)\circ'(y,w) = \bigl(x+\phi(\alpha)(y)+\bar{\mathsf{f}}'(\alpha,w),\ \alpha w\bigr). \tag{5.15}$$

Let $\Gamma\le\mathrm{E}(N)$ be as above. As in (4.9), there is a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \Gamma & \xrightarrow{\pi} & \widehat{\Gamma} & \longrightarrow & 1\\
 & & \mathrm{id}\big\downarrow & & \overline{\varphi}\big\downarrow & & \mathrm{id}\big\downarrow & & \\
1 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \overline{\varphi}(\Gamma) & \xrightarrow{\pi} & \widehat{\Gamma} & \longrightarrow & 1\\
 & & \cap & & \cap & & \cap & & \\
1 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathrm{E}(N') & \xrightarrow{\pi} & \widehat{N}\circ K & \longrightarrow & 1
\end{array} \tag{5.16}$$

In view of (4.11), (5.3) and (5.8), (5.9), the action (5.2) becomes

$$\begin{aligned}(n,\alpha)(x,w) &= \big(n+\phi(\alpha)(x)+\mu(\alpha)(\alpha w)+\nu(\alpha)(\alpha w),\ \alpha w\big)\\ &= \big(n+\phi(\alpha)(x)+\bar{\mathsf{f}}(\alpha,w)+\phi(\alpha)(\bar\lambda(w))-\bar\lambda(\alpha w),\ \alpha w\big)\\ &= \big(n+\phi(\alpha)(x)+\bar{\mathsf{f}}(\alpha,w)+\delta^1\bar\lambda(\alpha,w)-\bar\lambda(\alpha),\ \alpha w\big)\\ &=\big(n+\phi(\alpha)(x)+\bar{\mathsf{f}}'(\alpha,w)-\bar\lambda(\alpha),\ \alpha w\big)\\ &= (n-\bar\lambda(\alpha),\alpha)\circ'(x,w)=\overline{\varphi}(n,\alpha)\circ'(x,w),\end{aligned} \tag{5.17}$$

where $\circ'$ is defined in (5.14). Hence the action of (Γ, N) is equivalent to the infranil action of $\overline{\varphi}(\Gamma)$ on N' defined in (5.15).

On the other hand, there is an $\mathrm{E}(N')$-invariant complex structure J' on N' by Proposition 3.1 such that (N', J') is biholomorphic to $(\mathbb{C}\times\widehat{N}, J_0\times\widehat{J})$ by Corollary 3.2. By Proposition 4.1, for every $\alpha\in\widehat{\Gamma}$ there exists an element $\mu'(\alpha)\in \mathrm{hol}(\widehat{N},\mathbb{C})$ for which a holomorphic infranil action of $\overline{\varphi}(\Gamma)$ on (N', J') is obtained as

$$\overline{\varphi}(n,\alpha)\circ'(x,w)=\big(n+\phi(\alpha)(x)+\mu'(\alpha)(\alpha w),\ \alpha w\big). \tag{5.18}$$

Comparing this with (5.17), we obtain

$$\mu(\alpha)(\alpha w)+\nu(\alpha)(\alpha w)=\mu'(\alpha)(\alpha w). \tag{5.19}$$

For arbitrary $A\in T\mathbb{C}$ and $V\in T\widehat{N}$, calculate

$$\begin{aligned}(n,\alpha)_*J(A,V) = (n,\alpha)_*(J_0A,\widehat{J}V) &= \big(\phi(\alpha)(J_0A)+\mu(\alpha)_*(\alpha_*\widehat{J}V)+\nu(\alpha)_*(\alpha_*\widehat{J}V),\ \alpha_*\widehat{J}V\big)\\ &= \big(J_0\phi(\alpha)(A)+J_0\mu(\alpha)_*(\alpha_*V)+J_0\nu(\alpha)_*(\alpha_*V),\ \widehat{J}\alpha_*V\big)\\ &= \big(J_0\phi(\alpha)(A)+J_0\mu'(\alpha)_*(\alpha_*V),\ \widehat{J}\alpha_*V\big)\\ &= J'\big(\phi(\alpha)(A)+\mu'(\alpha)_*(\alpha_*V),\ \alpha_*V\big)=J'\overline{\varphi}(n,\alpha)_*(A,V).\end{aligned} \tag{5.20}$$

As $(n,\alpha)_*J=J(n,\alpha)_*$ on TN, it follows that $J'=J$ on $\mathbb{C}\times\widehat{N}=N=N'$. And hence the holomorphic action (Γ,N,J) is equivariantly biholomorphic to $(\varphi(\Gamma),N',J)$. Equivalently the quotient N/Γ is biholomorphic to the holomorphic infranil-manifold $N'/\overline{\varphi}(\Gamma)$. □

6. HOLOMORPHIC CLASSIFICATION

Let M be a holomorphic torus-Bott manifold of dimension $2n$. By Definition 2.1, $X_1=\mathbb{C}$. We assume inductively that X_{n-1} is biholomorphic to $\mathbb{C}^{n-1}$. By condition (2) of Definition 2.1, $\mathbb{C}\to X=X_n\to\widehat{X}=X_{n-1}$ is a holomorphic principal bundle. Thus by Corollary 3.2, (X,J) is biholomorphic to the product $(\mathbb{C}\times\widehat{X}, J_0\times\widehat{J})$ as a holomorphic bundle. Hence the action on the universal covering (X,π,J) is identified with a holomorphic Seifert action $(\mathbb{C}\times\widehat{X},\pi,J_0\times\widehat{J})$ as in (2.4).

Consider the associated group extension $1\to\mathbb{Z}^2\to\pi\to\widehat{\pi}\to 1$, which represents a 2-cocycle $[f]\in H_\phi(\widehat{\pi};\mathbb{Z}^2)$. As $(\pi,\mathbb{C}\times\widehat{X})$ is a holomorphic Seifert action, there is a holomorphic map $\chi(\alpha)\colon \widehat{N}\to\mathbb{C}$ for each $\alpha\in\widehat{\pi}$ such that

$$(n,\alpha)(x,w)=(n+\bar\phi(\alpha)(x)+\chi(\alpha)(\alpha w),\alpha w)\qquad \forall\,(n,\alpha)\in\pi,\quad \forall\,(x,w)\in\mathbb{C}\times\widehat{N}, \tag{6.1}$$

which satisfies

$$\delta[\widehat{\chi}]=[f]. \tag{6.2}$$

By Corollary 2.4, X/π is diffeomorphic to an infranil-manifold N/π. Suppose that $(\widehat{X},\widehat{\pi},\widehat{J})$ is equivariantly biholomorphic to $(\widehat{N},\widehat{\pi},\widehat{J})$ for which $\pi \le \mathrm{E}(N) = N \rtimes K$. Since $\phi\colon \widehat{\pi} \to \mathrm{Aut}(\mathbb{Z}^2)$ satisfies $\phi(\widehat{\pi}) \le \mathrm{U}(1)$ from (2.6), we may assume that K satisfies the requirement (3.2) of Proposition 3.1. (In fact, as N centralizes $\mathbb{C}$ and $N \rtimes K$ normalizes $\mathbb{C}$, the conjugation map $\rho\colon N \rtimes K \to \mathrm{GL}(2,\mathbb{R})$ satisfies $\rho(N \rtimes K) = \rho(K) \le \mathrm{O}(2)$ in general. Taking $\mathrm{U}(1) \le \mathrm{O}(2)$, we choose $K_0 \le K$ such that $\rho(K_0) \le \mathrm{U}(1)$ instead of K. As $\rho(\pi) = \phi(\widehat{\pi}) \le \mathrm{U}(1)$, it follows that $\pi \le N \rtimes K_0$, which satisfies the requirement obviously.)

By Proposition 3.1, there exists an $\mathrm{E}(N)$-invariant complex structure J such that $\pi \le \mathrm{E}_J(N)$, i.e. the action (N,π) is a holomorphic infranil action. As (N,J) is biholomorphic to $(\mathbb{C}\times\widehat{N}, J_0\times\widehat{J})$ by Corollary 3.2, Proposition 4.1 implies that there is a holomorphic map $\mu(\alpha)\colon \widehat{N} \to \mathbb{C}$ such that

$$(n,\alpha)(x,w) = \bigl(n + \bar{\phi}(\alpha)(x) + \mu(\alpha)(\alpha w),\ \alpha w\bigr). \tag{6.3}$$

It also follows that

$$\delta[\widehat{\mu}] = [f]. \tag{6.4}$$

As both $[\widehat{\chi}]$ and $[\widehat{\mu}]$ belong to $H^1_\phi(\widehat{\pi}, \mathrm{hol}(\widehat{N},\mathbb{C}))$, there exists an element $[\nu] \in H^1_\phi(\widehat{\pi}, \mathrm{hol}(\widehat{N},\mathbb{C}))$ such that

$$[\widehat{\mu}]^{-1}[\widehat{\chi}] = [\widehat{\nu}]. \tag{6.5}$$

This implies that $j(\chi(\alpha)(w)) = j(\mu(\alpha)(w) + \nu(\alpha)(w)) \in T^1_{\mathbb{C}}$ for all $w \in \widehat{N}$. We may assume that (up to a constant)

$$\chi = \mu + \nu\colon\ \widehat{\pi} \to \mathrm{hol}(\widehat{N},\mathbb{C}). \tag{6.6}$$

Theorem 6.1. *Let M be a holomorphic torus-Bott manifold of dimension $2n$ and (X,π,J) be its universal covering. There exists a nilpotent Lie group N' with $\mathrm{E}(N')$-invariant complex structure J such that the action (X,π,J) is equivariantly biholomorphic to a holomorphic infranil action (N',π',J) $(\pi' \le \mathrm{E}_J(N'))$. Specifically, a $2n$-dimensional holomorphic torus-Bott manifold M is biholomorphic to a holomorphic infranil-manifold N'/π'.*

Proof. Suppose inductively that $(\widehat{X},\widehat{\pi},\widehat{J})$ is equivariantly biholomorphic to $(\widehat{N},\widehat{\pi},\widehat{J})$. Then the action (X,π) is equivariantly biholomorphic to a holomorphic action (N,π,J) such that

$$(n,\alpha)(x,w) = \bigl(n + x + \mu(\alpha)(\alpha w) + \nu(\alpha)(\alpha w),\ \alpha\cdot w\bigr).$$

Applying Theorem 5.1 to this action, we find that there is a holomorphic infranil geometry $(\mathrm{E}_J(N'), N')$ such that the complex quotient N/π is biholomorphic to a holomorphic infranil-manifold N'/Γ' for a torsion-free discrete subgroup $\Gamma' \le \mathrm{E}_J(N')$. $\square$

7. APPLICATION

Let $M = M_n \to M_{n-1} \to \ldots \to M_1 \to \{\mathrm{pt}\}$ be a holomorphic torus-Bott tower as in (1.1). Each holomorphic fiber bundle induces a group extension $1 \to \mathbb{Z}^2 \to \pi_m \to \pi_{m-1} \to 1$ which represents a 2-cocycle in $H^2_\phi(\pi_{m-1};\mathbb{Z}^2)$, $m = 1,\ldots,n$ (see (2.2)).

Definition 7.1. A holomorphic torus-Bott tower is *of finite type* if each 2-cocycle has finite order in $H^2_\phi(\pi_{m-1};\mathbb{Z}^2)$. Otherwise (i.e. there exists a cocycle of infinite order), a holomorphic torus-Bott tower is said to be *of infinite type.*

7.1. Holomorphic torus-Bott manifold of finite type. Since $\mathrm{U}(n)$ is the maximal compact unitary subgroup in $\mathrm{GL}(n,\mathbb{C})$, the affine group $\mathrm{A}_{\mathbb{C}}(n) = \mathbb{C}^n \rtimes \mathrm{GL}(n,\mathbb{C})$ has the complex euclidean subgroup $\mathrm{E}_{\mathbb{C}}(n) = \mathbb{C}^n \rtimes \mathrm{U}(n)$. If Γ is a torsion-free discrete uniform subgroup in $\mathrm{E}_{\mathbb{C}}(n)$, then the quotient $\mathbb{C}^n/\Gamma$ is a compact complex euclidean space form. The group Γ is said to be a Bieberbach group.

Theorem 7.2. *If M is a $2n$-dimensional holomorphic torus-Bott manifold of finite type, then M is biholomorphic to a complex euclidean space form $\mathbb{C}^n/\Gamma$ $(\Gamma \leq \mathrm{E}_{\mathbb{C}}(n))$. Moreover, the holonomy group $L(\Gamma) \leq \mathrm{U}(n)$ is isomorphic to the product*

$$\begin{pmatrix} H_1 & & & \\ & H_2 & & \\ & & \ddots & \\ & & & H_n \end{pmatrix}$$

where H_i is either $\{1\}$, $\mathbb{Z}_2$, $\mathbb{Z}_4$ or $\mathbb{Z}_6$.

Proof. Put $(\pi, X) = (\pi_{n-1}, X_n)$ and $(\widehat{\pi}, \widehat{X}) = (\pi_{n-1}, X_{n-1})$. Let

$$(\mathbb{Z}^2, \mathbb{C}) \to (\pi, X) \xrightarrow{p} (\widehat{\pi}, \widehat{X}) \tag{7.1}$$

be an equivariant principal holomorphic bundle (cf. (2.3)). Inductively suppose that $\widehat{X}/\widehat{\pi}$ is biholomorphic to a complex euclidean space form $\mathbb{C}^{n-1}/\widehat{\Gamma}$ $(\widehat{\Gamma} \leq \mathrm{E}_{\mathbb{C}}(n-1))$. As $\widehat{\pi} \cong \widehat{\Gamma}$, $\widehat{\pi}$ has a normal free abelian subgroup $\mathbb{Z}^{2(n-1)}$ of finite index. Consider the commutative diagram as in (4.9):

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \pi & \longrightarrow & \widehat{\pi} & \longrightarrow & 1 \\ & & || & & \cup & & \cup & & \\ 1 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \Delta & \longrightarrow & \mathbb{Z}^{2(n-1)} & \longrightarrow & 1 \end{array} \tag{7.2}$$

Note that $\phi(\widehat{\pi}) \leq \mathrm{Aut}(\mathbb{Z}^2)$ is a finite cyclic group. Taking a finite index subgroup if necessary, we may assume that the lower sequence is a central group extension. The cocycle of $H^2_\phi(\widehat{\pi}; \mathbb{Z}^2)$ restricts to an element of a free abelian group $H^2(\mathbb{Z}^{2(n-1)}; \mathbb{Z}^2)$. Since the cocycle representing (7.2) is a torsion in $H^2_\phi(\pi_{m-1}; \mathbb{Z}^2)$ by the hypothesis, it is zero in $H^2(\mathbb{Z}^{2(n-1)}; \mathbb{Z}^2)$, i.e. the lower group extension splits; $\Delta \cong \mathbb{Z}^2 \times \mathbb{Z}^{2(n-1)} = \mathbb{Z}^{2n}$.

On the other hand, M is biholomorphic to a holomorphic infranil-manifold N/Γ for some $\Gamma \leq \mathrm{E}_J(\mathsf{N})$ by Theorem 6.1. In particular, Γ has a finite index subgroup Γ' isomorphic to $\mathbb{Z}^{2n}$. As Γ' is a discrete uniform subgroup of N, the Mal'cev uniqueness property implies that N is isomorphic to $\mathbb{C}^n$. (Note that N is isomorphic to a vector space $\mathbb{R}^{2n}$. The complex structure J on N is equivalent to the standard complex structure $J_0 = J_0 \times J_0$ on $\mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$ by Corollary 3.2. Thus (N, J) is holomorphically isomorphic to $\mathbb{C}^n$.) If we note that K belongs to $\mathrm{Aut}(\mathbb{C}^n) = \mathrm{GL}(n, \mathbb{C})$ in this case, it follows that $K = \mathrm{U}(n)$, so that $\mathrm{E}_J(\mathsf{N}) = \mathrm{E}_{\mathbb{C}}(n)$. Since $\Gamma \leq \mathrm{E}_{\mathbb{C}}(n)$, M is biholomorphic to a complex euclidean space form $\mathbb{C}^n/\Gamma$.

We may identify $M = \mathbb{C}^n/\Gamma$. Let $L\colon \mathrm{Aff}_{\mathbb{C}}(n) = \mathbb{C}^n \rtimes \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ be the holonomy homomorphism. It remains to describe the structure of the holonomy group $L(\Gamma)$ of $\mathbb{C}^n/\Gamma$. First of all note that $L(\Gamma) \leq \mathrm{U}(n)$. The (Bieberbach) group Γ has an extension as in (7.2):

$$1 \to \mathbb{Z}^2 \to \Gamma \xrightarrow{p} \widehat{\Gamma} \to 1 \tag{7.3}$$

where $\mathbb{C}^{n-1}/\widehat{\Gamma}$ is a $2(n-1)$-dimensional complex euclidean space. As Γ normalizes $\mathbb{C}$ $(\geq \mathbb{Z}^2)$, we have

$$L((n, \alpha)) = \left\{ \begin{pmatrix} \bar{\phi}(\alpha) & 0 \\ 0 & B_\alpha \end{pmatrix} \right\} \leq \mathrm{U}(n) \qquad \forall\, (n, \alpha) \in \Gamma. \tag{7.4}$$

If we recall that $\widehat{\Gamma} \leq \mathrm{E}_{\mathbb{C}}(n-1) = \mathbb{C}^n \rtimes \mathrm{U}(n-1)$, then the action of Γ_{n-1} on $\mathbb{C}^{n-1}$ is described as

$$\alpha(y) = (b_\alpha, B_\alpha)(y) = b_\alpha + B_\alpha(y), \qquad \alpha \in \widehat{\Gamma}, \quad y \in \mathbb{C}^{n-1}.$$

By the induction hypothesis we assume that $L(\widehat{\Gamma}) = \{B_\alpha\} \leq \prod_{i=2}^n H_i$ where each H_i is isomorphic to one of $\{1\}$, $\mathbb{Z}_2$, $\mathbb{Z}_4$ or $\mathbb{Z}_6$.

Noting that $H_1 = \phi(\{\alpha\}) = \{\pm 1\}$, $\{\pm\mathbf{i}\}$ or $\{e^{\pm\mathbf{i}\pi/3}\}$ from (2.6), we find from (7.4) that $L(\Gamma) \leq \prod_{i=1}^n H_i$. This proves the induction step. $\square$

Remark 7.3. By the hypothesis $[f] \in H^2_\phi(\widehat{\Gamma}; \mathbb{Z}^2)$ has finite order, say ℓ. Let $\ell \cdot f = \delta^1 \widetilde{\lambda}$ for some function $\widetilde{\lambda}\colon \widehat{\Gamma} \to \mathbb{Z}^2$. Putting $\lambda = \ell/\widetilde{\lambda}\colon \Gamma_{n-1} \to \mathbb{C}$, we obtain

$$f = \delta^1 \lambda. \tag{7.5}$$

We have another holomorphic Seifert action of Γ on $\mathbb{C}^n$ associated with the extension (7.3):

$$(n, \alpha)(x, y) = (n + \phi(\alpha)(x) + \lambda(\alpha), \alpha y) \qquad \forall\,(n, \alpha) \in \Gamma, \quad \forall\,(x, y) \in \mathbb{C}^n. \tag{7.6}$$

Then for $(n, \alpha) \in \Gamma$, the Seifert action (7.6) of Γ on $\mathbb{C} \times \mathbb{C}^{n-1} = \mathbb{C}^n$ is identified with the euclidean action:

$$(n, \alpha)\begin{bmatrix} x \\ y \end{bmatrix} = \left(\begin{bmatrix} n + \lambda(\alpha) \\ b_\alpha \end{bmatrix}, \begin{pmatrix} \phi(\alpha) & 0 \\ 0 & B_\alpha \end{pmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix}. \tag{7.7}$$

If we put

$$\rho((n, \alpha)) = \left(\begin{bmatrix} n + \lambda(\alpha) \\ b_\alpha \end{bmatrix}, \begin{pmatrix} \phi(\alpha) & 0 \\ 0 & B_\alpha \end{pmatrix}\right), \tag{7.8}$$

then this gives a faithful homomorphism $\rho\colon \Gamma \to \mathrm{E}_\mathbb{C}(n)$. We obtain a compact complex euclidean space form $\mathbb{C}^n/\rho(\Gamma)$. By the Bieberbach theorem, Γ is conjugate to $\rho(\Gamma)$ by some element $f \in \mathrm{A}(2n) = \mathbb{R}^{2n} \rtimes \mathrm{GL}(2n, \mathbb{R})$. Two complex euclidean space forms $\mathbb{C}^n/\Gamma$ and $\mathbb{C}^n/\rho(\Gamma)$ are affinely diffeomorphic. In general they are different holomorphic Bieberbach classes.

Remark 7.4. We have a similar result for an S^1-fibered nilBott manifold of finite type. In fact, it is diffeomorphic to a euclidean space form with holonomy group isomorphic to $(\mathbb{Z}_2)^s$, $0 \leq s \leq n$ (cf. [16]).

7.2. Kähler Bott tower. An example of finite type is a Kähler torus-Bott manifold, i.e. a torus-Bott manifold which admits a Kähler structure. More precisely, let $T^1_\mathbb{C} \to M_m \xrightarrow{p_m} M_{m-1}$ be a holomorphic torus-Bott tower as in (2.1). Suppose that

(1) each M_m supports a Kähler form Ω_m;

(2) $\mathbb{C} \to X_m \xrightarrow{p_m} X_{m-1}$ is the equivariant principal holomorphic bundle in which p_m is a Kähler submersion;

(3) $\mathbb{C}$ leaves Ω_m invariant ($m = 1, \dots, n$).

Then (2.1) is said to be a *Kähler Bott tower*. The top space $M = M_n$ is said to be a Kähler Bott manifold.

The following theorem is inspired by the result of Carrell [3] (see also [12]).

Theorem 7.5. *Let (M, Ω) be a Kähler Bott manifold. Then M is biholomorphic to the complex euclidean space form $\mathbb{C}^n/\Gamma$ where $L(\Gamma) = \prod_{i=1}^n H_i$.*

Proof. To apply Theorem 7.2, it suffices to show that each cocycle $[f]$ representing (2.2) is of finite order in $H^2_\phi(\pi_{m-1}; \mathbb{Z}^2)$. In fact, there is a central group extension $1 \to \mathbb{Z}^2 \to \Delta_m \xrightarrow{p_m} \Delta_{m-1} \to 1$ from (2.7). Put $T^1_\mathbb{C} = \mathbb{C}/\mathbb{Z}^2$, $Y_m = X_m/\Delta_m$ and $Y_{m-1} = X_{m-1}/\Delta_{m-1}$. Then M_m has a finite covering Y_m which admits a principal holomorphic fibration

$$T^1_\mathbb{C} \to Y_m \xrightarrow{q_m} Y_{m-1}. \tag{7.9}$$

Then it is proved in [3] (see also [12, Corollary 2.5]) that the Kähler isometric action of $T^1_\mathbb{C}$ is *homologically injective*, i.e. the orbit map $\mathrm{ev}(t) = ty$ at a point $y \in Y_m$ induces an injective homomorphism $\mathrm{ev}_*\colon H_1(T^1_\mathbb{C}; \mathbb{Z}) = \mathbb{Z}^2 \to H_1(Y_m; \mathbb{Z})$. This implies that Δ_m has a finite index normal splitting subgroup, so the representative cocycle of π_m in $H^2_\phi(\pi_{m-1};\mathbb{Z}^2)$ has finite order (see [5] for details). By Theorem 7.2, M is *biholomorphic* to a complex euclidean space form $\mathbb{C}^n/\Gamma$ with the holonomy group $L(\Gamma)=\prod_{i=1}^n H_i$. □

Remark 7.6. It follows from the result of Hasegawa [7] and Baues and Cortés [2] that a compact aspherical Kähler manifold with virtually solvable fundamental group is *biholomorphic* to a complex euclidean space form. As the fundamental group of a Kähler Bott manifold is virtually nilpotent by Lemma 2.3, the above theorem is obtained from this result except for the holonomy group characterization.

8. HOLOMORPHIC TORUS-BOTT TOWER OF INFINITE TYPE

We study a holomorphic torus-Bott tower of infinite type. It is hard to determine a *holomorphic classification* of holomorphic torus-Bott manifolds of *infinite type* in higher dimension. Recall the following facts about holomorphic torus-Bott manifolds of *infinite type*:

- The fundamental group is virtually nilpotent (but not abelian).
- A holomorphic torus-Bott manifold of infinite type is a non-Kähler manifold.

8.1. Four-dimensional holomorphic torus-Bott manifolds. It follows from the classification of complex surfaces that a four-dimensional holomorphic torus-Bott manifold is finitely covered by either $T^2_{\mathbb{C}}$ or $S^1\times\mathcal{N}/\Delta$ where $\mathcal{N}$ is a three-dimensional Heisenberg Lie group isomorphic to the 3×3 upper triangular unipotent matrices with lattice Δ.

Proposition 8.1. *A four-dimensional holomorphic torus-Bott manifold is biholomorphic to either* $T^2_{\mathbb{C}}/F$ *or* $S^1\times\mathcal{N}^3/\Delta$ *where F is a finite group of* $\mathrm{U}(2)$ *and Δ is a discrete uniform subgroup of* $\mathcal{N}\rtimes\mathrm{U}(1)$.

8.2. Six-dimensional examples of infinite type. As a special case of six-dimensional holomorphic torus-Bott manifolds of infinite type, there is a nontrivial holomorphic principal torus bundle over a complex 2-torus which is a holomorphic principal nilmanifold: $T^1_{\mathbb{C}}\to N_3/\Gamma\xrightarrow{q_3}T^2_{\mathbb{C}}$. Here N_3 is a two-step nilpotent Lie group with a left invariant complex structure. There is a classification of six-dimensional nilpotent Lie algebras with left invariant complex structure in [18, 19]. As b_1 is either 4 or 5 in this case except for $\mathbb{C}^3$, the classification gives

Proposition 8.2. *A six-dimensional holomorphic torus-Bott manifold over a four-dimensional complex euclidean space form is biholomorphic to the quotient of the following nilpotent Lie group by a cocompact subgroup acting properly discontinuously*:

- $\mathbb{C}^3$;
- $\mathcal{N}^3\times\mathcal{N}^3$ (*Lie algebra* $\mathfrak{h}_2$);
- $\mathbb{R}^+\times\mathcal{N}^5$ (*Lie algebra* $\mathfrak{h}_3$);
- *the Iwasawa Lie group* $\mathcal{L}_3$ (*Lie algebra* $\mathfrak{h}_5$);
- *the Nilpotent Lie group corresponding to* $\mathfrak{h}_4$;
- *the Nilpotent Lie group corresponding to* $\mathfrak{h}_6$;
- $\mathbb{R}^3\times\mathcal{N}^3$ (*Lie algebra* $\mathfrak{h}_8$).

Remark 8.3. Here $\mathcal{N}_8=\mathbb{R}^3\times\mathcal{N}^3$ is viewed as $\mathbb{R}\times\mathbb{R}\to\mathcal{N}_8\to\mathbb{R}^2\times\mathbb{C}$. There is another exact sequence $1\to\mathbb{C}\to\mathcal{N}_8\to\mathbb{R}^+\times\mathcal{N}^3\to 1$ such that $[\mathcal{N}_8,\mathcal{N}_8]=\mathbb{R}\le\mathbb{C}$. Note that this is a splitting exact sequence $\mathcal{N}_8=\mathbb{C}\times(\mathbb{R}^+\times\mathcal{N}^3)$ but the base space $\mathbb{R}^+\times\mathcal{N}^3$ is not $\mathbb{C}^2$.

It is interesting to find what non-Kähler geometric structure exists on a holomorphic torus-Bott manifold of infinite type. We have found two such classes in general dimension. The following result is obtained in [8, 13].

Theorem 8.4. (i) *A $(2n+2)$-dimensional compact infranil-manifold M admits a locally conformal Kähler structure if and only if $M = \mathbb{R} \times \mathcal{N}/\Gamma$ where $\mathcal{N}$ is the Heisenberg nilpotent Lie group and $\Gamma \leq \mathbb{R} \times (\mathcal{N} \rtimes \mathrm{U}(n))$ is a discrete cocompact subgroup. In this case M has the holomorphic torus fibration over the complex euclidean orbifold*:

$$T^1_{\mathbb{C}} \to M \to \mathbb{C}^n/\Gamma.$$

Some finite cover M' of $\mathbb{R} \times \mathcal{N}/\Gamma$ is a holomorphic torus-Bott manifold of infinite type:

$$M' \xrightarrow{T^1_{\mathbb{C}}} T^n_{\mathbb{C}} \to \ldots \to \{\mathrm{pt}\}.$$

(ii) *There exists a $2(2n+1)$-dimensional complex nilpotent Lie group $\mathcal{L} = \mathcal{L}_{2n+1}$ and a torsion-free discrete cocompact subgroup Γ of the semidirect product $\mathcal{L} \rtimes (\mathrm{Sp}(n) \cdot S^1)$ such that a $2(2n+1)$-dimensional complex infranil-manifold $\mathcal{L}/\Gamma$ admits a complex contact structure. The infranil-manifold $\mathcal{L}/\Gamma$ supports a holomorphic torus bundle over the quaternionic euclidean orbifold*:

$$T^1_{\mathbb{C}} \to \mathcal{L}/\Gamma \to \mathbb{H}^n/\Delta.$$

Moreover, some finite cover M' of $\mathcal{L}/\Gamma$ is a holomorphic torus-Bott manifold of infinite type:

$$M' \xrightarrow{T^1_{\mathbb{C}}} T^{2n}_{\mathbb{C}} \to \ldots \to \{\mathrm{pt}\}.$$

Here $\mathcal{L}_3$ is the Iwasawa complex nilpotent Lie group.

REFERENCES

1. O. Baues, “Infra-solvmanifolds and rigidity of subgroups in solvable linear algebraic groups,” Topology **43** (4), 903–924 (2004).
2. O. Baues and V. Cortés, “Aspherical Kähler manifolds with solvable fundamental group,” Geom. Dedicata **122**, 215–229 (2006).
3. J. B. Carrell, “Holomorphically injective complex toral actions,” in *Proc. 2nd Conf. on Compact Transformation Groups* (Springer, Berlin, 1972), Part II, Lect. Notes Math. **299**, pp. 205–236.
4. P. E. Conner and F. Raymond, “Actions of compact Lie groups on aspherical manifolds,” in *Topology of Manifolds: Proc. Univ. Georgia, 1969* (Markham, Chicago, 1971), pp. 227–264.
5. P. E. Conner and F. Raymond, “Injective operations of the toral groups,” Topology **10** (4), 283–296 (1971).
6. P. E. Conner and F. Raymond, “Holomorphic Seifert fibering,” in *Proc. 2nd Conf. on Compact Transformation Groups* (Springer, Berlin, 1972), Part II, Lect. Notes Math. **299**, pp. 124–204.
7. K. Hasegawa, “A note on compact solvmanifolds with Kähler structures,” Osaka J. Math. **43**, 131–135 (2006).
8. K. Hasegawa and Y. Kamishima, “Compact homogeneous locally conformally Kähler manifolds,” arXiv: 1312.2202 [math.CV].
9. F. Hirzebruch, *Topological Methods in Algebraic Geometry* (Springer, Berlin, 1966), Grundl. Math. Wiss. **131**.
10. Y. Kamishima, K. B. Lee, and F. Raymond, “The Seifert construction and its applications to infranilmanifolds,” Q. J. Math., Oxford, Ser. 2, **34**, 433–452 (1983).
11. Y. Kamishima and M. Nakayama, “Topology of nilBott tower of aspherical manifolds” (in preparation).
12. Y. Kamishima and M. Nakayama, “Torus actions and the Halperin–Carlsson conjecture,” arXiv: 1206.4790v1 [math.GT].
13. Y. Kamishima and A. Tanaka, “On complex contact similarity manifolds,” J. Math. Res. **5** (4), 1–10 (2013).
14. K. B. Lee and F. Raymond, *Seifert Fiberings* (Am. Math. Soc., Providence, RI, 2010), Math. Surv. Monogr. **166**.
15. S. Murakami, “Sur certains espaces fibrés principaux holomorphes admettant des connexions holomorphes,” Osaka Math. J. **11**, 43–62 (1959).
16. M. Nakayama, “On the S^1-fibred nilBott tower,” Osaka J. Math. **51**, 67–89 (2014); arXiv: 1110.1164 [math.AT].
17. S. Rollenske, “Geometry of nilmanifolds with left-invariant complex structure and deformations in the large,” Proc. London Math. Soc., Ser. 3, **99** (2), 425–460 (2009).
18. S. M. Salamon, “Complex structures on nilpotent Lie algebras,” J. Pure Appl. Algebra **157**, 311–333 (2001).
19. L. Ugarte, “Hermitian structures on six-dimensional nilmanifolds,” Transform. Groups **12** (1), 175–202 (2007).

This article was submitted by the authors in English