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Geometry of Compact Complex Manifolds with Maximal Torus Action

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Abstract—We study the geometry of compact complex manifolds M equipped with a *maximal* action of a torus $T=(S^1)^k$. We present two equivalent constructions that allow one to build any such manifold on the basis of special combinatorial data given by a simplicial fan Σ and a complex subgroup $H\subset T_{\mathbb{C}}=(\mathbb{C}^*)^k$. On every manifold M we define a canonical holomorphic foliation $\mathcal{F}$ and, under additional restrictions on the combinatorial data, construct a transverse Kähler form $\omega_{\mathcal{F}}$. As an application of these constructions, we extend some results on the geometry of moment–angle manifolds to the case of manifolds M.

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1. INTRODUCTION

Since the 1970s, toric varieties V_Σ have played a particularly important role in algebraic geometry [1–3]. Due to the existence of large groups of symmetries, toric varieties could be explicitly described in terms of combinatorial geometry. Numerous results that relate the geometric properties of toric varieties to the characteristics of the underlying combinatorial objects provide powerful tools for enumerative algebraic geometry, combinatorial geometry [4], number theory [5], and algebraic topology [6].

Until recently, the situation in the complex-analytic category was far less well studied. There were very few explicit examples of complex-analytic manifolds that admit torus actions and no formal notion of a “large group of symmetries.” However, since the early 2000s, several new families of complex manifolds have been discovered that admit a compact action of a torus $T=(S^1)^m$ (see [7–9]) and results about their complex geometry have been obtained [10–12]. In 2010, in [13] the authors constructed a large family of compact complex manifolds equipped with a torus action; this family includes all previous examples as special cases. Finally, in 2013, in [14] the notion of a *maximal torus action* was introduced and a construction that yields all compact complex manifolds equipped with a maximal torus action was presented.

In this paper we prove that the family of manifolds presented in [13] coincides with the set of compact complex manifolds proposed in [14]. Moreover, the approach of [13] turns out to be a complex-analytic analog of the Cox–Batyrev construction of toric varieties [3].

Despite the new explicit construction of compact complex manifolds equipped with a maximal torus action, there are very few results on their geometry. Since almost all of them are non-Kähler, most of the methods of complex geometry are not applicable. We introduce a canonical holomorphic foliation $\mathcal{F}$ on the manifolds under consideration and, under some restrictions on the underlying combinatorial data, construct a differential form $\omega_{\mathcal{F}}$ that is transverse Kähler with respect to the foliation $\mathcal{F}$. Thus, we generalize the results of [11] on the complex geometry of *moment–angle manifolds*. As an application of this construction, we prove some results on meromorphic functions and analytic subsets of compact complex manifolds equipped with a maximal torus action.

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2. MAXIMAL TORUS ACTIONS

Let M be a smooth compact manifold without boundary equipped with a smooth effective action of the torus $T = (S^1)^k$. In this section we are interested in certain topological constraints arising due to the existence of a torus action on the manifold M.

Theorem 2.1 [15] (see also [14]). *Let T be a torus acting effectively and smoothly on a smooth manifold M. Then for every $x \in M$ one has*

$$\dim M \geq \dim T + \dim T_x, \tag{2.1}$$

where $T_x \subset T$ is the stabilizer of x.

Theorem 2.1 justifies the introduction of the notion of a *maximal torus action* on a smooth manifold M (see [14]).

Definition 2.2 [14, Definition 2.1]. An effective action of a torus T on a smooth manifold M is said to be *maximal* if there exists an $x \in M$ such that inequality (2.1) turns into the equality:

$$\dim M = \dim T + \dim T_x. \tag{2.2}$$

The following proposition is the direct consequence of Theorem 2.1:

Proposition 2.3 [14, Lemma 2.2]. *Let T be a torus acting smoothly and effectively on a connected smooth manifold M. Assume that the induced action of a toric subgroup $T_0 \subset T$ is maximal. Then $T_0 = T$.*

Proposition 2.3 implies that a maximal action $T\colon M$ could not be extended to an action of a larger torus $T' \supset T$. Let us provide several examples of manifolds equipped with a maximal torus action.

Example 2.4. There are two possible extreme cases in equation (2.2):

(i) $\dim T_x = 0$ for some (and hence for any) point $x \in M$;

(ii) $\dim T_x = \dim T$ for some point x.

1. In case (i) one obtains only tori T acting on themselves by translations $T \times T \to T$. This action is free, i.e., for any point $x \in T$ the stabilizer T_x is trivial, $\dim T_x = 0$, and inequality (2.1) turns into the equality.

2. Case (ii) already provides a lot of interesting examples, including, in particular, compact symplectic manifolds equipped with a Hamiltonian action of a half-dimensional torus, which were classified by Delzant [16]. For instance, let $T = U(1)^n$ be a torus acting on a complex projective space $\mathbb{C}\mathrm{P}^n$ via coordinatewise multiplication in homogeneous coordinates:

$$(t_1, \dots, t_n) \cdot [z_0 : z_1 : \dots : z_n] = [z_0 : t_1 z_1 : \dots : t_n z_n].$$

In this case the point $x = [1 : 0 : \dots : 0]$ is fixed, i.e., $T_x = T$, so $\dim \mathbb{C}\mathrm{P}^n = \dim T + \dim T_x$ and the action is maximal.

3. Let $S^{2n-1} = \{z \subset \mathbb{C}^n \colon |z| = 1\}$ be a unit sphere in $\mathbb{C}^n$. The torus $T = U(1)^n$ acting on $\mathbb{C}^n$ via coordinatewise multiplication preserves the sphere. The stabilizer of the point $x = (1, 0, \dots, 0)$ is the coordinate subtorus $T_x = \{(1, z_2, \dots, z_n) \in T\}$, and inequality (2.1) again turns into an equality.

Note that the class of *smooth* manifolds equipped with a maximal torus action is very large and apparently does not have a complete description. In particular, given a manifold M equipped with a maximal torus T action and any manifold N with an effective torus T action, one can construct a new manifold $M \#_{T\cdot y} N$ by taking the equivariant connected sum along a free T orbit:

$$M \#_{T\cdot y} N = \big(M \setminus U(T \cdot y)\big) \cup_{\partial U(T\cdot y)} \big(N \setminus U(T \cdot y)\big), \tag{2.3}$$

where $U(T \cdot y)$ is an equivariant tubular neighborhood. Since the maximality of a torus action is provided by conditions at one point $x \in M$, the action of the torus T on $M \#_{T\cdot y} N$ is again maximal.

As we will see further, the situation in the complex-analytic category is fundamentally different. In particular, the results of [14] imply that all compact complex manifolds equipped with a maximal torus action have an explicit description similar to the description of smooth complete toric varieties [2]. We present the construction from [14] and an analog of the Cox–Batyrev construction [3] for manifolds equipped with a maximal torus action.

3. COMPLEX MANIFOLDS

Definition 3.1. A smooth action of a group G on an almost complex manifold M is said to *preserve an almost complex structure* J if for any $g \in G$ the differential of the multiplication $m_g\colon M \to M$ by g commutes with the operator of almost complex structure:

$$dm_g \circ J = J \circ dm_g.$$

In what follows, all groups act on *complex analytic* manifolds while preserving the corresponding almost complex structure.

A torus action $T\colon M$ defines a homomorphism from the Lie algebra $\mathfrak{t}$ of T to the Lie algebra of vector fields on the manifold, $\rho\colon \mathfrak{t} \to \mathcal{L}(M)$. This homomorphism could be complexified by means of the almost complex structure operator, $\rho_{\mathbb{C}}\colon \mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \oplus i\mathfrak{t} \to \mathcal{L}(M)$. In [14] it is proved that the integrability of the almost complex structure guarantees that $\rho_{\mathbb{C}}$ is a homomorphism of Lie algebras. Consequently, the group $\mathfrak{t}_{\mathbb{C}}$ acts on M via the exponential map. Since the lattice $N \subset \mathfrak{t}$ dual to the character lattice of T acts trivially on M, the action of $\mathfrak{t}_{\mathbb{C}}$ descends to an action of $\mathfrak{t}_{\mathbb{C}}/N$. This implies the following proposition:

Proposition 3.2 [14, Sect. 3]**.** *Let T be a torus acting on a complex manifold M. The action $T\colon M$ can be extended to a complexified action of the algebraic torus $T_{\mathbb{C}} \simeq (\mathbb{C}^*)^{\dim T}$ on M.*

Note that the action $T_{\mathbb{C}}\colon M$ is not necessarily effective. Thus let us introduce the following group:

$$H := \{h \in T_{\mathbb{C}} \mid hx = x \ \forall x \in M\}. \tag{3.1}$$

Since the torus action preserves the complex structure, H is a commutative complex subgroup. Moreover, since the compact part $T \subset T_{\mathbb{C}}$ acts effectively, it follows that $H \cap T = \{e\}$ and $H \simeq \mathbb{C}^k$ for some $k \in \mathbb{Z}$.

Let an action $T\colon M$ be maximal. In this case the subgroup $H \subset T_{\mathbb{C}}$ allows one to construct a *canonical holomorphic foliation* $\mathcal{F}$ on M, which turns out to be an extremely effective tool for studying the complex geometry of M.

Construction 3.3 (canonical holomorphic foliation)**.** The results of [14] imply that the subgroup $T_{\mathbb{C}}/H$ acts effectively on the manifold M with a dense open orbit on which the action is free. Let $\mathfrak{h} \subset \mathfrak{t}_{\mathbb{C}}$ be the Lie algebra of H and $\overline{\mathfrak{h}}$ be the complex conjugate of the Lie algebra $\mathfrak{h}$ with respect to the decomposition $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \oplus i\mathfrak{t}$. Note that both $\mathfrak{h}, \overline{\mathfrak{h}} \subset \mathfrak{t}_{\mathbb{C}}$ are complex Lie algebras.

The orbits of $H' = \exp \overline{\mathfrak{h}} \subset T_{\mathbb{C}}$ define a holomorphic foliation $\mathcal{F}$ on M, which is further referred to as *canonical.* Since $\mathfrak{h} \cap \mathfrak{t} = \{0\}$, the vector spaces $\mathfrak{h}$ and $\overline{\mathfrak{h}}$ are transverse; thus the group $H' \cap H$ is discrete and the leaves of $\mathcal{F}$ inside the open $T_{\mathbb{C}}/H$ orbit are isomorphic to $H'/(H \cap H') \simeq \mathbb{C}^{\dim_{\mathbb{C}} H}/\Lambda$, where $\Lambda \subset \mathbb{C}^{\dim_{\mathbb{C}} H}$ is some discrete subgroup. The dimension of the foliation $\mathcal{F}$ is

$$\dim_{\mathbb{C}} \mathcal{F} = \dim_{\mathbb{C}} H = \dim_{\mathbb{C}} T_{\mathbb{C}} - \dim_{\mathbb{C}} M = \frac{1}{2}(\dim T - \dim T_x).$$

Note that the leaves of the foliation $\mathcal{F}$ are not necessarily closed.

Example 3.4 (Hopf surface). Let $A\colon \mathbb{C}^2 \to \mathbb{C}^2$ be a semisimple linear operator with eigenvalues $\lambda_1 = e^{2\pi i q_1}$ and $\lambda_2 = e^{2\pi i q_2}$ such that $|\lambda_1|, |\lambda_2| > 1$. A *Hopf surface* is a manifold

$$\mathcal{H} := (\mathbb{C}^2 \setminus \{0\})/\Gamma,$$

where $\Gamma \simeq \mathbb{Z}$ is a group acting on $\mathbb{C}^2$ generated by A. It is straightforward to check that $\mathcal{H}$ is a complex manifold diffeomorphic to $S^3 \times S^1$.

The group $G = (\mathbb{C}^*)^2/\Gamma \simeq (S^1)^3 \times \mathbb{R}$ acts on $\mathcal{H}$ with a dense open orbit, and the toric subgroup

$$T \simeq (S^1)^3 \subset (\mathbb{C}^*)^2/\Gamma, \qquad (e^{2\pi i t_1}, e^{2\pi i t_2}, e^{2\pi i t_3}) \mapsto (e^{2\pi i (t_1 + q_1 t_3)}, e^{2\pi i (t_2 + q_2 t_3)}),$$

acts maximally: the stabilizer T_z of $z = [(1,0)] \in \mathcal{H}$ is $\{(1, e^{2\pi i t_2}, 1)\}$, so

$$\dim T + \dim T_z = \dim_{\mathbb{R}} \mathcal{H} = 4.$$

In this case $T_{\mathbb{C}} \simeq (\mathbb{C}^*)^3$ and, under the above identification $T \simeq (S^1)^3$, the kernel of the action $T_{\mathbb{C}} \colon \mathcal{H}$ is the subgroup $H = \{(e^{wq_1}, e^{wq_2}, e^{-w}) \mid w \in \mathbb{C}\}$ and $G = T_{\mathbb{C}}/H$.

4. QUOTIENTS OF NONSINGULAR TORIC VARIETIES

In this section we discuss in detail two equivalent constructions providing all compact complex manifolds equipped with a maximal torus action. Since both of them are closely related to the construction of *toric varieties*, we start with some basic facts required for their description and classification. A detailed introduction into the theory of toric varieties can be found in [2, 1].

4.1. Toric varieties.

Definition 4.1. Let $\mathbf{a}_1, \dots, \mathbf{a}_k \in N_{\mathbb{R}} \simeq \mathbb{R}^n$ be a set of vectors. A *polyhedral cone* spanned by the vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$ is the set

$$\sigma = \{\mu_1 \mathbf{a}_1 + \ldots + \mu_k \mathbf{a}_k \mid \mu_i \geq 0\}.$$

A cone σ is *strictly convex* if it does not contain a line. A *strictly convex* cone is *simplicial* if it is spanned by linearly independent vectors. A cone σ is *regular* if it is spanned by a part of a basis of some fixed lattice $N \subset N_{\mathbb{R}}$, $N \simeq \mathbb{Z}^n$.

The *dual cone* for a cone $\sigma \subset N_{\mathbb{R}}$ is the set

$$\check{\sigma} = \{\mathbf{u} \in N^*_{\mathbb{R}} \mid \langle \mathbf{u}, \mathbf{a} \rangle \geq 0 \ \forall \mathbf{a} \in \sigma\}.$$

Definition 4.2. A *fan* is a set of cones $\Sigma = \{\sigma_i\}_i$ such that

(1) any face of each cone is an element of the set;

(2) the intersection of any pair of cones is a face of each of them.

A fan Σ is *regular* if all its cones are regular, and is *complete* if $\bigcup_i \sigma_i = N_{\mathbb{R}}$.

Definition 4.3. A *toric variety* is a normal irreducible algebraic variety V containing an algebraic torus $T_{\mathbb{C}}$ as an open dense subset such that the action of the torus on itself extends to the whole variety.

Examples of toric varieties are $\mathbb{C}^n$, $\mathbb{C}\mathrm{P}^n$, and $\mathbb{C}^n \setminus \{0\}$.

The main result of the theory of toric varieties establishes a one-to-one correspondence between nonsingular toric varieties and regular fans in the Lie algebra $\mathfrak{t}$ of a compact torus $T \subset T_{\mathbb{C}}$. Namely, every smooth toric variety V can by obtained via the following construction:

Construction 4.4 (toric varieties). Let Σ be a fan in the Lie algebra $\mathfrak{t}$ of a compact torus T. Suppose the fan Σ is nonsingular with respect to the lattice $N \subset \mathfrak{t}$ dual to the character lattice.

For each cone $\sigma \in \Sigma$ we define an algebra $\mathbb{C}[\check{\sigma} \cap N^*]$ and an open chart $U_\sigma = \operatorname{Spec}\mathbb{C}[\check{\sigma} \cap N^*]$. The set of all charts U_σ is partially ordered by inclusion in the same way as the set of cones of Σ. Let us introduce the scheme

$$V_\Sigma := \varprojlim_{\sigma\in\Sigma} U_\sigma.$$

The scheme V_Σ turns out to be a nonsingular variety equipped with an action of the algebraic torus $T_\mathbb{C} = (\mathfrak{t}/N)_\mathbb{C}$, which acts with an open dense orbit. The variety V_Σ is compact if and only if the fan Σ is complete (see [2]).

4.2. Compact complex manifolds with maximal torus actions. There is a similar classification of compact complex manifolds equipped with a maximal torus action. We start with the construction from [13].

Construction 4.5 (quotient construction I)**.** Let $\mathcal{K}$ be a *simplicial complex* on the set of vertices $[m] = \{1,\dots,m\}$, i.e., a family of subsets of $[m]$ closed under the operation of taking subsets. Let $\Sigma_\mathcal{K}$ be a simplicial fan in $\mathbb{R}^m$:

$$\Sigma_\mathcal{K} = \bigcup_{I\in\mathcal{K}} \langle e_i \mid i \in I\rangle_{\mathbb{R}_\geq}, \tag{4.1}$$

where $e_1 \dots, e_m$ is the fixed basis of $\mathbb{R}^m$, I runs over all simplices of $\mathcal{K}$, and $\langle e_i \mid i \in I\rangle_{\mathbb{R}_\geq}$ is the cone spanned by the vectors e_i. Let $U(\mathcal{K}) := V_{\Sigma_\mathcal{K}}$ be the corresponding toric variety equipped with the action of the torus $T_\mathbb{C} \simeq (\mathbb{C}^*)^m$, and let $\mathfrak{t}$ be the Lie algebra of the compact torus $T \subset T_\mathbb{C}$. It is easy to check that $U(\mathcal{K})$ is the complement of the arrangement of coordinate subspaces of $\mathbb{C}^m$:

$$U(\mathcal{K}) = \mathbb{C}^m \setminus \bigcup_{J\notin\mathcal{K}} \{z_j = 0 \mid j \in J\}.$$

Let $\mathfrak{h} \subset \mathfrak{t}_\mathbb{C} = \mathfrak{t} \oplus i\mathfrak{t}$ be a complex subspace satisfying the following two conditions:

(a) the group $H = \exp\mathfrak{h} \subset T_\mathbb{C} \simeq (\mathbb{C}^*)^{[m]}$ intersects trivially the coordinate subtori of the form $(\mathbb{C}^*)^I$ for $I \in \mathcal{K}$;

(b) the projection $q\colon \mathfrak{t} \to \mathfrak{t}/p(\mathfrak{h})$, where $p\colon \mathfrak{t}_\mathbb{C} \to \mathfrak{t}$ is the natural projection on the real part, maps bijectively the fan $\Sigma_\mathcal{K}$ to the complete fan $q(\Sigma_\mathcal{K})$.

As proved in [13], the conditions on the subspace $\mathfrak{h}$ guarantee that the group H acts freely and properly on $U(\mathcal{K})$ and the orbit space $M = U(\mathcal{K})/H$ is a compact complex manifold equipped with a maximal torus action.

Another construction providing a large family of manifolds equipped with a maximal torus action is presented in [14]:

Construction 4.6 (quotient construction II)**.** Let V_Σ be a nonsingular toric variety equipped with an action of a torus $T_\mathbb{C}$; $\mathfrak{t}$ is the Lie algebra of the compact torus $T \subset T_\mathbb{C}$. Let $\mathfrak{h} \subset \mathfrak{t}_\mathbb{C} = \mathfrak{t} \oplus i\mathfrak{t}$ be a complex subspace satisfying the following two conditions:

(a) $\mathfrak{h} \cap \mathfrak{t} = \{0\}$, i.e., the restriction $p|_\mathfrak{h}$ of the projection $p\colon \mathfrak{t}_\mathbb{C} \to \mathfrak{t}$ is the inclusion;

(b) the projection $q\colon \mathfrak{t} \to \mathfrak{t}/p(\mathfrak{h})$ maps bijectively the fan Σ to the complete fan $q(\Sigma)$.

Consider the group $H := \exp\mathfrak{h} \subset T_\mathbb{C}$. It can be proved that conditions (a) and (b) imply that the group H acts on V_Σ freely and properly and the orbit space

$$M(\Sigma, \mathfrak{h}) := V_\Sigma/H$$

is a compact complex manifold equipped with a maximal torus T action.

A remarkable result of [14] is as follows:

Theorem 4.7 [14, Corollary 6.7]**.** *Any compact complex manifold equipped with a maximal torus action can be obtained via Construction* 4.6.

Note that the second construction operates with a larger family of toric varieties, while the first construction considers more general subgroups $H \subset T_{\mathbb{C}}$. Below we show that these two approaches are in fact equivalent.

Theorem 4.8. *Any compact complex manifold equipped with a maximal torus action can be obtained via Construction* 4.5.

Theorem 4.8 allows one to separate combinatorial (simplicial complex $\mathcal{K}$) and geometric (subspace $\mathfrak{h} \subset \mathbb{C}^m$) data from the data $(\Sigma, \mathfrak{h})$ defining the manifold $M(\Sigma, \mathfrak{h})$. Thus, Construction 4.5 is essentially an analog of the Cox–Batyrev construction of toric varieties [3].

Proof of Theorem 4.8. Let M be an arbitrary compact complex manifold equipped with a maximal torus action. According to Theorem 4.7 the manifold M is the quotient $M(\Sigma, \mathfrak{h}) = V_\Sigma/H$ for some fan $\Sigma \subset \mathfrak{t}$ and subspace $\mathfrak{h} \subset \mathfrak{t}_{\mathbb{C}}$.

It follows from the Cox–Batyrev construction [3] that any nonsingular toric variety V_Σ with an action of $T_{\mathbb{C}}$ is a G-quotient of $U(\mathcal{K})$ for some algebraic subgroup $G \subset (\mathbb{C}^*)^m$, where $\mathcal{K}$ is the partially ordered set of cones of Σ (which is a simplicial complex since Σ is regular) and $T_{\mathbb{C}} = (\mathbb{C}^*)^m/G$:

$$V_\Sigma = U(\mathcal{K})/G.$$

Let $\pi\colon (\mathbb{C}^*)^m \to (\mathbb{C}^*)^m/G$ be the natural projection. Then the manifold M is the orbit space of $\pi^{-1}(H) \subset (\mathbb{C}^*)^m$ acting on $U(\mathcal{K})$. This description almost coincides with Construction 4.5 except for the fact the group $\pi^{-1}(H)$ is not necessarily connected, i.e., it has the form $H' \times \Gamma$, where H' is connected and Γ is a finite abelian group. To correct this deficiency, we use the following simple proposition:

Proposition 4.9. *Let the group $\mathbb{C}^*$ act on a manifold M. Suppose that the subgroup $G_k \subset \mathbb{C}^*$ of k-th roots of unity acts freely on M. Then*

$$M/G_k \simeq (M \times \mathbb{C}^*)/\mathbb{C}^*,$$

where the group $\mathbb{C}^*$ acts on the manifold $M \times \mathbb{C}^*$ in the following manner:

$$w \cdot (m, z) = (w \cdot m, w^k z).$$

It follows from Proposition 4.9 that for some r the $H' \times \Gamma$ quotient of $U(\mathcal{K})$ coincides with the $H' \times (\mathbb{C}^*)^r$ quotient of $U(\mathcal{K}) \times (\mathbb{C}^*)^r$:

$$M = V_\Sigma/H = U(\mathcal{K})/(G \times H) = (U(\mathcal{K}) \times (\mathbb{C}^*)^r)/(H' \times (\mathbb{C}^*)^r).$$

Since the group actions $H\colon V_\Sigma$ and $G\colon U(\mathcal{K})$ are free, the group $H'' = H' \times (\mathbb{C}^*)^r$ acts freely as well; thus condition (a) of Construction 4.5 is satisfied. Condition (b) is also satisfied, since according to the Cox–Batyrev construction the fan $\Sigma_{\mathcal{K}}$ projects bijectively onto the fan Σ, which, in turn, according to condition (b) of Construction 4.6 projects bijectively onto the complete fan. □

In the example below, Constructions 4.5 and 4.6 coincide.

Example 4.10 (Hopf surface)**.** Let $\Sigma \subset \mathbb{R}^3$ be a fan with two one-dimensional cones spanned by the vectors $(1,0)$ and $(0,1)$. The corresponding toric variety V_Σ is $\mathbb{C}^2 \setminus \{0\} \times \mathbb{C}^*$.

Consider a complex subspace $\mathfrak{h} \subset \mathfrak{t}_{\mathbb{C}} \simeq \mathbb{C}^3$. It follows from conditions (a) and (b) of Construction 4.6 that $\dim_{\mathbb{C}} \mathfrak{h} = 1$, i.e., $\mathfrak{h} = \{(\alpha_1 z, \alpha_2 z, \alpha_3 z) \mid z \in \mathbb{C}\}$ for some $\alpha_i \in \mathbb{C}$. It is easy to check that $\mathfrak{h}$ satisfies conditions (a) and (b) if and only if the imaginary parts $\mathrm{Im}(\alpha_1/\alpha_3)$ and $\mathrm{Im}(\alpha_2/\alpha_3)$ have the same sign.

Given Σ and $H = \exp \mathfrak{h}$ as above, the manifold $M(\Sigma, \mathfrak{h})$ is

$$M(\Sigma, \mathfrak{h}) = V_\Sigma / H = (\mathbb{C}^2 \setminus \{0\} \times \mathbb{C}^*) / \{(e^{\alpha_1} z, e^{\alpha_2} z, e^{\alpha_3} z) \mid z \in \mathbb{C}\} = (\mathbb{C}^2 \setminus \{0\})/\Gamma,$$

where the generator of the group $\Gamma \simeq \mathbb{Z}$ acts on $\mathbb{C}^2 \setminus \{0\}$ via the coordinatewise multiplication by $(e^{2\pi i \alpha_1/\alpha_3}, e^{2\pi i \alpha_2/\alpha_3})$. Therefore, the manifold $M(\Sigma, \mathfrak{h})$ is a Hopf surface, and the conditions imposed on $\mathfrak{h}$ are equivalent to the conditions imposed on λ_i in Example 3.4.

5. COMPLEX GEOMETRY OF MANIFOLDS $M(\Sigma, \mathfrak{h})$

It follows from the general results on the cohomology ring of manifolds $M(\Sigma, \mathfrak{h})$ (see [17, Theorem 8.39; 7]) that almost all of them do not admit a symplectic structure: the top power of any element $\alpha \in H^2(M(\Sigma, \mathfrak{h}))$ is zero. Thus, most of the manifolds $M(\Sigma, \mathfrak{h})$ are non-Kähler. In this section we prove that despite the nonexistence of a Kähler structure, many of the manifolds $M(\Sigma, \mathfrak{h})$ admit a *transverse Kähler structure* $\omega_{\mathcal{F}}$, which vanishes along the canonical foliation $\mathcal{F}$ and is positive in the transverse directions. The form $\omega_{\mathcal{F}}$ turns out to be a powerful tool for studying the complex geometry of manifolds $M(\Sigma, \mathfrak{h})$. For example, in [11] the existence of the form $\omega_{\mathcal{F}}$ made it possible to describe all analytic subsets on certain *moment–angle manifolds*, i.e., manifolds $M(\Sigma, \mathfrak{h})$ corresponding to $\Sigma = \Sigma_{\mathcal{K}}$.

5.1. Transverse Kähler forms.

Definition 5.1. Let M be a complex manifold. A differential form $\omega \in \Lambda^{1,1}(M)$ is *transverse Kähler* with respect to the holomorphic foliation $\mathcal{F}$ if

(a) ω is closed, $d\omega = 0$;

(b) ω is nonnegative, i.e., $\omega(X, JX) \geq 0$ for any vector X;

(c) $\omega(X, JX) = 0$ if and only if the vector X is tangent to the foliation, $X \in T\mathcal{F}$.

Example 5.2 (Hopf surface)**.** Let $\mathcal{H}$ be a Hopf surface from Example 3.4 with identical λ_1 and λ_2. In this case the group Γ generated by (λ_1, λ_2) is a subgroup of $\mathbb{C}^*$ acting on $\mathbb{C}^2$ diagonally; therefore, $\mathcal{H}$ fibers over $\mathbb{C}\mathrm{P}^1$ with the fiber $\mathbb{C}^*/\Gamma$:

$$\mathbb{C}^2 \setminus \{0\} \xrightarrow{\Gamma} \mathcal{H} \xrightarrow{\mathbb{C}^*/\Gamma} \mathbb{C}\mathrm{P}_1.$$

Consider the differential form $\omega = \pi^* \omega_{\mathrm{FS}} \in \Lambda^{1,1}(\mathcal{H})$, where $\pi \colon \mathcal{H} \to \mathbb{C}\mathrm{P}^1$ is the projection and ω_{FS} is the Fubini–Study form on $\mathbb{C}\mathrm{P}^1$. Since the form ω_{FS} is positive, the form ω is transverse Kähler with respect to the foliation by the fibers of π.

Prior to constructing transverse Kähler forms on manifolds $M(\Sigma, \mathfrak{h})$, let us introduce some notions from convex geometry.

Definition 5.3. Let Σ be a complete fan in the vector space $N_{\mathbb{R}}$. Let us fix vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$ generating one-dimensional cones of Σ and a set of real numbers $b_1, \dots, b_m$. Consider m linear inequalities in the dual space $N_{\mathbb{R}}^*$:

$$\langle \mathbf{v}_i, \mathbf{u} \rangle + b_i \geq 0, \qquad i = 1, \dots, m, \tag{5.1}$$

where $\mathbf{u} \in N_{\mathbb{R}}^*$. For every cone $\sigma \in \Sigma$ of maximal dimension, let us define a *vertex* $\mathbf{u}_\sigma \in N_{\mathbb{R}}^*$ as the solution of the system of $\dim N_{\mathbb{R}}$ linear equations

$$\langle \mathbf{v}_i, \mathbf{u}_\sigma \rangle + b_i = 0, \qquad v_i \in \sigma, \tag{5.2}$$

where $\mathbf{v}_i$ runs over the generators of the cone σ. It follows from the completeness of Σ that system (5.2) has a unique solution.

A complete fan Σ is said to be *normal* if there exists a collection of numbers $\{b_i\}_1^m$ such that for every vertex $\mathbf{u}_\sigma$ and linear form $\langle \mathbf{v}_i, \mathbf{u}\rangle + b_i$

(a) $\langle \mathbf{v}_i, \mathbf{u}_\sigma\rangle + b_i \ge 0$;

(b) $\langle \mathbf{v}_i, \mathbf{u}_\sigma\rangle + b_i = 0$ if and only if $\mathbf{v}_i \in \sigma$.

In this case the fan Σ is also referred to as the *normal* fan of a polytope given by the system of inequalities (5.1).

A complete fan Σ is said to be *weakly normal* if there exists a collection of numbers $\{b_i\}_1^m$ such that for every vertex $\mathbf{u}_\sigma$ and linear form $\langle \mathbf{v}_i, \mathbf{u}\rangle + b_i$

(a) $\langle \mathbf{v}_i, \mathbf{u}_\sigma\rangle + b_i \ge 0$;

(b) the set defined by the system of inequalities (5.1) has the maximal dimension $\dim N^*_{\mathbb{R}}$.

Clearly, any normal fan is weakly normal. As follows from the example below, the converse is not true.

Example 5.4. In the vector space $V \simeq \mathbb{R}^3$ with the basis (e_1, e_2, e_3), consider the fan Σ whose one-dimensional cones are spanned by the seven vectors $\mathbf{v}_1 = -e_1$, $\mathbf{v}_2 = -e_2$, $\mathbf{v}_3 = -e_3$, $\mathbf{v}_4 = e_1 + e_2 + e_3$, $\mathbf{v}_5 = e_1 + e_2$, $\mathbf{v}_6 = e_2 + e_3$, and $\mathbf{v}_7 = e_1 + e_3$. It has ten maximal cones spanned by the following tuples of v_i's: $\{1,2,3\}$, $\{1,2,6\}$, $\{1,3,5\}$, $\{1,5,6\}$, $\{2,3,7\}$, $\{2,6,7\}$, $\{3,5,7\}$, $\{4,5,6\}$, $\{4,5,7\}$, and $\{4,6,7\}$.

This fan is presented in [2, Sect. 3.4] as an example of a nonnormal fan. Let us prove that it is weakly normal. Consider the collection of numbers $b_1 = b_2 = b_3 = 0$ and $b_4 = b_5 = b_6 = b_7 = 1$. It is easy to check that the vertices are $\mathbf{0}, -e_1^*, -e_2^*, -e_3^* \in V^*$ and every linear form $\mathbf{u} \mapsto \langle \mathbf{u}, \mathbf{v}_i\rangle + b_i$ is nonnegative on every vertex

The following important result in the theory of toric varieties holds:

Theorem 5.5 [2, Sect. 3.4]**.** *A nonsingular toric variety V_Σ is projective if and only if the fan Σ is normal.*

Theorem 5.5 implies that if a fan Σ is normal, then the manifold V_Σ admits a Kähler form ω, which is the curvature form of the line bundle $\mathcal{O}(1)$. As we will show below, a similar result holds for transverse Kähler forms on $M(\Sigma, \mathfrak{h})$.

Theorem 5.6. *Let $M(\Sigma, \mathfrak{h})$ be a manifold defined by Construction* 4.6*. Suppose that the fan $q(\Sigma)$ is weakly normal. Then for any $k \in \mathbb{N}$ there exists a C^k-smooth form $\omega_{\mathcal{F}}$ transverse Kähler with respect to the canonical foliation $\mathcal{F}$ on a dense open $T_{\mathbb{C}}/H$ orbit.*

Remark 5.7. Since the kernel spaces of the form $\omega_{\mathcal{F}}$ are required to coincide with the tangent spaces $T\mathcal{F}$ to the foliation $\mathcal{F}$ only on a dense open set, the normality condition in Theorem 5.5 can be relaxed to the weak normality.

Proof of Theorem 5.6. To prove the theorem, we use the following scheme:

(1) on every chart $U_\sigma \subset V_\Sigma$, we construct a C^{k+2}-smooth function $\Phi_\sigma\colon U_\sigma \to \mathbb{R}_>$;

(2) on every chart $U_\sigma \subset V_\Sigma$, we define the form $\omega_\sigma = dd^{\mathrm{c}} \log \Phi_\sigma$, where $d^{\mathrm{c}} = J \circ d \circ J$ is a real differential operator (J is the operator of the almost complex structure);

(3) we check that the forms ω_{σ_1} and ω_{σ_2} coincide on $U_{\sigma_1} \cap U_{\sigma_2}$, thus providing a nonnegative form ω on V_Σ;

(4) we check that the form ω descends to the from $\omega_{\mathcal{F}}$ on V_Σ/H;

(5) we prove that $\ker \omega_{\mathcal{F}} = T\mathcal{F}$ on $T_{\mathbb{C}}/H$.

1. Consider the character lattice $N^* \subset \mathfrak{t}^*$, and let $\mathbf{a}_1, \dots \mathbf{a}_m$ be the primitive generators of one-dimensional cones of Σ. Let us fix the cone σ spanned by the vectors $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_t}$. By definition,

an integral character $\mathbf{w} \in \check{\sigma} \cap N^*$ defines a regular function $\chi_{\mathbf{w}}\colon U_\sigma \to \mathbb{C}$. Similarly, any character $\mathbf{w} \in \check{\sigma}$ defines a continuous function

$$\chi^{\mathbb{R}}_{\mathbf{w}}\colon U_\sigma \to \mathbb{R}_{\geq}.$$

Extending arbitrarily the set of vectors $\{\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_t}\}$ to an integral basis $\{\mathbf{a}'_1, \dots, \mathbf{a}'_{\dim N}\}$ of the lattice N (this can be done since V_Σ is nonsingular and, hence, the fan Σ is regular), we can write the functions $\chi_{\mathbf{w}}$ and $\chi^{\mathbb{R}}_{\mathbf{w}}$ in the coordinates $z = (z_1, \dots, z_{\dim N})$ on $U_\sigma \simeq \mathbb{C}^{\dim\sigma} \times (\mathbb{C}^*)^{\dim N - \dim\sigma}$:

$$\chi_{\mathbf{w}}\colon (z_1, \dots, z_{\dim N}) \mapsto \prod_i z_i^{\langle \mathbf{w}, \mathbf{a}'_i\rangle}, \qquad \chi^{\mathbb{R}}_{\mathbf{w}}\colon (z_1, \dots, z_{\dim N}) \mapsto \prod_i |z_i|^{\langle \mathbf{w}, \mathbf{a}'_i\rangle}.$$

Note that the function $\chi^{\mathbb{R}}_{\mathbf{w}}$ is C^k-smooth if all the values $\langle \mathbf{w}, \mathbf{a}_{i_j}\rangle$ are equal to 0 or greater than k.

Recall that $q\colon \mathfrak{t} \to \mathfrak{t}/p(\mathfrak{h})$ is a natural projection. By the hypothesis of the theorem the fan $q(\Sigma)$ is weakly normal. Let $\mathbf{v}_i = q(\mathbf{a}_i)$ be the generators of its one-dimensional cones and $b_1, \dots, b_m$ be a collection of numbers defining the weakly normal structure. For the cone σ we fix a character $\mathbf{b}_\sigma \in \mathfrak{t}^*$ such that the equality $\langle \mathbf{b}_\sigma, \mathbf{a}_{i_j}\rangle = b_{i_j}$ holds for all generators $\mathbf{a}_{i_j}$ of σ.

For every vertex $\mathbf{u}_\tau \in (\mathfrak{t}/p(\mathfrak{h}))^*$, where $\tau \in \Sigma$ is a maximal cone, we define the character

$$\mathbf{w}_\tau = q^*(\mathbf{u}_\tau) + \mathbf{b}_\sigma.$$

Lemma 5.8. *The character $\mathbf{w}_\tau$ belongs to the cone $\check{\sigma} \subset \mathfrak{t}^*$.*

Proof. We have to check that for every vector $\mathbf{a} \in \sigma$ the value $\langle \mathbf{w}_\tau, \mathbf{a}\rangle$ is nonnegative. Since the cone σ is spanned by the vectors $\mathbf{a}_{i_s}$, it suffices to check this for its generators:

$$\langle \mathbf{w}_\tau, \mathbf{a}_{i_s}\rangle = \langle q^*(\mathbf{u}_\tau) + \mathbf{b}_\sigma, \mathbf{a}_{i_s}\rangle = \langle \mathbf{u}_\tau, q(\mathbf{a}_{i_s})\rangle + b_{i_s} = \langle \mathbf{u}_\tau, \mathbf{v}_{i_s}\rangle + b_{i_s} \geq 0,$$

where the last inequality holds due to the weak normality of $q(\Sigma)$. $\square$

So, every character $\mathbf{w}_\tau$ defines a nonnegative function $\chi^{\mathbb{R}}_{\mathbf{w}_\tau}$ on U_σ. Moreover, if $\sigma \subset \tau$, then the function is strictly positive, since the values $\langle \mathbf{w}_\tau, \mathbf{a}_{i_s}\rangle$ corresponding to the zero coordinates of $z \in U_\sigma$ vanish. Thus, the function

$$\Phi_\sigma = \sum_\tau \chi^{\mathbb{R}}_{\mathbf{w}_\tau}$$

is strictly positive on U_σ. Multiplying, if necessary, all characters $\mathbf{w}_\tau$ by a positive constant, one can guarantee that Φ_σ has any preassigned smoothness class.

2. Let us define the form $\omega_\sigma = dd^{\mathrm{c}} \log \Phi_\sigma$. According to the general result of [18, Theorem I.5.6], the function $\log \Phi_\sigma$ is *plurisubharmonic*; i.e., the form ω_σ is nonnegative.

3. Consider two functions $\log \Phi_{\sigma_1}$ and $\log \Phi_{\sigma_2}$ on $U_{\sigma_1} \cap U_{\sigma_2} = U_{\sigma_1 \cap \sigma_2}$. The definition of the functions Φ_{σ_i} implies that $\log \Phi_{\sigma_1} - \log \Phi_{\sigma_2} = \log \sum_\tau (\chi^{\mathbb{R}}_{\mathbf{b}}) = \log C \chi^{\mathbb{R}}_{\mathbf{b}}$, where C is the number of vertices and $\mathbf{b} = \mathbf{b}_{\sigma_1} - \mathbf{b}_{\sigma_2}$. Since the left-hand side is a well-defined function, the function $\chi^{\mathbb{R}}_{\mathbf{b}} = |z|^{\mathbf{b}}$ does not vanish on $U_{\sigma_1} \cap U_{\sigma_2}$. It follows from the Poincaré–Lelong formula [18, Theorem II.2.15] that $dd^{\mathrm{c}} \log \chi^{\mathbb{R}}_{\mathbf{b}} = 0$; therefore, the forms ω_{σ_1} and ω_{σ_2} coincide on $U_{\sigma_1} \cap U_{\sigma_2}$. Consequently, all the forms ω_σ are glued into a global form ω on V_Σ: $\omega|_{U_\sigma} = \omega_\sigma$.

4 and 5. The proof of these steps follows the proof of Theorem 4.6 in [11].

Lemma 5.9. *Let us consider a point z in the open part $T_{\mathbb{C}} \subset V_\Sigma$. The kernel of ω in $T_z V_\Sigma = T_z T_{\mathbb{C}} \simeq \mathfrak{t} \oplus J\mathfrak{t}$ is* $\ker q \oplus J \ker q$.

Proof. Consider the function $\Phi = \Phi_\sigma$. Since the function is constant along the toric part $\mathfrak{t}$ of $\mathfrak{t}_{\mathbb{C}}$, we have $\omega(\mathfrak{t}, J\mathfrak{t}) = 0$. Moreover, the form ω is J-invariant, $J \ker \omega|_{\mathfrak{t}} = \ker \omega|_{J\mathfrak{t}}$; therefore, $\ker \omega = \ker \omega|_{\mathfrak{t}} \oplus J \ker \omega|_{\mathfrak{t}}$.

To compute the kernel $\ker \omega|_{\mathfrak{t}}$, we find for every $\mathbf{v} \in \mathfrak{t}$

$$\frac{d^2}{d\lambda^2} \log \Phi(\exp \lambda \mathbf{v} \cdot z)|_{\lambda=0}.$$

Similarly to [11, Lemma 4.7] one obtains

$$\frac{d^2}{d\lambda^2} \log \Phi(\exp \lambda \mathbf{v} \cdot z)|_{\lambda=0} = \frac{1}{\Phi^2(z)} \left(\sum_{\tau_1, \tau_2} \chi^{\mathbb{R}}_{\tau_1}(z) \chi^{\mathbb{R}}_{\tau_2}(z) \langle \mathbf{w}_{\tau_1} - \mathbf{w}_{\tau_2}, \mathbf{v} \rangle^2 \right).$$

The right-hand side vanishes if and only if the values of all characters $\mathbf{w}_{\tau_1} - \mathbf{w}_{\tau_2}$ on the vector $\mathbf{v}$ are zero, or, equivalently, $\langle \mathbf{u}_{\tau_1} - \mathbf{u}_{\tau_2}, q(\mathbf{v}) \rangle = 0$. Condition (b) in Definition 5.3 of a weakly normal fan implies that this happens if and only if $q(\mathbf{v}) = 0$. Hence, for $z \in T_{\mathbb{C}}$ we have $\ker \omega|_{\mathfrak{t}} = \ker q$ and $\ker \omega = \ker q \oplus J \ker q = \mathfrak{h} \oplus \bar{\mathfrak{h}}$. □

It follows from the lemma that the form ω is *basic* with respect to the orbits of the H-action on the open part $T_{\mathbb{C}} \subset V_\Sigma$; i.e., for any vector $\mathbf{v} \in \mathfrak{h}$ and the corresponding fundamental vector field V we have $\mathcal{L}_V \omega|_{T_{\mathbb{C}}} = i_V \omega|_{T_{\mathbb{C}}} = 0$. For continuity reasons, ω is basic on the whole V_Σ; thus it descends to a form on V_Σ/H; i.e., there exists a form $\omega_{\mathcal{F}}$ on $M(\Sigma, \mathfrak{h}) = V_\Sigma/H$ such that $\omega = \pi^* \omega_{\mathcal{F}}$, where $\pi \colon V_\Sigma \to M(\Sigma, \mathfrak{h})$ is the natural projection. The kernels of the form $\omega_{\mathcal{F}}$ at the points of $T_{\mathbb{C}}/H$ coincide with the tangent spaces to the orbits of the group H' (see Construction 3.3). Thus the form $\omega_{\mathcal{F}}$ is transverse Kähler with respect to the foliation $\mathcal{F}$. The proof of Theorem 5.6 is complete. □

5.2. Meromorphic functions and analytic subsets. As an application of Theorems 4.8 and 5.6 we prove some results on the complex geometry of manifolds $M(\Sigma, \mathfrak{h})$.

Theorem 5.10. *Let M be a compact complex manifold equipped with a maximal torus action obtained via Construction* 4.5: $M = U(\mathcal{K})/H$ *with* $H \subset T_{\mathbb{C}}$. *Assume that*

(i) $U(\mathcal{K})$ *is simply connected*;

(ii) *the only liner function* $\mathbf{u} \in N^* \subset \mathfrak{t}^*$ *vanishing identically on* $p(\mathfrak{h})$ *is zero* (*here* N^* *is the character lattice of* T).

Then there are only finitely many analytic subsets of codimension 1 *on* M.

Corollary 5.11. *There are no nonconstant meromorphic functions on the manifolds satisfying the hypothesis of Theorem* 5.10.

The proof of both statements follows literally the proof of Theorem 4.15 and Corollary 4.16 in [11].

For a fixed Σ, the set of complex subspaces $\mathfrak{h} \subset \mathfrak{t}_{\mathbb{C}}$ satisfying conditions (a) and (b) of Construction 4.6 forms an open (in the ordinary topology) subset $\mathcal{M}_\Sigma$ of the complex Grassmannian $\mathrm{Gr}(\mathfrak{h}, \mathfrak{t}_{\mathbb{C}})$. The condition of normality of the fan $q(\Sigma)$ is, clearly, also open.

Definition 5.12. Some statement S is said to be true for the *general complex structure* on $M(\Sigma, \mathfrak{h})$ if the set of those subspaces $\mathfrak{h} \subset \mathcal{M}_\Sigma$ for which S does not hold has zero Lebesgue measure.

Theorem 5.13. *Let* $M(\Sigma, \mathfrak{h})$ *be a complex manifold endowed with a general complex structure such that the fan* $q(\Sigma)$ *is normal. Let* $Y \subset M(\Sigma, \mathfrak{h})$ *be an analytic subset. Then there are two possibilities*:

(i) Y *is the closure of an orbit*, $Y = \overline{T_{\mathbb{C}}/H \cdot x}$;

(ii) Y *is a compact torus contained in a leaf of the canonical foliation* $\mathcal{F}$.

Proof. According to the remark above, the set of $\mathfrak{h} \in \mathcal{M}_\Sigma$ such that the fan $q(\Sigma)$ is normal is open. Hence, for the general complex structure such that the fan $q(\Sigma)$ is normal, the hypothesis of Theorem 4.18 in [11] holds and we can repeat the proof. ☐

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