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Exact Solutions and Qualitative Features of Nonlinear Hyperbolic Reaction–Diffusion Equations with Delay

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Abstract—New classes of exact solutions to nonlinear hyperbolic reaction–diffusion equations with delay are described. All of the equations under consideration depend on one or two arbitrary functions of one argument, and the derived solutions contain free parameters (in certain cases, there can be any number of these parameters). The following solutions are found: periodic solutions with respect to time and space variable, solutions that describe the nonlinear interaction between a standing wave and a traveling wave, and certain other solutions. Exact solutions are also presented for more complex nonlinear equations in which delay arbitrarily depends on time. Conditions for the global instability of solutions to a number of reaction–diffusion systems with delay are derived. The generalized Stokes problem subject to the periodic boundary condition, which is described by a linear diffusion equation with delay, is solved.

Keywords: nonlinear reaction–diffusion equations with delay, exact solutions, generalized separable solutions, functional separable solutions, delay differential equations, global instability of solutions, generalized Stokes problem

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INTRODUCTION

The classical heat-conduction and diffusion models based on the Biot–Fourier–Fick law are widely used to describe non-steady-state thermal and reaction–diffusion processes in chemical engineering:

$$\mathbf{q} = -\lambda \nabla T, \qquad (1)$$

where $\mathbf{q}$ is the heat flux, T is temperature, λ is the thermal conductivity, and ∇ is the gradient operator. Law (1) leads to parabolic heat-conduction and diffusion equations [1–6]:

$$\frac{\partial T}{\partial t} = a\Delta T, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \qquad (2)$$

where t is time; $a = \lambda/(\rho c_p)$ is the thermal diffusivity; ρ is density; c_p is the specific heat of a body (medium) at constant pressure; x, y, and z are Cartesian coordinates; and Δ is the Laplace operator.

Parabolic equation (2) has a physically paradoxical property, i.e., an infinite disturbance propagation rate, which is not observed in nature. This leads to the need to develop heat- and mass-transfer models that result in a finite rate of heat or mass propagation. The Cattaneo–Vernotte model shown below is the most commonly used among them [7, 8]:

$$\mathbf{q} = -\lambda \nabla T - \tau \frac{\partial \mathbf{q}}{\partial t}, \qquad (3)$$

where τ is the relaxation time.

Model (3) leads to the following hyperbolic heat- and mass-transfer equations [7–11]:

$$\tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = a\Delta T, \qquad (4)$$

which have a finite disturbance propagation rate at $\tau > 0$. The thermal and diffusion relaxation times can vary in extremely wide limits from milliseconds (or less) to several tens of seconds [9–19] and should be taken into account in solving many heat- and mass-transfer problems. In the degenerate case, which corresponds to $\tau = 0$, Eq. (4) transforms into Eq. (2).

The second important feature of evolutionary processes, including heat- and mass-transfer processes with chemical conversions, is that, in the general case, the rate of variations in the desired quantities in chemical, biological, physicochemical, biochemical, and chemical engineering, as well as bioengineering, biomedical, ecological, and other systems, depends not only on the state at the given time point, but also on the entire previous evolution of the process [9, 13, 20]. These systems are called *hereditary systems*. In the particular case where the state of the system is only determined by a particular time point in the past, rather than the entire evolution of the system, the system is referred to as a delayed feedback system.

Systems with delayed feedback are frequently modeled by reaction–diffusion equations, in which the kinetic function F (the rate of chemical or biochemical reactions) depends on both the sought function $u = u(x, t)$ and the same function, but with the delayed argument $w = u(x, t - \tau)$. The special case of $F(u, w) = f(w)$ has a simple physical interpretation, i.e., heat- and mass-transfer processes in media with local nonequilibrium have inertial properties, i.e., the system does not react to action instantaneously at the given time point t, as in the classical local equilibrium case, but it reacts by the delay time τ later. In certain cases, delay can be the prescribed time function $\tau = \tau(t)$.

We now consider certain heat- and mass-transfer equations with delay. The simplest equation is the generalization of the classical diffusion equation, which includes the reaction term with delay (parabolic delay reaction–diffusion equation). In the one-dimensional case, this equation has the following form:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + F(u, w), \quad w = u(x, t - \tau),$$

where τ is the delay time and $F(u, w)$ is the kinetic function. Various properties and exact solutions to the above equation and systems of these equations are described in [21–31].

Another heat- and mass-transfer equation with delay is the differential-difference diffusion equation with a finite relaxation time as follows:

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left[G(u)\frac{\partial u}{\partial x}\right] + F(u, v), \quad v = u(x, t + \tau),$$

which is derived from the differential-difference model for a mass flux [17, 18]. A number of exact solutions to this nonlinear equation were obtained in [18, 19].

Since, when solving non-steady-state mass transfer problems in chemical engineering, it is necessary to take into account relaxation phenomena associated both with the finiteness of the rate of heat and mass transfer and with the finiteness of the times of chemical conversions and/or the microkinetic interaction between different phases that form a single transport macromedium, exact solutions to the following nonlinear hyperbolic reaction–diffusion equations are derived and analyzed in this study:

$$\varepsilon\frac{\partial^2 u}{\partial t^2} + \sigma\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + F(u, w), \quad w = u(x, t - \tau), \qquad (5)$$

where $a > 0$, $\varepsilon \geq 0$, and $\sigma \geq 0$ ($\varepsilon + \sigma \neq 0$). It should be noted that, as a particular case, at $\varepsilon = 0$, Eq. (5) includes parabolic equations with delay. More complex nonlinear reaction–diffusion equations with variable delay of the general form $\tau = \tau(t)$ will also be considered. The generalized Stokes problem subject to the periodic boundary condition will be solved for a linear diffusion equation with delay at $F(u, w) = -kw$.

Exact solutions to nonlinear reaction–diffusion equations with a variable transport coefficient $G(u)$ will also be presented in this study:

$$\varepsilon\frac{\partial^2 u}{\partial t^2} + \sigma\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[G(u)\frac{\partial u}{\partial x}\right] + F(u, w), \qquad (6)$$
$$w = u(x, t - \tau).$$

In the degenerate case, at $\varepsilon = 0$, i.e., for the parabolic equation, certain exact solutions to Eq. (6) were obtained in [32–34].

In addition, we will derive conditions for the instability of solutions to nonlinear systems of hyperbolic reaction–diffusion equations with delay of the special form. It will be shown that, when instability conditions are satisfied, initial value problems and certain initial-boundary value problems are ill-posed in the sense of Hadamard.

EXACT SOLUTIONS: METHODS FOR FINDING SOLUTIONS

Exact solutions to nonlinear differential equations promote the better understanding of the qualitative features of the processes under description (nonuniqueness, spatial localization, blowup regimes, etc.). It should be emphasized that delay substantially complicates the analysis of equations and is a factor that can lead to the instability of the systems being modeled [19, 30, 35, 36].

The term *exact solutions* with respect to the nonlinear delay of partial differential equations is used in the cases where a solution is expressed as follows [27–29]:

(i) The solution can be expressed in terms of elementary functions or can be represented in the closed form (the solution is expressed in terms of indefinite or definite integrals).

(ii) The solution can be expressed in terms of solutions to ordinary differential equations or delay ordinary differential equations (or systems of these equations).

(iii) The solution can be expressed in terms of solutions to linear partial differential equations.

The combinations of solutions from items (i)–(iii) are also allowable.

Remark 1. Solution methods and various applications of linear and nonlinear ordinary differential equations with delay, which are substantially simpler than nonlinear partial differential equations with delay, are described, e.g., in [37–40].

Remark 2. A number of exact solutions to certain nonlinear partial differential equations with delay (as well as systems of equations with delay), which are different from reaction–diffusion equations, are given in [36, 41, 42].

Remark 3. The numerical solving of various nonlinear equations and systems of equations with delay and difficulties that arise in this case are described in [43–46].

In this study, to seek exact solutions to nonlinear hyperbolic reaction–diffusion equations such as (5) and (6), we used various modifications of the methods of generalized and functional separation of variables [47–50] and the functional constraints method [28, 33, 51]. From this point on, intermediate calculations are generally omitted for the sake of brevity.

Remark 4. In the general case, Eqs. (5) and (6) admit evident exact solutions such as traveling wave solutions $u = U(z)$, where $z = kx + \lambda t$.

EXACT SOLUTIONS TO EQ. (5) WITH A KINETIC FUNCTION THAT DEPENDS ON THE RATIO w/u

We consider Eq. (5) in the following form:

$$\varepsilon\frac{\partial^2 u}{\partial t^2} + \sigma\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uF(w/u), \quad w = u(x, t-\tau), \tag{7}$$

where $F(z)$ is an arbitrary function.

1. Equation (7) yields the separable solution as the product of the functions of different arguments as follows:

$$u = [C_1\cos(\lambda x) + C_2\sin(\lambda x)]\psi(t), \tag{8}$$

where C_1, C_2, and λ are arbitrary constants and the function $\psi(t)$ in (8) is described by the following ordinary differential equation with delay:

$$\varepsilon\psi''(t) + \sigma\psi'(t) = -a\lambda^2\psi(t) + \psi(t)F(\psi(t-\tau)/\psi(t)). \tag{9}$$

Equation (9) yields the particular solution $\psi(t) = Ae^{\beta t}$, where A is an arbitrary constant and β is determined from the algebraic (or transcendental) equation

$$\varepsilon\beta^2 + \sigma\beta + a\lambda^2 - F(e^{-\beta\tau}) = 0.$$

2. Equation (7) yields another separable solution,

$$u = [C_1\exp(-\lambda x) + C_2\exp(\lambda x)]\psi(t), \tag{10}$$

where the function $\psi(t)$ is described by the following delay differential equation:

$$\varepsilon\psi''(t) + \sigma\psi'(t) = a\lambda^2\psi(t) + \psi(t)F(\psi(t-\tau)/\psi(t)). \tag{11}$$

Equation (11) yields the particular solution $\psi(t) = Ae^{\beta t}$, where β is determined from the algebraic (transcendental) equation

$$\varepsilon\beta^2 + \sigma\beta - a\lambda^2 - F(e^{-\beta\tau}) = 0.$$

3. Equation (7) also yields the solution

$$u = \exp(\alpha x + \beta t)\theta(z), \quad z = \lambda x + \gamma t, \tag{12}$$

where the function $\theta(z)$ is described by the following delay ordinary differential equation:

$$(a\lambda^2 - \varepsilon\gamma^2)\theta''(z) + (2a\alpha\lambda - 2\varepsilon\beta\gamma - \sigma\gamma)\theta'(z) + (a\alpha^2 - \varepsilon\beta^2 - \sigma\beta)\theta(z) + \theta(z)F(e^{-\beta\tau}\theta(z-\delta)/\theta(z)) = 0, \quad \delta = \gamma\tau.$$

This equation yields the particular solution $\theta(z) = Ae^{vz}$, where v is determined from the algebraic (transcendental) equation

$$(a\lambda^2 - \varepsilon\gamma^2)v^2 + (2a\alpha\lambda - 2\varepsilon\beta\gamma - \sigma\gamma)v + (a\alpha^2 - \varepsilon\beta^2 - \sigma\beta) + F(e^{-\beta\tau - v\delta}) = 0, \quad \delta = \gamma\tau.$$

Solution (12) is the nonlinear superposition of two different traveling waves.

4. Let the function

$$v = V_1(x, t, \sigma, b) \tag{13}$$

be any τ-periodic solution to the following linear hyperbolic equation:

$$\varepsilon\frac{\partial^2 v}{\partial t^2} + \sigma\frac{\partial v}{\partial t} = a\frac{\partial^2 v}{\partial x^2} + bv, \quad v(x,t) = v(x, t-\tau) \tag{14}$$

(from this point on, for the sake of brevity, the dependence of solutions (13) and (18) on the parameters ε and a, which appear in Eqs. (14) and (19), is not indicated explicitly). In that case, Eq. (7) yields the generalized separable solution

$$u = e^{ct}V_1(x, t, \sigma + 2\varepsilon c, b), \quad b = F\left(e^{-c\tau}\right) - \varepsilon c^2 - \sigma c, \tag{15}$$

where c is an arbitrary constant.

It can be shown that the general solution to Eq. (14) subject to the aforementioned condition of τ-periodicity with respect to time has the following form:

$$V_1(x,t,\sigma,b) = \sum_{n=0}^{\infty}\exp(-\lambda_n x)[A_n\cos(\beta_n t - \gamma_n x) + B_n\sin(\beta_n t - \gamma_n x)] + \sum_{n=1}^{\infty}\exp(\lambda_n x) \times [C_n\cos(\beta_n t + \gamma_n x) + D_n\sin(\beta_n t + \gamma_n x)], \tag{16}$$

$$\beta_n = \frac{2\pi n}{\tau},$$

$$\gamma_n = \left[\frac{\sqrt{(\varepsilon\beta_n^2 + b)^2 + \sigma^2\beta_n^2} + \varepsilon\beta_n^2 + b}{2a}\right]^{1/2}, \tag{17}$$

$$\lambda_n = \frac{\sigma\beta_n}{2a\gamma_n},$$

where A_n, B_n, C_n, and D_n are arbitrary constants at which series (16) and series for the derivatives $\frac{\partial^2 V_1}{\partial t^2}$, $\frac{\partial V_1}{\partial t}$, and $\frac{\partial^2 V_1}{\partial x^2}$ are convergent (the convergence can be ensured, e.g., if we set $A_n = B_n = C_n = D_n = 0$ at $n > N$, where N is any positive integer).

The following particular cases can be distinguished:

(i) τ-periodic (with respect to the time t) solutions (14) that decay at $x \to \infty$ are given by formulas (16) and (17) at $A_0 = B_0 = 0$, $C_n = D_n = 0$, and $n = 1, 2, \ldots$.

(ii) τ-periodic (with respect to the time t) solutions (14) bounded at $x \to \infty$ are given by formulas (16) and (17) at $C_n = D_n = 0$ and $n = 1, 2, \ldots$.

(iii) A stationary solution is given by formulas (16) and (17) at $A_n = B_n = C_n = D_n = 0$ and $n = 1, 2, \ldots$.

5. Let the function

$$v = V_2(x, t, \sigma, b) \tag{18}$$

be a τ-aperiodic solution to the following linear hyperbolic equation:

$$\varepsilon \frac{\partial^2 v}{\partial t^2} + \sigma \frac{\partial v}{\partial t} = a \frac{\partial^2 v}{\partial x^2} + bv, \quad v(x,t) = -v(x, t-\tau). \tag{19}$$

In that case, Eq. (7) yields the generalized separable solution

$$u = e^{ct} V_2(x, t, \sigma + 2\varepsilon c, b), \\ b = F\left(-e^{-c\tau}\right) - \varepsilon c^2 - \sigma c. \tag{20}$$

The general solution to Eq. (19) has the following form:

$$V_2(x,t,\sigma,b) = \sum_{n=1}^{\infty} \exp(-\lambda_n x)[A_n \cos(\beta_n t - \gamma_n x) + B_n \sin(\beta_n t - \gamma_n x)] + \sum_{n=1}^{\infty} \exp(\lambda_n x) \times [C_n \cos(\beta_n t + \gamma_n x) + D_n \sin(\beta_n t + \gamma_n x)], \tag{21}$$

$$\beta_n = \frac{\pi(2n-1)}{\tau}, \\ \gamma_n = \left[\frac{\sqrt{(\varepsilon\beta_n^2 + b)^2 + \sigma^2\beta_n^2} + \varepsilon\beta_n^2 + b}{2a}\right]^{1/2}, \\ \lambda_n = \frac{\sigma\beta_n}{2a\gamma_n}. \tag{22}$$

Solutions (τ-aperiodic with respect to the time t) that decay at $x \to \infty$ are given by formulas (21) and (22) at $C_n = D_n = 0$ and $n = 1, 2, \ldots$.

Solutions (16)–(17) and (21)–(22) are very similar. However, in the first solution, the first sum begins from $n = 0$ and, in the second solution, it begins from $n = 1$; the values of β_n are also different.

EXACT SOLUTIONS TO EQ. (5) WITH A KINETIC FUNCTION THAT DEPENDS ON THE DIFFERENCE $u - w$

We now consider Eq. (5) in the following form:

$$\varepsilon \frac{\partial^2 u}{\partial t^2} + \sigma \frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2} + bu + F(u - w), \\ w = u(x, t - \tau). \tag{23}$$

1. Equation (23) yields the separable solution as the sum of the functions of different arguments

$$u = \varphi(x) + \psi(t), \tag{24}$$

where

$$\varphi(x) = \begin{cases} C_1 \cos(\lambda x) + C_2 \sin(\lambda x), \\ \lambda = \sqrt{b/a} \text{ at } b > 0; \\ C_1 \exp(-\lambda x) + C_2 \exp(\lambda x), \\ \lambda = \sqrt{-b/a} \text{ at } b < 0, \end{cases} \tag{25}$$

and the function $\psi(t)$ is described by the following delay differential equation:

$$\varepsilon\psi''(t) + \sigma\psi'(t) = b\psi(t) + F(\psi(t) - \psi(t - \tau)). \tag{26}$$

2. At $b = 0$, Eq. (23) yields the separable solution that is quadratic with respect to x:

$$u = C_1 x^2 + C_2 x + \psi(t), \tag{27}$$

where the function $\psi(t)$ is described by the following delay differential equation:

$$\varepsilon\psi''(t) + \sigma\psi'(t) = 2C_1 a + F(\psi(t) - \psi(t - \tau)). \tag{28}$$

3. The solution to Eq. (23) that generalizes solution (24) has the form

$$u = \varphi(x) + \theta(z), \quad z = \beta x + \gamma t, \tag{29}$$

where the function $\varphi(x)$ is determined by formulas (25) and the function $\theta(z)$ is described by the following delay ordinary differential equation:

$$(\varepsilon\gamma^2 - a\beta^2)\theta''(z) + \sigma\gamma\theta'(z) = b\theta(z) + F(\theta(z) - \theta(z - \delta)), \quad \delta = \gamma\tau. \tag{30}$$

At $b > 0$, solution (29) describes the nonlinear interaction between a periodic standing wave and a traveling wave.

4. At $b = 0$, the solution to (23) that generalizes solution (27) has the form

$$u = C_1 x^2 + C_2 x + \theta(z), \; z = \beta x + \gamma t, \tag{31}$$

where the function $\theta(z)$ is described by the following delay ordinary differential equation:

$$(\varepsilon\gamma^2 - a\beta^2)\theta''(z) + \sigma\gamma\theta'(z) \\ = 2C_1 a + F(\theta(z) - \theta(z - \delta)), \quad \delta = \gamma\tau.$$

5. Equation (23) yields the degenerate generalized separable solution

$$u = t\varphi(x) + \psi(x),$$

where the function $\varphi(x)$ is determined by formulas (25) and the function $\psi(x)$ is described by the following linear inhomogeneous ordinary differential equation:

$$a\psi''(x) + b\psi(x) + F(\tau\varphi(x)) - \sigma\varphi(x) = 0.$$

More complex solutions to Eq. (23) can be derived using the following property.

Property 1 (nonlinear superposition of solutions). Let $u_0(x,t)$ be a solution to nonlinear equation (23) and $v = V_1(x, t; \sigma, b)$ be any τ-periodic solution to linear equation (14). In that case, the sum

$$u = u_0(x,t) + V_1(x, t; \sigma, b) \qquad (32)$$

is also the solution to Eq. (23). The general form of the function $V_1(x, t; \sigma, b)$ is determined by formulas (16)–(17).

For example, the traveling wave solution $u_0 = u_0(\alpha x + \beta t)$ can be used in (32) as the particular solution $u_0(x,t)$ to nonlinear equation (23).

We now consider Eq. (5) in the following form:

$$\begin{gathered}\varepsilon\frac{\partial^2 u}{\partial t^2} + \sigma\frac{\partial u}{\partial t} \\ = a\frac{\partial^2 u}{\partial x^2} + uF(u-w) + wG(u-w) + H(u-w),\end{gathered} \qquad (33)$$

where $F(z)$, $G(z)$, and $H(z)$ are arbitrary functions.

1. Equation (33) yields the solution

$$\begin{gathered}u = \xi_0 + V_1(x,t;\sigma, F(0)+G(0)), \\ \xi_0 = -\frac{H(0)}{F(0)+G(0)}.\end{gathered} \qquad (34)$$

The general form of the function $V_1(x, t; \sigma, b)$ is given by formulas (16) and (17).

2. Equation (33) has the generalized separable solution

$$\begin{gathered}u = \sum_{n=1}^{N}[\varphi_n(x)\cos(\beta_n t) + \psi_n(x)\sin(\beta_n t)] \\ + t\theta(x) + \xi(x), \quad \beta_n = \frac{2\pi n}{\tau},\end{gathered} \qquad (35)$$

where N is any positive integer and the functions $\varphi_n(x)$, $\psi_n(x)$, $\theta(x)$, and $\xi(x)$ are described by the following ordinary differential equations:

$$a\varphi_n'' + [F(\tau\theta) + G(\tau\theta)]\varphi_n + \varepsilon\beta_n^2\varphi_n - \sigma\beta_n\psi_n = 0,$$

$$a\psi_n'' + [F(\tau\theta) + G(\tau\theta)]\psi_n + \varepsilon\beta_n^2\psi_n + \sigma\beta_n\varphi_n = 0,$$

$$\alpha\theta'' + [F(\tau\theta) + G(\tau\theta)]\theta = 0,$$

$$a\xi'' + [F(\tau\theta) + G(\tau\theta)]\xi - [\sigma + \tau G(\tau\theta)]\theta + H(\tau\theta) = 0.$$

The third nonlinear equation of this system is independent and yields the solution $\theta = 0$. In this case, the other equations become linear equations with constant coefficients.

EXACT SOLUTIONS TO EQ. (5) WITH A KINETIC FUNCTION THAT DEPENDS ON A LINEAR COMBINATION OF w AND u

We now consider Eq. (5) in the following form:

$$\begin{gathered}\varepsilon\frac{\partial^2 u}{\partial t^2} + \sigma\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + bu + F(u-kw), \\ w = u(x, t-\tau), \quad k > 0.\end{gathered} \qquad (36)$$

Property 2 (generalizes property 1). Let $u_0(x,t)$ be a solution to nonlinear equation (36), and the function $v = V_1(x, t; \sigma, b)$ is any τ-periodic solution to linear hyperbolic equation (14). In that case, the sum

$$\begin{gathered}u = u_0(x,t) + e^{ct}V_1(x, t; \sigma + 2\varepsilon c, b - \varepsilon c^2 - \sigma c), \\ c = \frac{1}{\tau}\ln k\end{gathered} \qquad (37)$$

is also the solution to Eq. (36). The general form of the function $V_1(x, t; \sigma, b)$ is determined by formulas (16)–(17).

Property 2 makes it possible to derive a wide class of exact solutions to Eq. (36). The simplest particular solutions to Eq. (36) are the constant solutions $u_0 =$ const, which are found from the following algebraic (transcendental) equation:

$$bu_0 + F((1-k)u_0) = 0.$$

In the special case of $k = 1$, we have $u_0 = -F(0)/b$.

In the general case, for Eq. (36) with the arbitrary function $F(z)$, we can take the particular solutions $u_0(x,t)$ of the following forms in formula (37):

$$\begin{gathered}u_0 = \psi(t) \\ \text{(spatially homogeneous solution)}; \\ u_0 = \varphi(x) \\ \text{(stationary solution)}; \\ u_0 = \theta(z),\ z = \beta x + \gamma t \\ \text{(traveling wave solution)},\end{gathered} \qquad (38)$$

where the last solution generalizes the first two solutions. The traveling wave solution $u_0(x,t) = \theta(z)$, where $z = \beta x + \gamma t$, is described by the following delay equation:

$$\begin{gathered}(\varepsilon\gamma^2 - a\beta^2)\theta''(z) + \sigma\gamma\theta'(z) = b\theta(z) \\ + F(\theta(z) - k\theta(z-\delta)), \quad \delta = \gamma\tau.\end{gathered}$$

We now consider Eq. (5) in the following form:

$$\begin{gathered}\varepsilon\frac{\partial^2 u}{\partial t^2} + \sigma\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uF(u-kw) \\ + wG(u-kw) + H(u-kw), \quad k > 0.\end{gathered} \qquad (39)$$

1. Let ξ_0 be the root of the algebraic (transcendental) equation

$$\xi_0[F(\xi_0) + G(\xi_0)] + (1-k)H(\xi_0) = 0. \qquad (40)$$

In that case, Eq. (39) yields the following exact solution:

$$u = \frac{\xi_0}{1-k} + e^{ct}V_1(x,t;2\varepsilon c + \sigma, b),$$
$$c = \frac{1}{\tau}\ln k, \quad b = F(\xi_0) + \frac{1}{k}G(\xi_0) - \varepsilon c^2 - \sigma c, \tag{41}$$

where $v = V_1(x, t; \sigma, b)$ is any τ-periodic solution to Eq. (14). In the general case, the function $V_1(x, t; \sigma, b)$ is given by formulas (16) and (17). The different roots of Eq. (40) generate different solutions, such as (41) to Eq. (39).

2. Equation (39) also yields the solution

$$u = e^{ct}\left\{\theta(x) + \sum_{n=1}^{N}[\varphi_n(x)\cos(\beta_n t) + \psi_n(x)\sin(\beta_n t)]\right\} + \xi(x), \quad c = \frac{1}{\tau}\ln k, \quad \beta_n = \frac{2\pi n}{\tau}.$$

Here, N is any positive integer and the functions $\theta(x)$, $\varphi_n(x)$, $\psi_n(x)$, and $\xi(x)$ are described by the following system of ordinary differential equations:

$$a\varphi_n'' + \left[F(\eta) + \frac{1}{k}G(\eta) + \varepsilon(\beta_n^2 - c^2) - \sigma c\right]\times\varphi_n - (2\varepsilon c + \sigma)\beta_n\psi_n = 0,$$
$$a\psi_n'' + \left[F(\eta) + \frac{1}{k}G(\eta) + \varepsilon(\beta_n^2 - c^2) - \sigma c\right]\times\psi_n + (2\varepsilon c + \sigma)\beta_n\varphi_n = 0,$$
$$a\theta'' + \left[F(\eta) + \frac{1}{k}G(\eta) - \varepsilon c^2 - \sigma c\right]\theta = 0,$$
$$a\xi'' + [F(\eta) + G(\eta)]\xi + H(\eta) = 0,$$

where $\eta = (1 - k)\xi$. The last equation is independent.

We now consider Eq. (5) in the following form:

$$\varepsilon\frac{\partial^2 u}{\partial t^2} + \sigma\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + bu + F(u + kw),$$
$$w = u(x, t-\tau), \quad k > 0. \tag{42}$$

Property 3. Let $u_0(x,t)$ be some solution to nonlinear equation (42), and the function $v = V_2(x, t; \sigma, b)$ is any τ-aperiodic solution to linear hyperbolic equation (19). In that case, the sum

$$u = u_0(x, t) + e^{ct}V_2(x, t; \sigma + 2\varepsilon c, b - \varepsilon c^2 - \sigma c),$$
$$c = \frac{1}{\tau}\ln k, \tag{43}$$

is also the solution to Eq. (42). The general form of the function $V_2(x, t; \sigma, b)$ is determined by formulas (21)–(22).

Property 3 makes it possible to derive a wide class of exact solutions to Eq. (42). The simplest particular solutions to Eq. (42) are the constant solutions $u_0 =$ const, which are found from the following algebraic (transcendental) equation:

$$bu_0 + F((1 + k)u_0) = 0.$$

In the general case, for Eq. (42) with the arbitrary function $F(z)$, the aforementioned particular solutions of form (38) can be used in formula (43). In particular, using the traveling wave solution $u_0(x,t) = \theta(z)$, where $z = \beta x + \gamma t$, we derive the following delay ordinary differential equation:

$$(\varepsilon\gamma^2 - a\beta^2)\theta''(z) + \sigma\gamma\theta'(z) = b\theta(z) + F(\theta(z) + k\theta(z - \delta)), \quad \delta = \gamma\tau.$$

We now consider Eq. (5) in the following form:

$$\varepsilon\frac{\partial^2 u}{\partial t^2} + \sigma\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uF(u + kw)$$
$$+ wG(u + kw) + H(u + kw), \quad k > 0. \tag{44}$$

1. Let ξ_0 be the root of the algebraic (transcendental) equation

$$\xi_0[F(\xi_0) + G(\xi_0)] + (1 + k)H(\xi_0) = 0. \tag{45}$$

In that case, Eq. (44) yields the solution

$$u = \frac{\xi_0}{1+k} + e^{ct}V_2(x,t;2\varepsilon c + \sigma, b),$$
$$c = \frac{1}{\tau}\ln k, \quad b = F(\xi_0) - \frac{1}{k}G(\xi_0) - \varepsilon c^2 - \sigma c, \tag{46}$$

where $v = V_2(x, t; \sigma, b)$ is the τ-aperiodic solution to linear equation (19). In the general case, the function $V_2(x, t; \sigma, b)$ is given by formulas (21) and (22). The different roots of Eq. (45) generate different solutions such as (46) to Eq. (44).

2. Equation (44) also yields the solution

$$u = e^{ct}\sum_{n=1}^{N}[\varphi_n(x)\cos(\beta_n t) + \psi_n(x)\sin(\beta_n t)] + \xi(x),$$
$$c = \frac{1}{\tau}\ln k, \quad \beta_n = \frac{\pi(2n-1)}{\tau}.$$

Here, N is any positive integer and the functions $\varphi_n(x)$, $\psi_n(x)$, and $\xi(x)$ are described by the following system of ordinary differential equations:

$$a\varphi_n'' + \left[F(\eta) - \frac{1}{k}G(\eta) + \varepsilon(\beta_n^2 - c^2) - \sigma c\right]\varphi_n - (2\varepsilon c + \sigma)\beta_n\psi_n = 0,$$
$$a\psi_n'' + \left[F(\eta) - \frac{1}{k}G(\eta) + \varepsilon(\beta_n^2 - c^2) - \sigma c\right]\psi_n + (2\varepsilon c + \sigma)\beta_n\varphi_n = 0,$$
$$a\xi'' + [F(\eta) + G(\eta)]\xi + H(\eta) = 0,$$

where $\eta = (1 + k)\xi$. The last equation is independent.

GENERAL REACTION–DIFFUSION EQUATIONS WITH VARIABLE DELAY

We now consider the following more complex nonlinear hyperbolic reaction–diffusion equations with time-varying delay:

$$\varepsilon\frac{\partial^2 u}{\partial t^2}+\sigma\frac{\partial u}{\partial t}=a\frac{\partial^2 u}{\partial x^2}+F(u,w),\quad w=u(x,t-\tau(t)),\quad(47)$$

where $u = u(x, t)$, $a > 0$, $\varepsilon \ge 0$, $\sigma \ge 0$ $(\varepsilon + \sigma \ne 0)$, $F(u, w)$ is the kinetic function, and the delay time τ is considered to be the prescribed function of t. At $\varepsilon = 0$, parabolic equations of this type were considered in [15, 17].

We now consider Eq. (47) in the following form:

$$\varepsilon\frac{\partial^2 u}{\partial t^2}+\sigma\frac{\partial u}{\partial t}=a\frac{\partial^2 u}{\partial x^2}+uF(w/u),\quad w=u(x,t-\tau(t)).\quad(48)$$

1. Equation (48) yields solution (8) periodic with respect to x, where the function $\psi(t)$ is described by the following functional-differential equation:

$$\varepsilon\psi''(t)+\sigma\psi'(t)=-a\lambda^2\psi(t)+\psi(t)F(\psi(t-\tau(t))/\psi(t)).$$

2. Equation (48) also yields solution (10), where the function $\psi(t)$ is described by the following ordinary functional-differential equation:

$$\varepsilon\psi''(t)+\sigma\psi'(t)=a\lambda^2\psi(t)+\psi(t)F(\psi(t-\tau(t))/\psi(t)).$$

Equation (47) of the form

$$\varepsilon\frac{\partial^2 u}{\partial t^2}+\sigma\frac{\partial u}{\partial t}=a\frac{\partial^2 u}{\partial x^2}+bu+F(u-w),\quad w=u(x,t-\tau(t)),$$

yields solution (24)–(25), where the function $\psi(t)$ is described by the following ordinary functional-differential equation:

$$\varepsilon\psi''(t)+\sigma\psi'(t)=b\psi(t)+F(\psi(t)-\psi(t-\tau(t))).$$

EQUATIONS WITH A VARIABLE TRANSPORT COEFFICIENT

We now consider the following more complex nonlinear hyperbolic delay reaction–diffusion equation with a variable transport coefficient

$$\varepsilon\frac{\partial^2 u}{\partial t^2}+\sigma\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[G(u)\frac{\partial u}{\partial x}\right]+F(u,w),\quad w=u(x,t-\tau),\quad(49)$$

where $u = u(x, t)$, $\varepsilon \ge 0$, $\sigma \ge 0$ $(\varepsilon + \sigma \ne 0)$, $G(u)$ is the transport coefficient, and $F(u, w)$ is the kinetic function.

We consider Eq. (49) in the following form:

$$\varepsilon\frac{\partial^2 u}{\partial t^2}+\sigma\frac{\partial u}{\partial t}=a\frac{\partial}{\partial x}\left(u^n\frac{\partial u}{\partial x}\right)+bu^{n+1}+uF(w/u),\quad w=u(x,t-\tau).\quad(50)$$

1. At $b(n + 1) > 0$, Eq. (50) yields the separable solution

$$u=[C_1\cos(\lambda x)+C_2\sin(\lambda x)]^{1/(n+1)}\psi(t),$$

$$\lambda=\sqrt{b(n+1)/a},$$

where the function $\psi(t)$ is described by the following delay ordinary differential equation:

$$\varepsilon\psi''(t)+\sigma\psi'(t)=\psi(t)F(\psi(t-\tau)/\psi(t)).\quad(51)$$

Equation (51) yields the particular solution $\psi(t) = Ae^{\beta t}$, where β is determined from the algebraic (transcendental) equation

$$\varepsilon\beta^2+\sigma\beta-F(e^{-\beta\tau})=0.$$

2. At $b(n + 1) < 0$, Eq. (50) yields the following separable solution:

$$u=[C_1\exp(-\lambda x)+C_2\exp(\lambda x)]^{1/(n+1)}\psi(t),$$

$$\lambda=\sqrt{-b(n+1)/a},$$

where the function $\psi(t)$ is described by delay ordinary differential equation (51).

3. At $n = -1$, Eq. (50) yields the following separable solution:

$$u=C_1\exp\left(-\frac{b}{2a}x^2+C_2x\right)\psi(t),$$

where the function $\psi(t)$ is described by delay ordinary differential equation (51).

Equation (49) of the form

$$\varepsilon\frac{\partial^2 u}{\partial t^2}+\sigma\frac{\partial u}{\partial t}=a\frac{\partial}{\partial x}\left(u^n\frac{\partial u}{\partial x}\right)+uF(w/u)$$

yields the separable solution

$$u=\varphi(x)\psi(t),$$

where the functions $\varphi(x)$ and $\psi(t)$ satisfy the following ordinary differential equation and delay differential equation:

$$a(\varphi^n\varphi')'=b\varphi,\quad(52)$$

$$\varepsilon\psi''(t)+\sigma\psi'(t)=b\psi^{n+1}(t)+\psi(t)F(\psi(t-\tau)/\psi(t)),\quad(53)$$

where b is an arbitrary constant.

At $b = 0$, Eq. (53) transforms into (51), and Eq. (52) has the solution

$$\varphi(x)=\begin{cases}(C_1x+C_2)^{1/(n+1)} & \text{at } n\ne-1,\\ C_1\exp(C_2x) & \text{at } n=-1.\end{cases}$$

At $n \ne -2$ and $n \ne 0$, Eq. (52) has the following particular solution:

$$\varphi(x)=Ax^{2/n},\quad A=\left[\frac{bn^2}{2a(n+2)}\right]^{1/n}.$$

Equation (49) of the form

$$\varepsilon\frac{\partial^2 u}{\partial t^2}+\sigma\frac{\partial u}{\partial t}=a\frac{\partial}{\partial x}\left(u^n\frac{\partial u}{\partial x}\right)+uF(w/u)+u^{n+1}H(w/u)$$

yields the separable solution

$$u=e^{\lambda t}\varphi(x),$$

where λ is the solution to the algebraic (transcendental) equation

$$\varepsilon\lambda^2 + \sigma\lambda = F(e^{-\lambda\tau}),$$

and the function $\varphi(x)$ is described by the following ordinary differential equation:

$$a(\varphi^n\varphi')' + \varphi^{n+1}H(e^{-\lambda\tau}) = 0.$$

At $n \neq -1$, the substitution $\theta = \varphi^{n+1}$ leads to the second-order linear ordinary differential equation; at $n = -1$, the substitution $\theta = \ln\varphi$ should be made.

At $\varepsilon = 0$, $\sigma = 1$, and $k \neq 0$, Eq. (49) of the form

$$\varepsilon\frac{\partial^2 u}{\partial t^2} + \sigma\frac{\partial u}{\partial t} = a\frac{\partial}{\partial x}\left(u^n\frac{\partial u}{\partial x}\right) + cu^{n-2k+1} + u^{1-k}F(u^k - w^k)$$

yields the generalized separable solution

$$u = [\lambda x + \psi(t)]^{1/k}, \quad \lambda = \pm\sqrt{\frac{-ck^2}{a(n+1-k)}},$$

where $\psi(t)$ is described by the following delay ordinary differential equation:

$$\psi'(t) = kF(\psi(t) - \psi(t-\tau)).$$

At $\varepsilon \neq 0$, $\sigma = 0$, and $n \neq -1$, Eq. (49) of the form

$$\varepsilon\frac{\partial^2 u}{\partial t^2} + \sigma\frac{\partial u}{\partial t} = a\frac{\partial}{\partial x}\left(u^n\frac{\partial u}{\partial x}\right) + F(u^{n+1} - w^{n+1}) + u^{-2n-1}H(u^{n+1} - w^{n+1})$$

yields the generalized separable solution

$$u = (At + Bx^2 + C_1x + C_2)^{1/(n+1)},$$

$$B = -\frac{n+1}{2a}F(A\tau),$$

where the constant A is determined from the following algebraic (transcendental) equation:

$$\varepsilon nA^2 + (n+1)^2H(A\tau) = 0.$$

Equation (49) of the form

$$\varepsilon\frac{\partial^2 u}{\partial t^2} + \sigma\frac{\partial u}{\partial t} = a\frac{\partial}{\partial x}\left(u^{-1/2}\frac{\partial u}{\partial x}\right) + F(u^{1/2} - w^{1/2}) + u^{1/2}H(u^{1/2} - w^{1/2})$$

yields the generalized separable solution

$$u = [t\varphi(x) + \psi(x)]^2,$$

where the functions $\varphi(x)$ and $\psi(t)$ are described by the following ordinary differential equations:

$$2a\varphi'' + \varphi H(\tau\varphi) - 2\sigma\varphi^2 = 0,$$

$$2a\psi'' + \psi H(\tau\varphi) - 2\sigma\varphi\psi - 2\varepsilon\varphi^2 + F(\tau\varphi) = 0.$$

The particular solution to this system of equations has the form

$$\varphi = A, \quad \psi = \frac{2\varepsilon A^2 - F(A\tau)}{4a}x^2 + C_1x + C_2,$$

where the constant A is determined from the algebraic (transcendental) equation

$$H(A\tau) - 2\sigma A = 0.$$

We now consider Eq. (49) in the form

$$\varepsilon\frac{\partial^2 u}{\partial t^2} + \sigma\frac{\partial u}{\partial t} = a\frac{\partial}{\partial x}\left(e^{\beta u}\frac{\partial u}{\partial x}\right) + be^{\beta u} + F(u - w). \quad (54)$$

1. At $b = 0$, Eq. (54) has a solution in the following form of the sum of the functions of different arguments:

$$u = \frac{1}{\beta}\ln(C_1x^2 + C_2x + C_3) + \psi(t),$$

where the function $\psi(t)$ is described by the delay differential equation

$$\varepsilon\psi''(t) + \sigma\psi'(t) = \frac{2aC_1}{\beta}e^{\beta\psi(t)} + F(\psi(t) - \psi(t-\tau)).$$

2. At $b\beta > 0$, Eq. (54) yields another solution in the form of the sum of the functions of different arguments:

$$u = \frac{1}{\beta}\ln[C_1 + C_2\cos(\lambda x) + C_3\sin(\lambda x)] + \psi(t), \quad \lambda = \sqrt{b\beta/a},$$

where the function $\psi(t)$ is described by the delay differential equation

$$\varepsilon\psi''(t) + \sigma\psi'(t) = bC_1\beta e^{\beta\psi(t)} + F(\psi(t) - \psi(t-\tau)). \quad (55)$$

3. At $b\beta < 0$, Eq. (54) also yields a solution in the form of the sum of the functions of different arguments as follows:

$$u = \frac{1}{\beta}\ln[C_1 + C_2\exp(-\lambda x) + C_3\exp(\lambda x)] + \psi(t), \quad \lambda = \sqrt{-b\beta/a},$$

where the function $\psi(t)$ is described by delay differential equation (55).

At $\varepsilon = 0$, $\sigma \neq 0$, and $\gamma \neq \beta$, Eq. (49) of the form

$$\varepsilon\frac{\partial^2 u}{\partial t^2} + \sigma\frac{\partial u}{\partial t} = a\frac{\partial}{\partial x}\left(e^{\beta u}\frac{\partial u}{\partial x}\right) + ce^{(\beta-2\gamma)u} + e^{-\gamma u}F(e^{\gamma u} - e^{\gamma w})$$

has the generalized separable solution

$$u = \frac{1}{\gamma}\ln[\lambda x + \psi(t)], \quad \lambda = \pm\sqrt{\frac{c\gamma^2}{a(\gamma-\beta)}},$$

where the function $\psi(t)$ is described by the following delay differential equation:

$$\psi'(t) = \frac{\gamma}{\sigma}F(\psi(t) - \psi(t-\tau)).$$

At $\varepsilon \neq 0$ and $\sigma = 0$, Eq. (49) of the form

$$\varepsilon\frac{\partial^2 u}{\partial t^2} + \sigma\frac{\partial u}{\partial t} = a\frac{\partial}{\partial x}\left(e^{\beta u}\frac{\partial u}{\partial x}\right) + F(e^{\beta u} - e^{\beta w}) + e^{-2\beta u}H(e^{\beta u} - e^{\beta w})$$

yields the generalized separable solution

$$u = \frac{1}{\beta}\ln(At + Bx^2 + C_1x + C_2), \quad B = -\frac{\beta}{2a}F(A\tau),$$

where the constant A is determined from the following algebraic (transcendental) equation:

$$\varepsilon A^2 + \beta H(A\tau) = 0.$$

Equation (49) of the form

$$\varepsilon\frac{\partial^2 u}{\partial t^2} + \sigma\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[(a\ln u + b)\frac{\partial u}{\partial x}\right] - cu\ln u + uF(u/w),$$

yields the separable solution

$$u = \exp(\lambda x)\psi(t), \quad \lambda = \pm\sqrt{c/a},$$

where the function $\psi(t)$ is described by the following delay differential equation:

$$\varepsilon\psi''(t) + \sigma\psi'(t) = (a + b)\lambda^2\psi(t) + \psi(t)F(\psi(t)/\psi(t-\tau)).$$

The above equation yields the particular solution $\psi(t) = Ae^{\beta t}$, where β is determined from the algebraic (transcendental) equation

$$\varepsilon\beta^2 + \sigma\beta - (a + b)\lambda^2 - F(e^{\beta\tau}) = 0.$$

Equation (49) of the form

$$\varepsilon\frac{\partial^2 u}{\partial t^2} + \sigma\frac{\partial u}{\partial t} = a\frac{\partial}{\partial x}\left[g'(u)\frac{\partial u}{\partial x}\right] + b + \left[-\varepsilon A^2\frac{g''(u)}{(g'(u))^3} + \sigma A\frac{1}{g'(u)}\right]F(g(u) - g(w)),$$

where $g(z)$ and $F(z)$ are arbitrary functions and the prime denotes a derivative with respect to the corresponding argument, yields the functional separable solution in the implicit form

$$g(u) = At - \frac{b}{2a}x^2 + C_1x + C_2,$$

where A is determined from the algebraic (transcendental) equation $F(A\tau) = 1$.

LINEAR REACTION–DIFFUSION EQUATION WITH DELAY: EXACT SOLUTIONS

We now consider the following linear hyperbolic reaction–diffusion equation with delay:

$$\varepsilon\frac{\partial^2 u}{\partial t^2} + \sigma\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} - kw, \quad w = u(x, t-\tau), \quad k > 0, \tag{56}$$

where $u = u(x, t)$, $a > 0$, $\varepsilon \geq 0$, and $\sigma \geq 0$ ($\varepsilon + \sigma \neq 0$). Equation (56) is the particular case of some of the equations considered above and, consequently, yields the corresponding solutions. For example, Eq. (7) is reduced to form (56) by choosing the function $F(w/u) = -kw/u$ and Eq. (36) is transformed into (56) by choosing the function $F(u - w) = -b(u - w)$ and $b = -k$.

1. Separable solutions have the form

$$u = [A\cos(\mu x) + B\sin(\mu x)]f(t),$$

where $f(t)$ is described by the following delay differential equation:

$$\varepsilon f''(t) + \sigma f'(t) + a\mu^2 f(t) + kf(t - \tau) = 0. \tag{57}$$

The above equation yields exponential and trigonometric particular solutions. Methods for solving this equation are given, for example, in [31].

The other separable solutions have the form

$$u = [A\exp(-\mu x) + B\exp(\mu x)]f(t),$$

where the function $f(t)$ is described by Eq. (57), in which μ^2 should be replaced by $-\mu^2$.

2. Equation (56) yields the generalized separable solutions that are polynomial with respect to t as follows:

$$u(x,t) = \sum_{m=0}^{n} t^m\psi_m(x), \tag{58}$$

where the functions $\psi_m(t)$ are described by a system of linear ordinary differential equations. In particular, by setting $n = 2$ in (58), we have the solution

$$u(x, t) = t^2\psi_2(x) + t\psi_1(x) + \psi_0(x),$$

where the functions $\psi_m(t)$ are described by the following equations:

$$a\psi_2'' - k\psi_2 = 0,$$

$$a\psi_1'' - k\psi_1 = 2(\sigma - k\tau)\psi_2,$$

$$a\psi_0'' - k\psi_0 = (k\tau^2 + 2\varepsilon)\psi_2 + (\sigma - k\tau)\psi_1,$$

which are integrated.

3. Equation (56) yields the generalized separable solutions that are polynomial with respect to x:

$$u(x,t) = \sum_{m=0}^{n} x^m\varphi_m(t),$$

where the functions $\varphi_m(t)$ are described by the following system of delay differential equations:

$$\varepsilon\varphi_n''(t) + \sigma\varphi_n'(t) + k\varphi_n(t - \tau) = 0,$$

$$\varepsilon\varphi_{n-1}''(t) + \sigma\varphi_{n-1}'(t) + k\varphi_{n-1}(t - \tau) = 0,$$

$$\varepsilon\varphi_m''(t) + \sigma\varphi_m'(t) + k\varphi_m(t - \tau) = a(m + 1)(m + 2)\varphi_{m+2}(t), \quad m = 0, 1, \ldots, n - 2,$$

which can be solved sequentially.

4. We also have the following generalized separable solutions:

$$u(x, t) = \varphi(x)\cos(\omega t) + \psi(x)\sin(\omega t).$$

An example of these solutions is given in the next section.

LINEAR REACTION–DIFFUSION EQUATION WITH DELAY: THE STOKES PROBLEM

We consider the generalized Stokes problem for Eq. (56) without the initial conditions and with the periodic boundary conditions

$$u = A\cos(\omega t + \gamma) \text{ at } x = 0, \quad u \to 0 \text{ at } x \to \infty. \tag{59}$$

The solution to problem (56), (59) has the form

$$u = Ae^{-\lambda x}\cos(\omega t - \beta x + \gamma), \tag{60}$$

where the constants λ and β are determined from the following system of algebraic equations:

$$\sigma\omega - k\sin\omega\tau = 2a\beta\lambda, \quad \varepsilon\omega^2 - k\cos\omega\tau = a(\beta^2 - \lambda^2). \tag{61}$$

For the sake of convenience, we introduce the notation

$$C = (\sigma\omega - k\sin\omega\tau)/a, \; D = (\varepsilon\omega^2 - k\cos\omega\tau)/a \tag{62}$$

In that case, the solution to system of equations (61) can be represented as

$$\lambda = \left[\frac{(C^2 + D^2)^{1/2} - D}{2}\right]^{1/2}, \quad \beta = \left[\frac{(C^2 + D^2)^{1/2} + D}{2}\right]^{1/2}. \tag{63}$$

Solution (63) is not unique for system of equations (61). Other solutions are not considered, since they contain either negative or complex-valued λ and corresponding solutions (60) to Eq. (56) do not satisfy boundary conditions (59).

The problem under consideration is interesting because the damping decrement λ depends on the frequency ω and, at certain sets of values for the parameters ε, σ, a, k, and τ, vanishes at the frequencies ω_λ, which can be found from the first relationship in (61). The vanishing of the decrement λ leads to the fact that solution (60) ceases to decay and satisfy boundary condition (59) at $x \to \infty$. The function $\lambda(\omega)$ is continuous; therefore, similar phenomena, e.g., very slow decay, are observed at the frequencies close to ω_λ, although the solution itself still satisfies the boundary condition at infinity.

The damping decrement λ vanishes at $C = 0$, i.e., at the values of ω_λ that satisfy the equation $\sigma\omega_\lambda = k\sin\omega_\lambda\tau$, which can be represented as

$$\Omega_\lambda = \xi\sin\Omega_\lambda, \quad \Omega_\lambda = \omega_\lambda\tau, \quad \xi = k\tau/\sigma. \tag{64}$$

An analysis of Eq. (64) yields the following results:

(1) Equation (64) has the positive solutions Ω_λ only at $\xi > 1$.

(2) The number of the solutions Ω_λ increases with an increase in the value of ξ.

(3) All positive solutions Ω_λ satisfy the inequality $\Omega_\lambda \le \xi$ and lie in the following intervals: $2j\pi < \Omega_\lambda < (2j + 1)\pi, j = 0, 1, \ldots, \left[\frac{1}{2}N\right]$, where N is the number of solutions (here, $[A]$ denotes the largest integer that is less than or equal to A). The first interval contains one solution, the last interval contains one or two solutions, depending on the value of ξ, and the other intervals contain two solutions each.

At $\tau = 0$, we have a problem for the linear hyperbolic reaction–diffusion equation without delay. Its solution has the form

$$u = Ae^{-\lambda_0 x}\cos(\omega t - \beta_0 x + \gamma), \tag{65}$$

which is similar to the form of solution (60) to the problem with delay. The values of damping decrement λ_0 and shear coefficient β_0 are calculated using the formulas

$$\lambda_0 = \left[\frac{(C_0^2 + D_0^2)^{1/2} - D_0}{2}\right]^{1/2}, \quad \beta_0 = \left[\frac{(C_0^2 + D_0^2)^{1/2} + D_0}{2}\right]^{1/2}, \tag{66}$$

where the following notation was used:

$$C_0 = \sigma\omega/a, \; D_0 = (\varepsilon\omega^2 - k)/a. \tag{67}$$

An analysis of formulas (66) and (67) shows that there are no nonzero frequencies ω_λ at which $\lambda_0(\omega_\lambda) = 0$; therefore, solution (65) always satisfies boundary conditions (59). The function $\lambda_0(\omega)$ does not decrease at $\sigma^2 > 4\varepsilon k$, does not increase at $\sigma^2 < 4\varepsilon k$, and is constant (does not depend on ω) at $\sigma^2 = 4\varepsilon k$. The relationship $\beta_0(\omega)$ strictly increases at any sets of values for the parameters ε, σ, a, and k.

GLOBAL INSTABILITY OF SOLUTIONS TO CERTAIN DELAY REACTION–DIFFUSION SYSTEMS OF EQUATIONS

We now consider the following system of hyperbolic delay reaction–diffusion equations:

$$\varepsilon_1\frac{\partial^2 u_1}{\partial t^2} + \sigma_1\frac{\partial u_1}{\partial t} = a_1\frac{\partial^2 u_1}{\partial x^2} + bu_1 + F(u_1 - kw_1, u_2, w_2),$$
$$\varepsilon_2\frac{\partial^2 u_2}{\partial t^2} + \sigma_2\frac{\partial u_2}{\partial t} = a_2\frac{\partial^2 u_2}{\partial x^2} + G(u_1 - kw_1, u_2, w_2), \tag{68}$$
$$k > 0,$$

where $u_{1,2} = u_{1,2}(x,t)$, $w_{1,2} = u_{1,2}(x,t-\tau)$; $a_{1,2} > 0$, $\varepsilon_{1,2} \ge 0$, $\sigma_{1,2} \ge 0$ ($\varepsilon_{1,2} + \sigma_{1,2} \ne 0$), and F and G are the arbitrary functions of three arguments.

In the general case, system of equations (68) (at $\tau \ne 0$) yields the following simplest solutions: the stationary solution, the homogeneous solution (does not depend on x), and the traveling wave solution $u_1 = u_1(z)$, $u_2 = u_2(z)$, where $z = \alpha x + \beta t$. The stability of these and certain other solutions to different delay reaction–diffusion equations and systems of these equations is considered, for example, in [26–29].

Property 4 (generalizes property 2). Let

$$u_{10} = u_{10}(x,t), \quad u_{20} = u_{20}(x,t) \tag{69}$$

be the solution to system of equations (68). In that case, system of equations (68) also has the solution

$$u_1 = u_{10}(x,t) + e^{ct}V(x,t), \quad u_2 = u_{20}(x,t), \\ c = \frac{1}{\tau}\ln k, \quad k > 0, \tag{70}$$

where $V = V(x, t)$ is any τ-periodic solution to the following linear hyperbolic equation:

$$\varepsilon_1 \frac{\partial^2 V}{\partial t^2} + (\sigma_1 + 2\varepsilon_1 c)\frac{\partial V}{\partial t} \\ = a_1 \frac{\partial^2 V}{\partial x^2} + (b - \varepsilon_1 c^2 - \sigma_1 c)V, \quad V(x,t) = V(x,t-\tau). \tag{71}$$

The general form of the function

$$V(x,t) = V_1(x,t,\sigma_1 + 2\varepsilon_1 c,\, b - \varepsilon_1 c^2 - \sigma_1 c)$$

is given by formulas (16) and (17).

We use property 4 for deriving the conditions of instability for nonlinear reaction–diffusion system of equations (68). To accomplish this, we take the following stationary spatially periodic solution to Eq. (71):

$$V = \delta\sin(\gamma x + \mu), \\ \gamma = \sqrt{(b - \varepsilon_1 c^2 - \sigma_1 c)/a_1}, \quad b - \varepsilon_1 c^2 - \sigma_1 c \ge 0, \tag{72}$$

where δ and μ are arbitrary constants.

We analyze formulas (70) and (72). Let the following condition be satisfied:

$$k > 1, \quad \tau > 0, \quad b\tau^2 - \varepsilon_1(\ln k)^2 - \sigma_1\tau\ln k \ge 0. \tag{73}$$

In that case, at $0 \le t \le \tau$, solutions (69) and (70) differ little from each other for fairly small values of $\delta > 0$. However, at $t \to \infty$, these solutions unrestrictedly diverge due to the exponential factor in (70). This means that, when conditions (73) are satisfied, any solution $u_{10}(x,t)$, $u_{20}(x,t)$ to system of equations (68) is unstable.

Conditions (73) can be more clearly represented as

$$k > 1, \quad b > 0, \quad \tau \ge \tau_0, \\ \tau_0 = \frac{\ln k}{2b}\left(\sigma_1 + \sqrt{\sigma_1^2 + 4\varepsilon_1 b}\right). \tag{74}$$

Relationships (74) are the solution to inequality (73) for τ, and the branch

$$\tau \le \tau_0, \quad \tau_0 = \frac{\ln k}{2b}\left(\sigma_1 - \sqrt{\sigma_1^2 + 4\varepsilon_1 b}\right)$$

is not considered, since it contradicts the condition $\tau > 0$. The physical meaning of conditions (74) is that, in the region of parameters $k > 1$ and $b > 0$, instability arises due to delay, which should be fairly large, i.e., $\tau \ge \tau_0$.

Since the form of the kinetic functions F and G does not affect instability conditions (74) for system of equations (68), they are called the *global conditions of instability*. It should be emphasized that we are dealing with nonlinear instability, and all of the results derived above are exact (rather than linearized, as is the case in the theory of linear stability; various assumptions, expansions, and approximations that are characteristic of the majority of nonlinear theories are not used either).

INSTABILITY OF SOLUTIONS TO CERTAIN NONLINEAR INITIAL VALUE PROBLEMS

1. Let Eq. (69) be the solution to the Cauchy-type problem subject to the initial conditions

$$u_1 = u_{11}(x,t), \quad u_2 = u_{21}(x,t), \quad \partial_t u_1 = u_{12}(x,t), \\ \partial_t u_2 = u_{22}(x,t) \text{ at } 0 \le t \le \tau \tag{75}$$

for system of delay reaction–diffusion equations (68) over the entire range of the variable $-\infty < x < \infty$. From this point on, ∂_t denotes the partial derivative with respect to t.

It follows from property 4 that, at $k > 0$, system of equations (68) also has the solution that is determined by formulas (70) and (72). By designating this solution as $\tilde{u}_1, \tilde{u}_2$, we have

$$\tilde{u}_1 = u_{10} + \delta e^{ct}\sin(\gamma x + \mu), \quad \tilde{u}_2 = u_{20}, \tag{76}$$

where δ and μ are arbitrary constants, $c = \frac{1}{\tau}\ln k$, and the coefficient γ is determined in (72). Comparing solutions (69) and (76), as well as their derivatives with respect to t at $0 \le t \le \tau$, we have

$$|\tilde{u}_1 - u_{10}| \le \delta e^{c\tau}, \quad |\partial_t\tilde{u}_1 - \partial_t u_{10}| \le \delta c e^{c\tau}, \\ |\tilde{u}_2 - u_{20}| = 0, \quad |\partial_t\tilde{u}_2 - \partial_t u_{20}| = 0.$$

At fixed values of τ and k (at $k > 1$, which corresponds to $c > 0$), the differences between solutions (69) and (76) and their derivatives with respect to t can be made arbitrary small due to the choice of δ; i.e., the initial data for these solutions differ little at $0 \le t \le \tau$. On the other hand, when conditions (74) are satisfied and at $x = \frac{1}{\gamma}\left(\frac{\pi}{2} - \mu\right)$, we have

$$|\tilde{u}_1 - u_{10}| = \delta e^{c\tau} \to \infty \text{ at } t \to \infty,$$

i.e., when the global instability conditions are satisfied, initially close solutions to two Cauchy problems unrestrictedly diverge with the passage of time.

The specified instability of solutions to system of equations with delay (68) with respect to the initial data makes the Cauchy problem for (68) ill-posed in the sense of Hadamard (in the event that conditions (74) are satisfied). It should be noted that the instability is of a general character and does not depend on the form of functions F and G.

2. We show that, when conditions (74) are satisfied, there can be global instability in solutions to certain nonlinear initial-boundary value problems with boundary conditions of the first, second, and third kinds in the range of $0 \le x \le h$.

Let Eq. (69) be the solution to the initial-boundary value problem for a system of equations with delay (68) subject to initial conditions (75) and general boundary conditions of the first kind:

$$u_1(0,t) = \varphi_1(t), \quad u_2(0,t) = \varphi_2(t); \qquad u_1(h,t) = \psi_1(t), \quad u_2(h,t) = \psi_2(t), \tag{77}$$

where $h = \pi/\gamma$ and the coefficient γ is determined in (72).

At $\mu = 0$, formula (76) yields the solution to system of equations (68) that exactly satisfies boundary conditions (77). Due to the choice of δ, this solution can be made arbitrary close to solution (69) in the range of the initial data $0 \le t \le \tau$. However, when global instability conditions (74) are satisfied, initially close solutions (69) and (76) to the initial-boundary value problems under consideration exponentially diverge at $t \to \infty$ in the middle $x = h/2$ of the considered region. This instability of solutions to system of equations (68) with respect to the initial data makes the initial-boundary value problem for system of equations (68) ill-posed in the sense of Hadamard (in the event that conditions (74) are satisfied).

In the case of boundary conditions of the second kind, when derivatives with respect to the coordinate x are specified at the domain boundaries, solution (69) should be compared with the solution derived using property 4 and formula (72) at $\mu = \pi/2$.

SOME GENERALIZATIONS AND REMARKS

1. The system of equations

$$\begin{gathered}\varepsilon_1 \frac{\partial^2 u_1}{\partial t^2} + \sigma_1 \frac{\partial u_1}{\partial t} = a_1 \frac{\partial^2 u_1}{\partial x^2} + b_1 u_1 + d_1 w_1 \\ + F_1(u_1 - k w_1, u_2, w_2), \\ \varepsilon_2 \frac{\partial^2 u_2}{\partial t^2} + \sigma_2 \frac{\partial u_2}{\partial t} = a_2 \frac{\partial^2 u_2}{\partial x^2} + G(u_1 - k w_1, u_2, w_2), \\ k > 0,\end{gathered} \tag{78}$$

can be written as (68), where

$$b = b_1 + \frac{d_1}{k}, \quad F(z, u_2, w_2) = F_1(z, u_2, w_2) - \frac{d_1}{k} z.$$

Therefore, instability conditions for solutions to system of equations (78) are derived from (74) by replacing the parameter b with $b_1 + (d_1/k)$.

2. The derived results on instability extend to the following nonlinear multicomponent systems of equations:

$$\begin{gathered}\varepsilon_1 \frac{\partial^2 u}{\partial t^2} + \sigma_1 \frac{\partial u}{\partial t} \\ = a_1 \frac{\partial^2 u}{\partial x^2} + b_1 u + F(x, t, u - kw, u_2, w_2, \ldots, u_n, w_n), \\ \varepsilon_n \frac{\partial^2 u_n}{\partial t^2} + \sigma_n \frac{\partial u_n}{\partial t} \\ = a_n \frac{\partial^2 u_n}{\partial x^2} + G_n(x, t, u - kw, u_2, w_2, \ldots, u_n, w_n), \\ n = 2, \ldots, m; \quad k > 0,\end{gathered}$$

where $u = u(x, t)$, $w = u(x, t - \tau)$, $u_n = u_n(x, t)$, $w_n = u_n(x, t - \tau_n)$; F and G_n are arbitrary functions, and τ and τ_n are the delay times (which can be different).

3. We now consider the equation

$$\begin{gathered}\varepsilon \frac{\partial^2 u}{\partial t^2} + \sigma \frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2} \\ + bu - b\left(\frac{u - kw}{1 - k}\right) + F\left(\frac{u - kw}{1 - k}\right), \quad k \ne 1,\end{gathered} \tag{79}$$

where $b = \text{const}$ and F is an arbitrary function. The above equation is the particular case of system of equations (68). When there is no delay ($\tau = 0$ or $k = 0$), this equation transforms into the following equation:

$$\varepsilon \frac{\partial^2 u}{\partial t^2} + \sigma \frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2} + F(u). \tag{80}$$

Any well-posed Cauchy problem for Eq. (80) can be associated with the ill-posed initial value problem for Eq. (79) by appropriately choosing the parameters b, k, and τ. As an illustration, we can consider, for example, the diffusion equation with a first-order bulk chemical reaction, which corresponds to the particular case of Eq. (80) at $\varepsilon = 0$, $\sigma = 1$, and $F(u) = -c^2 u$.

CONCLUSIONS

New exact solutions to nonlinear reaction–diffusion equations with constant delay are derived that contain one or two arbitrary functions of one argument. Generalized and functional separable solutions are found, including periodic solutions with respect to time and space variable, solutions that describe the nonlinear interaction between a standing wave and a traveling wave, and certain other solutions. The methods of generalized and functional separation of variables and the functional constraints method were used to seek exact solutions. Certain exact solutions are described for more complex nonlinear reaction–diffusion equations such as those with variable delay of the general form $\tau = \tau(t)$ and with a variable transport coefficient and constant delay.

Conditions for the global nonlinear instability of solutions to systems of delay reaction–diffusion equations are derived. It is shown that, when instability conditions are satisfied, the respective initial value problems and initial-boundary value problems are ill-posed in the sense of Hadamard.

Certain exact solutions to the linear reaction–diffusion equation with constant delay are described. The generalized Stokes problem subject to the periodic boundary condition is formulated and solved, and a qualitative analysis of the solution is performed.

The derived exact solutions contain free parameters (in some cases, there can be any number of these parameters) and can be used to solve certain model problems and test approximate analytical and numerical methods for solving similar or more complex nonlinear delay partial differential equations.

NOTATION

A, B, C, D—arbitrary constants;

a—thermal diffusivity (in certain cases, diffusion coefficient);

c_p—specific heat at constant pressure;

F—kinetic function;

$\mathbf{q}$—heat flux;

T—temperature;

t—time;

$u = u(x, t)$—sought function (concentration at the time point t);

$w = u(x, t - \tau)$—sought function at the time point $t - \tau$;

x—spatial coordinate;

Δ—Laplace operator;

ε—coefficient in the hyperbolic equation in front of the highest time derivative;

λ—thermal conductivity;

ρ—density;

σ—coefficient in the hyperbolic equation in front of the first time derivative;

τ—delay or relaxation time;

ω—frequency;

∇ —gradient operator.

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