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SECOND-ORDER QUANTUM ARGUMENT SHIFTS IN Ugl_d

Y. Ikeda*

We describe an explicit formula for the second-order quantum argument shifts of an arbitrary central element of the universal enveloping algebra of a general linear Lie algebra. We identify the generators of the subalgebra generated by the quantum argument shifts up to the second order.

Keywords: universal enveloping algebra, Lie algebra, quantum argument shift method, deformation quantization

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1. Introduction

Let g be a complex Lie algebra. The Lie–Poisson bracket on the symmetric algebra $S(g)$ is the unique Poisson bracket extending the Lie bracket,

$$\begin{array}{ccc} S(g)\times S(g) & \xrightarrow{\text{Lie–Poisson bracket}} & S(g) \\ \uparrow & & \uparrow \\ g\times g & \xrightarrow{\text{Lie bracket}} & g \end{array}.$$

We suppose that ξ is an arbitrary element of the dual space g^* and let $\bar{\partial}_\xi$ denote the constant vector field in the direction ξ. We write $\overline{C}$ for the Poisson center of the symmetric algebra $S(g)$. We define $\overline{C}_\xi$ as the algebra generated by the set $\bigcup_{n=0}^{\infty}\bar{\partial}_\xi^n\overline{C}$. Mishchenko and Fomenko [1] showed the following theorem.

Theorem 1. *The algebra $\overline{C}_\xi$ is Poisson commutative.*

Vinberg [2] inquired whether the argument shift algebra $\overline{C}_\xi$ could be extended to a commutative subalgebra C_ξ of the universal enveloping algebra $U(g)$. Nazarov and Olshanski [3] constructed the quantum argument shift algebra C_ξ for any regular semisimple ξ in terms of (i) the Yangian in the case $g = gl_d(\mathbb{C})$ and (ii) the twisted Yangians in the orthogonal and symplectic cases. Tarasov [4] constructed the same quantum argument shift algebra for $g = gl_d(\mathbb{C})$ via the symmetrization map. The quantum argument shift algebra C_ξ is also constructed via the Feigin–Frenkel center for (i) any simple complex Lie algebra g and any regular ξ [5], [6], and (ii) any simple complex Lie algebra of type A or C and any ξ [7], [8].

So far, the argument shift operator $\bar{\partial}_\xi$ had not been quantized. Gurevich, Pyatov, and Saponov [9] defined the quantum derivations ∂^i_j on the universal enveloping algebra $Ugl_d(\mathbb{C})$. We found an explicit formula for the quantum derivations of appropriate elements [10] and showed a quantum analogue of the Mishchenko and Fomenko theorem [11].

*Lomonosov Moscow State University, Moscow, Russia, e-mail: yasushikeda@yahoo.com.

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In the following, we present an explicit formula for the quantum argument shifts of an arbitrary central element up to the second order (see Proposition 1). We also identify a reduced set of generators of the algebra generated by the quantum argument shifts up to the second order (see Corollary 1 and Theorem 5). This reduced set of generators provides an alternative to those given by Futorny and Molev [7]. Complex combinatorial formulas play an essential role here (see Theorem 4 and Proposition 4).

2. Preliminaries

We write δ for the identity matrix and let x^{T} be the transpose of a matrix x. We suppose that d is a nonnegative integer and let $M(d, A)$ denote the algebra of $d \times d$ matrices with entries in an algebra A. We write x^i_j for the (i, j) element of a $d \times d$ matrix x and

$$x^i = \begin{pmatrix} x^i_1 & \dots & x^i_d \end{pmatrix}, \qquad x_j = \begin{pmatrix} x^1_j \\ \vdots \\ x^d_j \end{pmatrix}$$

for the ith row vector and the jth column vector of the matrix x.

We define the generating matrix of the Lie algebra $gl_d = gl_d(\mathbb{C})$ as the $d \times d$ matrix e composed of the indeterminates e^i_j (generators of the Lie algebra gl_d). The universal enveloping algebra of the Lie algebra gl_d is the quotient algebra

$$Ugl_d = \mathbb{C}\langle e^i_j \rangle / \big(e^{i_1}_{j_1} e^{i_2}_{j_2} - e^{i_2}_{j_2} e^{i_1}_{j_1} - e^{i_2}_{j_1} \delta^{i_1}_{j_2} + \delta^{i_2}_{j_1} e^{i_1}_{j_2} \colon i_1, j_1, i_2, j_2 = 1, \dots, d\big),$$

where $\mathbb{C}\langle e^i_j \rangle$ denotes the free unital algebra on the indeterminates e^i_j and the denominator in the right-hand side denotes the ideal generated by the elements

$$\big\{e^{i_1}_{j_1} e^{i_2}_{j_2} - e^{i_2}_{j_2} e^{i_1}_{j_1} - e^{i_2}_{j_1} \delta^{i_1}_{j_2} + \delta^{i_2}_{j_1} e^{i_1}_{j_2} \colon i_1, j_1, i_2, j_2 = 1, \dots, d\big\}.$$

The following relation holds in the universal enveloping algebra Ugl_d:

$$[(e^n)^{i_1}_{j_1}, e^{i_2}_{j_2}] = [e^{i_1}_{j_1}, (e^n)^{i_2}_{j_2}] = (e^n)^{i_2}_{j_1} \delta^{i_1}_{j_2} - \delta^{i_2}_{j_1} (e^n)^{i_1}_{j_2}, \qquad n = 0, 1, 2, \dots. \tag{1}$$

This can be proved by induction.

Quantum derivations on the universal enveloping algebra Ugl_d were defined in [9]. We give a slightly modified definition of these operators as follows.

Definition 1. The quantum derivations on the universal enveloping algebra Ugl_d are the matrix elements of a unique homomorphism of unital complex algebras

$$Ugl_d \to M(d, Ugl_d), \qquad x \mapsto \partial x$$

such that $\partial \operatorname{tr}(\xi e) = \operatorname{tr}(\xi e) + \xi$ for any numerical matrix ξ.

We define the polynomials

$$f^{(n)}_{\pm}(x) = \sum_{m=0}^{n+1} \frac{1 \pm (-1)^{n-m}}{2} \binom{n}{m} x^m.$$

The following theorem is proved in [10].

Theorem 2. *The quantum derivations of the matrix elements* $(e^n)^i_j$ *are given by*

$$\partial(e^n)^i_j = \sum_{m=0}^{n} \big(f_+^{(n-m-1)}(e)_j (e^m)^i + f_-^{(n-m-1)}(e)(e^m)^i_j \big) = \\ = \sum_{m=0}^{n} \big((e^m)_j f_+^{(n-m-1)}(e)^i + (e^m)^i_j f_-^{(n-m-1)}(e) \big).$$

We write C for the center of the universal enveloping algebra Ugl_d. The center C is generated by the elements $\operatorname{tr} e, \operatorname{tr} e^2, \dots$.

We suppose that ξ is an arbitrary numerical matrix. The map $\partial_\xi = \operatorname{tr}(\xi\partial)$ is called the quantum argument shift operator in the direction ξ. We define C_ξ as the algebra generated by the set $\bigcup_{n=0}^{\infty} \partial_\xi^n C$. The following theorem is proved in [11], [12].

Theorem 3. *The algebra* C_ξ *is a quantum argument shift algebra in the direction* ξ*.*

3. Formulas for second-order quantum argument shifts

We present formulas for the second-order quantum argument shifts of central elements. Theorem 2 suffices for this purpose. We adopt the convention that $\operatorname{tr} e^{-1} = 1$ for simplicity of notation. The following formulas give the quantum argument shifts of an arbitrary central element up to the second order.

Proposition 1.

$$\partial\big(\operatorname{tr} e^{n_1} \operatorname{tr} e^{n_2} \dots\big) = \sum_{m_1=-1}^{n_1} \operatorname{tr} e^{m_1} \sum_{m_2=-1}^{n_2} \operatorname{tr} e^{m_2} \dots \prod_k f_-^{(n_k-m_k-1)}(e)$$

and

$$\partial\partial_\xi\big(\operatorname{tr} e^{n_1} \operatorname{tr} e^{n_2} \dots\big) = \sum_{m_1=-1}^{n_1} \operatorname{tr} e^{m_1} \sum_{m_2=-1}^{n_2} \operatorname{tr} e^{m_2} \dots \sum_{k_1=-1}^{n_1-m_1-1} f_-^{(k_1)}(e) \sum_{k_2=-1}^{n_2-m_2-1} f_-^{(k_2)}(e) \dots \\ \dots \partial \operatorname{tr}\Bigg(\xi \prod_\ell f_-^{(n_\ell-m_\ell-k_\ell-2)}(e) \Bigg) \tag{2}$$

for a finite product $\operatorname{tr} e^{n_1} \operatorname{tr} e^{n_2} \dots$.

Proof is by direct computation. We have

$$\partial \operatorname{tr} e^n = \sum_{m=0}^{n} \big(f_+^{(n-m-1)}(e) e^m + f_-^{(n-m-1)}(e) \operatorname{tr} e^m \big) = \\ = f_-^{(n)}(e) + \sum_{m=0}^{n} f_-^{(n-m-1)}(e) \operatorname{tr} e^m = \sum_{m=-1}^{n} f_-^{(n-m-1)}(e) \operatorname{tr} e^m$$

by Theorem 2 and the identity $\sum_{m=0}^{n} f_+^{(n-m-1)}(x) x^m = f_-^{(n)}(x)$. We obtain

$$\partial(\operatorname{tr} e^{n_1} \operatorname{tr} e^{n_2} \dots) = \partial(\operatorname{tr} e^{n_1}) \partial(\operatorname{tr} e^{n_2}) \dots = \\ = \sum_{m_1=-1}^{n_1} \operatorname{tr} e^{m_1} \sum_{m_2=-1}^{n_2} \operatorname{tr} e^{m_2} \dots \prod_k f_-^{(n_k-m_k-1)}(e). \tag{3}$$

We proceed to calculate the second-order quantum argument shifts

$$\partial_\xi(\operatorname{tr} e^{n_1} \operatorname{tr} e^{n_2} \ldots) = \sum_{m_1=-1}^{n_1} \operatorname{tr} e^{m_1} \sum_{m_2=-1}^{n_2} \operatorname{tr} e^{m_2} \ldots \operatorname{tr}\Bigl(\xi \prod_k f_-^{(n_k-m_k-1)}(e)\Bigr) \tag{4}$$

and

$$\partial\partial_\xi(\operatorname{tr} e^{n_1} \operatorname{tr} e^{n_2} \ldots) = \sum_{k_1=-1}^{n_1} \sum_{k_2=-1}^{n_2} \ldots \partial\Bigl(\prod_\ell \operatorname{tr} e^{k_\ell}\Bigr) \partial\Bigl(\operatorname{tr}\Bigl(\xi \prod_\ell f_-^{(n_\ell-k_\ell-1)}(e)\Bigr)\Bigr) =$$

$$= \sum_{k_1=-1}^{n_1} \sum_{k_2=-1}^{n_2} \ldots \sum_{m_1=-1}^{k_1} \operatorname{tr} e^{m_1} \sum_{m_2=-1}^{k_2} \operatorname{tr} e^{m_2} \ldots \prod_\ell f_-^{(k_\ell-m_\ell-1)}(e) \partial \operatorname{tr}\Bigl(\xi \prod_\ell f_-^{(n_\ell-k_\ell-1)}(e)\Bigr)$$

by formula (3). Because

$$\sum_{k_1=-1}^{n_1} \sum_{k_2=-1}^{n_2} \ldots \sum_{m_1=-1}^{k_1} \sum_{m_2=-1}^{k_2} \ldots = \sum_{m_1=-1}^{n_1} \sum_{m_2=-1}^{n_2} \ldots \sum_{k_1=m_1}^{n_1} \sum_{k_2=m_2}^{n_2} \ldots,$$

we arrive at formula (2). ■

We write $A[S]$ for the algebra generated by an algebra A and a set S contained in the quantum argument shift algebra C_ξ. We define

$$C_\xi^{(0)} = C, \qquad C_\xi^{(n)} = C_\xi^{(n-1)}[\partial_\xi^n C].$$

Formula (4) implies the following assertion.

Corollary 1. $C_\xi^{(1)} = C[\operatorname{tr}(\xi e^n) : n = 1, 2, \ldots]$.

We have

$$\partial_\xi^2(\operatorname{tr} e^{n_1} \operatorname{tr} e^{n_2} \ldots) = \sum_{m_1=-1}^{n_1} \operatorname{tr} e^{m_1} \sum_{m_2=-1}^{n_2} \operatorname{tr} e^{m_2} \ldots$$

$$\ldots \sum_{k_1=-1}^{n_1-m_1-1} \sum_{k_2=-1}^{n_2-m_2-1} \ldots \operatorname{tr}\Bigl(\xi \prod_\ell f_-^{(k_\ell)}(e)\, \partial \operatorname{tr}\Bigl(\xi \prod_\ell f_-^{(n_\ell-m_\ell-k_\ell-2)}(e)\Bigr)\Bigr) \tag{5}$$

by formula (2). Formula (5) implies the corollary.

Corollary 2. *The algebra $C_\xi^{(2)}$ is contained in the algebra generated by the algebra $C_\xi^{(1)}$ and the elements*

$$\operatorname{tr}(\xi e^m \partial \operatorname{tr}(\xi e^n)) + \operatorname{tr}(\xi e^n \partial \operatorname{tr}(\xi e^m)), \qquad m, n = 0, 1, 2, \ldots.$$

Proof. The elements of the form

$$\sum_{m_1=-1}^{n_1+1} \sum_{m_2=-1}^{n_2+1} \ldots \operatorname{tr}\Bigl(\xi \prod_k f_-^{(m_k)}(e)\, \partial \operatorname{tr}\Bigl(\xi \prod_k f_-^{(n_k-m_k)}(e)\Bigr)\Bigr)$$

belong to the additive monoid generated by the elements

$$\operatorname{tr}(\xi e^n \partial \operatorname{tr}(\xi e^n)), \quad \operatorname{tr}(\xi e^m \partial \operatorname{tr}(\xi e^n)) + \operatorname{tr}(\xi e^n \partial \operatorname{tr}(\xi e^m)), \qquad m, n = 0, 1, 2, \ldots.$$

Any element of $C_\xi^{(2)}$ is contained in the algebra generated by the algebra $C_\xi^{(1)}$ and the elements

$$\operatorname{tr}(\xi e^m \partial \operatorname{tr}(\xi e^n)) + \operatorname{tr}(\xi e^n \partial \operatorname{tr}(\xi e^m)), \qquad m, n = 0, 1, 2, \ldots. \quad ■$$

We suppose that m and n are nonnegative integers. We have

$$\operatorname{tr}(\xi e^m \partial \operatorname{tr}(\xi e^n)) = \operatorname{tr}\Big(\xi e^m \sum_{j=1}^{n+1} \big(f_+^{(n-j)}(e)\xi e^{j-1} + f_-^{(n-j)}(e)\operatorname{tr}(\xi e^{j-1})\big)\Big) =$$
$$= \sum_{j=1}^{n+1} \big(\operatorname{tr}(\xi e^m f_+^{(n-j)}(e)\xi e^{j-1}) + \operatorname{tr}(\xi e^m f_-^{(n-j)}(e))\operatorname{tr}(\xi e^{j-1})\big), \tag{6}$$

by Theorem 2 and thus

$$\operatorname{tr}(\xi e^m \partial \operatorname{tr}(\xi e^n)) = \sum_{j=1}^{n} \operatorname{tr}(\xi e^m f_+^{(n-j)}(e)\xi e^{j-1}) \mod C_\xi^{(1)} \tag{7}$$

by Corollary 1.

Definition 2. We define the $(m+n)\times n$ integer matrix $P_n^{(m)}$ as the coefficients of the polynomials

$$x^m f_+^{(n-j)}(x) = \sum_{i=1}^{m+n} (P_n^{(m)})_j^i x^{i-1}$$

and let $P_n = P_n^{(0)}$.

The matrix P_n is the submatrix of the matrix P_{n+1} in the top right corner, $P_{n+1} = \left(\begin{smallmatrix} * & P_n \\ 1 & 0 \end{smallmatrix}\right)$ and $P_n^{(m)} = \left(\begin{smallmatrix} 0 \\ P_n \end{smallmatrix}\right)$ (the first m row vectors are null). For instance, because

$$\begin{pmatrix} f_+^{(3)}(x) & f_+^{(2)}(x) & f_+^{(1)}(x) & f_+^{(0)}(x) \end{pmatrix} = \begin{pmatrix} 3x+x^3 & 1+x^2 & x & 1 \end{pmatrix} = \begin{pmatrix} x^0 & x^1 & x^2 & x^3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

we have $P_4 = \left(\begin{smallmatrix} 0 & 1 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{smallmatrix}\right)$.

Definition 3. We define

$$\tau_\xi(x) = \operatorname{tr}\left(\begin{pmatrix} \xi & \xi e & \dots & \xi e^{m-1} \end{pmatrix} x \begin{pmatrix} \xi \\ \xi e \\ \vdots \\ \xi e^{n-1} \end{pmatrix} \right) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_j^i \operatorname{tr}(\xi e^{i-1} \xi e^{j-1})$$

for any $m \times n$ numerical matrix x.

By formula (7), we now have

$$\operatorname{tr}(\xi e^m \partial \operatorname{tr}(\xi e^n)) = \tau_\xi(P_n^{(m)}) \mod C_\xi^{(1)}. \tag{8}$$

4. Generators of the algebra $C_\xi^{(2)}$

We give the reduced set of generators of the algebra $C_\xi^{(2)}$. The generators given in Corollary 2 can be expressed in terms of lower triangular matrices.

Definition 4. Let n be a nonnegative integer and x an $n\times n$ numerical matrix. We define the $n\times n$ lower triangular numerical matrix $\sigma(x)$ by the formula

$$\sigma(x)=\begin{pmatrix} x_1^1 & 0 & \cdots & 0\\ x_1^2+x_2^1 & x_2^2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ x_1^n+x_n^1 & x_2^n+x_n^2 & \cdots & x_n^n\end{pmatrix}=\sum_{i,j=1}^n x_j^i\delta_{\max\{i,j\}}\delta^{\min\{i,j\}}.$$

Proposition 2. $(\tau_\xi\circ\sigma)(x)=\tau_\xi(x)$ *for any numerical square matrix* x.

Proof. We suppose that m and n are nonnegative integers and let $(\zeta_1,\dots,\zeta_n)$ be a finite sequence of elements of the set $M(d,\mathbb{C})\sqcup\{e\}$. We have

$$\operatorname{tr}[\xi e^m,\zeta_1\dots\zeta_n]=\sum_{\zeta_k=e}\big(\operatorname{tr}(\zeta_1\dots\zeta_{k-1}e^m)\operatorname{tr}(\xi\zeta_{k+1}\dots\zeta_n)-\operatorname{tr}(\zeta_1\dots\zeta_{k-1})\operatorname{tr}(\xi e^m\zeta_{k+1}\dots\zeta_n)\big),$$

by the commutation relation (1), and thus

$$\begin{aligned}\operatorname{tr}[\xi e^m,\xi e^n]&=\sum_{k=1}^n\big(\operatorname{tr}(\xi e^{m+k-1})\operatorname{tr}(\xi e^{n-k})-\operatorname{tr}(\xi e^{k-1})\operatorname{tr}(\xi e^{m+n-k})\big)=\\&=\sum_{k=1}^n[\operatorname{tr}(\xi e^{m+k-1}),\operatorname{tr}(\xi e^{n-k})]=0,\end{aligned}\tag{9}$$

because the algebra $C_\xi^{(1)}=C\big[\operatorname{tr}(\xi e^n)\colon n=1,2,\dots\big]$ (see Corollary 1) is commutative by Theorem 3. We have

$$(\tau_\xi\circ\sigma)(x)=\sum_{i,j=1}^n x_j^i\operatorname{tr}(\xi e^{\max\{i,j\}-1}\xi e^{\min\{i,j\}-1})=\sum_{i,j=1}^n x_j^i\operatorname{tr}(\xi e^{i-1}\xi e^{j-1})=\tau_\xi(x)$$

for any $n\times n$ numerical matrix x by formula (9). ■

Proposition 3. *For any nonnegative integers m and n, we have*

$$\operatorname{tr}\big(\xi e^m\partial\operatorname{tr}(\xi e^n)\big)+\operatorname{tr}\big(\xi e^n\partial\operatorname{tr}(\xi e^m)\big)=(\tau_\xi\circ\sigma)\begin{pmatrix}0 & P_n^{\mathrm T}\\ P_m & 0\end{pmatrix}\mod C_\xi^{(1)}.$$

Proof. We have

$$\operatorname{tr}\big(\xi e^m\partial\operatorname{tr}(\xi e^n)\big)+\operatorname{tr}\big(\xi e^n\partial\operatorname{tr}(\xi e^m)\big)=\tau_\xi(P_n^{(m)})+\tau_\xi(P_m^{(n)})=\tau_\xi\left(\begin{smallmatrix} & n & m\\ m & 0 & 0\\ n & P_n & 0\end{smallmatrix}\right)+\tau_\xi\left(\begin{smallmatrix} & m & n\\ n & 0 & 0\\ m & P_m & 0\end{smallmatrix}\right)=$$

$$=(\tau_\xi\circ\sigma)\left(\begin{smallmatrix} & m & n\\ n & 0 & P_n^{\mathrm T}\\ m & 0 & 0\end{smallmatrix}\right)+(\tau_\xi\circ\sigma)\left(\begin{smallmatrix} & m & n\\ n & 0 & 0\\ m & P_m & 0\end{smallmatrix}\right)=(\tau_\xi\circ\sigma)\left(\begin{smallmatrix} & m & n\\ n & 0 & P_n^{\mathrm T}\\ m & P_m & 0\end{smallmatrix}\right)\mod C_\xi^{(1)}$$

by formula (8) and Proposition 2. ■

The following theorem plays an essential role in reducing the number of the generators given in Corollary 2 and Proposition 3. The proof is given in the Appendix.

Theorem 4. *For any nonnegative integers m and n, we have*

$$\sigma\begin{pmatrix} 0 & P_m^{\mathrm{T}} \\ P_{m+2n} & 0 \end{pmatrix} = \sum_{k=0}^{n}\left(\binom{2n-k}{k}+\binom{2n-k-1}{k-1}\right)P_{m+k}^{(m+k)}, \tag{10}$$

$$\sigma\begin{pmatrix} 0 & P_m^{\mathrm{T}} \\ P_{m+2n+1} & 0 \end{pmatrix} = \sum_{k=0}^{n}\binom{2n-k}{k}\left(P_{m+k+1}^{(m+k)}+P_{m+k}^{(m+k+1)}\right). \tag{11}$$

The following theorem is the main result in this paper.

Theorem 5. *The algebra $C_\xi^{(2)}$ is given by*

$$C_\xi^{(2)} = C_\xi^{(1)}\left[\tau_\xi(P_n^{(n)}), \tau_\xi(P_{n+1}^{(n)}) + \tau_\xi(P_n^{(n+1)}) \colon n = 1, 2, \dots\right].$$

Proof. The algebra $C_\xi^{(2)}$ is contained in the algebra

$$C_\xi^{(1)}\left[\tau_\xi(P_n^{(n)}), \tau_\xi(P_{n+1}^{(n)}) + \tau_\xi(P_n^{(n+1)}) \colon n = 1, 2, \dots\right]$$

by Proposition 3 and Theorem 4. We prove that the elements $\tau_\xi(P_n^{(n)})$ and $\tau_\xi(P_{n+1}^{(n)}) + \tau_\xi(P_n^{(n+1)})$ belong to the algebra

$$C_\xi^{(1)}\left[\partial_\xi^2 \operatorname{tr} e^n \colon n = 3, 4, \dots\right] \tag{12}$$

by induction on the nonnegative integer n. Suppose that the integer n is positive and the elements $\tau_\xi(P_m^{(m)})$, $\tau_\xi(P_{m+1}^{(m)}) + \tau_\xi(P_m^{(m+1)})$ belong to algebra (12) for any nonnegative integer $m < n$. The element $\tau_\xi(P_n^{(n)})$ belongs to algebra (12) because the element $\partial_\xi^2 \operatorname{tr} e^{2n+1} - (4n+2)\tau_\xi(P_n^{(n)})$ belongs to the submodule

$$\operatorname{span}_C\left\{\tau_\xi(P_m^{(m)})\right\}_{m=0}^{n-1} + \operatorname{span}_C\left\{\tau_\xi(P_{m+1}^{(m)}) + \tau_\xi(P_m^{(m+1)})\right\}_{m=0}^{n-1}$$

modulo $C_\xi^{(1)}$ by Theorem 4. Similarly, the element $\tau_\xi(P_{n+1}^{(n)}) + \tau_\xi(P_n^{(n+1)})$ belongs to algebra (12). ■

We compute the first several elements of the generators:

$$\begin{aligned}
&\tau_\xi(P_1^{(1)}) = \operatorname{tr}(\xi^2 e),\\
&\tau_\xi(P_2^{(1)}) + \tau_\xi(P_1^{(2)}) = \operatorname{tr}(2\xi^2 e^2 + \xi e \xi e),\\
&\tau_\xi(P_2^{(2)}) = \operatorname{tr}(\xi^2 e^3 + \xi e \xi e^2),\\
&\tau_\xi(P_3^{(2)}) + \tau_\xi(P_2^{(3)}) = \operatorname{tr}(2\xi^2 e^4 + 2\xi e \xi e^3 + \xi e^2 \xi e^2 + \xi^2 e^2),\\
&\tau_\xi(P_3^{(3)}) = \operatorname{tr}(\xi^2 e^5 + \xi e \xi e^4 + \xi e^2 \xi e^3 + \xi^2 e^3),\\
&\tau_\xi(P_4^{(3)}) + \tau_\xi(P_3^{(4)}) = \operatorname{tr}(2\xi^2 e^6 + 2\xi e \xi e^5 + 2\xi e^2 \xi e^4 + \xi e^3 \xi e^3 + 4\xi^2 e^4 + \xi e \xi e^3),\\
&\tau_\xi(P_4^{(4)}) = \operatorname{tr}(\xi^2 e^7 + \xi e \xi e^6 + \xi e^2 \xi e^5 + \xi e^3 \xi e^4 + 3\xi^2 e^5 + \xi e \xi e^4).
\end{aligned}$$

They form a commutative family together with the elements $\left\{\operatorname{tr}(\xi e^n) \colon n = 1, 2, \dots\right\}$ (see Theorem 3 and Corollary 1).

Appendix: Proof of Theorem 4

We note that relation (10) for $m+1$ implies the same relation for m and is therefore equivalent to the relation

$$\sigma(P_{2n}) = \sum_{m=1}^{n}\left(\binom{2n-m}{m} + \binom{2n-m-1}{m-1}\right)P_m^{(m)} \tag{13}$$

together with the relation for the first column vectors

$$\sigma\begin{pmatrix} 0 & P_{m+1}^{\mathrm{T}} \\ P_{m+2n+1} & 0 \end{pmatrix}_1^i = \sum_{k=0}^{n}\left(\binom{2n-k}{k} + \binom{2n-k-1}{k-1}\right)(P_{m+k+1}^{(m+k+1)})_1^i. \tag{14}$$

Relation (13) is equivalent to the combinatorial relation

$$\binom{2n_1+n_2+2n_3+1}{2n_3} + \binom{n_2+2n_3}{2n_3} = \\ = \sum_{n_4=0}^{n_3}\left(\binom{n_1+n_2+n_3+n_4+1}{2n_4} + \binom{n_1+n_2+n_3+n_4}{2n_4}\right)\binom{n_1+n_3-n_4}{2(n_3-n_4)}.$$

This follows by comparing the $(2n_1+n_2+2, n_2+1)$ element of the matrix $\sigma(P_{2n})$ with that of the matrix

$$\sum_{m=1}^{n}\left(\binom{2n-m}{m} + \binom{2n-m-1}{m-1}\right)P_m^{(m)},$$

for $n = n_1+n_2+n_3+1$.

Relation (14) is equivalent to the polynomial relation

$$f_+^{(m+2n)}(x) + f_+^{(m)}(x)x^{2n} = \sum_{i=1}^{m+2n+1}(P_{m+2n+1} + P_{m+1}^{(2n)})_1^i x^{i-1} = \\ = \sum_{k=0}^{n}\left(\binom{2n-k}{k} + \binom{2n-k-1}{k-1}\right)\sum_{i=1}^{m+2k+1}(P_{m+k+1}^{(k)})_1^i x^{i-1} = \\ = \sum_{k=0}^{n}\left(\binom{2n-k}{k} + \binom{2n-k-1}{k-1}\right)f_+^{(m+k)}(x)x^k.$$

Similar arguments apply to the case in (11). We thus arrive at the following proposition.

Proposition 4. 1. *Theorem 4 is equivalent to the following conditions.*

For any nonnegative integers n_1, n_2, and n_3, we have

$$\binom{2n_1+n_2+2n_3+1}{2n_3} + \binom{n_2+2n_3}{2n_3} = \\ = \sum_{n_4=0}^{n_3}\left(\binom{n_1+n_2+n_3+n_4+1}{2n_4} + \binom{n_1+n_2+n_3+n_4}{2n_4}\right)\binom{n_1+n_3-n_4}{2(n_3-n_4)}, \tag{15}$$

$$\binom{2n_1+n_2+2n_3+2}{2n_3} + \binom{n_2+2n_3}{2n_3} = \\ = \sum_{n_4=0}^{n_3}\binom{n_1+n_2+n_3+n_4+1}{2n_4}\left(\binom{n_1+n_3-n_4+1}{2(n_3-n_4)} + \binom{n_1+n_3-n_4}{2(n_3-n_4)}\right). \tag{16}$$

For any nonnegative integers m and n, we have

$$f_+^{(m+2n)}(x) + f_+^{(m)}(x)x^{2n} = \sum_{k=0}^{n} \left(\binom{2n-k}{k} + \binom{2n-k-1}{k-1} \right) f_+^{(m+k)}(x)x^k,$$

$$f_+^{(m+2n+1)}(x) + f_+^{(m)}(x)x^{2n+1} = \sum_{k=0}^{n} \binom{2n-k}{k} \big(f_+^{(m+k+1)}(x)x^k + f_+^{(m+k)}(x)x^{k+1} \big).$$

2. *Relation* (15) *is equivalent to the relation*

$$\sigma(P_{2n}) = \sum_{m=1}^{n} \left(\binom{2n-m}{m} + \binom{2n-m-1}{m-1} \right) P_m^{(m)}.$$

3. *Relation* (16) *is equivalent to the relation*

$$\sigma(P_{2n+1}) = \sum_{m=0}^{n} \binom{2n-m}{m} \big(P_{m+1}^{(m)} + P_m^{(m+1)} \big).$$

Proof of Theorem 4. We verify the corresponding conditions in Proposition 4 with Mathematica:

```
In[1]:= FullSimplify[Binomial[2n+m+2l+1,2l]+
 Binomial[m+2l,2l]-
 Sum[(Binomial[n+m+l+k+1,2k]+Binomial[n+m+l+k,2k])
 Binomial[n+l-k,2(l-k)],{k,0,l}],
 Element[n|m|l,Integers]&&n>=0&&m>=0&&l>=0]
Out[1]= 0
In[2]:= FullSimplify[Binomial[2n+m+2l+2,2l]+
 Binomial[m+2l,2l]-
 Sum[Binomial[n+m+l+k+1,2k](Binomial[n+l-k+1,2(l-k)]+
 Binomial[n+l-k,2(l-k)]),{k,0,l}],
 Element[n|m|l,Integers]&&n>=0&&m>=0&&l>=0]
Out[2]= 0
In[3]:= Fplus[n_][x_]:=((x+1)^n+(x-1)^n)/2
In[4]:= Simplify[Fplus[m+2n][x]+Fplus[m][x]x^(2n)-
 Sum[(Binomial[2n-k,k]+Binomial[2n-k-1,k-1])
 Fplus[m+k][x]x^k,{k,0,n}],
 Element[m|n,Integers]&&m>=0&&n>=0]
Out[4]= 0
In[5]:= Simplify[Fplus[m+2n+1][x]+Fplus[m][x]x^(2n+1)-
 Sum[Binomial[2n-k,k](Fplus[m+k+1][x]x^k+
 Fplus[m+k][x]x^(k+1)),{k,0,n}],
 Element[m|n,Integers]&&m>=0&&n>=0]
Out[5]= 0
```

■

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