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EXISTENCE AND STABILITY OF STATIONARY SOLUTIONS WITH BOUNDARY LAYERS IN A SYSTEM OF FAST AND SLOW REACTION–DIFFUSION–ADVECTION EQUATIONS WITH KPZ NONLINEARITIES

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The existence of stationary solutions of singularly perturbed systems of reaction–diffusion–advection equations is studied in the case of fast and slow reaction–diffusion–advection equations with nonlinearities containing the gradient of the squared sought function (KPZ nonlinearities). The asymptotic method of differential inequalities is used to prove the existence theorems. The boundary layer asymptotics of solutions are constructed in the case of Neumann and Dirichlet boundary conditions. The case of quasimonotone sources and systems without the quasimonotonicity requirement is also considered.

Keywords: singular perturbation, reaction–diffusion–advection equations, stationary solutions, KPZ nonlinearities, asymptotic method of differential inequalities, boundary layer, Lyapunov stability

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1. Introduction. Statement of the problem

We consider a system of fast and slow reaction–diffusion–advection equations with KPZ nonlinearities, which is a special case important for applications and allowing one to obtain constructive conditions for the existence and Lyapunov stability of solutions such as the stationary solutions of the corresponding parabolic problem

$$\begin{aligned}
&\mathcal{N}_u(u,v) := \varepsilon^2\frac{d^2u}{dx^2} - \varepsilon^2 A(u,x)\left(\frac{du}{dx}\right)^2 - g(u,v,x,\varepsilon) = 0,\\
&\mathcal{N}_v(u,v) := \frac{d^2v}{dx^2} - B(v,x)\left(\frac{dv}{dx}\right)^2 - f(u,v,x,\varepsilon) = 0, \qquad 0 < x < 1,
\end{aligned} \tag{1}$$

where $\varepsilon \in (0;\varepsilon_0]$ is a small parameter. Such systems naturally arise in modeling fast bimolecular reactions in the case where one of the sources (reaction, nonlinear source, interaction) is intensive (of the order of $1/\varepsilon^2$) and the other one is of the order of unity (see, e.g., [1]).

We assume that the function $u(x)$ satisfies one of the following versions of boundary conditions:

$$u'(0) = u^0, \qquad u'(1) = u^1, \tag{N}$$

$$u(0) = u^0, \qquad u(1) = u^1. \tag{D}$$

For the function $v(x)$, we pose the Dirichlet condition

$$v(0) = v^0, \qquad v(1) = v^1. \tag{2}$$

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In what follows, we respectively let (1.N) and (1.D) denote the problems where either the Neumann condition (N) or the Dirichlet condition (D) is imposed for $u(x)$. We omit the letter notation in the cases where the form of the form of the boundary condition is unimportant.

A special feature of the problem under study is the presence of terms containing the gradient of the sought function squared. The nonlinearities of this type are called Kardar–Parisi–Zhang (KPZ) nonlinearities and are widely used in modeling population dynamics processes (the squared gradient describes nonlocal interactions [2]), the free surface growth in the theory of polymers and in the nonlinear theory of thermal conductivity (see, e.g., [3] and the references therein). We note that this system of equations is also of definite theoretical interest: the square is the maximum (limit) exponent at which the Bernstein conditions for the nonlinearity growth are satisfied (the nonlinearity belongs to the class of Nagumo functions, see[4]–[9]). The stationary solutions with boundary and internal layers of the initial boundary value problem were considered in [10] only in the case of a fast equation.

Let the following conditions be satisfied.

Condition A1. *Let the functions $g(u,v,x,\varepsilon)$, $f(u,v,x,\varepsilon)$ be defined on the set $\overline{\Omega}_1 := (u,v,x,\varepsilon) \in I_u \times I_v \times [0;1] \times (0;\varepsilon_0]$, and let $A(u,x)$ and $B(v,x)$ be respectively defined on the sets $\overline{\Omega}_2 := (u,x) \in I_u \times [0;1]$ and $\overline{\Omega}_3 := (v,x) \in I_v \times [0;1]$ and be sufficiently smooth functions of their arguments.*

We consider the degenerate differential-algebraic system

$$\begin{aligned} &g(u,v,x,0) = 0, \\ &\frac{d^2v}{dx^2} - B(v,x)\left(\frac{dv}{dx}\right)^2 - f(u,v,x,0) = 0, \qquad 0 < x < 1. \end{aligned} \tag{3}$$

We require that the following solvability condition be satisfied for this system.

Condition A2. *Let the equation $g(u,v,x,0) = 0$ have a solution $u = \varphi(v,x)$ such that $g_u(\varphi(v,x),v,x,0) > 0$ for $(v,x) \in \overline{\Omega}_3$ and let the problem*

$$\begin{aligned} &\frac{d^2v}{dx^2} - B(v,x)\left(\frac{dv}{dx}\right)^2 - f(\varphi(v,x),v,x,0) = 0, \qquad 0 < x < 1, \\ &v(0) = v^0, \qquad v(1) = v^1, \end{aligned} \tag{4}$$

have an isolated solution $v = \bar{v}_0(x)$.

We set $\bar{u}_0(x) = \varphi(\bar{v}_0(x),x)$, $x \in [0,1]$. It also follows from Condition A2 that $\bar{g}_u(x) \equiv g_u(\bar{u}_0(x), \bar{v}_0(x),x,0) > 0$, $x \in [0,1]$ (here and hereafter, the bar over a function or over its derivative means that its value is taken at the point $(\bar{u}_0(x),\bar{v}_0(x),x,0)$).

In problem (1.D), we require the satisfaction of the standard condition that the boundary values u^0, u^1 belong to the domain of influence of a root of the degenerate equation.

Condition A3. *Let the following inequalities be satisfied*:

$$\begin{aligned} &\int_{\bar{u}_0(0)}^{\tilde{u}} g(s,\bar{v}_0(0),0,0)\exp\left(2\int_s^{\tilde{u}} A(\sigma,0)\,d\sigma\right) ds > 0 \qquad \text{for all} \quad \tilde{u} \in (\bar{u}_0(0),u^0], \\ &\int_{\bar{u}_0(1)}^{\tilde{u}} g(s,\bar{v}_0(1),1,0)\exp\left(2\int_s^{\tilde{u}} A(\sigma,1)\,d\sigma\right) ds > 0 \qquad \text{for all} \quad \tilde{u} \in (\bar{u}_0(1),u^1]. \end{aligned}$$

Below, we formulate additional conditions used to construct the asymptotics and to prove the existence of a solution with the constructed asymptotics.

2. Asymptotics of the solution

The formal asymptotic approximations of the solutions of problems (1.D) and (1.N) are constructed by the Vasil'eva method (see [7]) in the form

$$\begin{aligned} U(x,\varepsilon) &= \bar{u}(x,\varepsilon) + Lu(\tau,\varepsilon) + Ru(\eta,\varepsilon), \\ V(x,\varepsilon) &= \bar{v}(x,\varepsilon) + Lv(\tau,\varepsilon) + Rv(\eta,\varepsilon), \end{aligned} \tag{5}$$

where the regular parts become

$$\begin{aligned} \bar{u}(x,\varepsilon) &= \bar{u}_0(x) + \varepsilon\bar{u}_1(x) + \cdots + \varepsilon^n\bar{u}_n(x) + \cdots, \\ \bar{v}(x,\varepsilon) &= \bar{v}_0(x) + \varepsilon\bar{v}_1(x) + \cdots + \varepsilon^n\bar{v}_n(x) + \cdots, \end{aligned} \tag{6}$$

and the boundary parts in a neighborhood of $x = 0$ for u^0 and v^0 and in a neighborhood of $x = 1$ for u^1 and v^1 are

$$\begin{aligned} Lu(\tau,\varepsilon) &= Lu_0(\tau) + \varepsilon Lu_1(\tau) + \cdots + \varepsilon^n Lu_n(\tau) + \cdots, \\ Lv(\tau,\varepsilon) &= Lv_0(\tau) + \varepsilon Lv_1(\tau) + \cdots + \varepsilon^n Lv_n(\tau) + \cdots, \\ Ru(\eta,\varepsilon) &= Ru_0(\eta) + \varepsilon Ru_1(\eta) + \cdots + \varepsilon^n Ru_n(\eta) + \cdots, \\ Rv(\eta,\varepsilon) &= Rv_0(\eta) + \varepsilon Rv_1(\eta) + \cdots + \varepsilon^n Rv_n(\eta) + \cdots, \end{aligned} \tag{7}$$

where $\tau = x/\varepsilon$, $\eta = (1-x)/\varepsilon$ are extended variables in neighborhoods of the points $x = 0$ and $x = 1$.

We introduce the functions

$$\begin{aligned} G\left(\varepsilon\frac{du}{dx}, u(x), v(x), x, \varepsilon\right) &:= A(u,x)\left(\varepsilon\frac{du}{dx}\right)^2 + g(u,v,x,\varepsilon), \\ F\left(\frac{dv}{dx}, u(x), v(x), x, \varepsilon\right) &:= B(v,x)\left(\frac{dv}{dx}\right)^2 + f(u,v,x,\varepsilon). \end{aligned}$$

For these functions, we use the Vasil'eva representation, separating the regular and boundary components,

$$G = \overline{G} + LG + RG, \qquad F = \overline{F} + LF + RF,$$

where

$$\begin{aligned} \overline{G} &= G\left(\varepsilon\frac{d\bar{u}}{dx}(x,\varepsilon), \bar{u}(x,\varepsilon), \bar{v}(x,\varepsilon), x, \varepsilon\right), \\ LG &= G\left(\varepsilon\frac{d\bar{u}}{dx}(\tau\varepsilon,\varepsilon) + \frac{dLu}{d\tau}(\tau,\varepsilon), \bar{u}(\tau\varepsilon,\varepsilon) + Lu(\tau,\varepsilon), \bar{v}(\tau\varepsilon,\varepsilon) + Lv(\tau,\varepsilon), \tau\varepsilon, \varepsilon\right) - \\ &\quad - G\left(\varepsilon\frac{d\bar{u}}{dx}(\tau\varepsilon,\varepsilon), \bar{u}(\tau\varepsilon,\varepsilon), \bar{v}(\tau\varepsilon,\varepsilon), \tau\varepsilon, \varepsilon\right), \\ RG &= G\left(\varepsilon\frac{d\bar{u}}{dx}(1-\eta\varepsilon,\varepsilon) + \frac{dRu}{d\eta}(\tau,\varepsilon), \bar{u}(1-\eta\varepsilon,\varepsilon) + Ru(\eta,\varepsilon), \bar{v}(1-\eta\varepsilon,\varepsilon) + Rv(\eta,\varepsilon), 1-\eta\varepsilon, \varepsilon\right) - \\ &\quad - G\left(\varepsilon\frac{d\bar{u}}{dx}(1-\eta\varepsilon,\varepsilon), \bar{u}(1-\eta\varepsilon,\varepsilon), \bar{v}(1-\eta\varepsilon,\varepsilon), 1-\eta\varepsilon, \varepsilon\right), \end{aligned} \tag{8}$$

and the terms in the representation for F are similar. Further, the original system standardly splits into regularly perturbed equations for regular and boundary layer parts of the asymptotics (for the regular part, the differential operator in the first approximation is subordinate, i.e., the first equation is considered as a finite equation):

$$\begin{aligned} &\varepsilon^2\frac{d^2\bar{u}}{dx^2} = \overline{G}, && \frac{d^2\bar{v}}{dx^2} = \overline{F}, \\ &\frac{d^2 Lu}{d\tau^2} = LG, && \frac{d^2 Ru}{d\eta^2} = RG, \\ &\frac{d^2 Lv}{d\tau^2} = \varepsilon^2 LF, && \frac{d^2 Rv}{d\eta^2} = \varepsilon^2 RF. \end{aligned} \tag{9}$$

These equations are related by the boundary conditions supplemented with the standard conditions of decrease in the extended argument at infinity for the boundary functions: in the case of Neumann boundary conditions,

$$\begin{aligned} &\frac{dLu}{d\tau}(0,\varepsilon) + \varepsilon\frac{d\bar{u}}{dx}(0,\varepsilon) = \varepsilon u^0, \qquad \frac{dRu}{d\eta}(0,\varepsilon) + \varepsilon\frac{d\bar{u}}{dx}(1,\varepsilon) = \varepsilon u^1, \\ &Lu(+\infty,\varepsilon) = 0, \qquad Ru(+\infty,\varepsilon) = 0, \end{aligned}$$

in the case of Dirichlet boundary conditions,

$$\begin{aligned} &Lu(0,\varepsilon) + \bar{u}(0,\varepsilon) = u^0, \qquad Ru(0,\varepsilon) + \bar{u}(1,\varepsilon) = u^1, \\ &Lu(+\infty,\varepsilon) = 0, \qquad Ru(+\infty,\varepsilon) = 0 \end{aligned}$$

and

$$\begin{aligned} &Lv(0,\varepsilon) + \bar{v}(0,\varepsilon) = v^0, \qquad Rv(0,\varepsilon) + \bar{v}(1,\varepsilon) = v^1, \\ &Lv(+\infty,\varepsilon) = 0, \qquad Rv(+\infty,\varepsilon) = 0. \end{aligned} \tag{10}$$

The coefficients of asymptotic representation (5) are determined in the following order. At the kth step, we first determine the boundary functions of the v component, then find the functions $\bar{u}_k$ and $\bar{v}_k$, and then determine the boundary functions of the u component. It follows from Eqs. (9) and the conditions at infinity that $Lv_k(\tau) = Rv_k(\eta) = 0$ for $k = 0, 1, 2$ in the case of Neumann boundary conditions and for $k = 0, 1$ in the case of Dirichlet boundary conditions. The regular part of the asymptotics, i.e., the functions $\bar{u}_0(x)$ and $\bar{v}_0(x)$, are determined from the degenerate system defined in Condition A2.

In the case of the Dirichlet condition, the problems for Lu_0 and Ru_0 become

$$\begin{aligned} &\frac{d^2 Lu_0}{d\tau^2} = A(\bar{u}_0(0) + Lu_0(\tau), 0)\left(\frac{dLu_0}{d\tau}\right)^2 + g(\bar{u}_0(0) + Lu_0(\tau), \bar{v}_0(0), 0, 0), \\ &\frac{d^2 Ru_0}{d\eta^2} = A(\bar{u}_0(1) + Ru_0(\eta), 1)\left(\frac{dRu_0}{d\eta}\right)^2 + g(\bar{u}_0(1) + Ru_0(\eta), \bar{v}_0(1), 1, 0), \\ &Lu_0(0) = u^0 - \bar{u}_0(0), \quad Ru_0(0) = u^1 - \bar{u}_0(1), \quad Lu_0(+\infty) = 0, \quad Ru_0(+\infty) = 0. \end{aligned} \tag{11}$$

It is well known that the solvability of these problems is guaranteed by Condition A3. In this case, there exists a unique monotone solution of each problem. The solutions of problems (11) are determined in quadratures and have a standard exponential estimate (see, e.g., [8]). In the case of Neumann boundary conditions, these problems have zero solutions.

Because $Lv_1(\tau) = Rv_1(\eta) = 0$, the functions $\bar{u}_1(x)$ and $\bar{v}_1(x)$ in the regular part of the asymptotics can be found from the boundary value problem for the linear differential-algebraic system of equations with zero boundary conditions:

$$\begin{aligned} &\bar{g}_u\bar{u}_1 + \bar{g}_v\bar{v}_1 + \bar{g}_\varepsilon = 0, \\ &\frac{d^2\bar{v}_1}{dx^2} - 2\overline{B}\frac{d\bar{v}_0}{dx}\frac{d\bar{v}_1}{dx} - \overline{B}_v\left(\frac{d\bar{v}_0}{dx}\right)^2\bar{v}_1 - \bar{f}_u\bar{u}_1 - \bar{f}_v\bar{v}_1 - \bar{f}_\varepsilon = 0, \qquad 0 < x < 1, \\ &\bar{v}_1(0) = 0, \qquad \bar{v}_1(1) = 0. \end{aligned} \tag{12}$$

Investigating the identity $g(\varphi(v,x),v,x,0)=0$ in Condition A2, we obtain a relation between $\bar{g}_u$ and $\bar{g}_v$,

$$\bar{g}_u\bar{\varphi}_v+\bar{g}_v=0.$$

We express $\bar{u}_1$ from the first equation of the system and, taking this relation into account, substitute it in the second differential equation. We then obtain the problem

$$\begin{aligned}&\frac{d^2\bar{v}_1}{dx^2}-2\overline{B}\frac{d\bar{v}_0}{dx}\frac{d\bar{v}_1}{dx}-\left(\overline{B}_v\left(\frac{d\bar{v}_0}{dx}\right)^2+\bar{f}_v+\bar{\varphi}_v\bar{f}_u\right)\bar{v}_1=f_1, \qquad 0<x<1,\\ &\bar{v}_1(0)=0, \qquad \bar{v}_1(1)=0,\end{aligned} \tag{13}$$

where f_1 is a known function. The linear differential operator of problem (13) is not self-adjoint (it reduces to the divergence form by a well-known change of variables). We now formulate a condition that ensures the existence and uniqueness of the solution of the boundary value problem and hence of differential-algebraic system (12) (also see Theorem 3 in [11]).

Condition A4. *Assume that the inequality*

$$\overline{B}_v\left(\frac{d\bar{v}_0}{dx}\right)^2+\bar{f}_v+\bar{\varphi}_v\bar{f}_u>-\lambda_0$$

holds for all $x\in[0,1]$*, where* λ_0 *is the principal eigenvalue of the problem*

$$\begin{aligned}&\frac{d^2\tilde{\Psi}}{dx^2}-2\overline{B}\frac{d\bar{v}_0}{dx}\frac{d\tilde{\Psi}}{dx}+\lambda\tilde{\Psi}=0, \qquad 0<x<1,\\ &\tilde{\Psi}(0)=0, \qquad \tilde{\Psi}(1)=0.\end{aligned} \tag{14}$$

The existence of a positive principal eigenvalue λ_0 and of the corresponding positive eigenfunction $\tilde{\Psi}(x)$, $x\in(0,1)$, of problem (14) is a will-known result (see [12], Theorem 4.3).

For Neumann boundary condition, we obtain the following problems for Lu_1 and Ru_1:

$$\begin{aligned}&\frac{d^2Lu_1}{d\tau^2}=\bar{g}_u(0)Lu_1,\\ &\frac{d^2Ru_1}{d\eta^2}=\bar{g}_u(1)Ru_1,\\ &\frac{dLu_1}{d\tau}(0)=u^0-\frac{d\bar{u}_0}{dx}(0), \qquad \frac{dRu_1}{d\eta}(0)=u^1-\frac{d\bar{u}_0}{dx}(1),\\ &Lu_1(+\infty)=0, \qquad Ru_1(+\infty)=0.\end{aligned} \tag{15}$$

Each of equations (15) is an equation with constant coefficients. Their solutions up to a factor are exponential functions with the respective exponents $-\sqrt{\bar{g}_u(0)}\tau$ and $-\sqrt{\bar{g}_u(1)}\eta$.

For the Dirichlet condition, the problems for Lu_1 and Ru_1 become

$$\begin{aligned}&\frac{d^2Lu_1}{d\tau^2}-2A(\bar{u}_0(0)+Lu_0(\tau),0)\frac{dLu_0}{d\tau}\frac{dLu_1}{d\tau}-\left(\frac{\partial A}{\partial u}(\bar{u}_0(0)+Lu_0(\tau),0)\times\right.\\ &\qquad\left.\times\left(\frac{dLu_0}{d\tau}\right)^2+\frac{\partial g}{\partial u}(\bar{u}_0(0)+Lu_0(\tau),\bar{v}_0(0),0,0)\right)Lu_1=Lg_1(\tau),\\ &\frac{d^2Ru_1}{d\eta^2}-2A(\bar{u}_0(1)+Ru_0(\eta),1)\frac{dRu_0}{d\eta}\frac{dRu_1}{d\eta}-\left(\frac{\partial A}{\partial u}(\bar{u}_0(1)+Ru_0(\eta),1)\times\right.\\ &\qquad\left.\times\left(\frac{dRu_0}{d\eta}\right)^2+\frac{\partial g}{\partial u}(\bar{u}_0(1)+Ru_0(\eta),\bar{v}_0(1),1,0)\right)Ru_1=Rg_1(\eta),\\ &Lu_1(0)=-u_1(0), \qquad Ru_1(0)=-u_1(1), \qquad Lu_1(+\infty)=0, \qquad Ru_1(+\infty)=0,\end{aligned} \tag{16}$$

where Lg_1 and Rg_1 are known exponentially decreasing functions, standardly expressed in terms of the coefficients of the asymptotic approximation obtained at the preceding stage. In particular, Lg_1 has the form

$$\begin{aligned} Lg_1(\tau) = &\left(\frac{\partial A}{\partial u}(\bar{u}_0(0)+Lu_0(\tau),0)\left(\frac{dLu_0}{d\tau}\right)^2 + \frac{\partial g}{\partial u}(\bar{u}_0(0)+Lu_0(\tau),\bar{v}_0(0),0,0)\right)\left(\frac{d\bar{u}_0}{dx}(0)\tau+\bar{u}_1(0)\right) + \\ &+\left(\frac{\partial A}{\partial x}(\bar{u}_0(0)+Lu_0(\tau),0)\left(\frac{dLu_0}{d\tau}\right)^2 + \frac{\partial g}{\partial x}(\bar{u}_0(0)+Lu_0(\tau),\bar{v}_0(0),0,0)\right)\tau + \\ &+\frac{\partial g}{\partial \varepsilon}(\bar{u}_0(0)+Lu_0(\tau),\bar{v}_0(0),0,0) + 2A(\bar{u}_0(0)+Lu_0(\tau),0)\frac{d\bar{u}_0}{dx}(0)\frac{dLu_0}{d\tau} + \\ &+\frac{\partial g}{\partial v}(\bar{u}_0(0)+Lu_0(\tau),\bar{v}_0(0),0,0)\left(\frac{d\bar{v}_0}{dx}(0)\tau+\bar{v}_1(0)\right). \end{aligned} \tag{17}$$

The solutions of problems (16) can be obtained explicitly; for example, for $Lu_1(\tau)$, we have

$$Lu_1(\tau) = -\bar{u}_1(0)\frac{\tilde{v}(\tau)}{\tilde{v}(0)} - \tilde{v}(\tau)\int_0^\tau \frac{1}{p(s)(\tilde{v}(s))^2}\int_s^{+\infty} p(\eta)\tilde{v}(\kappa)Lg_1(\kappa)\,d\kappa\,ds, \tag{18}$$

where

$$p(\xi) = \exp\left(-2\int_0^\xi A(\bar{u}_0(0)+Lu_0(y),0)\tilde{v}(y)\,dy\right), \qquad \tilde{v}(\tau) = \frac{dLu_0}{d\tau}.$$

The boundary functions of the u component in the next orders are determined from similar problems (with the same differential operator), and their solutions can be obtained in explicit form.

The boundary functions of the v component of an order $k \geqslant 3$ in the case of Neumann boundary conditions and of and order $k \geqslant 2$ in the case of Dirichlet boundary conditions are determined from inhomogeneous equations whose solutions can also be written explicitly. The problems for Lv_k and Rv_k are

$$\frac{d^2Lv_k}{d\tau^2} = LF_{k-2}(\tau), \qquad \frac{d^2Rv_k}{d\eta^2} = RF_{k-2}(\eta),$$

where LF_{k-2}, RF_{k-2} are the coefficients of ε^{k-2} in the expansion of LF and RF in a power series in ε. In the case of the Neumann condition, we obtain $LF_0 = 0$ because $Lu_0 = 0$, $Lv_0 = 0$, and $Lv_1 = 0$. Therefore, we also have $Lv_2 = 0$. We similarly obtain $Rv_2 = 0$. For Lv_3, we have the problem

$$\begin{aligned} &\frac{d^2Lv_3}{d\tau^2} = LF_1(\tau) = f_u(\bar{u}_0(0),\bar{v}_0(0),0,0)Lu_1(\tau), \\ &Lv_3(\infty) = 0. \end{aligned} \tag{19}$$

For Lv_2, in the case of the Dirichlet condition,

$$LF_0(\tau) = f(\bar{u}_0(0)+Lu_0(\tau),\bar{v}_0(0),0,0) - f(\bar{u}_0(0),\bar{v}_0(0),0,0)$$

is an exponentially decreasing nonzero function, and the function Lv_2 is determined from the problem

$$\frac{d^2Lv_2}{d\tau^2} = LF_0(\tau), \qquad Lv_2(\infty) = 0. \tag{20}$$

Obviously, after double integration of exponentially decreasing functions $LF_0(\tau)$, $LF_1(\tau)$ in (19) and (20), we also obtain exponentially decreasing functions. An arbitrary linear part arising in these functions is identically zero due to the conditions $Lv_2(\infty) = 0$ and $Lv_3(\infty) = 0$.

Thus, the conditions at infinity are sufficient for uniquely determining the functions $Lv_2(\tau)$ and $Lv_3(\tau)$ in problems (19) and (20). We similarly determine $Rv_2(\eta)$ and $Rv_3(\eta)$. The boundary functions of the v component in the next orders are determined from similar problems similar to (19) and (20), and the solutions of these problems are also obtained by double integration.

The functions $\bar{u}_k(x)$ and $\bar{v}_k(x)$ in the regular part of the asymptotics in the next orders in ε are determined from the boundary value problems with the same differential-algebraic operator as in the problem for $\bar{u}_1(x)$ and $\bar{v}_1(x)$.

The process of finding the coefficients of asymptotics (5) can be extended to any order in ε. It standardly follows from the method for constructing the asymptotics that the nth-order partial sums $U_n(x,\varepsilon)$ for the u component and $V_n(x,\varepsilon)$ for the v component satisfy the first equation of system (1) with a discrepancy $O(\varepsilon^{n+1})$ and the second equation with a discrepancy $O(\varepsilon^{n-1})$.

3. Existence and asymptotics of the stationary solution

To prove the existence of a solution in each of the cases discussed in what follows, we use the asymptotic method of differential inequalities (see survey [6] and the references therein). The main idea of this method is to modify the obtained asymptotic form so as to obtain the lower and upper solutions of the problem under study. We recall the definition of upper and lower solutions.

Definition 1. Functions

$$\beta(x,\varepsilon) = (\beta^u(x,\varepsilon), \beta^v(x,\varepsilon)) \quad \text{and} \quad \alpha(x,\varepsilon) = (\alpha^u(x,\varepsilon), \alpha^v(x,\varepsilon))$$

are called an upper and a lower solution of problem (1) if they satisfy the following conditions.

1. The ordering condition: $\alpha^{u,v}(x) \leqslant \beta^{u,v}(x,\varepsilon)$, $x \in [0;1]$.
2. The action of the operator on the upper and lower solutions:
for all $x \in (0;1)$, $\alpha^v(x,\varepsilon) \le v \le \beta^v(x,\varepsilon)$,

$$\mathcal{N}_u(\beta^u, v) := \varepsilon^2 \frac{d^2\beta^u}{dx^2} - \varepsilon^2 A(\beta^u, x)\left(\frac{d\beta^u}{dx}\right)^2 - g(\beta^u, v, x, \varepsilon) \le 0,$$

$$0 \le \mathcal{N}_u(\alpha^u, v) := \varepsilon^2 \frac{d^2\alpha^u}{dx^2} - \varepsilon^2 A(\alpha^u, x)\left(\frac{d\alpha^u}{dx}\right)^2 - g(\alpha^u, v, x, \varepsilon),$$

and for all $x \in (0;1)$, $\alpha^u(x,\varepsilon) \le u \le \beta^u(x,\varepsilon)$,

$$\mathcal{N}_v(u, \beta^v) := \frac{d^2\beta^v}{dx^2} - B(\beta^v, x)\left(\frac{d\beta^v}{dx}\right)^2 - f(u, \beta^v, x, \varepsilon) \le 0,$$

$$0 \le \mathcal{N}_v(u, \alpha^v) := \frac{d^2\alpha^v}{dx^2} - B(\alpha^v, x)\left(\frac{d\alpha^v}{dx}\right)^2 - f(u, \alpha^v, x, \varepsilon).$$

3. The condition on the boundary:

$$\alpha^v(0,\varepsilon) \le v^0 \le \beta^v(0,\varepsilon), \qquad \alpha^v(1,\varepsilon) \le v^1 \le \beta^v(1,\varepsilon),$$

in the case of Neumann boundary conditions,

$$\frac{d\alpha^u}{dx}(0,\varepsilon) \ge u^0 \ge \frac{d\beta^u}{dx}(0,\varepsilon), \qquad \frac{d\alpha^u}{dx}(1,\varepsilon) \le u^1 \le \frac{d\beta^u}{dx}(1,\varepsilon),$$

and in the case of Dirichlet boundary conditions,

$$\alpha^u(0,\varepsilon) \le u^0 \le \beta^u(0,\varepsilon), \qquad \alpha^u(1,\varepsilon) \le u^1 \le \beta^u(1,\varepsilon).$$

It is well known (see, e.g., [13] and the references therein) that if there exists a lower and an upper solution of problem (1), then this problem has a solution $(u(x,\varepsilon), v(x,\varepsilon))$ such that, for all $x \in [0,1]$,

$$\begin{aligned} &\alpha^u(x,\varepsilon) \le u(x,\varepsilon) \le \beta^u(x,\varepsilon), \\ &\alpha^v(x,\varepsilon) \le v(x,\varepsilon) \le \beta^v(x,\varepsilon). \end{aligned} \tag{21}$$

3.1. Neumann boundary conditions. We consider problem (1.N) with the following quasimonotonicity condition.

Condition A5. *Assume that the vector function* (g,f) *is quasimonotone nonincreasing in* (u,v) *in the domain of definition for a sufficiently small* $\varepsilon > 0$.

This condition means that $g_v \le 0$ for a fixed u and $f_u \le 0$ for a fixed v in their range.

We consider the differential-algebraic system

$$\begin{aligned} &\bar{g}_u(x)\gamma_1 + \bar{g}_v(x)\gamma_2 = h_1(x), \\ &\frac{d^2\gamma_2}{dx^2} - 2\overline{B}\frac{d\bar{v}_0}{dx}\frac{d\gamma_2}{dx} - \overline{B}_v\left(\frac{d\bar{v}_0}{dx}\right)^2\gamma_2 - [\bar{f}_u(x)\gamma_1 + \bar{f}_v(x)\gamma_2] = h_2(x), \quad x \in (0,1), \\ &\gamma_2(0) > 0, \qquad \gamma_2(1) > 0, \end{aligned} \tag{22}$$

where $h_1(x) > 0$, $h_2(x) < 0$ for $x \in [0,1]$. The following result holds.

Lemma 1. *Under Conditions* A1, A2, A4, *and* A5, *differential-algebraic system* (22) *has a solution* $\gamma_1(x) > 0$, $\gamma_2(x) > 0$.

To prove the lemma, we express $\gamma_1(x)$ in terms of $\gamma_2(x)$ and obtain a problem for $\gamma_2(x)$ (similar to problem (12) for $\bar{v}_1(x)$),

$$\begin{aligned} &\frac{d^2\gamma_2}{dx^2} - 2\overline{B}\frac{d\bar{v}_0}{dx}\frac{d\bar{\gamma}_2}{dx} - \left(\overline{B}_v\left(\frac{d\bar{v}_0}{dx}\right)^2 + \bar{f}_v + \bar{\varphi}_v\bar{f}_u\right)\gamma_2 = h(x), \qquad 0 < x < 1, \\ &\gamma_2(0) > 0, \qquad \gamma_2(1) > 0, \end{aligned} \tag{23}$$

where $h(x)$ is a known function with $h(x) < 0$ for $x \in [0,1]$ due to Condition A5 ($\bar{f}_u(x) \le 0$) and the condition $h_1(x) > 0$. Obviously, $\alpha = 0$ is a lower solution of problem (23). One can show that for a sufficiently large positive constant M and a positive eigenfunction $W(x)$, the function $\beta = MW(x)$ corresponding to the principal eigenvalue of the problem

$$\begin{aligned} &\frac{d^2\Psi}{dx^2} - 2\overline{B}\frac{d\bar{v}_0}{dx}\frac{d\Psi}{dx} + k\Psi = 0, \qquad -\delta < x < 1+\delta, \quad \delta > 0, \\ &\Psi(-\delta) = 0, \qquad \Psi(1+\delta) = 0, \end{aligned} \tag{24}$$

for sufficiently small δ is an upper solution of problem (23). The condition $\gamma_2(x) > 0$ and the first equation in the differential-algebraic system imply that $\gamma_1(x) > 0$.

Theorem 1N. *If Conditions* A1, A2, A4, *and* A5 *are satisfied, then a solution* $(u(x,\varepsilon), v(x,\varepsilon))$ *of problem* (1) *exists for sufficiently small* ε *and has the asymptotic representation*

$$\begin{aligned} &u(x,\varepsilon) = \sum_{k=0}^{n}\varepsilon^k\bar{u}_k(x) + \sum_{k=0}^{n}\varepsilon^k Lu_k(\tau) + \sum_{k=0}^{n}\varepsilon^k Ru_k(\eta) + O(\varepsilon^{n+1}), \\ &v(x,\varepsilon) = \sum_{k=0}^{n}\varepsilon^k\bar{v}_k(x) + \sum_{k=0}^{n}\varepsilon^k Lv_k(\tau) + \sum_{k=0}^{n}\varepsilon^k Rv_k(\eta) + O(\varepsilon^{n+1}), \qquad x \in [0,1]. \end{aligned}$$

Proof. We choose a lower and an upper solution of problem (1.N), $(\alpha^u_{n+1}, \alpha^v_{n+1})$ and $(\beta^u_{n+1}, \beta^v_{n+1})$ as a modification of the formal asymptotics of the order $(n+1)$. For the slow component v, these are the functions

$$\begin{aligned}\alpha^v_{n+1}(x,\varepsilon) &= \sum_{k=0}^{n+1} \varepsilon^k \bar{v}_k(x) + \sum_{k=3}^{n+3} \varepsilon^k (Lv_k(\tau) + Rv_k(\eta)) - \varepsilon^{n+1}\gamma_2(x),\\ \beta^v_{n+1}(x,\varepsilon) &= \sum_{k=0}^{n+1} \varepsilon^k \bar{v}_k(x) + \sum_{k=3}^{n+3} \varepsilon^k (Lv_k(\tau) + Rv_k(\eta)) + \varepsilon^{n+1}\gamma_2(x),\end{aligned} \tag{25}$$

and for the fast component u, the functions

$$\begin{aligned}\alpha^u_{n+1}(x,\varepsilon) &= \sum_{k=0}^{n+1} \varepsilon^k (\bar{u}_k(x) + Lu_k(\tau) + Ru_k(\eta)) - \varepsilon^{n+1}\gamma_1(x) - \varepsilon^{n+2}[e^{-\kappa\tau} + e^{-\kappa\eta}],\\ \beta^u_{n+1}(x,\varepsilon) &= \sum_{k=0}^{n+1} \varepsilon^k (\bar{u}_k(x) + Lu_k(\tau) + Ru_k(\eta)) + \varepsilon^{n+1}\gamma_1(x) + \varepsilon^{n+2}[e^{-\kappa\tau} + e^{-\kappa\eta}].\end{aligned} \tag{26}$$

In expressions (25) and (26), the positive functions $\gamma_1(x)$ and $\gamma_2(x)$ are defined in Lemma 1. The standard exponentially decreasing extra terms in the lower and upper solutions ensure the satisfaction of the boundary inequalities. Ordering condition 1 (see Definition 1) is then obviously satisfied. The differential inequalities can be verified by substitution. For the upper solution, by the quasimonotonicity condition, the following differential inequalities must be satisfied:

$$\mathcal{N}_u(\beta^u_{n+1}(x,\varepsilon), \beta^v_{n+1}(x,\varepsilon)) \le 0, \qquad \mathcal{N}_v(\beta^u_{n+1}(x,\varepsilon), \beta^v_{n+1}(x,\varepsilon)) \le 0. \tag{27}$$

Substituting β^u_{n+1} and β^v_{n+1} defined in (25) and (26) in (27), after simple transformations based on the use of equations for the terms of the formal asymptotics, we obtain

$$\begin{aligned}&\mathcal{N}_u(\beta^u_{n+1}(x,\varepsilon), \beta^v_{n+1}(x,\varepsilon)) = -\varepsilon^{n+1}[\bar{g}_u(x)\gamma_1 + \bar{g}_v(x)\gamma_2] + O(\varepsilon^{n+2}) = -\varepsilon^{n+1}h_1(x) + O(\varepsilon^{n+2}) \le -c\varepsilon^{n+1},\\ &\mathcal{N}_v(\beta^u_{n+1}(x,\varepsilon), \beta^v_{n+1}(x,\varepsilon)) = \varepsilon^{n+1}\left(\frac{d^2\gamma_2}{dx^2} - 2\overline{B}\frac{d\bar{v}_0}{dx}\frac{d\gamma_2}{dx} - \overline{B}_v\left(\frac{d\bar{v}_0}{dx}\right)^2\gamma_2 - \bar{f}_u\gamma_1 - \bar{f}_v\gamma_2\right) + O(\varepsilon^{n+2}) =\\ &\qquad = \varepsilon^{n+1}h_2(x) + O(\varepsilon^{n+2}) \le -c\varepsilon^{n+1}\end{aligned} \tag{28}$$

for sufficiently small ε due to Lemma 1. The differential inequalities for the lower solution can be verified similarly. Thus, all conditions for determining the lower and upper solutions are satisfied. The solution of problem (1) exists and satisfies the inequalities

$$\begin{aligned}\alpha^u_{n+1}(x,\varepsilon) \le u(x,\varepsilon) \le \beta^u_{n+1}(x,\varepsilon),\\ \alpha^v_{n+1}(x,\varepsilon) \le v(x,\varepsilon) \le \beta^v_{n+1}(x,\varepsilon),\end{aligned}$$

which imply the estimate in Theorem 1N.

We now consider problem (1.N) under the following quasimonotonicity condition.

Condition A5*. *Assume that the vector function (g, f) is quasimonotone nondecreasing in (u, v) in the domain of definition for sufficiently small ε.*

This condition means that $g_v \ge 0$ for a fixed u and $f_u \ge 0$ for a fixed v in their range. In this case, due to the quasimonotonicity condition, the following differential inequalities must be satisfied for the upper solution:

$$\mathcal{N}_u(\beta^u_{n+1}(x,\varepsilon), \alpha^v_{n+1}(x,\varepsilon)) \le 0, \qquad \mathcal{N}_v(\alpha^u_{n+1}(x,\varepsilon), \beta^v_{n+1}(x,\varepsilon)) \le 0. \tag{29}$$

In this case, the key role in the proof of an analogue of Theorem 1N is played by the positivity of the solutions of the boundary value problem for the differential-algebraic system

$$
\begin{aligned}
&\bar{g}_u(x)\gamma_1 - \bar{g}_v(x)\gamma_2 = h_1(x),\\
&\frac{d^2\gamma_2}{dx^2} - 2\overline{B}\frac{d\bar{v}_0}{dx}\frac{d\gamma_2}{dx} - \overline{B}_v\left(\frac{d\bar{v}_0}{dx}\right)^2\gamma_2 + \bar{f}_u(x)\gamma_1 - \bar{f}_v(x)\gamma_2 = h_2(x), \qquad x \in (0,1),\\
&\gamma_2(0) > 0, \qquad \gamma_2(1) > 0,
\end{aligned}
$$

where $h_1(x) > 0$, $h_2(x) < 0$ for $x \in [0,1]$. The proof is completely similar to the proof of Lemma 1. Thus, the following theorem similar to Theorem 1N holds.

Theorem 2N. *If Conditions* A1, A2, A4, *and* A5* *are satisfied, then a solution* $(u(x,\varepsilon), v(x,\varepsilon))$ *of problem* (1) *exists for sufficiently small* ε *and has the asymptotic representation*

$$
u(x,\varepsilon) - U_n(x,\varepsilon) = O(\varepsilon^{n+1}), \qquad v(x,\varepsilon) - V_n(x,\varepsilon) = O(\varepsilon^{n+1}), \qquad x \in [0,1],
$$

where $U_n(x,\varepsilon)$ *and* $V_n(x,\varepsilon)$ *are* n*th-order partial sums of the asymptotics of problem* (1.N) *constructed in Sec.* 2.

We consider problem (1.N) in the case where the quasimonotonicity condition is not satisfied. This means that g_v for a fixed u and f_u for a fixed v change the sign in the domain of definition (between the lower and upper solutions). In this case, the following differential inequalities (see Definition 1) must be satisfied for the upper solution:

$$
\begin{aligned}
&\mathcal{N}_u(\beta^u_{n+1}(x,\varepsilon), v) \le 0, \qquad \alpha^v_{n+1}(x,\varepsilon) \le v \le \beta^v_{n+1}(x,\varepsilon),\\
&\mathcal{N}_v(u, \beta^v_{n+1}(x,\varepsilon)) \le 0, \qquad \alpha^u_{n+1}(x,\varepsilon) \le u \le \beta^u_{n+1}(x,\varepsilon).
\end{aligned} \tag{30}
$$

These differential inequalities, just as in the cases considered above, are ensured by the positivity of solutions of the differential-algebraic system that takes the form

$$
\begin{aligned}
&\bar{g}_u(x)\gamma_1 - s_1\bar{g}_v(x)\gamma_2 = h_1(x),\\
&\frac{d^2\gamma_2}{dx^2} - 2\overline{B}\frac{d\bar{v}_0}{dx}\frac{d\gamma_2}{dx} - \overline{B}_v\left(\frac{d\bar{v}_0}{dx}\right)^2\gamma_2 + s_2\bar{f}_u(x)\gamma_1 - \bar{f}_v(x)\gamma_2 = h_2(x), \quad x \in (0,1),\\
&\gamma_2(0) > 0, \qquad \gamma_2(1) > 0,
\end{aligned} \tag{31}
$$

with some choice of $h_1(x) > 0$ and $h_2(x) < 0$ for $x \in [0,1]$, $s_i \in [-1,1]$. Following the scheme of the proof of Lemma 1, we can show that $\gamma_1(x)$ and $\gamma_2(x)$ are positive if the following condition is satisfied.

Condition A5.** *Assume the condition*

$$
\overline{B}_v\left(\frac{d\bar{v}_0}{dx}\right)^2 + \bar{f}_v - |\bar{\varphi}_v\bar{f}_u| > -\lambda_0
$$

to be satisfied for all $x \in [0,1]$, *where* λ_0 *is the principal eigenvalue of the problem defined in Condition* A4.

Theorem 3N. *If Conditions* A1, A2, A4, *and* A5** *are satisfied, then a solution* $(u(x,\varepsilon), v(x,\varepsilon))$ *of problem* (1) *exists for sufficiently small* ε *and has the asymptotic representation*

$$
u(x,\varepsilon) - U_n(x,\varepsilon) = O(\varepsilon^{n+1}), \qquad v(x,\varepsilon) - V_n(x,\varepsilon) = O(\varepsilon^{n+1}), \qquad x \in [0,1],
$$

where $U_n(x,\varepsilon)$ *and* $V_n(x,\varepsilon)$ *are* n*th-order partial sums of the asymptotics of problem* (1.N) *constructed in Sec.* 2.

3.2. Dirichlet boundary conditions. In the case of Dirichlet boundary conditions for the u component, we consider problem (1.D) following a similar strategy with necessary variations in the structure of the upper and lower solutions. If quasimonotonicity Condition A5 is satisfied, then we have the following theorem.

Theorem 1D. *If Conditions* A1–A5 *are satisfied, then a solution* $(u(x,\varepsilon), v(x,\varepsilon))$ *of problem* (1) *exists for sufficiently small* ε *and has the asymptotic representation*

$$u(x,\varepsilon) - U_n(x,\varepsilon) = O(\varepsilon^{n+1}), \qquad v(x,\varepsilon) - V_n(x,\varepsilon) = O(\varepsilon^{n+1}), \qquad x \in [0,1],$$

where $U_n(x,\varepsilon)$ *and* $V_n(x,\varepsilon)$ *are nth-order partial sums of the asymptotics of problem* (1.D) *constructed in Sec.* 2.

The proof of Theorem 1D is similar to the proof of Theorem 1N. The lower and upper solutions of problem (1.D) are given by

$$\begin{aligned} \alpha^u_{n+1}(x,\varepsilon) &= \sum_{k=0}^{n+1} \varepsilon^k(\bar u_k(x) + Lu_k(\tau) + Ru_k(\eta)) - \varepsilon^{n+1}\gamma_1(x) - \varepsilon^{n+1}(Lu_\alpha(\tau) + Ru_\alpha(\eta)), \\ \beta^u_{n+1}(x,\varepsilon) &= \sum_{k=0}^{n+1} \varepsilon^k(\bar u_k(x) + Lu_k(\tau) + Ru_k(\eta)) + \varepsilon^{n+1}\gamma_1(x) + \varepsilon^{n+1}(Lu_\beta(\tau) + Ru_\beta(\eta)) \end{aligned} \tag{32}$$

and

$$\begin{aligned} \alpha^v_{n+1}(x,\varepsilon) &= \sum_{k=0}^{n+1} \varepsilon^k \bar v_k(x) + \sum_{k=2}^{n+3} \varepsilon^k(Lv_k(\tau) + Rv_k(\eta)) - \varepsilon^{n+1}\gamma_2(x) - \varepsilon^{n+3}(Lv_\alpha(\tau) + Rv_\alpha(\eta)), \\ \beta^v_{n+1}(x,\varepsilon) &= \sum_{k=0}^{n+1} \varepsilon^k \bar v_k(x) + \sum_{k=2}^{n+3} \varepsilon^k(Lv_k(\tau) + Rv_k(\eta)) + \varepsilon^{n+1}\gamma_2(x) + \varepsilon^{n+3}(Lv_\beta(\tau) + Rv_\beta(\eta)), \end{aligned}$$

where the positive functions $\gamma_1(x)$ and $\gamma_2(x)$ are defined in Lemma 1 and the positive functions $Lu_\alpha(\tau)$, $Ru_\alpha(\eta)$, $Lv_\alpha(\tau)$, and $Rv_\alpha(\eta)$ are to be found from the problems

$$\begin{aligned} &\frac{d^2 Lu_\alpha}{d\tau^2} - 2A(\bar u_0(0) + Lu_0(\tau), 0)\frac{dLu_0}{d\tau}\frac{dLu_\alpha}{d\tau} - \biggl(\frac{\partial A}{\partial u}(\bar u_0(0) + \\ &\qquad\qquad + Lu_0(\tau), 0)\biggl(\frac{dLu_0}{d\tau}\biggr)^2 + \frac{\partial g}{\partial u}(\bar u_0(0) + Lu_0(\tau), \bar v_0(0), 0, 0)\biggr)Lu_\alpha = \\ &\qquad = Lg_\alpha(\tau) - C_0 e^{-\kappa_0\tau} \equiv \psi_\alpha(\tau), \\ &\frac{d^2 Ru_\alpha}{d\eta^2} - 2A(\bar u_0(1) + Ru_0(\eta), 1)\frac{dRu_0}{d\eta}\frac{dRu_\alpha}{d\eta} - \biggl(\frac{\partial A}{\partial u}(\bar u_0(1) + \\ &\qquad\qquad + Ru_0(\eta), 1)\biggl(\frac{dRu_0}{d\eta}\biggr)^2 + \frac{\partial g}{\partial u}(\bar u_0(1) + Ru_0(\eta), \bar v_0(1), 1, 0)\biggr)Ru_\alpha = \\ &\qquad = Rg_\alpha(\eta) - C_1 e^{-\kappa_1\eta} \equiv \psi_\alpha(\eta), \\ &\frac{d^2 Lv_\alpha}{d\tau^2} = Lf_\alpha(\tau), \qquad \frac{d^2 Rv_\alpha}{d\eta^2} = Rf_\alpha(\eta), \\ &Lu_\alpha(0) = 0, \qquad Ru_\alpha(0) = 0, \qquad Lu_\alpha(+\infty) = 0, \qquad Ru_\alpha(+\infty) = 0, \\ &Lv_\alpha(+\infty) = 0, \qquad Rv_\alpha(+\infty) = 0, \end{aligned} \tag{33}$$

where C_0, C_1, κ_0, and κ_1 are some positive constants chosen such that $\psi_\alpha(\tau)$ and $\psi_\alpha(\eta)$ are negative and $Lg_\alpha(\tau)$, $Rg_\alpha(\eta)$, $Lf_\alpha(\tau)$, and $Rf_\alpha(\eta)$ are known standard exponentially decreasing functions arising after substituting the modified lower solutions in the regular parts of the asymptotics. For example, $Lg_\alpha(\tau)$ and $Lf_\alpha(\tau)$ are given by

$$\begin{aligned}
Lg_\alpha(\tau) = {} & \left(\frac{\partial A}{\partial u}(\bar{u}_0(0) + Lu_0(\tau), 0)\left(\frac{dLu_0}{d\tau}\right)^2\right)\gamma_1(0) + \\
& + \left(\frac{\partial g}{\partial u}(\bar{u}_0(0) + Lu_0(\tau), \bar{v}_0(0), 0, 0) - \frac{\partial g}{\partial u}(\bar{u}_0(0), \bar{v}_0(0), 0, 0)\right)\gamma_1(0) + \\
& + \left(\frac{\partial g}{\partial v}(\bar{u}_0(0) + Lu_0(\tau), \bar{v}_0(0), 0, 0) - \frac{\partial g}{\partial v}(\bar{u}_0(0), \bar{v}_0(0), 0, 0)\right)\gamma_2(0), \\
Lf_\alpha(\tau) = {} & \left(\frac{\partial f}{\partial u}(\bar{u}_0(0) + Lu_0(\tau), \bar{v}_0(0), 0, 0) - \frac{\partial f}{\partial u}(\bar{u}_0(0), \bar{v}_0(0), 0, 0)\right) \times \\
& \times (\gamma_1(0) + Lu_\alpha(\tau)) + \left(\frac{\partial f}{\partial v}(\bar{u}_0(0) + Lu_0(\tau), \bar{v}_0(0), 0, 0)\right)\gamma_2(0) - \\
& - \frac{\partial f}{\partial v}(\bar{u}_0(0), \bar{v}_0(0), 0, 0)\gamma_2(0).
\end{aligned} \tag{34}$$

The functions $Rg_\alpha(\eta)$, $Rf_\alpha(\eta)$ have a similar form. The functions $Lu_\beta(\tau)$ and $Ru_\beta(\eta)$ in the upper solution are determined from similar problems, with $\psi_\beta(\tau)$ and $\psi_\beta(\eta)$ being negative. The exponentially decreasing positive corrections to the lower and upper solutions of the u component are determined by formulas similar to (18). The exponential corrections to the lower and upper solutions of the v component are determined by double integration, and they are chosen such that, when verifying differential inequalities (27), the coefficient of ε^{n+1} containing the functions $Lu_0(\tau)$, $Lu_\alpha(\tau)$, $Ru_0(\eta)$, and $Ru_\alpha(\eta)$ vanish. The functions $Lv_\beta(\tau)$ and $Rv_\beta(\eta)$ are determined similarly.

Differential inequalities (27) can be verified standardly. Ordering condition 1 in Definition 1 is obviously satisfied. For the upper solution, by Lemma 1, we have

$$\begin{aligned}
\mathcal{N}_u(\beta^u_{n+1}(x,\varepsilon), \beta^v_{n+1}(x,\varepsilon)) &= -\varepsilon^{n+1}[\bar{g}_u(x)\gamma_1 + \bar{g}_v(x)\gamma_2] - \varepsilon^{n+1}(C_0 e^{-\kappa_0\tau} + C_1 e^{-\kappa_1\eta}) + O(\varepsilon^{n+2}) = \\
&= -\varepsilon^{n+1}h_1(x) - \varepsilon^{n+1}(C_0 e^{-\kappa_0\tau} + C_1 e^{-\kappa_1\eta}) + O(\varepsilon^{n+2}) \le -c\varepsilon^{n+1}, \\
\mathcal{N}_v(\beta^u_{n+1}(x,\varepsilon), \beta^v_{n+1}(x,\varepsilon)) &= \varepsilon^{n+1}\left(\frac{d^2\gamma_2}{dx^2} - 2\overline{B}\frac{d\bar{v}_0}{dx}\frac{d\gamma_2}{dx} - \overline{B}_v\left(\frac{d\bar{v}_0}{dx}\right)^2\gamma_2 - \bar{f}_u\gamma_1 - \bar{f}_v\gamma_2\right) + O(\varepsilon^{n+2}) = \\
&= \varepsilon^{n+1}h_2(x) + O(\varepsilon^{n+2}) \le -c\varepsilon^{n+1}
\end{aligned} \tag{35}$$

for sufficiently small for sufficiently small ε. The differential inequalities for the lower solution can be verified similarly.

If quasimonotonicity Condition A5* is satisfied, then the following theorem holds.

Theorem 2D. *If Conditions* A1–A5* *are satisfied, then a solution* $(u(x,\varepsilon), v(x,\varepsilon))$ *of problem* (1) *exists for sufficiently small* ε *and has the asymptotic representation*

$$u(x,\varepsilon) - U_n(x,\varepsilon) = O(\varepsilon^{n+1}), \qquad v(x,\varepsilon) - V_n(x,\varepsilon) = O(\varepsilon^{n+1}), \qquad x \in [0,1],$$

where $U_n(x,\varepsilon)$ *and* $V_n(x,\varepsilon)$ *are nth-order partial sums of the asymptotics of problem* (1.D) *constructed in Sec.* 2.

The verification of differential inequalities in this case practically repeats that in the proof of Theorem 1D.

We next consider the case where the quasimonotonicity condition is not satisfied. In this case, inequalities (30) must be satisfied for the upper solution. Proceeding as in the proof of Theorem 1D, we see that expressions (35) have the same form as in the proof of Theorem 1D. But the functions $\gamma_1(x)$ and $\gamma_2(x)$ are then determined as solutions of differential-algebraic system (31) under Condition A5**. These functions are also used to determine boundary layer corrections for the upper and lower solutions in (32). The following theorem holds.

Theorem 3D. *If Conditions* A1–A5** *are satisfied, then a solution* $(u(x,\varepsilon), v(x,\varepsilon))$ *of problem* (1) *exists for sufficiently small* ε *and has the asymptotic representation*

$$u(x,\varepsilon) - U_n(x,\varepsilon) = O(\varepsilon^{n+1}), \qquad v(x,\varepsilon) - V_n(x,\varepsilon) = O(\varepsilon^{n+1}), \qquad x \in [0,1],$$

where $U_n(x,\varepsilon)$ *and* $V_n(x,\varepsilon)$ *are* n*th-order partial sums of the asymptotics of problem* (1.D) *constructed in Sec.* 2.

4. Asymptotic stability of solutions

The solution of boundary value problems (1.D) or (1.N), whose existence is proved in the theorems in the preceding section, can be regarded as stationary solutions of the corresponding initial boundary value parabolic problem for the system

$$\begin{aligned} &L_u(u,v) := \varepsilon^2 \frac{\partial^2 u}{\partial x^2} - \varepsilon^2 A(u,x)\left(\frac{\partial u}{\partial x}\right)^2 - \frac{\partial u}{\partial t} - g(u,v,x,\varepsilon) = 0, \\ &L_v(u,v) := \frac{\partial^2 v}{\partial x^2} - B(v,x)\left(\frac{\partial v}{\partial x}\right)^2 - \frac{\partial v}{\partial t} - f(u,v,x,\varepsilon) = 0, \quad 0 < x < 1,\ t > 0, \\ &u(x,0,\varepsilon) = u^0(x,\varepsilon), \qquad v(x,0,\varepsilon) = v^0(x,\varepsilon), \qquad x \in [0,1], \end{aligned} \tag{36}$$

with prescribed boundary conditions for these problems. We let $u_s(x,\varepsilon)$ and $v_s(x,\varepsilon)$ denote these solutions. The Lyapunov stability of these solutions treated as stationary solutions of problem (36) obviously follows from the fact that the lower and upper solutions of the boundary value problem are the lower and upper solutions of problem (36) under the condition

$$\alpha^u_{n+1}(x,\varepsilon) \le u^0(x,\varepsilon) \le \beta^u_{n+1}(x,\varepsilon), \qquad \alpha^v_{n+1}(x,\varepsilon) \le v^0(x,\varepsilon) \le \beta^v_{n+1}(x,\varepsilon).$$

The proof of the asymptotic Lyapunov stability of the solutions $u_s(x,\varepsilon)$ and $v_s(x,\varepsilon)$ as stationary solutions of problem (36) is based on the use of the approach that is efficient in many classes of problems and amounts to constructing the upper and lower solutions of a special structure (see [6] and the references therein). We seek the upper and lower solutions of problem (36) in the form

$$\begin{aligned} &U_\beta(x,t,\varepsilon) = u_s(x,\varepsilon) + (\beta^u_{n+1}(x,\varepsilon) - u_s(x,\varepsilon))e^{-\lambda\varepsilon t}, \\ &U_\alpha(x,t,\varepsilon) = u_s(x,\varepsilon) + (\alpha^u_{n+1}(x,\varepsilon) - u_s(x,\varepsilon))e^{-\lambda\varepsilon t}, \\ &V_\beta(x,t,\varepsilon)) = v_s(x,\varepsilon) + (\beta^v_{n+1}(x,\varepsilon) - v_s(x,\varepsilon))e^{-\lambda\varepsilon t}, \\ &V_\alpha(x,\varepsilon) = v_s(x,\varepsilon) + (\alpha^v_{n+1}(x,\varepsilon) - v_s(x,\varepsilon))e^{-\lambda\varepsilon t}, \qquad x \in (0,1), \quad t \in \mathbb{R}^+, \end{aligned} \tag{37}$$

where $(\alpha^u_{n+1}(x,\varepsilon),\, \alpha^v_{n+1}(x,\varepsilon))$ and $(\beta^u_{n+1}(x,\varepsilon),\, \beta^v_{n+1}(x,\varepsilon))$ are the lower and upper solutions of this problem and $\lambda > 0$ is a constant. The standard transformations based on the use of equations for the terms of the asymptotics of stationary solutions, differential inequalities for stationary lower and upper solutions, and the estimates for the difference of the derivatives of the asymptotic solutions

$$\left|\frac{du_s(x,\varepsilon)}{dx} - \frac{dU_n(x,\varepsilon)}{dx}\right| = O(\varepsilon^n), \qquad \left|\frac{dv_s(x,\varepsilon)}{dx} - \frac{dV_n(x,\varepsilon)}{dx}\right| = O(\varepsilon^{n+1}), \tag{38}$$

which follow from the theorem proved in [7] for the general boundary value problem (see the proof of a similar estimate in [14] for details), show that the corresponding differential inequalities are satisfied for the upper and lower solutions determined by expressions (37). In particular, after the substitution and under quasimonotonicity Condition A5 for the stationary solution, as defined in Theorem 1N, we have the upper solutions

$$\begin{aligned} L_u(U_\beta, V_\beta) &= e^{-\lambda\varepsilon t}(\mathcal{N}_u(\beta^u_{n+1}(x,\varepsilon), \beta^v_{n+1}(x,\varepsilon)) + \varepsilon\lambda(\beta^u_{n+1} - u_s) + O(\varepsilon^{2n+2})) < 0, \\ L_v(U_\beta, V_\beta) &= e^{-\lambda\varepsilon t}(\mathcal{N}_v(\beta^u_{n+1}(x,\varepsilon), \beta^v_{n+1}(x,\varepsilon)) + \varepsilon\lambda(\beta^v_{n+1} - v_s) + O(\varepsilon^{2n+2})) < 0 \end{aligned} \tag{39}$$

for sufficiently small ε and $\lambda > 0$ for $n \geq 0$ due to differential inequalities (35) for $\mathcal{N}_u(\beta^u_{n+1}(x,\varepsilon), \beta^v_{n+1}(x,\varepsilon))$ and $\mathcal{N}_v(\beta^u_{n+1}(x,\varepsilon), \beta^v_{n+1}(x,\varepsilon))$, because $(\beta^u_{n+1} - u_s) = O(\varepsilon^{n+1})$ and $(\beta^v_{n+1} - v_s) = O(\varepsilon^{n+1})$. The differential inequality for $(U_\alpha(x,t,\varepsilon), V_\alpha(x,t,\varepsilon))$ can be verified similarly.

The verification of the conditions of problem (1.D) is completely similar. It follows from (39) that a solution of problem (36) exists under the condition $\alpha^u_1(x,\varepsilon) \leq u^0(x,\varepsilon) \leq \beta^u_1(x,\varepsilon)$, $\alpha^v_1(x,\varepsilon) \leq v^0(x,\varepsilon) \leq \beta^v_1(x,\varepsilon)$, where the lower and upper solutions are as in (37). The local uniqueness of the solution of boundary value problem (1) follows from the uniqueness theorem for the solution of the problem and the structure of the lower and upper solutions in (37).

Theorem 1NS. *Under Conditions* A1, A2, A4, *and* A5, *for sufficiently small* ε, *the stationary solution* $(u_s(x,\varepsilon), v_s(x,\varepsilon))$ *is asymptotically Lyapunov stable as a solution of problem* (36), *with the stability domain not less than*

$$[\alpha^u_1(x,\varepsilon); \beta^u_1(x,\varepsilon)] \times [\alpha^v_1(x,\varepsilon); \beta^v_1(x,\varepsilon)],$$

and is locally unique as a solution of problem (1) *in this domain.*

Theorem 1DS. *Under Conditions* A1–A5, *for sufficiently small* ε, *the stationary solution* $(u_s(x,\varepsilon), v_s(x,\varepsilon))$ *is asymptotically Lyapunov stable as a solution of problem* (36) *with the stability domain not less than*

$$[\alpha^u_1(x,\varepsilon); \beta^u_1(x,\varepsilon)] \times [\alpha^v_1(x,\varepsilon); \beta^v_1(x,\varepsilon)],$$

and is locally unique as a solution of problem (1) *in this domain.*

For stationary solutions whose existence was proved in Theorems 2N, 3N, 2D, and 3D, the corresponding analogues of these theorems also hold.

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