

THE FOCUSING COUPLED MODIFIED KORTEWEG–DE VRIES EQUATION WITH NONZERO BOUNDARY CONDITIONS: THE RIEMANN–HILBERT PROBLEM AND SOLITON CLASSIFICATION

Xinxin Ma*

The focusing coupled modified Korteweg–de Vries equation with nonzero boundary conditions is investigated by the Riemann–Hilbert approach. Three symmetries are formulated to derive compact exact solutions. The solutions include six different types of soliton solutions and breathers, such as dark–dark, bright–bright, kink–dark–dark, kink–bright–bright solitons, and a breather–breather solution.

Keywords: focusing coupled modified Korteweg–de Vries equation, nonzero boundary condition, dark–dark soliton, kink soliton, Riemann–Hilbert problem

DOI: 10.1134/S004057792404007X

1. Introduction

The modified Korteweg–de Vries (mKdV) equation was first introduced by Miura [1] as a natural extension of the KdV equation. It appears in many physical contexts, such as meandering ocean currents [2], acoustic waves in a certain anharmonic lattice [3], the Alfvén waves in a cold collision-free plasma [4], and so on. As a consequence, the equation has been studied extensively and many important results have been obtained [5]–[12].

In recent years, multicomponent integrable models have attracted considerable attention because of their physical significance in many fields such as optical fibers [13], [14], Bose–Einstein condensate [15], [16], and nonlinear optics [13]. An extension to the multicomponent equations is needed to describe different physical phenomena. For example, we know that the scalar nonlinear Schrödinger (NLS) equation controls the propagation of light pulses with fixed polarization, but when we consider the polarization effect in the pulse propagation along an optical fiber [17], we should use its two-component version. And when it comes to the effects of polarization or anisotropy in birefringent fibers [18], the short-pulse equation characterizing the propagation of ultrashort optical pulses in nonlinear media should be replaced by its two-component counterpart to describe the propagation of ultrashort pulses in birefringent fibers. Moreover, multicomponent integrable systems are rich in mathematical structures, and therefore their studies from different standpoints offer completely different experiences and challenges. Generalizations of the mKdV equation to the multicomponent case have been proposed and studied by several authors [19]–[24].

*Department of Mathematics, China University of Mining and Technology, Beijing, China,
e-mail: xinxinma2021@gmail.com.

Prepared from an English manuscript submitted by the author; for the Russian version, see *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 219, No. 1, pp. 80–113, April, 2024. Received September 19, 2023. Revised November 13, 2023. Accepted November 14, 2023.

In this paper, we consider the focusing coupled modified Korteweg–de Vries (fcmKdV) equation

$$\mathbf{q}_t + \mathbf{q}_{xxx} + 3\|\mathbf{q}\|^2\mathbf{q}_x + 3\mathbf{q}^T\mathbf{q}_x\mathbf{q} = \mathbf{0}, \quad \mathbf{q} = (q_1, q_2)^T, \quad (1.1)$$

where the superscripts T and $\|\cdot\|$ represent the matrix transpose and the standard Euclidean norm, and $\mathbf{0}$ denotes a zero matrix of the appropriate size. In 2017, multisoliton solutions of the fcmKdV equation decaying rapidly for large $|x|$ were obtained in [25] by using the inverse scattering transform (IST) method. In 2020, Wang et al. [26] obtained the nonlinear stability of breather solutions of the fcmKdV equation.

To better understand nonlinear phenomena of integrable systems, it is important to use various methods to find their exact solutions, among which the Riemann–Hilbert approach is always considered as an effective tool for solving initial-value problems. The Riemann–Hilbert approach is the modern version of the IST method. It has become evident that this method is also applicable to the long-time asymptotics of solutions to a wide range of integrable systems [27]–[29]. In addition, integrable nonlinear systems with nonzero boundary conditions (NZBCs) have numerous applications. Recent experiments with Bose–Einstein condensates have shown a proliferation of dark–dark vector solitons in the dynamics of two miscible superfluids experiencing fast counterflow in a narrow (cigar-shaped) condensate [16]. Problems with NZBCs are also relevant for the modulational instability in optics and the generation of rogue waves [30], [31]. In this paper, we use IST to solve the fcmKdV equation with the NZBCs

$$\lim_{x \rightarrow \pm\infty} \mathbf{q}(x, t) = \mathbf{q}_{\pm}, \quad \|\mathbf{q}_{\pm}\| = q_0 > 0, \quad (1.2)$$

where $\mathbf{q}_{\pm} = (q_{1\pm}, q_{2\pm})^T$. We note that the defocusing cmKdV equation with NZBCs was considered in [19]. Compared with the latter problem, we show that the focusing one in this paper has a more complicated spectral structure and richer classification of the discrete spectrum, leading to abundance of soliton solutions. We also want to explore the soliton classification for this problem.

This paper is organized as follows. In Sec. 2, for the potential $\mathbf{q}(x, t)$ such that $\mathbf{q}(\cdot, t) - \mathbf{q}_{\pm} \in L^1(\mathbb{R})$, the Jost eigenfunctions and scattering coefficients are defined in the direct problem. The auxiliary eigenfunctions that can comprise a complete set of analytic eigenfunctions are given. The symmetries and asymptotic behavior of the eigenfunctions and scattering coefficients are systematically obtained. In Sec. 3, a matrix Riemann–Hilbert problem (RHP) for the inverse problem is formulated, and the trace formula and the asymptotic phase difference are given. In Sec. 4, a compact matrix-form formula for general soliton solutions is derived in the reflectionless case, including an arbitrary combination of six different types of solitons. The conclusions are given in Sec. 5.

2. Direct problem

The Lax pair for Eq. (1.1) is given by

$$\begin{aligned} \varphi_x &= \mathbf{X}\varphi, & \mathbf{X}(x, t, k) &= ik\boldsymbol{\sigma} + \mathbf{Q}, \\ \varphi_t &= \mathbf{T}\varphi, & \mathbf{T}(x, t, k) &= 4k^2\mathbf{Q} + 2ik(2k^2\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{Q}^2 + \mathbf{Q}_x\boldsymbol{\sigma}) + 2\mathbf{Q}^3 - [\mathbf{Q}, \mathbf{Q}_x] - \mathbf{Q}_{xx}, \end{aligned} \quad (2.1)$$

where

$$\boldsymbol{\sigma} = \text{diag}(-1, 1, 1), \quad \mathbf{Q}(x, t) = \begin{pmatrix} 0 & \mathbf{q}^T \\ -\mathbf{q} & 0 \end{pmatrix}, \quad (2.2)$$

and $[\cdot, \cdot]$ denotes matrix commutator.

2.1. Jost eigenfunctions and the scattering matrix. To introduce Jost solutions, we must first study the asymptotic spectral problem for (2.1) as $x \rightarrow \pm\infty$,

$$\begin{aligned}\phi_{\pm,x} &= \mathbf{X}_{\pm}\phi_{\pm}, & \mathbf{X}_{\pm} &= ik\sigma + \mathbf{Q}_{\pm}, \\ \phi_{\pm,t} &= \mathbf{T}_{\pm}\phi_{\pm}, & \mathbf{T}_{\pm} &= 4k^2\mathbf{Q}_{\pm} + 2ik(2k^2\sigma + \sigma\mathbf{Q}_{\pm}^2) + 2\mathbf{Q}_{\pm}^3,\end{aligned}\tag{2.3}$$

where

$$\mathbf{Q}_{\pm} = \begin{pmatrix} 0 & \mathbf{q}_{\pm}^{\text{T}} \\ -\mathbf{q}_{\pm} & 0 \end{pmatrix}.$$

The eigenvalues of \mathbf{X}_{\pm} are ik and $\pm i\lambda$, where

$$\lambda(k) = \sqrt{k^2 + q_0^2}.\tag{2.4}$$

We note that $\lambda(k)$ is multivalued. As usual, this situation can be obviated by introducing the two-sheeted Riemann surface defined by (2.4). Its branch points are the values of k for which $\lambda(k) = 0$, i.e., $k = \pm iq_0$. This Riemann surface is constructed by gluing two copies along the branch cut $i[-q_0, q_0]$. We define $\lambda(k)$ such that $\text{Im } \lambda(k) > 0$ is the upper-half plane on the sheet I and the lower-half plane on the sheet II, and $\text{Im } \lambda(k) < 0$ is the lower-half plane on the sheet I and the upper-half plane on the sheet II.

For convenience, we introduce the uniformization variable as

$$z = k + \lambda,\tag{2.5}$$

whose inverse map is given by

$$k = \frac{1}{2}(z - \hat{z}), \quad \lambda = \frac{1}{2}(z + \hat{z}),\tag{2.6}$$

where $\hat{z} := q_0^2/z$. Let C_0 be the circle of radius q_0 centered at the origin in the complex z plane. Then the branch cut on either sheet is mapped onto C_0 : the first sheet \mathbb{C}_{I} is mapped onto the exterior of C_0 , and the second sheet \mathbb{C}_{II} is mapped onto the interior of C_0 . We also have $z(k, \lambda_{\text{I}})z(k, \lambda_{\text{II}}) = q_0^2$.

We next perform the IST on the complex z plane. On either sheet of the Riemann surface, we can take the asymptotic eigenvector matrices in the form

$$\mathbf{E}_{\pm}(z) = \begin{pmatrix} 1 & 0 & -iq_0/z \\ -i\mathbf{q}_{\pm}/z & \mathbf{q}_{\pm}^{\perp}/q_0 & \mathbf{q}_{\pm}/q_0 \end{pmatrix},\tag{2.7}$$

whence

$$\begin{aligned}\mathbf{E}_{\pm}^{-1}\mathbf{X}_{\pm}\mathbf{E}_{\pm} &= i\mathbf{\Lambda}_1, & \mathbf{\Lambda}_1(z) &= \text{diag}(-\lambda, k, \lambda), \\ \mathbf{E}_{\pm}^{-1}\mathbf{T}_{\pm}\mathbf{E}_{\pm} &= i\mathbf{\Lambda}_2, & \mathbf{\Lambda}_2(z) &= \text{diag}(-2\lambda(2k^2 - q_0^2), 4k^3, 2\lambda(2k^2 - q_0^2)),\end{aligned}\tag{2.8}$$

where $\mathbf{q}_{\pm}^{\perp} = (q_{2\pm}, -q_{1\pm})^{\text{T}}$. For future reference, we note that

$$\det \mathbf{E}_{\pm}(z) = 1 + \frac{q_0^2}{z^2} := \gamma(z).\tag{2.9}$$

The continuous spectrum $k \in (\mathbb{R} \cup i[-q_0, q_0])$ consists of all values of k (on either sheet) such that $\lambda(k) \in \mathbb{R}$. The corresponding set is $\Sigma = \mathbb{R} \cup C_0$ in the complex z plane. In that way, for $z \in \Sigma$, we can define $\varphi_{\pm}(x, t, z)$ as the Jost eigenfunctions with the boundary conditions

$$\varphi_{\pm}(x, t, z) = \mathbf{E}_{\pm}(z)e^{i\Theta(x,t,z)} + o(1), \quad x \rightarrow \pm\infty,\tag{2.10}$$

where $\Theta(x, t, z) = \mathbf{\Lambda}_1(z)x + \mathbf{\Lambda}_2(z)t = \text{diag}(\theta_1(x, t, z), \theta_2(x, t, z), -\theta_1(x, t, z))$. As usual, we introduce the modified Jost eigenfunctions

$$\mu_{\pm}(x, t, z) = \varphi_{\pm}(x, t, z)e^{-i\Theta(x, t, z)}, \quad (2.11)$$

such that

$$\lim_{x \rightarrow \pm\infty} \mu_{\pm}(x, t, z) = \mathbf{E}_{\pm}(z), \quad z \in \Sigma. \quad (2.12)$$

The spectral problem for $\mu_{\pm}(x, t, z)$ is defined by

$$\begin{aligned} (\mathbf{E}_{\pm}^{-1}\mu_{\pm})_x + i[\mathbf{E}_{\pm}^{-1}\mu_{\pm}, \mathbf{\Lambda}_1] &= \mathbf{E}_{\pm}^{-1}\Delta\mathbf{Q}_{\pm}\mu_{\pm}, \\ (\mathbf{E}_{\pm}^{-1}\mu_{\pm})_t + i[\mathbf{E}_{\pm}^{-1}\mu_{\pm}, \mathbf{\Lambda}_2] &= \mathbf{E}_{\pm}^{-1}\Delta\mathbf{T}_{\pm}\mu_{\pm}, \end{aligned} \quad (2.13)$$

where $\Delta\mathbf{Q}_{\pm} = \mathbf{Q} - \mathbf{Q}_{\pm}$ and $\Delta\mathbf{T}_{\pm} = \mathbf{T} - \mathbf{T}_{\pm}$. Using these linear equations, it can be proved that $\mu_{\pm}(x, t, z)$ are well defined.

Theorem 2.1. *Let $\mathbf{q}(\cdot, t) - \mathbf{q}_{\pm} \in L^1(\mathbb{R})$. Then the columns of $\mu_{\pm}(x, t, z)$ can be analytically extended to the corresponding domains in the complex z plane,*

$$\begin{aligned} \mu_{-,1} : D_1, \quad \mu_{-,2} : \mathbb{C}_-, \quad \mu_{-,3} : D_4, \\ \mu_{+,1} : D_2, \quad \mu_{+,2} : \mathbb{C}_+, \quad \mu_{+,3} : D_3, \end{aligned} \quad (2.14)$$

where the analytic domains D_1, \dots, D_4 and $\mathbb{C}_+, \mathbb{C}_-$ are

$$\begin{aligned} D_1 &= \{z \mid \text{Im } z > 0, |z| > q_0\}, & D_2 &= \{z \mid \text{Im } z < 0, |z| > q_0\}, \\ D_3 &= \{z \mid \text{Im } z < 0, |z| < q_0\}, & D_4 &= \{z \mid \text{Im } z > 0, |z| < q_0\}, \\ \mathbb{C}_- &= \{z \mid \text{Im } z < 0\}, & \mathbb{C}_+ &= \{z \mid \text{Im } z > 0\}. \end{aligned} \quad (2.15)$$

We next introduce the scattering matrix in a standard way. Because $\text{tr } \mathbf{X} = ik$ and $\text{tr } \mathbf{T} = 4ik^3$, it follows from the Abel theorem that

$$\frac{\partial}{\partial x} \det[\varphi_{\pm}(x, t, z)e^{-i\Theta(x, t, z)}] = \frac{\partial}{\partial t} \det[\varphi_{\pm}(x, t, z)e^{-i\Theta(x, t, z)}] = 0, \quad (2.16)$$

which implies that

$$\det \varphi_{\pm}(x, t, z) = \gamma(z)e^{i\theta_2(x, t, z)}, \quad z \in \Sigma. \quad (2.17)$$

As a result, for $z \in \Sigma_0 := \Sigma \setminus \{\pm iq_0\}$, $\varphi_+(x, t, z)$ and $\varphi_-(x, t, z)$ are two fundamental solutions of linear system (2.1), and hence a 3×3 scattering matrix $\mathbf{A}(z)$ must exist such that

$$\varphi_-(x, t, z) = \varphi_+(x, t, z)\mathbf{A}(z). \quad (2.18)$$

Because of the explicit time dependence of boundary condition (2.10), $\mathbf{A}(z)$ is independent of time. Combining (2.18) with (2.17) yields

$$\det \mathbf{A}(z) = 1, \quad z \in \Sigma_0. \quad (2.19)$$

We set $\mathbf{B}(z) := \mathbf{A}^{-1}(z) = (b_{ij}(z))$ for convenience. Furthermore, we obtain the following theorem.

Theorem 2.2. *Under the same hypotheses as in Theorem 2.1, the scattering coefficients can be analytically extended to the corresponding domains in the complex z plane:*

$$\begin{aligned} a_{11}(z) : D_1, \quad a_{22}(z) : \mathbb{C}_-, \quad a_{33}(z) : D_4, \\ b_{11}(z) : D_2, \quad b_{22}(z) : \mathbb{C}_+, \quad b_{33}(z) : D_3. \end{aligned} \quad (2.20)$$

2.2. Auxiliary eigenfunctions. To formulate a complete RHP, four more eigenfunctions for (2.1) that are respectively analytic in four fundamental domains must be constructed. For this, as in [32], we turn to the adjoint problem

$$\tilde{\varphi}_x = \tilde{\mathbf{X}}(z)\tilde{\varphi}, \quad \tilde{\varphi}_t = \tilde{\mathbf{T}}(z)\tilde{\varphi}, \quad (2.21)$$

where $\tilde{\mathbf{X}}(z) = \mathbf{X}^*(z^*)$ and $\tilde{\mathbf{T}}(z) = \mathbf{T}^*(z^*)$. As before, for $z \in \Sigma$, we define the adjoint Jost eigenfunctions $\tilde{\varphi}_\pm(x, t, z)$ for (2.21) satisfying boundary conditions

$$\tilde{\varphi}_\pm(x, t, z) = \mathbf{E}_\pm^*(z^*)e^{-i\Theta(x, t, z)} + o(1), \quad x \rightarrow \pm\infty. \quad (2.22)$$

Under the same assumption as in Theorem 2.1, by introducing the modified adjoint Jost eigenfunctions $\tilde{\mu}_\pm(x, t, z) = \tilde{\varphi}_\pm(x, t, z)e^{i\Theta(x, t, z)}$, it can be shown that the columns of $\tilde{\mu}_\pm(x, t, z)$ can be analytically extended to the corresponding domains:

$$\begin{aligned} \tilde{\mu}_{-,1} &: D_2, & \tilde{\mu}_{-,2} &: \mathbb{C}_+, & \tilde{\mu}_{-,3} &: D_3, \\ \tilde{\mu}_{+,1} &: D_1, & \tilde{\mu}_{+,2} &: \mathbb{C}_-, & \tilde{\mu}_{+,3} &: D_4. \end{aligned} \quad (2.23)$$

For $z \in \Sigma$, we have $\det \tilde{\varphi}_\pm(x, t, z) = \gamma(z)e^{-i\theta_2(x, t, z)}$. Consequently, for $z \in \Sigma_0$, we introduce the corresponding scattering matrix as

$$\tilde{\varphi}_-(x, t, z) = \tilde{\varphi}_+(x, t, z)\tilde{\mathbf{A}}(z). \quad (2.24)$$

By simple calculation, we can obtain the following fact.

Proposition 2.1. *If $\tilde{\mathbf{p}}(x, t, z)$ and $\tilde{\mathbf{q}}(x, t, z)$ are any two solutions to adjoint problem (2.21), then*

$$\mathbf{r}(x, t, z) = e^{i\theta_2(x, t, z)}[\tilde{\mathbf{p}} \times \tilde{\mathbf{q}}](x, t, z) \quad (2.25)$$

is a solution of the original Lax pair (2.1), where \times denotes the usual cross product.

Recalling the definition of the adjoint problem, we conclude that

$$\tilde{\varphi}_\pm(x, t, z) = \varphi_\pm^*(x, t, z^*), \quad z \in \Sigma. \quad (2.26)$$

By the above results, we can build four new solutions of the original spectral problem (2.1), referred to as the auxiliary eigenfunctions, as

$$\begin{aligned} \chi_1(x, t, z) &= e^{i\theta_2(x, t, z)}[\varphi_{+,1}^* \times \varphi_{-,2}^*](x, t, z^*), & z \in D_1, \\ \chi_2(x, t, z) &= e^{i\theta_2(x, t, z)}[\varphi_{-,1}^* \times \varphi_{+,2}^*](x, t, z^*), & z \in D_2, \\ \chi_3(x, t, z) &= e^{i\theta_2(x, t, z)}[\varphi_{+,2}^* \times \varphi_{-,3}^*](x, t, z^*), & z \in D_3, \\ \chi_4(x, t, z) &= e^{i\theta_2(x, t, z)}[\varphi_{-,2}^* \times \varphi_{+,3}^*](x, t, z^*), & z \in D_4. \end{aligned} \quad (2.27)$$

The corresponding modified auxiliary eigenfunctions are defined as

$$\begin{aligned} m_j(x, t, z) &= \chi_j(x, t, z)e^{i\theta_1(x, t, z)}, & z \in D_j, \quad j = 1, 2, \\ m_l(x, t, z) &= \chi_l(x, t, z)e^{-i\theta_1(x, t, z)}, & z \in D_l, \quad l = 3, 4. \end{aligned} \quad (2.28)$$

By using above results again, we establish the following relation in a straightforward way.

Lemma 2.1. For all cyclic permutations of the indices $j, l,$ and m with $z \in \Sigma,$

$$\varphi_{\pm,j}(x, t, z) = e^{i\theta_2(x,t,z)} \frac{[\varphi_{\pm,l}^* \times \varphi_{\pm,m}^*](x, t, z^*)}{\gamma_j(z)}, \quad (2.29)$$

where $\gamma_1(z) = \gamma_3(z) = 1$ and $\gamma_2(z) = \gamma(z).$

It is useful to note that

$$(\det \varphi_{\pm})\mathbf{I} = (\varphi_{\pm,2} \times \varphi_{\pm,3}, \varphi_{\pm,3} \times \varphi_{\pm,1}, \varphi_{\pm,1} \times \varphi_{\pm,2})\varphi_{\pm}^T, \quad z \in \Sigma_0, \quad (2.30)$$

where \mathbf{I} denotes the 3×3 identity matrix. Thanks to this identity and scattering relation (2.18), we have the following corollary.

Corollary 2.1. The scattering matrices $\mathbf{A}(z)$ and $\tilde{\mathbf{A}}(z)$ satisfy the relation

$$\tilde{\mathbf{A}}(z) = \mathbf{\Gamma}(z)[\mathbf{A}^{-1}(z)]^T \mathbf{\Gamma}^{-1}(z), \quad z \in \Sigma_0, \quad (2.31)$$

where $\mathbf{\Gamma}(z) = \text{diag}(1, \gamma(z), 1).$

Applying scattering relation (2.24) and relation (2.29) to the auxiliary eigenfunctions (2.27) and then using (2.31) yields the following statement.

Corollary 2.2. For $z \in \Sigma_0,$ the Jost eigenfunctions $\varphi_{\pm,1}(x, t, z), \varphi_{\pm,3}(x, t, z)$ have the decompositions

$$\begin{aligned} \varphi_{-,1}(x, t, z) &= \frac{1}{a_{22}(z)}[\chi_3(z) + a_{21}(z)\varphi_{-,2}(z)] = \frac{1}{a_{33}(z)}[\chi_4(z) + a_{31}(z)\varphi_{-,3}(z)], \\ \varphi_{-,3}(x, t, z) &= \frac{1}{a_{11}(z)}[\chi_1(z) + a_{13}(z)\varphi_{-,1}(z)] = \frac{1}{a_{22}(z)}[\chi_2(z) + a_{23}(z)\varphi_{-,2}(z)], \\ \varphi_{+,1}(x, t, z) &= \frac{1}{b_{33}(z)}[\chi_3(z) + b_{31}(z)\varphi_{+,3}(z)] = \frac{1}{b_{22}(z)}[\chi_4(z) + b_{21}(z)\varphi_{+,2}(z)], \\ \varphi_{+,3}(x, t, z) &= \frac{1}{b_{11}(z)}[\chi_2(z) + b_{13}(z)\varphi_{+,1}(z)] = \frac{1}{b_{22}(z)}[\chi_1(z) + b_{23}(z)\varphi_{+,2}(z)]. \end{aligned} \quad (2.32)$$

2.3. Symmetries. Symmetries of the eigenfunctions and scattering coefficients determine the distribution of the discrete spectrum, degrees of freedom of the soliton parameters, and solvability of the problem.

2.3.1. First symmetry. We first consider the transformation $z \mapsto z^*.$ The following result is a straightforward consequence of the fact that $\mathbf{Q}^\dagger = -\mathbf{Q}$ and $\sigma\mathbf{Q} = -\mathbf{Q}\sigma,$ where \dagger denotes the conjugate transpose.

Lemma 2.2. The Jost eigenfunctions $\varphi_{\pm}(x, t, z)$ admit the symmetry relation

$$\varphi_{\pm}^\dagger(x, t, z^*) = \mathbf{C}(z)\varphi_{\pm}^{-1}(x, t, z), \quad z \in \Sigma, \quad (2.33)$$

where $\mathbf{C}(z) = \text{diag}(\gamma(z), 1, \gamma(z)).$

Using identity (2.30) and symmetry (2.33), substituting the (2.32) into it, and applying the Schwartz reflection principle, we obtain the following lemma.

Lemma 2.3. *The Jost eigenfunctions satisfy the symmetry relations*

$$\begin{aligned}
\varphi_{-,1}^*(z^*) &= \frac{e^{-i\theta_2(x,t,z)}}{a_{22}(z)} [\varphi_{-,2} \times \chi_2](x, t, z), & z \in D_2, \\
\varphi_{-,2}^*(z^*) &= \begin{cases} \frac{e^{-i\theta_2(x,t,z)}}{\gamma(z)a_{11}(z)} [\chi_1 \times \varphi_{-,1}](x, t, z), & z \in D_1, \\ \frac{e^{-i\theta_2(x,t,z)}}{\gamma(z)a_{33}(z)} [\varphi_{-,3} \times \chi_4](x, t, z), & z \in D_4, \end{cases} \\
\varphi_{-,3}^*(z^*) &= \frac{e^{-i\theta_2(x,t,z)}}{a_{22}(z)} [\chi_3 \times \varphi_{-,2}](x, t, z), & z \in D_3, \\
\varphi_{+,1}^*(z^*) &= \frac{e^{-i\theta_2(x,t,z)}}{b_{22}(z)} [\varphi_{+,2} \times \chi_1](x, t, z), & z \in D_1, \\
\varphi_{+,2}^*(z^*) &= \begin{cases} \frac{e^{-i\theta_2(x,t,z)}}{\gamma(z)b_{33}(z)} [\varphi_{+,3} \times \chi_3](x, t, z) & z \in D_3, \\ = \frac{e^{-i\theta_2(x,t,z)}}{\gamma(z)b_{11}(z)} [\chi_2 \times \varphi_{+,1}](x, t, z), & z \in D_2, \end{cases} \\
\varphi_{+,3}^*(z^*) &= \frac{e^{-i\theta_2(x,t,z)}}{b_{22}(z)} [\chi_4 \times \varphi_{+,2}](x, t, z), & z \in D_4.
\end{aligned} \tag{2.34}$$

We now explore the impact of the first symmetry on the scattering matrix. The next lemma follows from symmetry (2.33) and scattering relation (2.18).

Lemma 2.4. *For $z \in \Sigma_0$, the relation between the scattering matrices $\mathbf{A}(z)$ and $\mathbf{B}(z)$ is*

$$\mathbf{A}^\dagger(z^*) = \mathbf{C}(z)\mathbf{B}(z)\mathbf{C}^{-1}(z). \tag{2.35}$$

Recalling the analyticity of the scattering coefficients, we conclude from the Schwartz reflection principle that

$$\begin{aligned}
b_{11}(z) &= a_{11}^*(z^*), & z \in D_2, \\
b_{22}(z) &= a_{22}^*(z^*), & z \in \mathbb{C}_+, \\
b_{33}(z) &= a_{33}^*(z^*), & z \in D_3.
\end{aligned} \tag{2.36}$$

2.3.2. Second symmetry. We next discuss the transformation $z \mapsto -z^*$. It is easy to prove the following lemma.

Lemma 2.5. *The Jost eigenfunctions $\varphi_\pm(x, t, z)$ have the symmetry*

$$\varphi_\pm(x, t, z) = \varphi_\pm^*(x, t, -z^*), \quad z \in \Sigma. \tag{2.37}$$

We now discuss the symmetry of the scattering matrix in the context of the second symmetry. Substituting (2.37) in scattering relation (2.18), we obtain the following statement.

Lemma 2.6. *For $z \in \Sigma_0$, the scattering matrices satisfy the symmetries*

$$\mathbf{A}(z) = \mathbf{A}^*(-z^*), \quad \mathbf{B}(z) = \mathbf{B}^*(-z^*). \tag{2.38}$$

Then the Schwartz reflection principle allows us to conclude that

$$\begin{aligned} a_{11}(z) &= a_{11}^*(-z^*), & z \in D_1, \\ a_{22}(z) &= a_{22}^*(-z^*), & z \in \mathbb{C}_-, \\ a_{33}(z) &= a_{33}^*(-z^*), & z \in D_4. \end{aligned} \tag{2.39}$$

Finally, we combine symmetries (2.37) with definitions (2.27) of the auxiliary eigenfunctions to arrive at the following lemma.

Lemma 2.7. *The auxiliary eigenfunctions admit the symmetry relations*

$$\chi_j(x, t, z) = \chi_j^*(x, t, -z^*), \quad z \in D_j, \quad j = 1, \dots, 4. \tag{2.40}$$

2.3.3. Third symmetry. Finally, we study the transformation $z \mapsto -\hat{z}$.

Lemma 2.8. *The Jost eigenfunctions $\varphi_{\pm}(x, t, z)$ satisfy the symmetry*

$$\varphi_{\pm}(x, t, z) = \varphi_{\pm}(x, t, -\hat{z})\mathbf{\Pi}(z), \quad z \in \Sigma, \tag{2.41}$$

where

$$\mathbf{\Pi}(z) = \begin{pmatrix} 0 & 0 & -iq_0/z \\ 0 & 1 & 0 \\ -iq_0/z & 0 & 0 \end{pmatrix}.$$

Combining symmetry (2.41) with scattering relation (2.18) yields the following lemma.

Lemma 2.9. *For $z \in \Sigma_0$, the scattering matrices $\mathbf{A}(z)$ and $\mathbf{B}(z)$ admit the symmetries*

$$\mathbf{A}(-\hat{z}) = \mathbf{\Pi}(z)\mathbf{A}(z)\mathbf{\Pi}^{-1}(z), \quad \mathbf{B}(-\hat{z}) = \mathbf{\Pi}(z)\mathbf{B}(z)\mathbf{\Pi}^{-1}(z). \tag{2.42}$$

Using the Schwartz reflection principle, we conclude that

$$\begin{aligned} a_{11}(z) &= a_{33}(-\hat{z}), & z \in D_1, \\ a_{22}(z) &= a_{22}(-\hat{z}), & z \in \mathbb{C}_-. \end{aligned} \tag{2.43}$$

To see how the third symmetry affects the symmetry of the auxiliary eigenfunctions, we substitute (2.41) in (2.27).

Lemma 2.10. *The auxiliary eigenfunctions obey the symmetries*

$$\begin{aligned} \chi_1(x, t, z) &= -\frac{iq_0}{z} \chi_4(x, t, -\hat{z}), & z \in D_1, \\ \chi_2(x, t, z) &= -\frac{iq_0}{z} \chi_3(x, t, -\hat{z}), & z \in D_2. \end{aligned} \tag{2.44}$$

2.3.4. Reflection coefficients. We note that we cannot exclude the possible existence of zeros for $a_{11}(z)$ and $b_{22}(z)$ along Σ . To solve the RHP, we focus on the potentials without spectral singularities [33]. For $z \in \Sigma$, there are three reflection coefficients in the inverse problem, which are given by

$$\rho_1(z) = \frac{a_{21}(z)}{a_{11}(z)}, \quad \rho_2(z) = \frac{a_{31}(z)}{a_{11}(z)}, \quad \rho_3(z) = \frac{a_{32}(z)}{a_{22}(z)}. \quad (2.45)$$

They have the following symmetries. We first consider the effect from the first symmetry:

$$\rho_1(z) = \gamma(z) \frac{b_{12}^*(z^*)}{b_{11}^*(z^*)}, \quad \rho_2(z) = \frac{b_{13}^*(z^*)}{b_{11}^*(z^*)}, \quad \rho_3(z) = \frac{1}{\gamma(z)} \frac{b_{23}^*(z^*)}{b_{22}^*(z^*)}. \quad (2.46)$$

Next, the second symmetry leads to the relations

$$\rho_j(z) = \rho_j^*(-z^*), \quad j = 1, 2, 3. \quad (2.47)$$

Finally, due to the third symmetry, we obtain that

$$\begin{aligned} \rho_1(-\hat{z}) &= \frac{iz}{q_0} \frac{a_{23}(z)}{a_{33}(z)} = \frac{iz\gamma(z)}{q_0} \frac{b_{32}^*(z^*)}{b_{33}^*(z^*)}, & \rho_2(-\hat{z}) &= \frac{a_{13}(z)}{a_{33}(z)} = \frac{b_{31}^*(z^*)}{b_{33}^*(z^*)}, \\ \rho_3(-\hat{z}) &= -\frac{iq_0}{z} \frac{a_{12}(z)}{a_{22}(z)} = -\frac{iq_0}{z\gamma(z)} \frac{b_{21}^*(z^*)}{b_{22}^*(z^*)}. \end{aligned} \quad (2.48)$$

2.4. Asymptotic behavior. We next study the asymptotic behavior of the eigenfunctions and scattering coefficients as $k \rightarrow \infty$. Because $z(\infty_I) = \infty$ and $z(\infty_{II}) = 0$, we discuss the asymptotics both as $z \rightarrow \infty$ and as $z \rightarrow 0$. Specifically, substituting the corresponding standard Wentzel–Kramers–Brillouin expansions in differential equations (2.13) with boundary conditions (2.12) leads to the following lemma.

Lemma 2.11. *As $z \rightarrow \infty$, the columns of the modified Jost eigenfunctions $\mu_{\pm}(x, t, z)$ in the corresponding analyticity domains have the following asymptotic behavior:*

$$\mu_{\pm,1}(x, t, z) = \left(\begin{array}{c} 1 + \frac{1}{z} \int_{\pm\infty}^x i[q_0^2 - \|\mathbf{q}(y, t)\|^2] dy \\ -i\mathbf{q}(x, t)/z \end{array} \right) + o(z^{-2}), \quad (2.49a)$$

$$(\mu_{\pm,2}(x, t, z), \mu_{\pm,3}(x, t, z)) = \left(\begin{array}{cc} 0 & 0 \\ \mathbf{q}_{\pm}^{\perp}/q_0 & \mathbf{q}_{\pm}/q_0 \end{array} \right) + o(z^{-1}). \quad (2.49b)$$

Using the last two terms in (2.49a), we obtain the following proposition.

Proposition 2.2. *The solution $\mathbf{q}(x, t) = (q_1(x, t), q_2(x, t))^T$ of the fcmKdV equation with NZBCs can be expressed as*

$$q_j(x, t) = i \lim_{z \rightarrow \infty} [z\mu_{\pm}(x, t, z)]_{(j+1)1}. \quad (2.50)$$

Lemma 2.12. *As $z \rightarrow 0$, the columns of the modified Jost eigenfunctions $\mu_{\pm}(x, t, z)$ in the corresponding analyticity domains have the following asymptotic behavior:*

$$\begin{aligned} (\mu_{\pm,1}(x, t, z), \mu_{\pm,3}(x, t, z)) &= \left(\begin{array}{cc} 0 & -iq_0/z \\ -i\mathbf{q}_{\pm}^{\perp}/z & \mathbf{0} \end{array} \right) + o(1), \\ \mu_{\pm,2}(x, t, z) &= \left(\begin{array}{c} 0 \\ \mathbf{q}_{\pm}^{\perp}/q_0 \end{array} \right) + o(z). \end{aligned} \quad (2.51)$$

Substituting the above asymptotics in representations (2.28) for the modified auxiliary eigenfunctions yields the following corollary.

Corollary 2.3. *The modified auxiliary eigenfunctions $m_j(x, t, z)$ ($j = 1, \dots, 4$) in the corresponding analyticity domains have the following asymptotics:*

$$\begin{aligned} (m_1, m_2, m_3, m_4) &= \begin{pmatrix} 0 & 0 & \mathbf{q}_+^T \mathbf{q}_- / q_0^2 & \mathbf{q}_+^T \mathbf{q}_- / q_0^2 \\ \mathbf{q}_- / q_0 & \mathbf{q}_+ / q_0 & \mathbf{0} & \mathbf{0} \end{pmatrix} + o(z^{-1}), & z \rightarrow \infty, \\ (m_1, m_2, m_3, m_4) &= \begin{pmatrix} -i\mathbf{q}_+^T \mathbf{q}_- / (zq_0) & -i\mathbf{q}_+^T \mathbf{q}_- / (zq_0) & 0 & 0 \\ \mathbf{0} & \mathbf{0} & -i\mathbf{q}_+ / z & -i\mathbf{q}_- / z \end{pmatrix} + o(1), & z \rightarrow 0. \end{aligned} \quad (2.52)$$

Next, we investigate the asymptotic behavior of the scattering coefficients. Combining the asymptotics of the Jost eigenfunctions in Lemmas 2.11 and 2.12 with scattering relation (2.18), we arrive at the following statements.

Corollary 2.4. *As $z \rightarrow \infty$, the scattering coefficients in the corresponding analyticity domains have the asymptotic behavior*

$$\begin{aligned} a_{11}(z) &= 1 + o(z^{-1}), & a_{22}(z) &= \frac{\mathbf{q}_+^T \mathbf{q}_-}{q_0^2} + o(z^{-1}), & a_{33}(z) &= \frac{\mathbf{q}_+^T \mathbf{q}_-}{q_0^2} + o(z^{-1}), \\ b_{11}(z) &= 1 + o(z^{-1}), & b_{22}(z) &= \frac{\mathbf{q}_+^T \mathbf{q}_-}{q_0^2} + o(z^{-1}), & b_{33}(z) &= \frac{\mathbf{q}_+^T \mathbf{q}_-}{q_0^2} + o(z^{-1}). \end{aligned} \quad (2.53)$$

Corollary 2.5. *As $z \rightarrow 0$, the scattering coefficients in the corresponding analyticity domains have the asymptotics*

$$a_{11}(z) = \frac{\mathbf{q}_+^T \mathbf{q}_-}{q_0^2} + o(z), \quad a_{22}(z) = \frac{\mathbf{q}_+^T \mathbf{q}_-}{q_0^2} + o(z), \quad a_{33}(z) = 1 + o(z), \quad (2.54a)$$

$$b_{11}(z) = \frac{\mathbf{q}_+^T \mathbf{q}_-}{q_0^2} + o(z), \quad b_{22}(z) = \frac{\mathbf{q}_+^T \mathbf{q}_-}{q_0^2} + o(z), \quad b_{33}(z) = 1 + o(z). \quad (2.54b)$$

3. Inverse problem

In what follows, we formulate the corresponding RHP and then obtain a reconstruction formula and the trace formula.

3.1. Riemann–Hilbert problem. We start by introducing the piecewise meromorphic function

$$\begin{aligned} \mathbf{M}(x, t, z) &= \left(\frac{\mu_{-,1}(x, t, z)}{a_{11}(z)}, \mu_{+,2}(x, t, z), \frac{m_1(x, t, z)}{b_{22}(z)} \right), & z \in D_1, \\ \mathbf{M}(x, t, z) &= \left(\mu_{+,1}(x, t, z), \frac{\mu_{-,2}(x, t, z)}{a_{22}(z)}, \frac{m_2(x, t, z)}{b_{11}(z)} \right), & z \in D_2, \\ \mathbf{M}(x, t, z) &= \left(\frac{m_3(x, t, z)}{b_{33}(z)}, \frac{\mu_{-,2}(x, t, z)}{a_{22}(z)}, \mu_{+,3}(x, t, z) \right), & z \in D_3, \\ \mathbf{M}(x, t, z) &= \left(\frac{m_4(x, t, z)}{b_{22}(z)}, \mu_{+,2}(x, t, z), \frac{\mu_{-,3}(x, t, z)}{a_{33}(z)} \right), & z \in D_4. \end{aligned} \quad (3.1)$$

We next study the jump conditions of the matrix function $\mathbf{M}(x, t, z)$ across the contour Σ . Thus, we set

$$\mathbf{M}_{\pm}(x, t, z) = \lim_{\substack{\hat{z} \rightarrow z \\ \hat{z} \in D_{\pm}}} \mathbf{M}(x, t, \hat{z}), \quad z \in \Sigma,$$

where $D_+ = D_1 \cup D_3$, $D_- = D_2 \cup D_4$. Here, $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4$, where Σ_j is the boundary of $\overline{D}_j \cap \overline{D}_{(j+1) \bmod 4}$. We define the orientation of the boundary Σ_j , $j = 1, \dots, 4$, such that D_+ is always on its left side. Then by combining scattering relation (2.18) with decompositions (2.32), we conclude that the following lemma holds.

Lemma 3.1. *The function $\mathbf{M}(x, t, z)$ satisfies the jump conditions*

$$\mathbf{M}_+(x, t, z) = \mathbf{M}_-(x, t, z)\mathbf{G}(x, t, z), \quad z \in \Sigma, \quad (3.2)$$

where

$$\mathbf{G}(x, t, z) = e^{i\Theta(x, t, z)}\mathbf{G}_j(z)e^{-i\Theta(x, t, z)}, \quad z \in \Sigma_j, \quad j = 1, \dots, 4, \quad (3.3)$$

with

$$\begin{aligned} \mathbf{G}_1(z) &= \begin{pmatrix} 1 + \frac{1}{\gamma(z)}|\rho_1(z)|^2 + |\rho_2(z)|^2 & \frac{1}{\gamma(z)}\rho_1^*(z) & \rho_2^*(z) - \rho_1^*(z)\rho_3^*(z) \\ \rho_1(z) & 1 & -\gamma(z)\rho_3^*(z) \\ \rho_2(z) - \rho_1(z)\rho_3(z) & -\rho_3(z) & 1 + \gamma(z)|\rho_3(z)|^2 \end{pmatrix}, \quad z \in \Sigma_1, \\ \mathbf{G}_2(z) &= \begin{pmatrix} 1 - \rho_2^*(z^*)\rho_2(\hat{z}) & 0 & \rho_2^*(z^*) \\ 0 & 1 & 0 \\ -\rho_2(\hat{z}) & 0 & 1 \end{pmatrix}, \quad z \in \Sigma_2, \\ \mathbf{G}_3(z) &= \begin{pmatrix} 1 + |\rho_2(\hat{z})|^2 & -\frac{b_{12}(z)}{a_{22}(z)a_{33}(z)} & -\rho_2^*(\hat{z}) \\ -\frac{a_{21}(z)}{b_{22}(z)b_{33}(z)} & \frac{1}{a_{22}(z)b_{22}(z)} & \gamma(z)\rho_3^*(z) \\ -\rho_2(\hat{z}) & \rho_3(z) & 1 \end{pmatrix}, \quad z \in \Sigma_3, \\ \mathbf{G}_4(z) &= \begin{pmatrix} 1 - \rho_2(z)\rho_2^*(\hat{z}^*) & 0 & -\rho_2^*(\hat{z}^*) \\ 0 & 1 & 0 \\ \rho_2(z) & 0 & 1 \end{pmatrix}, \quad z \in \Sigma_4. \end{aligned} \quad (3.4)$$

The following symmetry holds:

$$\mathbf{G}(x, t, z) = \mathbf{C}^{-1}(z)\mathbf{G}^\dagger(x, t, z^*)\mathbf{C}(z), \quad z \in \Sigma. \quad (3.5)$$

Moreover, using the asymptotics of the eigenfunctions and scattering coefficients with the restriction that \mathbf{q}_+ is parallel to \mathbf{q}_- , we derive the following result.

Lemma 3.2. *The matrix $\mathbf{M}(x, t, z)$ admits the normalization conditions*

$$\begin{aligned} \mathbf{M}(x, t, z) &= \mathbf{M}_\infty + o(z^{-1}), \quad z \rightarrow \infty, \\ \mathbf{M}(x, t, z) &= -\frac{i}{z}\mathbf{M}_0 + o(1), \quad z \rightarrow 0, \end{aligned} \quad (3.6)$$

where

$$\mathbf{M}_\infty = \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{0} & \mathbf{q}_+^\perp/q_0 & \mathbf{q}_+/q_0 \end{pmatrix}, \quad \mathbf{M}_0 = \begin{pmatrix} 0 & 0 & q_0 \\ \mathbf{q}_+ & \mathbf{0} & \mathbf{0} \end{pmatrix}. \quad (3.7)$$

The next lemma follows from (2.36), (2.39), and (2.43).

Lemma 3.3. *If $z_0 \in D_1$, then*

$$\begin{aligned}
b_{22}(z_0) = 0 &\iff a_{22}(z_0^*) = 0 \iff a_{22}(-\hat{z}_0^*) = 0 \iff b_{22}(-\hat{z}_0) = 0 \iff b_{22}(-z_0^*) = 0 \iff \\
&\iff a_{22}(-z_0) = 0 \iff a_{22}(\hat{z}_0) = 0 \iff b_{22}(\hat{z}_0^*) = 0, \\
a_{11}(z_0) = 0 &\iff b_{11}(z_0^*) = 0 \iff b_{33}(-\hat{z}_0^*) = 0 \iff a_{33}(-\hat{z}_0) = 0 \iff a_{11}(-z_0^*) = 0 \iff \\
&\iff b_{11}(-z_0) = 0 \iff b_{33}(\hat{z}_0) = 0 \iff a_{33}(\hat{z}_0^*) = 0.
\end{aligned} \tag{3.8}$$

These relations indicate that it suffices to study the zeros of $a_{11}(z)$ and $b_{22}(z)$ for $z \in D_1$. In this paper, we consider the zeros in the simple cases. Hence, there are six possible types of discrete eigenvalues:

1. $\{\xi_n\}_{n=1}^{N_1}$ is the set of all type-I eigenvalues; $a_{11}(\xi_n) = 0$ and $b_{22}(\xi_n) \neq 0$ for $\text{Re } \xi_n = 0$;
2. $\{\omega_n\}_{n=1}^{N_2}$ is the set of all type-II eigenvalues; $a_{11}(\omega_n) = 0$ and $b_{22}(\omega_n) \neq 0$ for $\text{Re } \omega_n \neq 0$;
3. $\{\eta_n\}_{n=1}^{N_3}$ is the set of all type-III eigenvalues; $a_{11}(\eta_n) \neq 0$ and $b_{22}(\eta_n) = 0$ for $\text{Re } \eta_n = 0$;
4. $\{z_n\}_{n=1}^{N_4}$ is the set of all type-IV eigenvalues; $a_{11}(z_n) \neq 0$ and $b_{22}(z_n) = 0$ for $\text{Re } z_n \neq 0$;
5. $\{\nu_n\}_{n=1}^{N_5}$ is the set of all type-V eigenvalues; $a_{11}(\nu_n) = b_{22}(\nu_n) = 0$ for $\text{Re } \nu_n = 0$;
6. $\{\zeta_n\}_{n=1}^{N_6}$ is the set of all type-VI eigenvalues; $a_{11}(\zeta_n) = b_{22}(\zeta_n) = 0$ for $\text{Re } \zeta_n \neq 0$.

Lemma 3.4. *The meromorphic matrix $\mathbf{M}(x, t, z)$ satisfies the residue conditions*

$$\begin{aligned}
\text{Res}_{z=\xi_n} \mathbf{M}(x, t, z) &= A_n(\mathbf{M}_3(\xi_n), \mathbf{0}, \mathbf{0}), \\
\text{Res}_{z=-\hat{\xi}_n^*} \mathbf{M}(x, t, z) &= -\frac{q_0^2}{(\hat{\xi}_n^*)^2} A_n^*(\mathbf{M}_3(-\hat{\xi}_n^*), \mathbf{0}, \mathbf{0}), \\
\text{Res}_{z=-\hat{\xi}_n^*} \mathbf{M}(x, t, z) &= -A_n^*(\mathbf{0}, \mathbf{0}, \mathbf{M}_1(\hat{\xi}_n^*)), \\
\text{Res}_{z=-\hat{\xi}_n^*} \mathbf{M}(x, t, z) &= \frac{q_0^2}{\hat{\xi}_n^2} A_n(\mathbf{0}, \mathbf{0}, \mathbf{M}_1(-\hat{\xi}_n));
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
\text{Res}_{z=\omega_n} \mathbf{M}(x, t, z) &= B_n(\mathbf{M}_3(\omega_n), \mathbf{0}, \mathbf{0}), \\
\text{Res}_{z=-\hat{\omega}_n^*} \mathbf{M}(x, t, z) &= -\frac{q_0^2}{(\hat{\omega}_n^*)^2} B_n^*(\mathbf{M}_3(-\hat{\omega}_n^*), \mathbf{0}, \mathbf{0}), \\
\text{Res}_{z=\hat{\omega}_n^*} \mathbf{M}(x, t, z) &= -B_n^*(\mathbf{0}, \mathbf{0}, \mathbf{M}_1(\hat{\omega}_n^*)), \\
\text{Res}_{z=-\hat{\omega}_n^*} \mathbf{M}(x, t, z) &= \frac{q_0^2}{\hat{\omega}_n^2} B_n(\mathbf{0}, \mathbf{0}, \mathbf{M}_1(-\hat{\omega}_n)), \\
\text{Res}_{z=-\hat{\omega}_n^*} \mathbf{M}(x, t, z) &= -B_n^*(\mathbf{M}_3(-\hat{\omega}_n^*), \mathbf{0}, \mathbf{0}), \\
\text{Res}_{z=\hat{\omega}_n^*} \mathbf{M}(x, t, z) &= \frac{q_0^2}{\hat{\omega}_n^2} B_n(\mathbf{M}_3(\hat{\omega}_n), \mathbf{0}, \mathbf{0}), \\
\text{Res}_{z=-\omega_n} \mathbf{M}(x, t, z) &= B_n(\mathbf{0}, \mathbf{0}, \mathbf{M}_1(-\omega_n)), \\
\text{Res}_{z=\hat{\omega}_n^*} \mathbf{M}(x, t, z) &= -\frac{q_0^2}{(\hat{\omega}_n^*)^2} B_n^*(\mathbf{0}, \mathbf{0}, \mathbf{M}_1(\hat{\omega}_n^*));
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
\operatorname{Res}_{z=\eta_n} \mathbf{M}(x, t, z) &= C_n(\mathbf{0}, \mathbf{0}, \mathbf{M}_2(\eta_n)), \\
\operatorname{Res}_{z=-\hat{\eta}_n^*} \mathbf{M}(x, t, z) &= \frac{iq_0^3}{(\eta_n^*)^3} \frac{C_n^*}{\gamma(\eta_n^*)}(\mathbf{0}, \mathbf{M}_1(-\hat{\eta}_n^*), \mathbf{0}), \\
\operatorname{Res}_{z=\eta_n^*} \mathbf{M}(x, t, z) &= -\frac{C_n^*}{\gamma(\eta_n^*)}(\mathbf{0}, \mathbf{M}_3(\eta_n^*), \mathbf{0}), \\
\operatorname{Res}_{z=-\hat{\eta}_n} \mathbf{M}(x, t, z) &= \frac{iq_0}{\eta_n} C_n(\mathbf{M}_2(-\hat{\eta}_n), \mathbf{0}, \mathbf{0});
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
\operatorname{Res}_{z=z_n} \mathbf{M}(x, t, z) &= D_n(\mathbf{0}, \mathbf{0}, \mathbf{M}_2(z_n)), \\
\operatorname{Res}_{z=-\hat{z}_n^*} \mathbf{M}(x, t, z) &= \frac{iq_0^3}{(z_n^*)^3} \frac{D_n^*}{\gamma(z_n^*)}(\mathbf{0}, \mathbf{M}_1(-\hat{z}_n^*), \mathbf{0}), \\
\operatorname{Res}_{z=z_n^*} \mathbf{M}(x, t, z) &= -\frac{D_n^*}{\gamma(z_n^*)}(\mathbf{0}, \mathbf{M}_3(z_n^*), \mathbf{0}), \\
\operatorname{Res}_{z=-\hat{z}_n} \mathbf{M}(x, t, z) &= \frac{iq_0}{z_n} D_n(\mathbf{M}_2(-\hat{z}_n), \mathbf{0}, \mathbf{0}), \\
\operatorname{Res}_{z=-\hat{z}_n^*} \mathbf{M}(x, t, z) &= -D_n^*(\mathbf{0}, \mathbf{0}, \mathbf{M}_2(-z_n^*)), \\
\operatorname{Res}_{z=\hat{z}_n} \mathbf{M}(x, t, z) &= \frac{iq_0^3}{z_n^3} \frac{D_n}{\gamma(z_n)}(\mathbf{0}, \mathbf{M}_1(\hat{z}_n), \mathbf{0}), \\
\operatorname{Res}_{z=-z_n} \mathbf{M}(x, t, z) &= \frac{D_n}{\gamma(z_n)}(\mathbf{0}, \mathbf{M}_3(-z_n), \mathbf{0}), \\
\operatorname{Res}_{z=\hat{z}_n^*} \mathbf{M}(x, t, z) &= \frac{iq_0}{z_n^*} D_n^*(\mathbf{M}_2(\hat{z}_n^*), \mathbf{0}, \mathbf{0});
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
\operatorname{Res}_{z=\nu_n} \mathbf{M}(x, t, z) &= E_n(\mathbf{M}_2(\nu_n), \mathbf{0}, \mathbf{0}), \\
\operatorname{Res}_{z=-\hat{\nu}_n^*} \mathbf{M}(x, t, z) &= \frac{iq_0^3}{(\nu_n^*)^3} \frac{E_n^*}{\gamma(\nu_n^*)}(\mathbf{0}, \mathbf{M}_3(-\hat{\nu}_n^*), \mathbf{0}), \\
\operatorname{Res}_{z=\nu_n^*} \mathbf{M}(x, t, z) &= -\frac{E_n^*}{\gamma(\nu_n^*)}(\mathbf{0}, \mathbf{M}_1(\nu_n^*), \mathbf{0}), \\
\operatorname{Res}_{z=-\hat{\nu}_n} \mathbf{M}(x, t, z) &= \frac{iq_0}{\nu_n} E_n(\mathbf{0}, \mathbf{0}, \mathbf{M}_2(-\hat{\nu}_n));
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
\operatorname{Res}_{z=\zeta_n} \mathbf{M}(x, t, z) &= F_n(\mathbf{M}_2(\zeta_n), \mathbf{0}, \mathbf{0}), \\
\operatorname{Res}_{z=-\hat{\zeta}_n^*} \mathbf{M}(x, t, z) &= \frac{iq_0^3}{(\zeta_n^*)^3} \frac{F_n^*}{\gamma(\zeta_n^*)}(\mathbf{0}, \mathbf{M}_3(-\hat{\zeta}_n^*), \mathbf{0}), \\
\operatorname{Res}_{z=\zeta_n^*} \mathbf{M}(x, t, z) &= -\frac{F_n^*}{\gamma(\zeta_n^*)}(\mathbf{0}, \mathbf{M}_1(\zeta_n^*), \mathbf{0}), \\
\operatorname{Res}_{z=-\hat{\zeta}_n} \mathbf{M}(x, t, z) &= \frac{iq_0}{\zeta_n} F_n(\mathbf{0}, \mathbf{0}, \mathbf{M}_2(-\hat{\zeta}_n)), \\
\operatorname{Res}_{z=-\hat{\zeta}_n^*} \mathbf{M}(x, t, z) &= -F_n^*(\mathbf{M}_2(-\zeta_n^*), \mathbf{0}, \mathbf{0}), \\
\operatorname{Res}_{z=\hat{\zeta}_n} \mathbf{M}(x, t, z) &= \frac{iq_0^3}{\zeta_n^3} \frac{F_n}{\gamma(\zeta_n)}(\mathbf{0}, \mathbf{M}_3(\hat{\zeta}_n), \mathbf{0}), \\
\operatorname{Res}_{z=-\zeta_n} \mathbf{M}(x, t, z) &= \frac{F_n}{\gamma(\zeta_n)}(\mathbf{0}, \mathbf{M}_1(-\zeta_n), \mathbf{0}), \\
\operatorname{Res}_{z=\hat{\zeta}_n^*} \mathbf{M}(x, t, z) &= \frac{iq_0}{\zeta_n^*} F_n^*(\mathbf{0}, \mathbf{0}, \mathbf{M}_2(\hat{\zeta}_n^*))
\end{aligned} \tag{3.14}$$

with the normalization constants

$$\begin{aligned} A_n &= \frac{a_n b_{22}(\xi_n)}{a'_{11}(\xi_n)} e^{-2i\theta_1(\xi_n)}, & B_n &= \frac{b_n b_{22}(\omega_n)}{a'_{11}(\omega_n)} e^{-2i\theta_1(\omega_n)}, & C_n &= \frac{c_n}{b'_{22}(\eta_n)} e^{i(\theta_1+\theta_2)(\eta_n)}, \\ D_n &= \frac{d_n}{b'_{22}(z_n)} e^{i(\theta_1+\theta_2)(z_n)}, & E_n &= \frac{e_n}{a'_{11}(\nu_n)} e^{i(\theta_2-\theta_1)(\nu_n)}, & F_n &= \frac{f_n}{a'_{11}(\zeta_n)} e^{i(\theta_2-\theta_1)(\zeta_n)}, \end{aligned}$$

where $a_n, b_n, c_n, d_n, e_n,$ and f_n are constants satisfying

$$a_n = a_n^*, \quad c_n = c_n^*, \quad e_n = e_n^*, \quad (3.15)$$

and where $n = 1, \dots, N_1$ for equations involving ξ_n , $n = 1, \dots, N_2$ for equations involving ω_n , $n = 1, \dots, N_3$ for equations involving η_n , $n = 1, \dots, N_4$ for equations involving z_n , $n = 1, \dots, N_5$ for equations involving ν_n , and $n = 1, \dots, N_6$ for equations involving ζ_n .

When there are no zeros of $a_{11}(z)$ and $b_{22}(z)$, that is, the RHP is regular, the uniqueness of its solution is equivalent to the vanishing Lemma [34], [35].

Lemma 3.5. *Suppose that no discrete spectrum is present, or the RHP has no poles. Then the RHP defined in (3.1)–(3.7) has a unique solution.*

However, the RHP defined in (3.1)–(3.15) has many singularities. According to [36], the RHP with poles can be mapped into a regular one. This RHP can be regularized by subtracting the leading terms of the asymptotic behavior and the pole contributions from the jump conditions. Then by the Plemelj's formula [37], we obtain the following theorem.

Theorem 3.1. *The general solution of the RHP is given by*

$$\begin{aligned} \mathbf{M}(x, t, z) &= \mathbf{E}_+(z) + \sum_{n=1}^{N_1} \left(\frac{\text{Res}_{z=\xi_n} \mathbf{M}}{z - \xi_n} + \frac{\text{Res}_{z=\xi_n^*} \mathbf{M}}{z - \xi_n^*} + \frac{\text{Res}_{z=-\hat{\xi}_n} \mathbf{M}}{z + \hat{\xi}_n} + \frac{\text{Res}_{z=-\hat{\xi}_n^*} \mathbf{M}}{z + \hat{\xi}_n^*} \right) + \\ &+ \sum_{n=1}^{N_2} \left(\frac{\text{Res}_{z=\omega_n} \mathbf{M}}{z - \omega_n} + \frac{\text{Res}_{z=-\omega_n^*} \mathbf{M}}{z + \omega_n^*} + \frac{\text{Res}_{z=\omega_n^*} \mathbf{M}}{z - \omega_n^*} + \frac{\text{Res}_{z=-\omega_n} \mathbf{M}}{z + \omega_n} + \right. \\ &\quad \left. + \frac{\text{Res}_{z=-\hat{\omega}_n^*} \mathbf{M}}{z + \hat{\omega}_n^*} + \frac{\text{Res}_{z=\hat{\omega}_n} \mathbf{M}}{z - \hat{\omega}_n} + \frac{\text{Res}_{z=-\hat{\omega}_n} \mathbf{M}}{z + \hat{\omega}_n} + \frac{\text{Res}_{z=\hat{\omega}_n^*} \mathbf{M}}{z - \hat{\omega}_n^*} \right) + \\ &+ \sum_{n=1}^{N_3} \left(\frac{\text{Res}_{z=\eta_n} \mathbf{M}}{z - \eta_n} + \frac{\text{Res}_{z=\eta_n^*} \mathbf{M}}{z - \eta_n^*} + \frac{\text{Res}_{z=-\hat{\eta}_n} \mathbf{M}}{z + \hat{\eta}_n} + \frac{\text{Res}_{z=-\hat{\eta}_n^*} \mathbf{M}}{z + \hat{\eta}_n^*} \right) + \\ &+ \sum_{n=1}^{N_4} \left(\frac{\text{Res}_{z=z_n} \mathbf{M}}{z - z_n} + \frac{\text{Res}_{z=-z_n^*} \mathbf{M}}{z + z_n^*} + \frac{\text{Res}_{z=z_n^*} \mathbf{M}}{z - z_n^*} + \frac{\text{Res}_{z=-z_n} \mathbf{M}}{z + z_n} + \right. \\ &\quad \left. + \frac{\text{Res}_{z=-\hat{z}_n^*} \mathbf{M}}{z + \hat{z}_n^*} + \frac{\text{Res}_{z=\hat{z}_n} \mathbf{M}}{z - \hat{z}_n} + \frac{\text{Res}_{z=-\hat{z}_n} \mathbf{M}}{z + \hat{z}_n} + \frac{\text{Res}_{z=\hat{z}_n^*} \mathbf{M}}{z - \hat{z}_n^*} \right) + \\ &+ \sum_{n=1}^{N_5} \left(\frac{\text{Res}_{z=\nu_n} \mathbf{M}}{z - \nu_n} + \frac{\text{Res}_{z=\nu_n^*} \mathbf{M}}{z - \nu_n^*} + \frac{\text{Res}_{z=-\hat{\nu}_n} \mathbf{M}}{z + \hat{\nu}_n} + \frac{\text{Res}_{z=-\hat{\nu}_n^*} \mathbf{M}}{z + \hat{\nu}_n^*} \right) + \\ &+ \sum_{n=1}^{N_6} \left(\frac{\text{Res}_{z=\zeta_n} \mathbf{M}}{z - \zeta_n} + \frac{\text{Res}_{z=-\zeta_n^*} \mathbf{M}}{z + \zeta_n^*} + \frac{\text{Res}_{z=\zeta_n^*} \mathbf{M}}{z - \zeta_n^*} + \frac{\text{Res}_{z=-\zeta_n} \mathbf{M}}{z + \zeta_n} + \right. \\ &\quad \left. + \frac{\text{Res}_{z=-\hat{\zeta}_n} \mathbf{M}}{z + \hat{\zeta}_n} + \frac{\text{Res}_{z=\hat{\zeta}_n} \mathbf{M}}{z - \hat{\zeta}_n} + \frac{\text{Res}_{z=-\hat{\zeta}_n^*} \mathbf{M}}{z + \hat{\zeta}_n^*} + \frac{\text{Res}_{z=\hat{\zeta}_n^*} \mathbf{M}}{z - \hat{\zeta}_n^*} \right) + \\ &+ \frac{1}{2\pi i} \int_{\Sigma} \frac{\mathbf{M}_-(x, t, \zeta)(\mathbf{G}(x, t, \zeta) - \mathbf{I})}{\zeta - z} d\zeta, \quad z \in \mathbb{C} \setminus \Sigma. \end{aligned} \quad (3.16)$$

Using the symmetries of the eigenfunctions, we can simplify the columns of $\mathbf{M}(x, t, z)$ as

$$\begin{aligned}
\mathbf{M}_1(z) &= \mathbf{E}_{+,1}(z) + \sum_{n=1}^{\mathcal{N}_1} \frac{K_n \mathbf{M}_3(k_n)}{z - k_n} + \sum_{n=1}^{\mathcal{N}_2} \frac{L_n \mathbf{M}_2(l_n)}{z - l_n} + \frac{1}{2\pi i} \int_{\Sigma} \frac{\mathbf{M}_-(\mathbf{G} - \mathbf{I})_1(\zeta)}{\zeta - z} d\zeta, \\
\mathbf{M}_2(z) &= \mathbf{E}_{+,2}(z) - \sum_{n=1}^{\mathcal{N}_2} \frac{L_n^* \mathbf{M}_1(l_n^*)}{z + \hat{l}_n^* - \hat{z} - l_n^*} + \frac{1}{2\pi i} \int_{\Sigma} \frac{\mathbf{M}_-(\mathbf{G} - \mathbf{I})_2(\zeta)}{\zeta - z} d\zeta, \\
\mathbf{M}_3(z) &= \mathbf{E}_{+,3}(z) - \sum_{n=1}^{\mathcal{N}_1} \frac{K_n^* \mathbf{M}_1(k_n^*)}{z - k_n^*} + \sum_{n=1}^{\mathcal{N}_2} \frac{i q_0 L_n \mathbf{M}_2(l_n)}{l_n(z + \hat{l}_n)} + \\
&\quad + \frac{1}{2\pi i} \int_{\Sigma} \frac{\mathbf{M}_-(\mathbf{G} - \mathbf{I})_3(\zeta)}{\zeta - z} d\zeta,
\end{aligned} \tag{3.17}$$

where k_n , K_n and l_n , L_n are given in Appendix A.1 with $\mathcal{N}_1 = 2N_1 + 4N_2$ and $\mathcal{N}_2 = N_3 + 2N_4 + N_5 + 2N_6$.

From formula (2.50), we deduce the following theorem.

Theorem 3.2 (reconstruction formula). *The solution $\mathbf{q}(x, t) = (q_1(x, t), q_2(x, t))^T$ of the fcmKdV equation (1.1) with NZBCs (1.2) is reconstructed as*

$$q_j(x, t) = q_{j+} + i \sum_{n=1}^{\mathcal{N}_1} K_n \mathbf{M}_{(j+1)3}(k_n) + i \sum_{n=1}^{\mathcal{N}_2} L_n \mathbf{M}_{(j+1)2}(l_n) - \frac{1}{2\pi} \int_{\Sigma} [\mathbf{M}_-(\mathbf{G} - \mathbf{I})]_{(j+1)1}(z) dz. \tag{3.18}$$

3.2. Trace formula and the asymptotic phase difference. We next reconstruct the analytic scattering coefficients in terms of scattering data. Thus can be achieved by constructing another RHP, as shown in Appendix A.2.

Lemma 3.6 (trace formula). *The analytic scattering coefficients $a_{11}(z)$ and $b_{22}(z)$ can be expressed as*

$$\begin{aligned}
b_{22}(z) &= \delta \prod_{n=1}^{N_3} \frac{(z - \eta_n)(z + \hat{\eta}_n)}{(z - \eta_n^*)(z + \hat{\eta}_n^*)} \prod_{n=1}^{N_4} \frac{(z - z_n)(z + z_n^*)(z + \hat{z}_n)(z - \hat{z}_n^*)}{(z - z_n^*)(z + z_n)(z + \hat{z}_n^*)(z - \hat{z}_n)} \times \\
&\quad \times \prod_{n=1}^{N_5} \frac{(z - \nu_n)(z + \hat{\nu}_n)}{(z - \nu_n^*)(z + \hat{\nu}_n^*)} \prod_{n=1}^{N_6} \frac{(z - \zeta_n)(z + \zeta_n^*)(z + \hat{\zeta}_n)(z - \hat{\zeta}_n^*)}{(z - \zeta_n^*)(z + \zeta_n)(z + \hat{\zeta}_n^*)(z - \hat{\zeta}_n)} \times \\
&\quad \times \exp \left[-\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{J_0(\zeta)}{\zeta - z} d\zeta \right], \quad z \in \mathbb{C}_+,
\end{aligned} \tag{3.19a}$$

$$\begin{aligned}
a_{11}(z) &= \prod_{n=1}^{N_1} \frac{(z - \xi_n)(z + \hat{\xi}_n^*)}{(z - \xi_n^*)(z + \hat{\xi}_n)} \prod_{n=1}^{N_2} \frac{(z - \omega_n)(z + \omega_n^*)(z + \hat{\omega}_n^*)(z - \hat{\omega}_n)}{(z - \omega_n^*)(z + \omega_n)(z + \hat{\omega}_n)(z - \hat{\omega}_n^*)} \times \\
&\quad \times \prod_{n=1}^{N_3} \frac{z + \hat{\eta}_n}{z + \hat{\eta}_n^*} \prod_{n=1}^{N_4} \frac{(z + \hat{z}_n)(z - \hat{z}_n^*)}{(z + \hat{z}_n^*)(z - \hat{z}_n)} \prod_{n=1}^{N_5} \frac{z - \nu_n}{z - \nu_n^*} \prod_{n=1}^{N_6} \frac{(z - \zeta_n)(z + \zeta_n^*)}{(z - \zeta_n^*)(z + \zeta_n)} \times \\
&\quad \times \exp \left[\frac{1}{2\pi i} \int_{\Sigma} \frac{J(\zeta)}{\zeta - z} d\zeta \right], \quad z \in D_1,
\end{aligned} \tag{3.19b}$$

where $\delta = \pm 1$ and the matrices $J_0(z)$ and $J(z)$ are respectively defined in (A.5b) and (A.9a)–(A.9d).

Next, comparing the asymptotic behavior of the trace formula for $a_{11}(z)$ as $z \rightarrow 0$ with the result in (2.54a), we obtain that

$$\delta = (-1)^{N_3+N_5} \exp\left[\frac{1}{2\pi i} \int_{\Sigma} \frac{J(z)}{z} dz\right]. \quad (3.20)$$

Furthermore, we decompose the above integral term into

$$\begin{aligned} H_1 &= \int_{-\infty}^{-q_0} \frac{J_1(z)}{z} dz, & I_1 &= \int_{-\pi}^{-\pi/2} iJ_2(q_0 e^{i\alpha}) d\alpha, \\ H_3 &= \int_{q_0}^0 \frac{J_3(z)}{z} dz, & I_3 &= \int_{\pi}^{\pi/2} iJ_4(q_0 e^{i\alpha}) d\alpha, \\ H_2 &= \int_{q_0}^{\infty} \frac{J_1(z)}{z} dz, & I_2 &= \int_{-\pi/2}^0 iJ_2(q_0 e^{i\alpha}) d\alpha, \\ H_4 &= \int_0^{-q_0} \frac{J_3(z)}{z} dz, & I_4 &= \int_{\pi/2}^0 iJ_4(q_0 e^{i\alpha}) d\alpha. \end{aligned} \quad (3.21)$$

Applying the symmetries of the reflection coefficients to the jump matrices $J_j(z)$ shows that $H_1 = -H_2$, $H_3 = -H_4$, $I_1 = -I_4$, and $I_2 = -I_3$. It hence follows that

$$\delta = (-1)^{N_3+N_5}. \quad (3.22)$$

Corollary 3.1 (asymptotic phase difference). *The potential $\mathbf{q}(x, t)$ has the same boundary condition $\mathbf{q}_+ = \mathbf{q}_-$ at infinity when $N_3 + N_5$ is even. The potential $\mathbf{q}(x, t)$ has the opposite boundary condition $\mathbf{q}_+ = -\mathbf{q}_-$ at infinity when $N_3 + N_5$ is odd.*

4. Reflectionless potential and soliton solutions

We now explicitly construct the potential $\mathbf{q}(x, t)$ in the case of a reflectionless potential, that is, $\rho_1(z) = \rho_2(z) = \rho_3(z) = 0$. In this case, there is no jump across the contour Σ in the RHP, allowing the solution $\mathbf{q}(x, t)$ to be found from a closed algebraic system (3.17).

Theorem 4.1. *In the case of a reflectionless potential, the solution $\mathbf{q}(x, t)$ of the fcmKdV equation (1.1) with NZBCs (1.2) can be expressed as*

$$\mathbf{q}(x, t) = \frac{1}{\det \mathbf{G}} \begin{pmatrix} \det \mathbf{G}_1^{\text{aug}} \\ \det \mathbf{G}_2^{\text{aug}} \end{pmatrix}, \quad \mathbf{G}_k^{\text{aug}} = \begin{pmatrix} q_{k+} & \mathbf{Y}^{\text{T}} \\ \mathbf{B}_k & \mathbf{G} \end{pmatrix}, \quad k = 1, 2, \quad (4.1)$$

with the $(\mathcal{N}_1 + \mathcal{N}_2) \times (\mathcal{N}_1 + \mathcal{N}_2)$ matrix $\mathbf{G} = \mathbf{I} + \mathbf{F}$, $\mathbf{B}_k = (B_{k1}, \dots, B_{k(\mathcal{N}_1 + \mathcal{N}_2)})^{\text{T}}$, $\mathbf{Y} = (Y_1, \dots, Y_{(\mathcal{N}_1 + \mathcal{N}_2)})^{\text{T}}$ and

$$Y_n = \begin{cases} iK_n, & n = 1, \dots, \mathcal{N}_1, \\ iL_{n-\mathcal{N}_1}, & n = \mathcal{N}_1 + 1, \dots, \mathcal{N}_1 + \mathcal{N}_2, \end{cases} \quad (4.2)$$

$$B_{kn} = \begin{cases} \frac{q_{k+}}{q_0} + \sum_{m=1}^{\mathcal{N}_1} \frac{iq_{k+} K_m^*}{k_m^* (k_n - k_m^*)}, & n = 1, \dots, \mathcal{N}_1, \\ \frac{(-1)^k q_{k+}}{q_0} + \sum_{m=1}^{\mathcal{N}_2} \frac{iq_{k+} L_m^*}{l_m^* (l_{n-\mathcal{N}_1} + \hat{l}_m^* - \hat{l}_{n-\mathcal{N}_1} - l_m^*)}, & n = \mathcal{N}_1 + 1, \dots, \mathcal{N}_1 + \mathcal{N}_2, \end{cases} \quad (4.3)$$

$$F_{nj} = \begin{cases} \sum_{m=1}^{\mathcal{N}_1} \frac{K_m^* K_j}{(k_n - k_m^*)(k_m^* - k_j)}, & n, j = 1, \dots, \mathcal{N}_1, \\ -\frac{i q_0 L_{j-\mathcal{N}_1}}{l_{j-\mathcal{N}_1} (k_n + \hat{l}_{j-\mathcal{N}_1})} + \sum_{m=1}^{\mathcal{N}_1} \frac{K_m^* L_{j-\mathcal{N}_1}}{(k_n - k_m^*)(k_m^* - l_{j-\mathcal{N}_1})}, & n = 1, \dots, \mathcal{N}_1, \quad j = \mathcal{N}_1 + 1, \dots, \mathcal{N}_1 + \mathcal{N}_2, \\ \sum_{m=1}^{\mathcal{N}_2} \frac{L_m^* K_j}{(l_{n-\mathcal{N}_1} + \hat{l}_m^* - \hat{l}_{n-\mathcal{N}_1} - l_m^*)(l_m^* - k_j)}, & n = \mathcal{N}_1 + 1, \dots, \mathcal{N}_1 + \mathcal{N}_2, \quad j = 1, \dots, \mathcal{N}_1, \\ \sum_{m=1}^{\mathcal{N}_2} \frac{L_m^* L_{j-\mathcal{N}_1}}{(l_{n-\mathcal{N}_1} + \hat{l}_m^* - \hat{l}_{n-\mathcal{N}_1} - l_m^*)(l_m^* - l_{j-\mathcal{N}_1})}, & n, j = \mathcal{N}_1 + 1, \dots, \mathcal{N}_1 + \mathcal{N}_2, \end{cases} \quad (4.4)$$

where $\bar{k} = 3 - k$.

Next, we focus on the soliton solutions corresponding to the six different types of discrete eigenvalues and explore their properties.

4.1. Type-I solitons. In the case $N_1 = 1$ and $N_2 = N_3 = N_4 = N_5 = N_6 = 0$, in general, we take the discrete eigenvalue and the proportionality constant in the form

$$\xi_1 = iZ, \quad a_1 = e^{\xi + i(m-1)\pi}, \quad Z > q_0, \quad \xi \in \mathbb{R}, \quad m = 0, 1.$$

Substituting these quantities in the general formula (4.1) yields the explicit one-soliton solution

$$\mathbf{q}(x, t) = \frac{\cosh \chi + (-1)^m \operatorname{sech}(\ln(Z/q_0)) \cosh(2 \ln(Z/q_0))}{\cosh \chi + (-1)^m \operatorname{sech}(\ln(Z/q_0))} \mathbf{q}_+, \quad (4.5)$$

where

$$\chi = -\frac{q_0^6}{Z^3} t + \left(Z - \frac{q_0^2}{Z} \right) (3q_0^2 t - x) + Z^3 t + \xi.$$

The solutions are completely determined by m and the vector \mathbf{q}_+ . Some examples of such one-soliton solutions describing the dark–dark and bright–bright solitons are shown in Figs. 1 and 2. The soliton classification in this case is given in detail in Sec. 4.7.

4.2. Type-II solitons. In the case $N_2 = 1$ and $N_1 = N_3 = N_4 = N_5 = N_6 = 0$, in general, we take the soliton parameters in the form

$$\omega_1 = Z e^{i\theta}, \quad b_1 = e^{\alpha + i\beta}, \quad Z > q_0, \quad \theta \in \left(0, \frac{\pi}{2} \right) \cup \left(\frac{\pi}{2}, \pi \right), \quad \alpha, \beta \in \mathbb{R}.$$

From (4.1), we can obtain the breather–breather solutions of the fcmKdV equation. An example of such a solution is plotted in Fig. 3.

4.3. Type-III solitons. In the case $N_3 = 1$ and $N_1 = N_2 = N_4 = N_5 = N_6 = 0$, in general, we take the discrete eigenvalue and proportionality constant in the form

$$\eta_1 = iZ, \quad c_1 = e^{\eta + i(m-1)\pi}, \quad Z > q_0, \quad \eta \in \mathbb{R}, \quad m = 0, 1.$$

Substituting these parameters in the general formula (4.1) generates the one-soliton solution

$$\mathbf{q}(x, t) = -\tanh \left(\chi + \ln \frac{Z \sqrt{Z^2 + q_0^2}}{Z^2 - q_0^2} \right) \mathbf{q}_+ - (-1)^m \frac{\sqrt{Z^2 + q_0^2}}{Z} \operatorname{sech} \left(\chi + \ln \frac{Z \sqrt{Z^2 + q_0^2}}{Z^2 - q_0^2} \right) \mathbf{q}_+^\perp, \quad (4.6)$$

where

$$\chi = \frac{q_0^6}{Z^3} t + \frac{q_0^2}{Z} (3q_0^2 t - x) + \eta.$$

The solutions are also completely determined by m and \mathbf{q}_+ . Some examples of such solitons describing the kink–bright–dark and kink–dark–bright solitons are given in Figs. 4 and 5. In this paper, the pair of a kink–bright soliton for the component $q_1(x, t)$ and a kink–dark soliton for the component $q_2(x, t)$ is simply referred to as the kink–bright–dark soliton, and so on. The soliton classification is given in detail in Sec. 4.7.

4.4. Type-IV solitons. In the case $N_4 = 1$ and $N_1 = N_2 = N_3 = N_5 = N_6 = 0$, in general, we take the soliton parameters in the form

$$z_1 = Ze^{i\theta}, \quad d_1 = e^{\alpha+i\beta}, \quad Z > q_0, \quad \theta \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right), \quad \alpha, \beta \in \mathbb{R}.$$

From (4.1), we can obtain the W (type)–S (type) or the breather–breather solutions for the fcmKdV equation. Examples of such solutions are shown in Figs. 6 and 7.

4.5. Type-V solitons. In the case $N_5 = 1$ and $N_1 = N_2 = N_3 = N_4 = N_6 = 0$, in general, we take the discrete eigenvalue and proportionality constant in the form

$$\nu_1 = iZ, \quad e_1 = e^{\nu+i(m-1)\pi}, \quad Z > q_0, \quad \nu \in \mathbb{R}, \quad m = 0, 1.$$

Substituting these quantities in (4.1) yields the explicit one-soliton solution

$$\mathbf{q}(x, t) = -\tanh\left(\chi + \ln \frac{Z}{\sqrt{Z^2 + q_0^2}}\right) \mathbf{q}_+ + (-1)^m \frac{\sqrt{Z^2 + q_0^2}}{q_0} \operatorname{sech}\left(\chi + \ln \frac{Z}{\sqrt{Z^2 + q_0^2}}\right) \mathbf{q}_+^\perp, \quad (4.7)$$

where

$$\chi = Z^3 t + Z(3q_0^2 t - x) + \nu.$$

The type-V solitons are similar to those of type-III, and are also determined by m and \mathbf{q}_+ . Some examples of such soliton solutions describing the kink–dark–dark and kink–bright–bright solitons are plotted in Figs. 8 and 9. The soliton classification is given in Sec. 4.7.

4.6. Type-VI solitons. In the case $N_6 = 1$ and $N_1 = N_2 = N_3 = N_4 = N_5 = 0$, in general, we take the soliton parameters in the form

$$\zeta_1 = Ze^{i\theta}, \quad f_1 = e^{\alpha+i\beta}, \quad Z > q_0, \quad \theta \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right), \quad \alpha, \beta \in \mathbb{R}.$$

From (4.1), we can obtain the breather–breather solution for the fcmKdV equation. An example of such a solution is shown in Fig. 10.

4.7. Discussions. First, the results for the soliton solutions admit the asymptotic phase difference, that is, only when $N_3 + N_5$ is odd, there are kink solitons. The soliton classification of type-I, III, and V discrete eigenvalues in terms of m and \mathbf{q}_+ is summarized in respective Tables 1, 2, and 3. In addition, the similarities between type-III and V solitons can be traced to their expressions in (4.6) and (4.7): their bright-soliton part always follows an orthogonal polarization to that of the dark-soliton part. And their solutions are any combinations of kink–dark or kink–bright solitons, where kink–dark solitons have rich complex soliton dynamics in Bose–Einstein condensates [38], [39] and nano-optical fibers [40]. Thus, we expect that our results could be beneficial to the development of the one-dimensional two-component Bose–Einstein condensate systems. Finally, type-II, IV, and VI solutions all have breather–breather solutions.

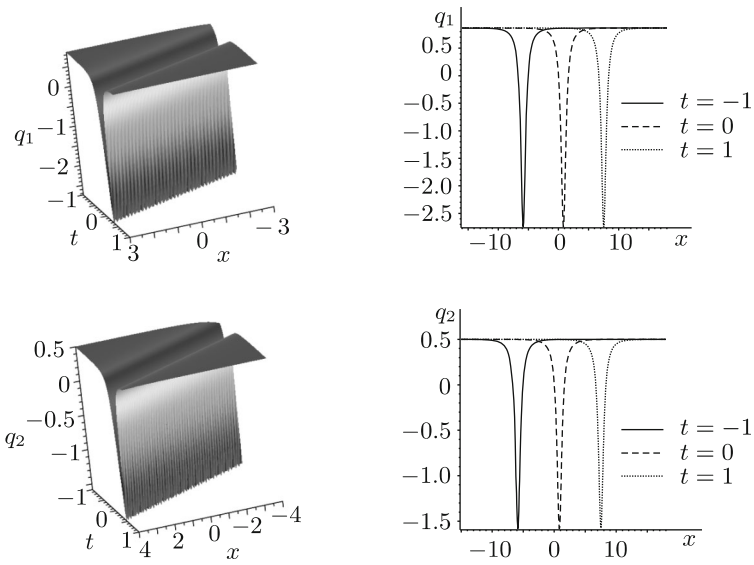


Fig. 1. One dark–dark soliton of the fcmKdV equation with NZBCs obtained by taking $m = 1$, $Z = 3/2$, $\xi = \ln 2$, $\mathbf{q}_+ = (\sqrt{3}/2, 1/2)^T$, and $q_0 = 1$.

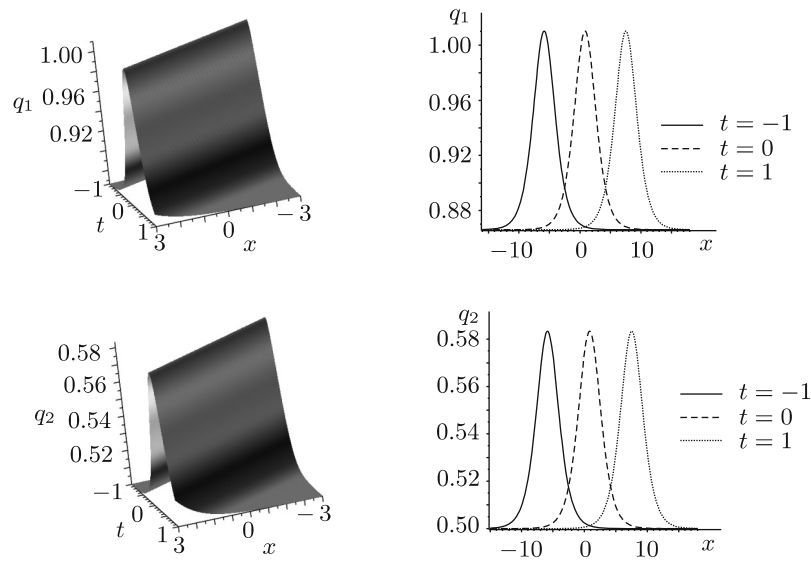


Fig. 2. One bright–bright soliton of the fcmKdV equation with NZBCs obtained by taking $m = 0$, $Z = 3/2$, $\xi = \ln 2$, $\mathbf{q}_+ = (\sqrt{3}/2, 1/2)^T$, and $q_0 = 1$.

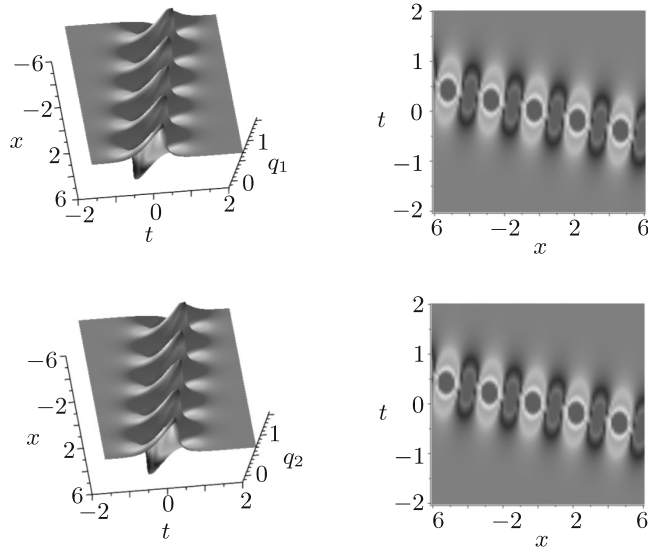


Fig. 3. One breather–breather solution of the fcmKdV equation with NZBCs obtained by taking $\omega_1 = 2 + i/2$, $\alpha = 0$, $\beta = \pi/4$, $\mathbf{q}_+ = (\sqrt{2}/2, \sqrt{2}/2)^T$, and $q_0 = 1$.

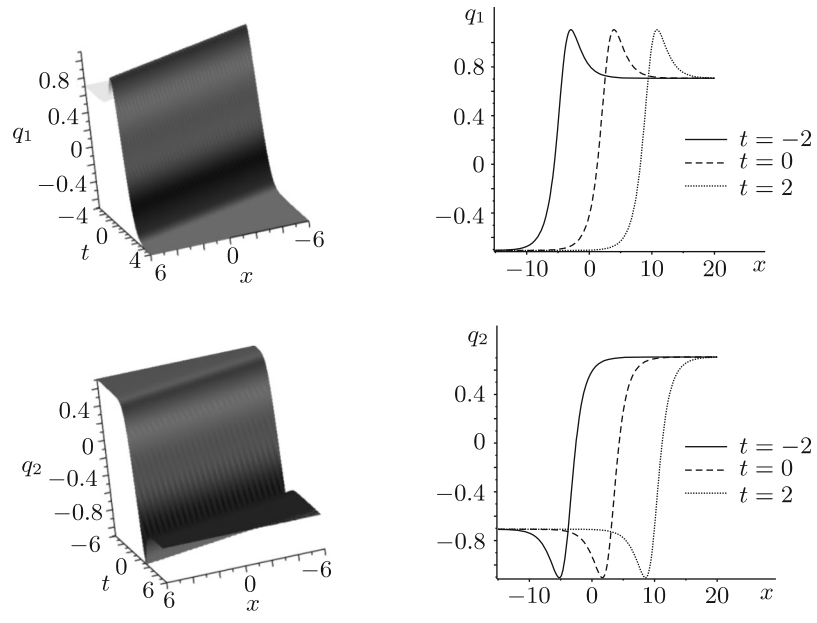


Fig. 4. One kink–bright–dark soliton of the fcmKdV equation with NZBCs obtained by taking $m = 1$, $Z = 3/2$, $\eta = \ln 3$, $\mathbf{q}_+ = (\sqrt{2}/2, \sqrt{2}/2)^T$, and $q_0 = 1$.

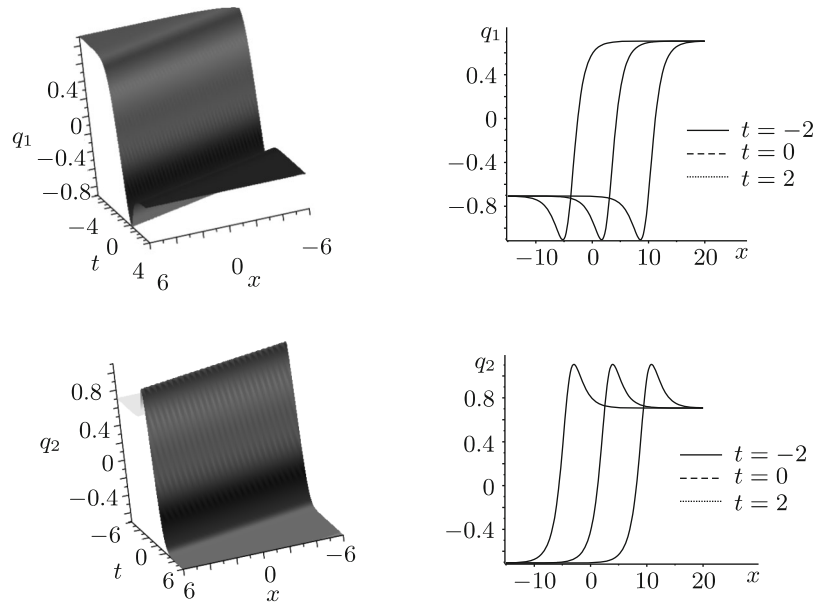


Fig. 5. One kink–dark–bright soliton of the fcmKdV equation with NZBCs obtained by taking $m = 0$, $Z = 3/2$, $\eta = \ln 3$, $\mathbf{q}_+ = (\sqrt{2}/2, \sqrt{2}/2)^T$, and $q_0 = 1$.

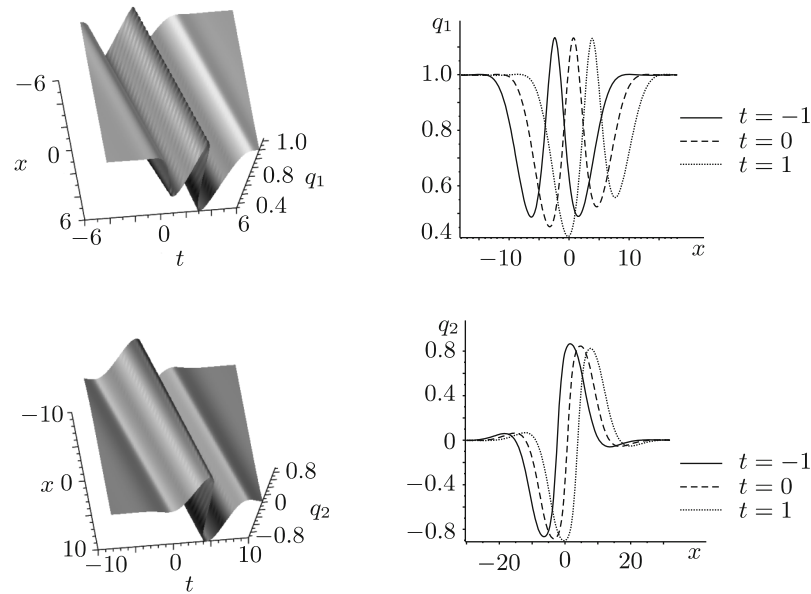


Fig. 6. One W (type)–S (type) soliton of the fcmKdV equation with NZBCs obtained by taking $z_1 = 2 + 2i$, $\alpha = \beta = 0$, $\mathbf{q}_+ = (1, 0)^T$, and $q_0 = 1$.

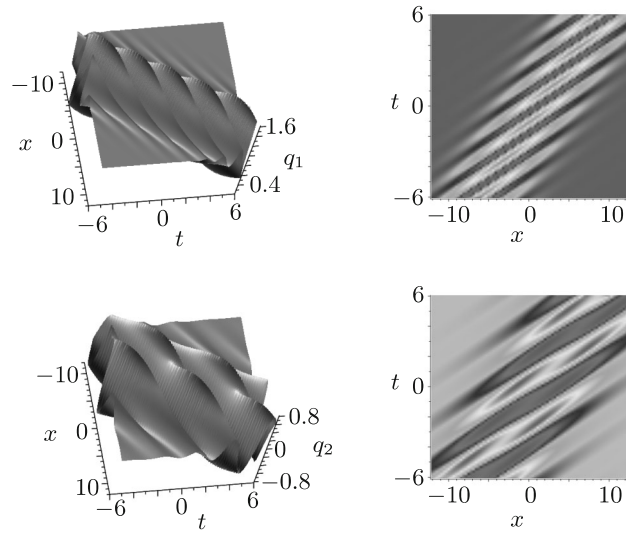


Fig. 7. One breather–breather solution of the fcmKdV equation with NZBCs obtained by taking $z_1 = 1 + i/2$, $\alpha = \beta = 0$, $\mathbf{q}_+ = (1, 0)^T$, and $q_0 = 1$.

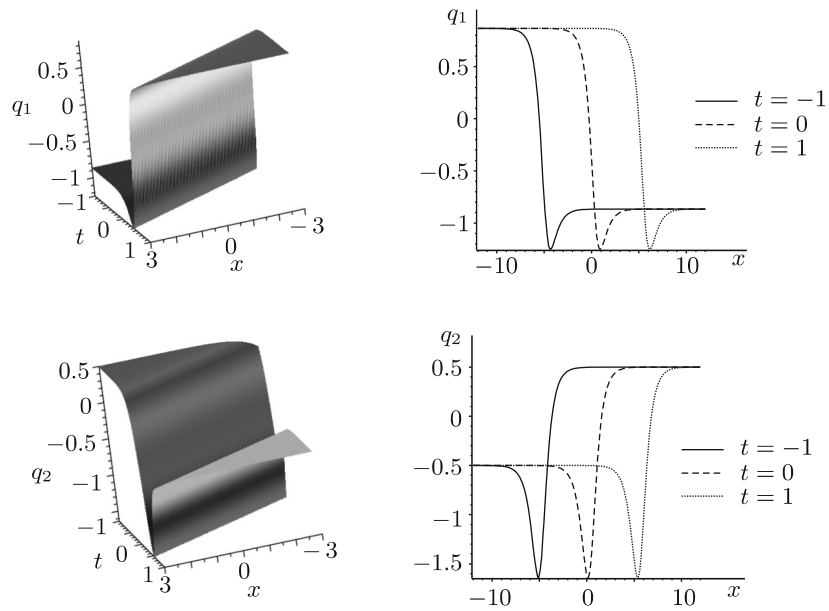


Fig. 8. One kink–dark–dark soliton of the fcmKdV equation with NZBCs obtained by taking $m = 1$, $Z = 3/2$, $\nu = \ln 2$, $\mathbf{q}_+ = (-\sqrt{3}/2, 1/2)^T$, and $q_0 = 1$.

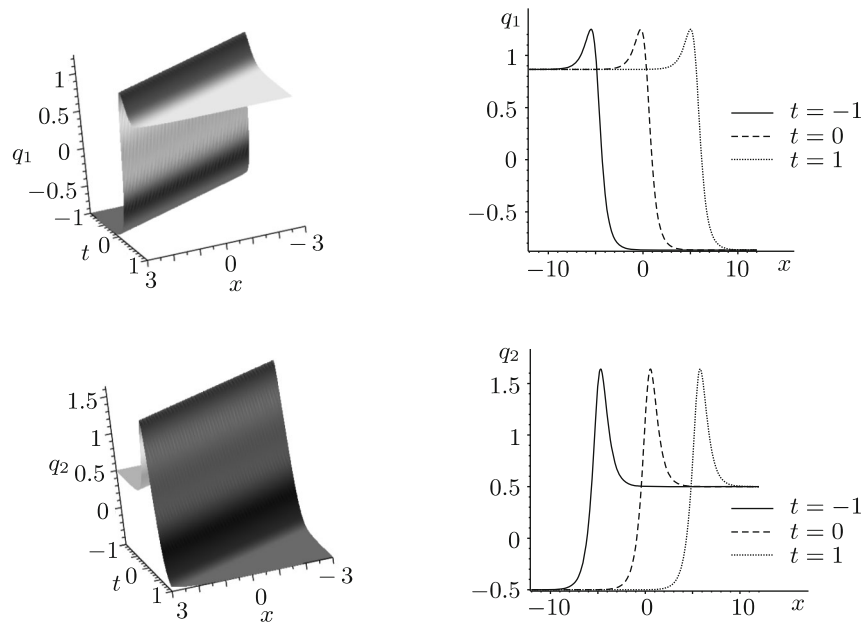


Fig. 9. One kink–bright–bright soliton of the fcmKdV equation with NZBCs obtained by taking $m = 0$, $Z = 3/2$, $\nu = \ln 2$, $\mathbf{q}_+ = (-\sqrt{3}/2, 1/2)^T$, and $q_0 = 1$.

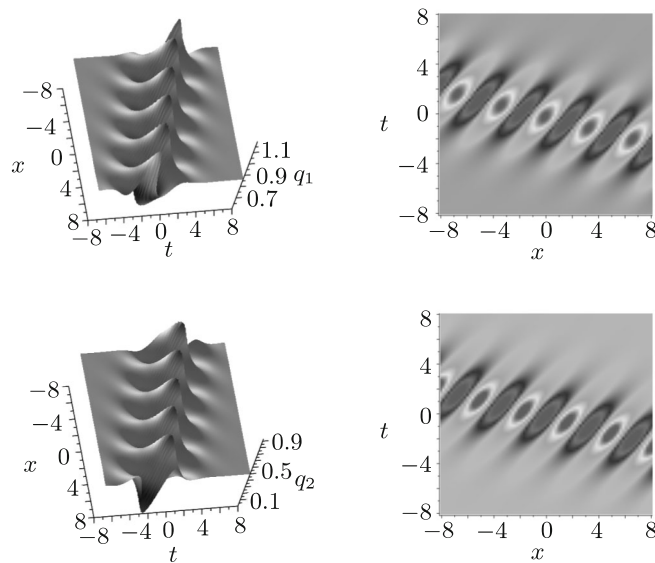


Fig. 10. One breather–breather solution of the fcmKdV equation with NZBCs obtained by taking $\zeta_1 = 3/2 + i/4$, $\alpha = \beta = 0$, $\mathbf{q}_+ = (\sqrt{3}/2, 1/2)^T$, and $q_0 = 1$.

Table 1. Soliton classification of type I

	$q_{1+} > 0$	$q_{1+} > 0$	$q_{1+} < 0$	$q_{1+} < 0$
	$q_{2+} > 0$	$q_{2+} < 0$	$q_{2+} > 0$	$q_{2+} < 0$
$m = 1$	dark–dark	dark–bright	bright–dark	bright–bright
$m = 0$	bright–bright	bright–dark	dark–bright	dark–dark

Table 2. Soliton classification of type III

	$q_{1+} > 0$	$q_{1+} > 0$	$q_{1+} < 0$	$q_{1+} < 0$
	$q_{2+} > 0$	$q_{2+} < 0$	$q_{2+} > 0$	$q_{2+} < 0$
$m = 1$	kink–bright–dark	kink–dark–dark	kink–bright–bright	kink–dark–bright
$m = 0$	kink–dark–bright	kink–bright–bright	kink–dark–dark	kink–bright–dark

Table 3. Soliton classification of type V

	$q_{1+} > 0$	$q_{1+} > 0$	$q_{1+} < 0$	$q_{1+} < 0$
	$q_{2+} > 0$	$q_{2+} < 0$	$q_{2+} > 0$	$q_{2+} < 0$
$m = 1$	kink–dark–bright	kink–bright–bright	kink–dark–dark	kink–bright–dark
$m = 0$	kink–bright–dark	kink–dark–dark	kink–bright–bright	kink–dark–bright

Of course, (4.1) allows us to easily generate the multisoliton solutions by including any combinations of any number of the six types of solitons. For example, Fig. 11 shows the interaction between two dark–dark solitons, Fig. 12 shows the interaction between two bright–bright solitons, Fig. 13 shows the interaction between kink–dark–bright and W (type)–S (type) solitons, and Fig. 14 describes the interaction between kink–dark–bright and breather–breather solutions.

5. Conclusions

In this paper, we applied the IST to the fcmKdV equation with NZBCs. This problem has three symmetries, resulting in six different types of discrete eigenvalues. Thus, six types of soliton solutions are obtained. Furthermore, the soliton classification for the types I, III, and V are constructed. Several two-soliton solutions having interesting structures are displayed.

We note that this paper is more involved than the defocusing counterpart in [19] due to the four fundamental domains of analyticity instead of two. Specifically, four new auxiliary eigenfunctions and a more complex RHP including four jump relations had to be formulated. In addition, although they both have three symmetries, the focusing case has more discrete eigenvalues, which leads to more diverse solutions. Within the framework of the IST method with NZBCs, we also note that the differences between the focusing Manakov system [41] and the fcmKdV equation are mainly manifested in four aspects: 1) the Jost eigenfunctions of the fcmKdV equation have three symmetries, with a total of $4N_1 + 8N_2 + 4N_3 + 8N_4 + 4N_5 + 8N_6$ discrete eigenvalues; 2) the fcmKdV equation has a simpler asymptotic phase difference, which is direct information to characterize solitons; 3) the potential of the fcmKdV equation is real, and the formula of its exact solutions is more compact; 4) the fcmKdV equation has richer soliton solutions and breathers, including dark–dark, bright–bright, W (type)–S (type), kink solitons, and breather–breather solutions.

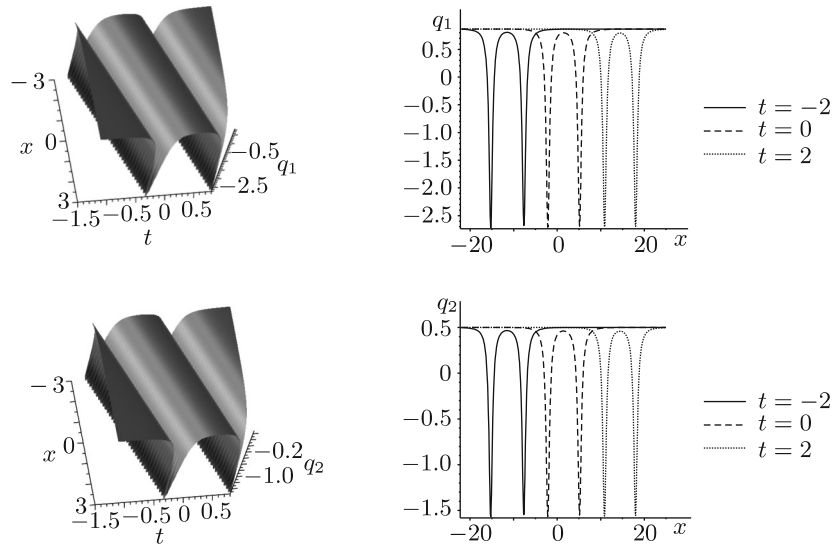


Fig. 11. One two-dark-dark soliton of the fcmKdV equation with NZBCs obtained by taking $N_1 = 2$, $N_2 = N_3 = N_4 = N_5 = N_6 = 0$, $\xi_1 = 3i/2$, $\xi_2 = 4i/3$, $a_1 = 2$, $a_2 = -3$, $\mathbf{q}_+ = (\sqrt{3}/2, 1/2)^T$, and $q_0 = 1$.

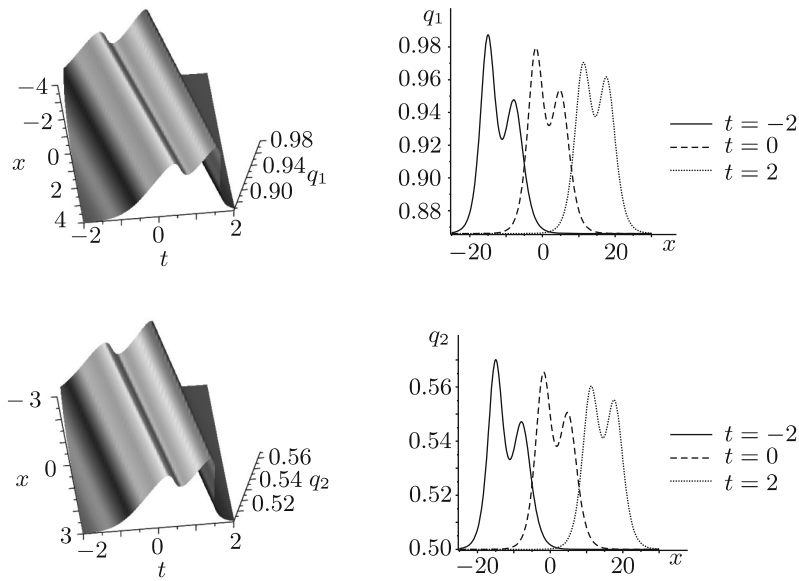


Fig. 12. One two-bright-bright soliton of the fcmKdV equation with NZBCs obtained by taking $N_1 = 2$, $N_2 = N_3 = N_4 = N_5 = N_6 = 0$, $\xi_1 = 3i/2$, $\xi_2 = 4i/3$, $a_1 = -2$, $a_2 = 3$, $\mathbf{q}_+ = (\sqrt{3}/2, 1/2)^T$, and $q_0 = 1$.

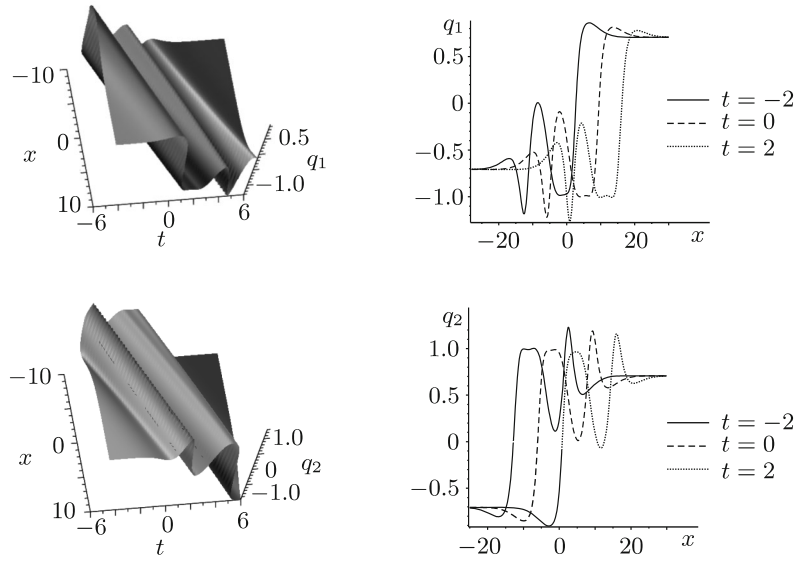


Fig. 13. One kink–dark-bright, W (type)–S (type) soliton of the fcmKdV equation with NZBCs obtained by taking $N_3 = N_4 = 1$, $N_1 = N_2 = N_5 = N_6 = 0$, $\eta_1 = 2i$, $c_1 = -1$, $z_1 = 1 + 2i$, $d_1 = 1$, $\mathbf{q}_+ = (\sqrt{2}/2, \sqrt{2}/2)^T$, and $q_0 = 1$.

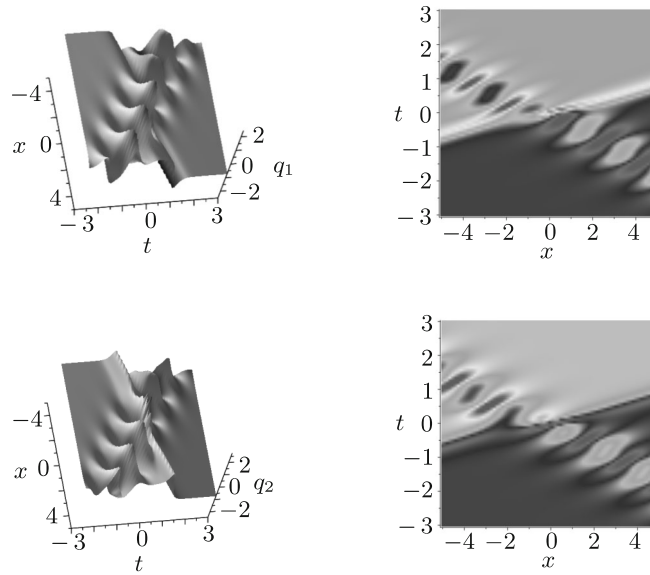


Fig. 14. One kink–dark-bright, breather–breather soliton of the fcmKdV equation with NZBCs obtained by taking $N_5 = N_6 = 1$, $N_1 = N_2 = N_3 = N_4 = 0$, $\nu_1 = 3i/2$, $e_1 = 1$, $\zeta_1 = 3/2 + i$, $f_1 = 1$, $\mathbf{q}_+ = (\sqrt{2}/2, \sqrt{2}/2)^T$, and $q_0 = 1$.

Appendix A

A.1. The RHP solution. The coefficients k_n , K_n , and l_n , L_n appearing in Theorem 3.1 are defined as follows:

$$\begin{aligned} \text{for } n = 1, \dots, N_1 \\ k_n = \xi_n, \quad K_n = A_n, \end{aligned} \tag{A.1a}$$

$$\begin{aligned} \text{for } n = N_1 + 1, \dots, 2N_1 \\ k_n = -\hat{\xi}_{n-N_1}^*, \quad K_n = -\frac{q_0^2}{(\xi_{n-N_1}^*)^2} A_{n-N_1}^*, \end{aligned} \tag{A.1b}$$

$$\begin{aligned} \text{for } n = 2N_1 + 1, \dots, 2N_1 + N_2 \\ k_n = \omega_{n-2N_1}, \quad K_n = B_{n-2N_1}, \end{aligned} \tag{A.1c}$$

$$\begin{aligned} \text{for } n = 2N_1 + N_2 + 1, \dots, 2N_1 + 2N_2 \\ k_n = -\hat{\omega}_{n-2N_1-N_2}^*, \quad K_n = -\frac{q_0^2}{(\omega_{n-2N_1-N_2}^*)^2} B_{n-2N_1-N_2}^*, \end{aligned} \tag{A.1d}$$

$$\begin{aligned} \text{for } n = 2N_1 + 2N_2 + 1, \dots, 2N_1 + 3N_2 \\ k_n = -\omega_{n-2N_1-2N_2}^*, \quad K_n = -B_{n-2N_1-2N_2}^*, \end{aligned} \tag{A.1e}$$

$$\begin{aligned} \text{for } n = 2N_1 + 3N_2 + 1, \dots, \mathcal{N}_1 \\ k_n = \hat{\omega}_{n-2N_1-3N_2}, \quad K_n = \frac{q_0^2}{\omega_{n-2N_1-3N_2}^2} B_{n-2N_1-3N_2}, \end{aligned} \tag{A.1f}$$

$$\begin{aligned} \text{for } n = 1, \dots, N_3 \\ l_n = -\hat{\eta}_n, \quad L_n = -\frac{i l_n}{q_0} C_n, \end{aligned} \tag{A.2a}$$

$$\begin{aligned} \text{for } n = N_3 + 1, \dots, N_3 + N_4 \\ l_n = -\hat{z}_{n-N_3}, \quad L_n = -\frac{i l_n}{q_0} D_{n-N_3}, \end{aligned} \tag{A.2b}$$

$$\begin{aligned} \text{for } n = N_3 + N_4 + 1, \dots, N_3 + 2N_4 \\ l_n = \hat{z}_{n-N_3-N_4}^*, \quad L_n = \frac{i l_n}{q_0} D_{n-N_3-N_4}^*, \end{aligned} \tag{A.2c}$$

$$\begin{aligned} \text{for } n = N_3 + 2N_4 + 1, \dots, N_3 + 2N_4 + N_5 \\ l_n = \nu_{n-N_3-2N_4}, \quad L_n = E_{n-N_3-2N_4}, \end{aligned} \tag{A.2d}$$

$$\begin{aligned} \text{for } n = N_3 + 2N_4 + N_5 + 1, \dots, N_3 + 2N_4 + N_5 + N_6 \\ l_n = \zeta_{n-N_3-2N_4-N_5}, \quad L_n = F_{n-N_3-2N_4-N_5}, \end{aligned} \tag{A.2e}$$

$$\begin{aligned} \text{for } n = N_3 + 2N_4 + N_5 + N_6 + 1, \dots, \mathcal{N}_2 \\ l_n = -\hat{\zeta}_{n-N_3-2N_4-N_5-N_6}^*, \quad L_n = -F_{n-N_3-2N_4-N_5-N_6}^*. \end{aligned} \tag{A.2f}$$

A.2. Trace formula. The scattering coefficients $a_{22}(z)$ and $b_{22}(z)$ has a jump across \mathbb{R} . We recall that $a_{22}(z)$ is analytic in \mathbb{C}_- and has simple zeros

$$\{\eta_n^*, -\hat{\eta}_n^*\}_{n=1}^{N_3}, \quad \{z_n^*, -z_n, -\hat{z}_n^*, \hat{z}_n\}_{n=1}^{N_4}, \quad \{\nu_n^*, -\hat{\nu}_n^*\}_{n=1}^{N_5}, \quad \{\zeta_n^*, -\zeta_n, -\hat{\zeta}_n^*, \hat{\zeta}_n\}_{n=1}^{N_6},$$

while $b_{22}(z)$ is analytic in \mathbb{C}_+ and has simple zeros

$$\{\eta_n, -\hat{\eta}_n\}_{n=1}^{N_3}, \quad \{z_n, -z_n^*, -\hat{z}_n, \hat{z}_n^*\}_{n=1}^{N_4}, \quad \{\nu_n, -\hat{\nu}_n\}_{n=1}^{N_5}, \quad \{\zeta_n, -\zeta_n^*, -\hat{\zeta}_n, \hat{\zeta}_n^*\}_{n=1}^{N_6}.$$

We therefore define

$$\begin{aligned} \beta(z) &= \delta b_{22}(z) \prod_{n=1}^{N_3} \frac{(z - \eta_n^*)(z + \hat{\eta}_n^*)}{(z - \eta_n)(z + \hat{\eta}_n)} \prod_{n=1}^{N_4} \frac{(z - z_n^*)(z + z_n)(z + \hat{z}_n^*)(z - \hat{z}_n)}{(z - z_n)(z + z_n^*)(z + \hat{z}_n)(z - \hat{z}_n^*)} \times \\ &\quad \times \prod_{n=1}^{N_5} \frac{(z - \nu_n^*)(z + \hat{\nu}_n^*)}{(z - \nu_n)(z + \hat{\nu}_n)} \prod_{n=1}^{N_6} \frac{(z - \zeta_n^*)(z + \zeta_n)(z + \hat{\zeta}_n^*)(z - \hat{\zeta}_n)}{(z - \zeta_n)(z + \zeta_n^*)(z + \hat{\zeta}_n)(z - \hat{\zeta}_n^*)}, \quad z \in \mathbb{C}_+, \end{aligned} \quad (\text{A.3a})$$

$$\begin{aligned} \beta(z) &= \delta a_{22}(z) \prod_{n=1}^{N_3} \frac{(z - \eta_n)(z + \hat{\eta}_n)}{(z - \eta_n^*)(z + \hat{\eta}_n^*)} \prod_{n=1}^{N_4} \frac{(z - z_n)(z + z_n^*)(z + \hat{z}_n)(z - \hat{z}_n^*)}{(z - z_n^*)(z + z_n)(z + \hat{z}_n^*)(z - \hat{z}_n)} \times \\ &\quad \times \prod_{n=1}^{N_5} \frac{(z - \nu_n)(z + \hat{\nu}_n)}{(z - \nu_n^*)(z + \hat{\nu}_n^*)} \prod_{n=1}^{N_6} \frac{(z - \zeta_n)(z + \zeta_n^*)(z + \hat{\zeta}_n)(z - \hat{\zeta}_n^*)}{(z - \zeta_n^*)(z + \zeta_n)(z + \hat{\zeta}_n^*)(z - \hat{\zeta}_n)}, \quad z \in \mathbb{C}_-. \end{aligned} \quad (\text{A.3b})$$

It is easy to find that $\beta(z) \rightarrow 1$ as $z \rightarrow \infty$ and has no poles. Combining the fact that $\beta_+(z)\beta_-(z) = a_{22}(z)b_{22}(z)$ with the 2, 2 element of the identity $\mathbf{B}(z)\mathbf{A}(z) = \mathbf{I}$, we obtain that

$$\ln \beta_+(z) + \ln \beta_-(z) = -J_0(z), \quad z \in \mathbb{R}, \quad (\text{A.4})$$

where

$$\beta_{\pm}(x, t, z) = \lim_{\substack{\hat{z} \rightarrow z \\ \hat{z} \in \mathbb{C}_{\pm}}} \beta(x, t, \hat{z}), \quad z \in \mathbb{R}, \quad (\text{A.5a})$$

$$J_0(z) = \ln \left[1 + \gamma(z)\rho_3(z)\rho_3(-z) + \frac{z^2\gamma(z)}{q_0^2}\rho_3(\hat{z})\rho_3(-\hat{z}) \right]. \quad (\text{A.5b})$$

Then applying the Plemelj formula to (A.4) yields

$$\beta(z) = \exp \left[-\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{J_0(\zeta)}{\zeta - z} d\zeta \right], \quad z \in \mathbb{C}_+. \quad (\text{A.6})$$

The first half of the proof is completed by substituting (A.6) in (A.3a).

We recall that $a_{11}(z)$, $b_{11}(z)$, $b_{33}(z)$, and $a_{33}(z)$ are respectively analytic in D_1 , D_2 , D_3 , and D_4 . For this reason, we can solve the problem by establishing a RHP as in Sec. 3.1. Based on the jump conditions, we see from the identity $\mathbf{B}(z)\mathbf{A}(z) = \mathbf{I}$ that

$$\begin{aligned} b_{31}(z)a_{13}(z) + b_{32}(z)a_{23}(z) + b_{33}(z)a_{33}(z) &= 1, \\ b_{11}(z)a_{11}(z) + b_{12}(z)a_{21}(z) + b_{13}(z)a_{31}(z) &= 1, \\ b_{22}(z) &= a_{11}(z)a_{33}(z) - a_{31}(z)a_{13}(z), \\ a_{22}(z) &= b_{11}(z)b_{33}(z) - b_{31}(z)b_{13}(z). \end{aligned}$$

It is worth noting that $b_{22}(z)$ and $a_{22}(z)$ are involved in the last two relations. Thus, the $a_{11}(z)$ trace formula is related to the type-III and IV discrete eigenvalues.

Recalling the definition of the oriented contour Σ , we define

$$\begin{aligned} \tilde{\beta}(z) &= a_{11}(z) \prod_{n=1}^{N_1} \frac{(z - \xi_n^*)(z + \hat{\xi}_n)}{(z - \xi_n)(z + \hat{\xi}_n^*)} \prod_{n=1}^{N_2} \frac{(z - \omega_n^*)(z + \omega_n)(z + \hat{\omega}_n)(z - \hat{\omega}_n^*)}{(z - \omega_n)(z + \omega_n^*)(z + \hat{\omega}_n^*)(z - \hat{\omega}_n)} \prod_{n=1}^{N_3} \frac{z + \hat{\eta}_n^*}{z + \hat{\eta}_n} \times \\ &\quad \times \prod_{n=1}^{N_4} \frac{(z + \hat{z}_n^*)(z - \hat{z}_n)}{(z + \hat{z}_n)(z - \hat{z}_n^*)} \prod_{n=1}^{N_5} \frac{z - \nu_n^*}{z - \nu_n} \prod_{n=1}^{N_6} \frac{(z - \zeta_n^*)(z + \zeta_n)}{(z - \zeta_n)(z + \zeta_n^*)}, \quad z \in D_1, \end{aligned} \quad (\text{A.7a})$$

$$\begin{aligned} \tilde{\beta}(z) &= b_{11}(z) \prod_{n=1}^{N_1} \frac{(z - \xi_n)(z + \hat{\xi}_n^*)}{(z - \xi_n^*)(z + \hat{\xi}_n)} \prod_{n=1}^{N_2} \frac{(z - \omega_n)(z + \omega_n^*)(z + \hat{\omega}_n^*)(z - \hat{\omega}_n)}{(z - \omega_n^*)(z + \omega_n)(z + \hat{\omega}_n)(z - \hat{\omega}_n^*)} \prod_{n=1}^{N_3} \frac{z + \hat{\eta}_n}{z + \hat{\eta}_n^*} \times \\ &\times \prod_{n=1}^{N_4} \frac{(z + \hat{z}_n)(z - \hat{z}_n^*)}{(z + \hat{z}_n^*)(z - \hat{z}_n)} \prod_{n=1}^{N_5} \frac{z - \nu_n}{z - \nu_n^*} \prod_{n=1}^{N_6} \frac{(z - \zeta_n)(z + \zeta_n^*)}{(z - \zeta_n^*)(z + \zeta_n)}, \quad z \in D_2, \end{aligned} \quad (\text{A.7b})$$

$$\begin{aligned} \tilde{\beta}(z) &= \delta b_{33}(z) \prod_{n=1}^{N_1} \frac{(z - \xi_n^*)(z + \hat{\xi}_n)}{(z - \xi_n)(z + \hat{\xi}_n^*)} \prod_{n=1}^{N_2} \frac{(z - \omega_n^*)(z + \omega_n)(z + \hat{\omega}_n)(z - \hat{\omega}_n^*)}{(z - \omega_n)(z + \omega_n^*)(z + \hat{\omega}_n^*)(z - \hat{\omega}_n)} \prod_{n=1}^{N_3} \frac{z - \eta_n}{z - \eta_n^*} \times \\ &\times \prod_{n=1}^{N_4} \frac{(z + z_n^*)(z - z_n)}{(z + z_n)(z - z_n^*)} \prod_{n=1}^{N_5} \frac{z + \hat{\nu}_n}{z + \hat{\nu}_n^*} \prod_{n=1}^{N_6} \frac{(z + \hat{\zeta}_n)(z - \hat{\zeta}_n^*)}{(z + \hat{\zeta}_n^*)(z - \hat{\zeta}_n)}, \quad z \in D_3, \end{aligned} \quad (\text{A.7c})$$

$$\begin{aligned} \tilde{\beta}(z) &= \delta a_{33}(z) \prod_{n=1}^{N_1} \frac{(z - \xi_n)(z + \hat{\xi}_n^*)}{(z - \xi_n^*)(z + \hat{\xi}_n)} \prod_{n=1}^{N_2} \frac{(z - \omega_n)(z + \omega_n^*)(z + \hat{\omega}_n^*)(z - \hat{\omega}_n)}{(z - \omega_n^*)(z + \omega_n)(z + \hat{\omega}_n)(z - \hat{\omega}_n^*)} \prod_{n=1}^{N_3} \frac{z - \eta_n^*}{z - \eta_n} \times \\ &\times \prod_{n=1}^{N_4} \frac{(z + z_n)(z - z_n^*)}{(z + z_n^*)(z - z_n)} \prod_{n=1}^{N_5} \frac{z + \hat{\nu}_n^*}{z + \hat{\nu}_n} \prod_{n=1}^{N_6} \frac{(z + \hat{\zeta}_n^*)(z - \hat{\zeta}_n)}{(z + \hat{\zeta}_n)(z - \hat{\zeta}_n^*)}, \quad z \in D_4. \end{aligned} \quad (\text{A.7d})$$

Thus, $\tilde{\beta}(z) \rightarrow 1$ as $z \rightarrow \infty$ and has no poles. Also, the following jump conditions hold:

$$\ln \tilde{\beta}_+(z) + \ln \tilde{\beta}_-(z) = J(z), \quad z \in \Sigma, \quad (\text{A.8})$$

where the jump matrices $J_j(z) = J(z)|_{z \in \Sigma_j}$ are

$$J_1(z) = -\ln \left[1 + \frac{1}{\gamma(z)} \rho_1(z) \rho_1(-z) + \rho_2(z) \rho_2(-z) \right], \quad z \in \Sigma_1, \quad (\text{A.9a})$$

$$J_2(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{J_0(\zeta)}{\zeta - z} d\zeta - \ln[1 - \rho_2(-z) \rho_2(\hat{z})], \quad z \in \Sigma_2, \quad (\text{A.9b})$$

$$J_3(z) = -\ln \left[1 + \frac{q_0^2}{z^2 \gamma(z)} \rho_1(\hat{z}) \rho_1(-\hat{z}) + \rho_2(\hat{z}) \rho_2(-\hat{z}) \right], \quad z \in \Sigma_3, \quad (\text{A.9c})$$

$$J_4(z) = -\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{J_0(\zeta)}{\zeta - z} d\zeta - \ln[1 - \rho_2(-\hat{z}) \rho_2(z)], \quad z \in \Sigma_4. \quad (\text{A.9d})$$

Applying the Plemelj formula to (A.8), we obtain the solutions of the RHP in the form

$$\tilde{\beta}(z) = \exp \left[\frac{1}{2\pi i} \int_{\Sigma} \frac{J(\zeta)}{\zeta - z} d\zeta \right], \quad z \in D_1 \cup D_3. \quad (\text{A.10})$$

Substituting the above result in (A.7a) yields Eq. (3.19a).

Funding. This work was supported by the National Natural Science Foundation of China (NNSFC) (grant Nos. 12171474 and 11931017) and the Yue Qi Young Scholar Project, China University of Mining & Technology, Beijing (grant No. 00-800015Z1201).

Conflict of interest. The author of this work declares that she has no conflicts of interest.

REFERENCES

1. R. M. Miura, "Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation," *J. Math. Phys.*, **9**, 1202–1204 (1968).
2. E. A. Ralph and L. Pratt, "Predicting eddy detachment for an equivalent barotropic thin jet," *J. Nonlinear Sci.*, **4**, 355–374 (1994).

3. H. Ono, “Soliton fission in anharmonic lattices with reflectionless inhomogeneity,” *J. Phys. Soc. Japan*, **61**, 4336–4343 (1992).
4. A. H. Khater, O. H. El-Kalaawy, and D. K. Callebaut, “Bäcklund transformations and exact solutions for Alfvén solitons in a relativistic electron-positron plasma,” *Phys. Scr.*, **58**, 545–548 (1998).
5. G. Q. Zhang and Z. Y. Yan, “Focusing and defocusing mKdV equations with nonzero boundary conditions: Inverse scattering transforms and soliton interactions,” *Phys. D*, **410**, 132521, 22 pp. (2020).
6. H. Ono, “On a modified Korteweg–de Vries equation,” *J. Phys. Soc. Japan*, **37**, 882–882 (1974).
7. M. Wadati, “The exact solution of the modified Korteweg–de Vries equation,” *J. Phys. Soc. Japan*, **32**, 1681–1681 (1972).
8. M. Wadati and K. Ohkuma, “Multiple-pole solutions of the modified Korteweg–de Vries equation,” *J. Phys. Soc. Japan*, **51**, 2029–2035 (1982).
9. F. Demontis, “Exact solutions of the modified Korteweg–de Vries equation,” *Theoret. and Math. Phys.*, **168**, 886–897 (2011).
10. N. Cheemaa, A. R. Seadawy, T. G. Sugati, and D. Baleanu, “Study of the dynamical nonlinear modified Korteweg–de Vries equation arising in plasma physics and its analytical wave solutions,” *Results Phys.*, **19**, 103480, 8 pp. (2020).
11. H. Schamel, “A modified Korteweg–de Vries equation for ion acoustic waves due to resonant electrons,” *J. Plasma Phys.*, **9**, 377–387 (1973).
12. J. Y. Zhu, D. W. Zhou, and X. G. Geng, “ $\bar{\partial}$ -problem and Cauchy matrix for the mKdV equation with self-consistent sources,” *Phys. Scr.*, **89**, 065201, 7 pp. (2014).
13. B. F. Feng, K. Maruno, and Y. Ohta, “Integrable semi-discretization of a multi-component short pulse equation,” *J. Math. Phys.*, **56**, 043502, 15 pp. (2015).
14. Y. Matsuno, “A novel multi-component generalization of the short pulse equation and its multisoliton solutions,” *J. Math. Phys.*, **52**, 123702, 22 pp. (2011).
15. T. Busch and J. R. Anglin, “Dark-bright solitons in inhomogeneous Bose–Einstein condensates,” *Phys. Rev. Lett.*, **87**, 010401, 4 pp. (2001).
16. M. A. Hofer, J. J. Chang, C. Hamner, and P. Engels, “Dark-dark solitons and modulational instability in miscible two-component Bose–Einstein condensates,” *Phys. Rev. A*, **84**, 041605, 4 pp. (2011).
17. Q.-H. Park and H. J. Shin, “Darboux transformation and Crum’s formula for multi-component integrable equations,” *Phys. D*, **157**, 1–15 (2001).
18. D. V. Kartashov, A. V. Kim, and S. A. Skobelev, “Soliton structures of a wave field with an arbitrary number of oscillations in nonresonance media,” *JETP Lett.*, **78**, 276–280 (2003).
19. Y. Xiao, E. G. Fan, and P. Liu, “Inverse scattering transform for the coupled modified Korteweg–de Vries equation with nonzero boundary conditions,” *J. Math. Anal. Appl.*, **504**, 125567, 31 pp. (2021).
20. S.-F. Tian, “Initial-boundary value problems for the coupled modified Korteweg–de Vries equation on the interval,” *Commun. Pure Appl. Anal.*, **17**, 923–957 (2018).
21. X. G. Geng, M. M. Chen, and K. D. Wang, “Long-time asymptotics of the coupled modified Korteweg–de Vries equation,” *J. Geom. Phys.*, **142**, 151–167 (2019).
22. X. G. Geng, Y. Y. Zhai, and H. H. Dai, “Algebro-geometric solutions of the coupled modified Korteweg–de Vries hierarchy,” *Adv. Math.*, **263**, 123–153 (2014).
23. T. Tsuchida and M. Wadati, “The coupled modified Korteweg–de Vries equations,” *J. Phys. Soc. Japan*, **67**, 1175–1187 (1998).
24. H. Q. Zhang, B. Tian, T. Xu, H. Li, C. Zhang, and H. Zhang, “Lax pair and Darboux transformation for multi-component modified Korteweg–de Vries equations,” *J. Phys. A: Math. Theor.*, **41**, 355210, 13 pp. (2008).
25. J. P. Wu and X. G. Geng, “Inverse scattering transform and soliton classification of the coupled modified Korteweg–de Vries equation,” *Commun. Nonlinear Sci. Numer. Simul.*, **53**, 83–93 (2017).
26. J. Q. Wang, L. X. Tian, B. L. Guo, and Y. N. Zhang, “Nonlinear stability of breather solutions to the coupled modified Korteweg–de Vries equations,” *Commun. Nonlinear Sci. Numer. Simul.*, **90**, 105367, 14 pp. (2020).
27. K. D. Wang, X. G. Geng, M. M. Chen, and B. Xue, “Riemann–Hilbert approach and long-time asymptotics for the three-component derivative nonlinear Schrödinger equation,” *Anal. Math. Phys.*, **12**, 71, 33 pp. (2022).

28. A. B. de Monvel, A. Kostenko, D. Shepelsky, and G. Teschl, “Long-time asymptotics for the Camassa–Holm equation,” *SIAM J. Math. Anal.*, **41**, 1559–1588 (2009).
29. P. A. Deift and X. Zhou, “Long-time asymptotics for integrable systems. Higher order theory,” *Commun. Math. Phys.*, **165**, 175–191 (1994).
30. V. E. Zakharov and L. A. Ostrovsky, “Modulation instability: The beginning,” *Phys. D*, **238**, 540–548 (2009).
31. D. Bilman and P. D. Miller, “A robust inverse scattering transform for the focusing nonlinear Schrödinger equation,” *Commun. Pure Appl. Math.*, **72**, 1722–1805 (2019).
32. D. J. Kaup, “The three-wave interaction — a nondispersive phenomenon,” *Stud. Appl. Math.*, **55**, 9–44 (1976).
33. X. Zhou, “Direct and inverse scattering transforms with arbitrary spectral singularities,” *Commun. Pure Appl. Math.*, **42**, 895–938 (1989).
34. H. Liu, J. Shen, and X. G. Geng, “Inverse scattering transformation for the N -component focusing nonlinear Schrödinger equation with nonzero boundary conditions,” *Lett. Math. Phys.*, **113**, 23, 47 pp. (2023).
35. X. Zhou, “The Riemann–Hilbert problem and inverse scattering,” *SIAM J. Math. Anal.*, **20**, 966–986 (1989).
36. A. S. Fokas and A. R. Its, “The linearization of the initial-boundary value problem of the nonlinear Schrödinger equation,” *SIAM J. Math. Anal.*, **27**, 738–764 (1996).
37. G. Biondini and G. Kovačič, “Inverse scattering transform for the focusing nonlinear Schrödinger equation with nonzero boundary conditions,” *J. Math. Phys.*, **55**, 031506, 22 pp. (2014).
38. A. Mohamadou, E. Wamba, D. Lissouck, and T. C. Kofane, “Dynamics of kink-dark solitons in Bose–Einstein condensates with both two- and three-body interactions,” *Phys. Rev. E*, **85**, 046605, 8 pp. (2012).
39. Y. Li, Y.-H. Qin, L.-C. Zhao et al., “Vector kink-dark complex solitons in a three-component Bose–Einstein condensate,” *Commun. Theor. Phys.*, **73**, 055502, 7 pp. (2021).
40. K. S. Al-Ghafri, E. V. Krishnan, and A. Biswas, “W-shaped and other solitons in optical nanofibers,” *Results Phys.*, **23**, 103973, 15 pp. (2021).
41. D. Kraus, G. Biondini, and G. Kovačič, “The focusing Manakov system with nonzero boundary conditions,” *Nonlinearity*, **28**, 3101–3151 (2015).

Publisher’s Note. Pleiades Publishing remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.