QUANTIFYING THE IRREVERSIBILITY OF CHANNELS

Shunlong Luo[*][†] **and Yuan Sun**[‡]

In contrast to unitary evolutions, which are reversible, generic quantum processes (operations and quantum channels) are often irreversible. However, the degree of irreversibility is different for different channels, and it is desirable to have a quantitative characterization of irreversibility. In this paper, by exploiting the channel–state duality implemented by the Jamiołkowski–Choi isomorphism, we quantify the irreversibility of channels via entropy of the Jamiołkowski–Choi states of the corresponding channels and compare it with the notions of entanglement fidelity and entropy exchange. General properties of a reasonable measure of irreversibility are discussed from an intuitive perspective, and entropic measures of irreversibility are introduced. Several relations between irreversibility, entanglement fidelity, the degree of nonunitality, and decorrelating power are established. Some measures of irreversibility for a variety of prototypical channels are evaluated explicitly, revealing some information-theoretic aspects of the structure of channels from the perspective of irreversibility.

1. Introduction

Irreversibility is ubiquitous in nature and emerges frequently in many physical processes. The second law of thermodynamics, stating that the total entropy of a system either increases or remains constant in any spontaneous process, characterizes some fundamentally irreversible aspects of natural phenomena [1]–[4].

In the quantum world, irreversibility arises naturally from the system–environment interaction and quantum measurements, which lead to information leakage into the environment that cannot be recovered due to the uncontrollability of the environment. This causes decoherence and dissipation of open system dynamics [5]–[10].

In this paper, we are concerned with the irreversibility of quantum channels in the context of super-operators, i.e., maps sending quantum states (described by density operators) to quantum states [11]–[13]. In a rather broad and widely used setup, general quantum processes are often described by quantum channels (i.e., linear trace-preserving completely positive maps of quantum states, henceforth abbreviated as channels). These physical processes are usually divided into two categories: unitary evolutions (which are reversible) and open system dynamics (which are irreversible). While irreversible aspects of channels have been widely studied with the main focus on qualitative perspectives such as quantum error correction and

[*]Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China, e-mail: luosl@amt.ac.cn.

[†]School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, China.

[‡]School of Mathematical Sciences, Nanjing Normal University, Nanjing, China.

quantum control, they are relatively less studied from a quantitative perspective. This calls for quantitative investigations of the irreversibility of channels.

To proceed in a rigorous way, we recall that a quantum process described by a channel $\mathcal{E}$ is (physically) reversible if there exists a channel $\mathcal{D}$ such that the composition $\mathcal{D} \circ \mathcal{E} = \mathcal{I}$ is the identity channel in the system. Otherwise, it is termed irreversible. Evidently, any unitary channel $\mathcal{E}_U(\rho) = U\rho U^\dagger$ (with U a unitary operator) is reversible, and the corresponding reverse is $\mathcal{D}(\rho) = \mathcal{E}_{U^\dagger}(\rho) = U^\dagger \rho U$. Indeed, $(\mathcal{E}_{U^\dagger} \circ \mathcal{E}_U)(\rho) = U^\dagger(U\rho U^\dagger)U = \rho$ for any state ρ. On the other extreme, for a d-dimensional system, it is intuitively clear that the completely depolarizing channel $\mathcal{E}_{\mathrm{cde}}(\rho) = \mathbf{1}/d$ (for any state ρ) is among the most irreversible channels. Here, $\mathbf{1}$ denotes the identity operator on the system Hilbert space. Intermediate between these two extremes are many irreversible channels, the random unitary channels being prominent and important examples.

Many features of irreversibility have been studied in the literature from various aspects. For example, decoherence and dissipation have been extensively investigated in quantum measurements and the quantum-to-classical transition [5]–[10]. Entanglement fidelity has been introduced to characterize the quality of channels and quantum gates in preserving states [14]–[20]. Information and entropy change has been discussed for the coupling of systems to environments [21], [22]. Markovian/non-Markovian properties of open system dynamics have attracted great interest with the emergence of quantum information theory [23]–[27]. Entropy production has been extensively investigated in quantum dynamical semigroups and irreversible processes [28]–[41].

In view of the channel–state duality, for any channel, there is a corresponding Jamiołkowski–Choi state [42]–[44]. Because irreversibility is intimately related to a change in entropy, it is desirable to study the entropy of Jamiołkowski–Choi states and to see to what extent this entropy can be used to quantify the irreversibility of the corresponding channel. Inspired by this general idea, in this paper, by using the channel–state duality [42]–[45], we quantify the irreversibility of channels in terms of entropy of the associated Jamiołkowski–Choi states, which is connected to the entropy produced by the channels. We remark that the general idea of relating a change in entropy to irreversibility is well known. To pursue further quantitative and specific links between them, it is crucial to find appropriate entropic quantities of channels. Our key point here is to use the entropy of the Jamiołkowski–Choi states determined by the channels. We relate the measure of irreversibility to the notions of entanglement fidelity, entropy exchange, nonunitality, and decorrelating power [14]–[20], and illustrate the results by several important channels.

The remainder of this paper is organized as follows. In Sec. 2, we discuss irreversibility from an axiomatic standpoint and postulate some basic requirements for a reasonable measure of irreversibility. Somewhat complementary to irreversibility, we also discuss entanglement fidelity, which quantifies the capability of a channel to preserve states and is thus intimately related to the degree of reversibility. In Sec. 3, we propose to use the entropy of the Jamiołkowski–Choi states of channels as a measure of irreversibility and clarify its basic properties. In Sec. 4, we make a comparative study of irreversibility and entanglement fidelity. In Sec. 5, we relate irreversibility to the degree of nonunitality (deviation from the unital map). In Sec. 6, we establish an intuitive and succinct relation connecting irreversibility and decorrelating power. In Sec. 7, we evaluate the irreversibility of some widely used channels, and shed light on these channels from the perspective of irreversibility. In particular, as a simple illustrative application, we apply the results to the scenario of teleportation. We conclude with a summary and discussion in Sec. 8. In the appendix, we present a detailed proof of the main results, discuss an alternative measure of irreversibility in terms of the Tsallis entropy, which is easier to compute, and summarize the results in a table for comparison. For simplicity, we only work with finite-dimensional systems.

2. Basic features of irreversibility

We consider a channel

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger \tag{1}$$

on a quantum system described by a d-dimensional complex Hilbert space H, where E_k are the Kraus operators of $\mathcal{E}$ satisfying $\sum_k E_k^\dagger E_k = \mathbf{1}$ to ensure the trace-preserving property of $\mathcal{E}$. If, moreover, $\sum_k E_k E_k^\dagger = \mathbf{1}$, then the channel is unital (i.e., leaving the maximally mixed state invariant, $\mathcal{E}(\mathbf{1}/d) = \mathbf{1}/d$). Clearly, the map $\mathcal{E}$ defined by Eq. (1) actually makes sense for any operator X on H: $\mathcal{E}(X) = \sum_k E_k X E_k^\dagger$.

We want to quantify the degree of irreversibility of the channel $\mathcal{E}$. For this, we proceed from an axiomatic consideration motivated by physical intuition. We postulate the following simple and mandatory conditions for a reasonable measure $S(\mathcal{E})$ of irreversibility of the channel $\mathcal{E}$.

1. $S(\mathcal{E}) \geqslant 0$, which becomes an equality if and only if $\mathcal{E}$ is a unitary channel in the sense that $\mathcal{E}(\rho) = U\rho U^\dagger$ for some unitary operator U on the system Hilbert space H. This is motivated by the fact that unitary channels are reversible and generate no irreversibility because there is no information leakage to the environment: the system is closed under unitary evolution. On the other extreme, for a fixed d-dimensional system, $S(\mathcal{E})$ attains its maximum value for the completely depolarizing channel $\mathcal{E}_{\mathrm{cde}}(\rho) = \mathbf{1}/d$ (for any state ρ). This channel completely loses the original information of the state ρ and should be considered among the maximally irreversible.

2. $S(\,\cdot\,)$ is concave in the sense that

$$S(p_1 \mathcal{E}_1 + p_2 \mathcal{E}_2) \geqslant p_1 S(\mathcal{E}_1) + p_2 S(\mathcal{E}_2)$$

   for any $p_1, p_2 \geqslant 0$, $p_1 + p_2 = 1$, and any channels $\mathcal{E}_1$ and $\mathcal{E}_2$ on the system Hilbert space H. This is motivated by the intuition that the mixing of channels loses information and generates more irreversibility on average.

3. $S(\,\cdot\,)$ is invariant under composition with unitary channels in the sense that

$$S(\mathcal{E}_U \circ \mathcal{E}) = S(\mathcal{E} \circ \mathcal{E}_U) = S(\mathcal{E})$$

   for any channel $\mathcal{E}_U(\rho) = U\rho U^\dagger$ with U any unitary operator on the system Hilbert space H.

4. $S(\,\cdot\,)$ is ancilla-independent in the sense that

$$S(\mathcal{I}^a \otimes \mathcal{E}) = S(\mathcal{E}),$$

   where $\mathcal{I}^a$ is the identity channel on any ancilla system a.

5. $S(\,\cdot\,)$ is monotonic in the sense that

$$S(\mathcal{F} \circ \mathcal{E}) \geqslant S(\mathcal{E})$$

   for any unital channel $\mathcal{F}$.

It is conceivable that there may exist many quantities satisfying the above requirements, and it is desirable to seek those that are easy to compute and at the same time have intuitive physical significance. Taking these points into account, we provide some entropic quantifiers of the irreversibility of channels.

Complementary to the irreversibility in some sense, it is also desirable to investigate the capability of a channel to preserve states. This leads to the celebrated notion of entanglement fidelity [14], and is somewhat related to the reversibility of channels. Among all channels on a system, it is straightforward and intuitive to regard the identity channel $\mathcal{I}$ as the channel with the maximal fidelity. Thus, the deviation of a channel $\mathcal{E}$ from $\mathcal{I}$ can be used to quantify the fidelity of $\mathcal{E}$. We elaborate on the relations between irreversibility and entanglement fidelity in Sec. 4.

3. Irreversibility via entropy of Jamiołkowski–Choi states

In this section, we quantify the irreversibility of a channel via the entropy of the Jamiołkowski–Choi state determined by the channel, for which the Jamiołkowski–Choi isomorphism plays a key role [42]–[45]. Any channel $\mathcal{E}$ on a system Hilbert space H is isomorphic to the bipartite state

$$J_{\mathcal{E}} = (\mathcal{I} \otimes \mathcal{E})(|\Phi^+\rangle\langle\Phi^+|), \tag{2}$$

where $\mathcal{I}$ is the identity channel on an ancilla system $H^a = H$ and

$$|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_i |i\rangle \otimes |i\rangle \tag{3}$$

is a maximally entangled state on $H^a \otimes H = H \otimes H$ associated with an orthonormal basis $\{|i\rangle \colon i = 1, 2, \ldots, d\}$ of H. Thus, studying the channel $\mathcal{E}$ is formally equivalent to studying the Jamiołkowski–Choi state $J_{\mathcal{E}}$. In particular, we can connect the entropy of the latter to the irreversibility of the former, as is discussed in what follows.

For any state ρ, its von Neumann entropy

$$S(\rho) = -\operatorname{tr} \rho \ln \rho$$

is a key quantity in quantum information theory [11]. Despite its alternative name of quantum entropy, it actually quantifies the mixedness (statistical uncertainty) of a state ρ. If we take ρ to be the Jamiołkowski–Choi state $J_{\mathcal{E}}$ (defined in Eq. (2)) of the channel $\mathcal{E}$, we obtain the entropy

$$S(J_{\mathcal{E}}) = -\operatorname{tr} J_{\mathcal{E}} \ln J_{\mathcal{E}}.$$

It is obvious that $S(J_{\mathcal{E}})$ can be regarded as the entropy exchange of the channel $\mathcal{E}$ in the maximally mixed state $\mathbf{1}/d$ [14]. We emphasize that this quantity is quite different from the entropy production, which is widely studied in both classical and quantum thermodynamics [40].

As a basic feature of a channel $\mathcal{E}$, it is desirable to show that $S(J_{\mathcal{E}})$ is independent of the choice of the orthonormal basis $\{|i\rangle \colon i = 1, 2, \ldots, d\}$ of H. To prove this, we let $\mathcal{E}$ be a channel with Kraus operators $\{E_k \colon k = 1, 2, \ldots, n\}$; then $J_{\mathcal{E}}$ can be written as

$$J_{\mathcal{E}} = \sum_{k=1}^n |\xi_k\rangle\langle\xi_k| = (|\xi_1\rangle, |\xi_2\rangle, \ldots, |\xi_n\rangle) \begin{pmatrix} \langle\xi_1| \\ \langle\xi_2| \\ \vdots \\ \langle\xi_n| \end{pmatrix}, \tag{4}$$

where $|\xi_k\rangle = (\mathbf{1} \otimes E_k)|\Phi^+\rangle \in H \otimes H$. Let $W = (w_{kl})$ be the Gram matrix with entries defined as $w_{kl} = \langle\xi_k|\xi_l\rangle = \frac{1}{d}\operatorname{tr}(E_k^\dagger E_l)$; then W is apparently independent of the choice of the orthonormal basis $\{|i\rangle \colon i = 1, 2, \ldots, d\}$, and can be further written as

$$W = \begin{pmatrix} \langle\xi_1| \\ \langle\xi_2| \\ \vdots \\ \langle\xi_n| \end{pmatrix} (|\xi_1\rangle, |\xi_2\rangle, \ldots, |\xi_n\rangle). \tag{5}$$

By Eqs. (4) and (5), we have $S(J_{\mathcal{E}}) = S(W)$, which implies that $S(J_{\mathcal{E}})$ is independent of the choice of the orthonormal basis.

After the above preparations, we introduce the quantity

$$S(\mathcal{E}) = \frac{1}{2} S(J_{\mathcal{E}}) \tag{6}$$

as a quantifier of the irreversibility of a channel $\mathcal{E}$. The factor $1/2$ is for later convenience and for ensuring the upper bound $\ln d$ for a d-dimensional system (see Eq. (7)), and originates from the fact that $J_{\mathcal{E}}$ is a composite "system + ancilla" state, which doubles the dimension of the system. The same symbol S used for both the von Neumann entropy and the irreversibility of channels should not cause confusion because their arguments respectively refer to states and channels.

The measure of irreversibility $S(\mathcal{E})$ has the following properties consistent with requirements 1–5 in Sec. 2.

Proposition 1. *Let $\mathcal{E}$ be a channel on a d-dimensional quantum system with a Hilbert space H.*

1. *We have*
$$0 \leqslant S(\mathcal{E}) \leqslant \ln d. \tag{7}$$

   Moreover, $S(\mathcal{E}) = 0$ if and only if $\mathcal{E}$ is a unitary channel, and $S(\mathcal{E})$ attains the maximal value $\ln d$ if and only if $\mathcal{E}$ is the completely depolarizing channel $\mathcal{E}_{\mathrm{cde}}(\rho) = \mathbf{1}/d$ for any state ρ.

2. *$S(\cdot)$ is concave in the sense that*

$$S(p_1 \mathcal{E}_1 + p_2 \mathcal{E}_2) \geqslant p_1 S(\mathcal{E}_1) + p_2 S(\mathcal{E}_2)$$

   for $p_1, p_2 \geqslant 0$, $p_1 + p_2 = 1$, and any channels $\mathcal{E}_1$ and $\mathcal{E}_2$.

3. *$S(\cdot)$ is invariant under composition with unitary channels in the sense that*

$$S(\mathcal{E}_U \circ \mathcal{E}) = S(\mathcal{E} \circ \mathcal{E}_U) = S(\mathcal{E}),$$

   where $\mathcal{E}_U(\rho) = U\rho U^\dagger$ with U any unitary operator on H.

4. *$S(\cdot)$ is ancilla-independent in the sense that*

$$S(\mathcal{I}^a \otimes \mathcal{E}) = S(\mathcal{E}),$$

   where $\mathcal{I}^a$ is the identity channel on any ancilla system Hilbert space H^a.

5. *$S(\cdot)$ is monotonic in the sense that*
$$S(\mathcal{F} \circ \mathcal{E}) \geqslant S(\mathcal{E})$$

   for any unital channel $\mathcal{F}$.

6. *$S(\cdot)$ is additive under tensor product in the sense that*

$$S(\mathcal{E}^a \otimes \mathcal{E}^b) = S(\mathcal{E}^a) + S(\mathcal{E}^b),$$

   where $\mathcal{E}^a$ and $\mathcal{E}^b$ are channels on respective systems a and b.

Proof is given in Appendix A.

Being simple and significant extensions of unitary channels, random unitary channels constitute an important class of channels and have many nice properties. Their degree of irreversibility has the following quantification.

Proposition 2. *For any random unitary channel*

$$\mathcal{E}_{\mathrm{ru}}(\rho) = \sum_k p_k U_k \rho U_k^\dagger$$

with U_k being any unitary operators and $p_k > 0$, $\sum_k p_k = 1$, we have

$$0 \leqslant S(\mathcal{E}_{\mathrm{ru}}) \leqslant \min\left\{\frac{1}{2}H(\{p_k\}), \ln d\right\},$$

where $H(\{p_k\}) = -\sum_k p_k \ln p_k$ is the Shannon entropy of the probability distribution $\{p_k\}$. Moreover,

1. *$S(\mathcal{E}_{\mathrm{ru}}) = 0$ if and only if $\mathcal{E}_{\mathrm{ru}}$ is a unitary channel;*

2. *$S(\mathcal{E}_{\mathrm{ru}}) = \frac{1}{2}H(\{p_k\})$ if and only if the unitary operators U_k are pairwise orthogonal in the sense that $\mathrm{tr}(U_k^\dagger U_l) = 0$ for any $k \neq l$;*

3. *$S(\mathcal{E}_{\mathrm{ru}}) = \ln d$ if and only if $\mathcal{E}_{\mathrm{ru}}$ is the completely depolarizing channel, i.e., $\mathcal{E}_{\mathrm{ru}}(\rho) = \mathcal{E}_{\mathrm{cde}}(\rho) = \mathbf{1}/d$ for any state ρ.*

Proof is given in Appendix B.

To illustrate Proposition 2, we consider the generalized depolarizing channel

$$\mathcal{E}(\rho) = \sum_{k=0}^{3} p_k \sigma_k \rho \sigma_k$$

on a qubit system, where $p_k \geqslant 0$, $\sum_k p_k = 1$, $\sigma_0 = \mathbf{1}$, and σ_j, $j = 1, 2, 3$, are the Pauli matrices. Direct calculation shows that

$$J_{\mathcal{E}} = \frac{1}{2}\begin{pmatrix} p_0 + p_3 & 0 & 0 & p_0 - p_3 \\ 0 & p_1 + p_2 & p_1 - p_2 & 0 \\ 0 & p_1 - p_2 & p_1 + p_2 & 0 \\ p_0 - p_3 & 0 & 0 & p_0 + p_3 \end{pmatrix}$$

with eigenvalues p_k, $k = 0, 1, 2, 3$. Thus,

$$S(\mathcal{E}) = -\frac{1}{2}\sum_k p_k \ln p_k.$$

In particular, the conventional depolarizing channel

$$\mathcal{E}_{\mathrm{de}}(\rho) = (1 - 3p)\rho + p\sum_{k=1}^{3} \sigma_k \rho \sigma_k, \qquad 0 \leqslant p \leqslant \frac{1}{3},$$

corresponds to the generalized depolarizing channel with $p_0 = 1 - 3p$, $p_1 = p_2 = p_3 = p$. In this case, we have

$$S(\mathcal{E}_{\mathrm{de}}) = -\frac{1}{2}\big((1 - 3p)\ln(1 - 3p) + 3p\ln p\big).$$

4. Comparison between irreversibility and entanglement fidelity

We consider a channel $\mathcal{E}$, defined by Eq. (1), on a d-dimensional system with a Hilbert space H. We take an ancilla system Hilbert space $H^a = H$ and let $|\Phi^+\rangle$ be any maximally entangled state defined by Eq. (3). Then the entanglement fidelity

$$F(\mathcal{E}) = F(|\Phi^+\rangle\langle\Phi^+|, (\mathcal{I} \otimes \mathcal{E})(|\Phi^+\rangle\langle\Phi^+|)) = \langle\Phi^+|\mathcal{I} \otimes \mathcal{E}(|\Phi^+\rangle\langle\Phi^+|)|\Phi^+\rangle,$$

which was first introduced in [14], quantifies how well the entanglement (correlations) between the system of interest and the ancilla system is preserved by the (local) channel $\mathcal{E}$ [14]–[20]. Here, $F(\rho, \sigma) = (\text{tr}(\rho^{1/2}\sigma\rho^{1/2})^{1/2})^2$ is the fidelity between the states ρ and σ.

By the use of the channel–state duality, the entanglement fidelity can be represented as the fidelity between the Jamiołkowski–Choi states $J_{\mathcal{E}}$ and $J_{\mathcal{I}}$, i.e.,

$$F(\mathcal{E}) = F(J_{\mathcal{E}}, J_{\mathcal{I}}).$$

Direct calculation shows that [46]

$$F(\mathcal{E}) = \frac{1}{d^2}\sum_k |\text{tr}\, E_k|^2,$$

which is intrinsically related to the average fidelity [47]

$$\overline{F}(\mathcal{E}) = \int F(|\phi\rangle\langle\phi|, \mathcal{E}(|\phi\rangle\langle\phi|))\, d\phi = \frac{1}{d+1} + \frac{1}{d(d+1)}\sum_k |\text{tr}\, E_k|^2$$

as

$$F(\mathcal{E}) = \left(1 + \frac{1}{d}\right)\overline{F}(\mathcal{E}) - \frac{1}{d}.$$

Here, $d\phi$ is the normalized Haar measure on all pure states in the system Hilbert space H.

Several nice properties of the entanglement fidelity are summarized as follows. Let $\mathcal{E}$ be a channel on a d-dimensional quantum system with a Hilbert space H.

1. $0 \leqslant F(\mathcal{E}) \leqslant 1$. Moreover, $F(\mathcal{E}) = 0$ if and only if all Kraus operators E_k of the channel $\mathcal{E}$ have a vanishing trace, and $F(\mathcal{E}) = 1$ if and only if $\mathcal{E}$ is the identity channel.

2. $F(\mathcal{E})$ is affine in the sense that

$$F(p_1\mathcal{E}_1 + p_2\mathcal{E}_2) = p_1 F(\mathcal{E}_1) + p_2 F(\mathcal{E}_2)$$

for $p_1, p_2 \geqslant 0$, $p_1 + p_2 = 1$, and any channels $\mathcal{E}_1$ and $\mathcal{E}_2$.

3. $F(\mathcal{E})$ is unitarily covariant in the sense that

$$F(\mathcal{U} \circ \mathcal{E} \circ \mathcal{U}^\dagger) = F(\mathcal{E})$$

for any unitary channel $\mathcal{U}(\rho) = U\rho U^\dagger$, where $\mathcal{U}^\dagger(\rho) = U^\dagger\rho U$ is its dual map and $\circ$ is the composition of maps. However, $F(\mathcal{E})$ is in general *not* invariant under composition with unitary channels in the sense that

$$F(\mathcal{E}_U \circ \mathcal{E}) = F(\mathcal{E} \circ \mathcal{E}_U) \neq F(\mathcal{E}),$$

where $\mathcal{E}_U(\rho) = U\rho U^\dagger$ for any unitary operator U on the system Hilbert space.

4. $F(\mathcal{E})$ is multiplicative under tensor product in the sense that

$$F(\mathcal{E}^a \otimes \mathcal{E}^b) = F(\mathcal{E}^a)F(\mathcal{E}^b),$$

for the channels $\mathcal{E}^a$, $\mathcal{E}^b$, and $\mathcal{E}^a \otimes \mathcal{E}^b$ on systems a and b and the composite system ab. In particular, when $\mathcal{E}^a = \mathcal{I}^a$, we have

$$F(\mathcal{I}^a \otimes \mathcal{E}^b) = F(\mathcal{E}^b),$$

which can be interpreted as the ancilla-independence of the entanglement fidelity.

The irreversibility $S(\mathcal{E})$ and the entanglement fidelity $F(\mathcal{E})$ have the following trade-off relation.

Proposition 3. *For any channel $\mathcal{E}$ on a d-dimensional system, we have*

$$\sqrt{\ln d - S(\mathcal{E})} + \sqrt{1 - F(\mathcal{E})} \geqslant 1 - \frac{1}{d^2}. \qquad (8)$$

A more transparent interpretation is to regard $R(\mathcal{E}) = \ln d - S(\mathcal{E})$ as a measure of reversibility, and $G(\mathcal{E}) = 1 - F(\mathcal{E})$ as a measure of infidelity; then (8) is equivalent to

$$\sqrt{R(\mathcal{E})} + \sqrt{G(\mathcal{E})} \geqslant 1 - \frac{1}{d^2}.$$

This proposition establishes a constraint between these two quantities. The reversibility and infidelity cannot be simultaneously small, which is consistent with our physical intuition.

Proof is given in Appendix C.

5. Degree of nonunitality of channels

We recall that a channel $\mathcal{E}$ is unital if $\mathcal{E}(\mathbf{1}/d) = \mathbf{1}/d$. Nonunitality refers to the deviation of the state $\mathcal{E}(\mathbf{1}/d)$ from the maximally mixed state $\mathbf{1}/d$. In terms of the relative entropy $S(\rho|\sigma) = \operatorname{tr}\rho(\ln\rho - \ln\sigma)$, a natural figure of merit for the nonunitality of $\mathcal{E}$ can be defined as

$$N(\mathcal{E}) = S(\mathcal{E}(\mathbf{1}/d)|\mathbf{1}/d) = \ln d - S(\mathcal{E}(\mathbf{1}/d)). \qquad (9)$$

This quantity has the following properties.

1. $0 \leqslant N(\mathcal{E}) \leqslant \ln d$. Moreover, $N(\mathcal{E}) = 0$ if and only if $\mathcal{E}$ is a unital channel, and $N(\mathcal{E}) = \ln d$ if and only if $\mathcal{E}$ is a channel satisfying $\mathcal{E}(\rho) = |\phi\rangle\langle\phi|$ for some pure state $|\phi\rangle$.

2. $N(\mathcal{E})$ is convex in the sense that

$$N(p_1\mathcal{E}_1 + p_2\mathcal{E}_2) \leqslant p_1 N(\mathcal{E}_1) + p_2 N(\mathcal{E}_2)$$

for $p_1, p_2 \geqslant 0$, $p_1 + p_2 = 1$, and any channels $\mathcal{E}_1$ and $\mathcal{E}_2$.

3. $N(\mathcal{E})$ is unitary invariant in the sense that

$$N(\mathcal{E}_U \circ \mathcal{E}) = N(\mathcal{E} \circ \mathcal{E}_U) = N(\mathcal{E}),$$

where $\mathcal{E}_U(\rho) = U\rho U^\dagger$ with U any unitary operator on H.

4. $N(\mathcal{E})$ is nonincreasing in the sense that

$$N(\mathcal{F} \circ \mathcal{E}) \leqslant N(\mathcal{E})$$

for any unital channel $\mathcal{F}$.

5. For any channels $\mathcal{E}^a$ and $\mathcal{E}^b$ on systems a and b, we have

$$N(\mathcal{E}^a \otimes \mathcal{E}^b) = N(\mathcal{E}^a) + N(\mathcal{E}^b).$$

In particular, if $\mathcal{E}^a = \mathcal{I}^a$ is an identity channel, we have

$$N(\mathcal{I}^a \otimes \mathcal{E}^b) = N(\mathcal{E}^b).$$

The above properties readily follow from the properties of von Neumann entropy.

We recall the Araki–Lieb inequality

$$|S(\rho^a) - S(\rho^b)| \leqslant S(\rho^{ab})$$

for any bipartite state ρ^{ab} on a composite system ab with reduced states ρ^a and ρ^b on respective parties a and b. By taking $\rho^{ab} = J_\mathcal{E}$, we derive a relation between irreversibility and nonunitality,

$$N(\mathcal{E}) \leqslant 2S(\mathcal{E}). \tag{10}$$

Thus, the degree of nonunitality is dominated by the degree of irreversibility, or, in other words, nonunitality accounts for part of the irreversibility.

6. Decorrelating power of channels

For a bipartite state ρ^{ab} on a composite system ab, the total correlations in ρ^{ab} can be quantified by the quantum mutual information

$$I(\rho^{ab}) = S(\rho^{ab}|\rho^a \otimes \rho^b) = S(\rho^a) + S(\rho^b) - S(\rho^{ab}). \tag{11}$$

Among all channels, the identity channel $\mathcal{I}$ is the channel that neither generates nor disturbs correlations. Thus, by the Jamiołkowski–Choi isomorphism, the difference of correlations between the Jamiołkowski–Choi states $J_\mathcal{I}$ and $J_\mathcal{E}$ associated with the channels $\mathcal{I}$ and $\mathcal{E}$ can be used to quantify the decorrelating power of the channel $\mathcal{E}$. Thus, we introduce

$$D(\mathcal{E}) = I(J_\mathcal{I}) - I(J_\mathcal{E}) \tag{12}$$

as a quantity characterizing the decorrelating power of $\mathcal{E}$.

In [48], the total loss of correlations was decomposed into classical and quantum parts. Here, we obtain a new decomposition for the loss of correlations: nonunitality and irreversibility. We state the result as follows.

Proposition 4. *We have*

$$D(\mathcal{E}) = N(\mathcal{E}) + 2S(\mathcal{E}), \tag{13}$$

which implies that the total correlations disturbed by $\mathcal{E}$ can be divided into two parts: one part quantifies the nonunitality of $\mathcal{E}$ and the other quantifies the irreversibility of $\mathcal{E}$.

Proof is given in Appendix D.

The decorrelating power $D(\mathcal{E})$ has the following properties.

Proposition 5. 1. *For any channel $\mathcal{E}$ on a d-dimensional quantum system, we have*

$$0 \leqslant D(\mathcal{E}) \leqslant 2 \ln d. \tag{14}$$

Moreover, $D(\mathcal{E}) = 0$ if and only if $\mathcal{E}$ is a unitary channel, and $D(\mathcal{E})$ attains the maximal value $2 \ln d$ if and only if $\mathcal{E}$ is a replacement channel, i.e., $\mathcal{E}(\rho) = \mathcal{E}(\mathbf{1}/d)$ for any state ρ.

2. *$D(\cdot)$ is concave in the sense that*

$$D(p_1 \mathcal{E}_1 + p_2 \mathcal{E}_2) \geqslant p_1 D(\mathcal{E}_1) + p_2 D(\mathcal{E}_2)$$

for $p_1, p_2 \geqslant 0$, $p_1 + p_2 = 1$, and any channels $\mathcal{E}_1$ and $\mathcal{E}_2$.

3. *$D(\cdot)$ is invariant under composition with unitary channels in the sense that*

$$D(\mathcal{E}_U \circ \mathcal{E}) = D(\mathcal{E} \circ \mathcal{E}_U) = D(\mathcal{E}),$$

where $\mathcal{E}_U(\rho) = U \rho U^\dagger$ with U any unitary operator on H.

4. *$D(\cdot)$ is ancilla-independent in the sense that*

$$D(\mathcal{I}^a \otimes \mathcal{E}) = D(\mathcal{E}),$$

where $\mathcal{I}^a$ is the identity channel on any ancilla system a.

5. *$D(\cdot)$ is monotonic in the sense that*
$$D(\mathcal{F} \circ \mathcal{E}) \geqslant D(\mathcal{E})$$

for any channel $\mathcal{F}$.

6. *$D(\cdot)$ is additive under tensor product in the sense that*

$$D(\mathcal{E}^a \otimes \mathcal{E}^b) = D(\mathcal{E}^a) + D(\mathcal{E}^b),$$

where $\mathcal{E}^a$, $\mathcal{E}^b$ are channels on the respective systems a and b.

Proof is given in Appendix E.

7. Evaluating the irreversibility of various channels

In this section, we evaluate the irreversibility of several basic and important channels and make some comparative studies. These quantitative results shed light on the structure of various channels from the perspective of irreversibility.

7.1. Unitary channel. For any unitary channel

$$\mathcal{E}_U(\rho) = U \rho U^\dagger,$$

where U is any unitary operator on a d-dimensional quantum system, we have

$$J_{\mathcal{E}_U} = |\Phi_U^+\rangle\langle\Phi_U^+|$$

where $|\Phi_U^+\rangle = \frac{1}{\sqrt{d}} \sum_i |i\rangle \otimes |i_U\rangle$ and $|i_U\rangle = U|i\rangle$ for $\{|i\rangle : i = 1, 2, \ldots, d\}$ is an orthonormal basis of the system Hilbert space. Consequently, $S(\mathcal{E}_U) = 0$, as it should be.

7.2. Lüders channel. For the Lüders channel [49]

$$\Pi(\rho) = \sum_{k=1}^{m} \Pi_k \rho \Pi_k$$

on a d-dimensional system determined by a Lüders (projective) measurement

$$\Pi = \left\{ \Pi_k = \sum_{j=n_{k-1}+1}^{n_k} |j\rangle\langle j| \colon k = 1, 2, \ldots, m \right\}, \qquad \sum_k \Pi_k = \mathbf{1},$$

where

$$\Pi_k^\dagger = \Pi_k, \quad \Pi_k \Pi_l = \delta_{kl} \Pi_k, \qquad n_k = \sum_{j=1}^{k} \operatorname{tr} \Pi_j \ \text{for}\ k \geqslant 1, \quad n_0 = 0,$$

we have

$$J_\Pi = \frac{1}{d} \sum_{k=1}^{m} \sum_{n_{k-1}+1 \leqslant j,l \leqslant n_k} |jj\rangle\langle ll|.$$

It is obvious that $\operatorname{tr} \Pi_k/d, k = 1, 2, \ldots, m$, are the nonzero eigenvalues of J_Π and

$$S(\Pi) = \frac{1}{2}\ln d - \frac{1}{2d} \sum_{k=1}^{m} (\operatorname{tr} \Pi_k) \ln(\operatorname{tr} \Pi_k).$$

It is easy to verify that the Lüders channels with the maximal irreversibility are

$$\mathcal{E}_{\mathrm{vN}}(\rho) = \sum_k \Pi_k \rho \Pi_k$$

which are induced by any von Neumann measurement $\{\Pi_k \colon k = 1, 2, \ldots, d\}$ (i.e., $\operatorname{tr} \Pi_k = 1$ for all k). In this case,

$$S(\mathcal{E}_{\mathrm{vN}}) = \frac{1}{2}\ln d.$$

This reveals an interesting feature of von Neumann measurements: the irreversibility of any von Neumann measurement stands exactly half the way between the two extreme values of irreversibility (the minimum 0 and the maximum $\ln d$).

7.3. Channel induced by SIC-POVM. We recall that a SIC-POVM (symmetric informationally complete positive operator-valued measure) on a d-dimensional system is a set of d^2 rank-one operators $E_k = \frac{1}{d}|\phi_k\rangle\langle\phi_k|, k = 1, 2, \ldots, d^2$, satisfying the relations [50]

$$|\langle \phi_k | \phi_l \rangle|^2 = \begin{cases} 1, & k = l, \\ \dfrac{1}{d+1}, & k \neq l. \end{cases}$$

Any SIC-POVM naturally induces a channel

$$\mathcal{E}_{\mathrm{sic}}(\rho) = \sum_k \sqrt{E_k}\, \rho \sqrt{E_k}.$$

By straightforward calculation, we obtain

$$J_{\mathcal{E}_{\mathrm{sic}}} = \frac{1}{d^2} \sum_k |\bar{\phi}_k\rangle\langle\bar{\phi}_k| \otimes |\phi_k\rangle\langle\phi_k|.$$

436

Here, $|\bar{\phi}\rangle$ denotes the complex conjugate of the vector $|\phi\rangle$ with respect to the basis $\{|i\rangle\colon i = 1, 2, \ldots, d\}$ of H. Let

$$|\psi_1\rangle = \frac{1}{d}\sum_k |\bar{\phi}_k\rangle \otimes |\phi_k\rangle, \qquad |\psi_l\rangle = \frac{1}{\sqrt{d^2+1}}(d|\bar{\phi}_l\rangle \otimes |\phi_l\rangle - |\psi_1\rangle), \quad l = 2, 3, \ldots, d^2.$$

It is then easy to verify that $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_{d^2}\rangle$ are linearly independent and

$$J_{\mathcal{E}_{\mathrm{sic}}}|\psi_1\rangle = \frac{1}{d}|\psi_1\rangle, \quad J_{\mathcal{E}_{\mathrm{sic}}}|\psi_l\rangle = \frac{1}{d(d+1)}|\psi_l\rangle, \qquad l = 2, 3, \ldots, d^2.$$

Thus, $J_{\mathcal{E}_{\mathrm{sic}}}$ has two different eigenvalues $1/d$ and $1/d(d+1)$ with the multiplicities respective 1 and $d^2 - 1$, and the irreversibility of $\mathcal{E}_{\mathrm{sic}}$ can be evaluated as

$$S(\mathcal{E}_{\mathrm{sic}}) = \frac{1}{2}\ln d + \frac{d-1}{2d}\ln(d+1).$$

For large d, this approximates the maximum $\ln d$, which implies that the irreversibility of the channel induced by a SIC-POVM in higher dimensions is asymptotically maximal.

7.4. The Werner–Holevo channel.

The Werner–Holevo channel [51]

$$\mathcal{E}_{\mathrm{WH}}(\rho) = \frac{1}{d-1}(\mathbf{1} - \rho^{\mathrm{T}})$$

provides a counterexample to the additivity conjecture for the output purity of channels [51]. Here, ρ^{T} is the transpose of ρ in an orthonormal basis $\{|i\rangle\colon i = 1, 2, \ldots, d\}$ of H. It is known that a Kraus representation of $\mathcal{E}_{\mathrm{WH}}$ is

$$\mathcal{E}_{\mathrm{WH}}(\rho) = \frac{1}{2(d-1)}\sum_{i,j}(|i\rangle\langle j| - |j\rangle\langle i|)\rho(|i\rangle\langle j| - |j\rangle\langle i|)^{\dagger}.$$

Direct calculation shows that

$$J_{\mathcal{E}_{\mathrm{WH}}} = \frac{1}{d(d-1)}\left(\mathbf{1}\otimes\mathbf{1} - \sum_{i,j}|ij\rangle\langle ji|\right),$$

which is a Werner state and has two different eigenvalues 0 and $2/(d^2 - d)$ with the respective multiplicities $(d^2 + d)/2$ and $(d^2 - d)/2$. Thus, the irreversibility of $\mathcal{E}_{\mathrm{WH}}$ can be directly evaluated as

$$S(\mathcal{E}_{\mathrm{WH}}) = \frac{1}{2}\ln\frac{d^2 - d}{2}.$$

In particular, if $d = 2$, the Werner–Holevo channel reduces to the unitary channel

$$\mathcal{E}_{\mathrm{WH}}(\rho) = \mathrm{tr}(\rho)\mathbf{1} - \rho^{\mathrm{T}} = \sigma_2\rho\sigma_2$$

where σ_2 is the second Pauli matrix. In this case, $S(\mathcal{E}_{\mathrm{WH}}) = 0$.

7.5. Completely depolarizing channel.

For the completely depolarizing channel

$$\mathcal{E}_{\mathrm{cde}}(\rho) = \frac{1}{d}\sum_k X_k\rho X_k = \frac{1}{d}\mathbf{1},$$

where $\{X_k\colon k = 1, 2, \ldots, d^2\}$ is an orthonormal basis of $L(H)$ consisting of all Hermitian operators on the system H, we have

$$J_{\mathcal{E}_{\mathrm{cde}}} = \frac{1}{d^2}\mathbf{1}\otimes\mathbf{1},$$

whence $S(\mathcal{E}_{\mathrm{cde}}) = \ln d$. Therefore, in view of Eq. (7), we see that the completely depolarizing channel attains the maximal value of irreversibility. This is reasonable, as is already suggested by the name of this channel.

7.6. Completely decoherent channel. We recall that a completely decoherent channel on a qubit system is defined as [52], [53], [48]

$$\mathcal{E}_{cd}(\rho) = M * \rho,$$

where M is a nonnegative-definite matrix with all diagonal elements being 1 (i.e., a correlation matrix), and $*$ denotes the Hadamard (entry-wise) product of matrices. For simplicity, we consider the 2×2 matrix

$$M = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}, \qquad -1 \leqslant \alpha \leqslant 1.$$

It is easy to verify that the channel can be expressed as $\mathcal{E}_{cd}(\rho) = \sum_k E_k \rho E_k^\dagger$ with the Kraus operators

$$E_1 = \begin{pmatrix} \sqrt{1 - |\alpha|} & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{1 - |\alpha|} \end{pmatrix}, \qquad E_3 = \sqrt{|\alpha|} \begin{pmatrix} 1 & 0 \\ 0 & \operatorname{sgn}\alpha \end{pmatrix}.$$

Direct calculation shows that

$$J_{\mathcal{E}_{cd}} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 1 \end{pmatrix}.$$

The nonzero eigenvalues of $J_{\mathcal{E}_{cd}}$ are $(1 + \alpha)/2$, $(1 - \alpha)/2$ and the irreversibility of $\mathcal{E}_{cd}$ can be evaluated as

$$S(\mathcal{E}_{cd}) = -\frac{1 + \alpha}{4} \ln \frac{1 + \alpha}{2} - \frac{1 - \alpha}{4} \ln \frac{1 - \alpha}{2}.$$

7.7. Amplitude-damping channel. For the amplitude-damping channel

$$\mathcal{E}_{ad}(\rho) = \sum_k E_k \rho E_k^\dagger$$

on a qubit system with the Kraus operators

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - p} \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}, \qquad 0 \leqslant p \leqslant 1,$$

we have

$$J_{\mathcal{E}_{ad}} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & \sqrt{1 - p} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 \\ \sqrt{1 - p} & 0 & 0 & 1 - p \end{pmatrix}$$

which has the nonzero eigenvalues $p/2$ and $1 - p/2$. Consequently,

$$S(\mathcal{E}_{ad}) = -\frac{p}{4} \ln \frac{p}{2} - \frac{2 - p}{4} \ln \frac{2 - p}{2}.$$

We see that $S(\mathcal{E}_{ad})$ is increasing with the parameter p.

438

7.8. Phase-damping channel. For the phase-damping channel

$$\mathcal{E}_{\mathrm{pd}}(\rho) = \sum_k E_k \rho E_k^\dagger$$

on a qubit system with the Kraus operators

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix}, \qquad 0 \leqslant p \leqslant 1,$$

we have

$$J_{\mathcal{E}_{\mathrm{pd}}} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & \sqrt{1-p} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{1-p} & 0 & 0 & 1 \end{pmatrix}$$

which has the nonzero eigenvalues $p' = (1 + \sqrt{1-p})/2$ and $1 - p'$. Consequently,

$$S(\mathcal{E}_{\mathrm{pd}}) = -\frac{1}{2}\big(p' \ln p' + (1-p') \ln(1-p')\big).$$

We see that $S(\mathcal{E}_{\mathrm{pd}})$ is also an increasing function of the parameter p.

In this context, it is interesting to compare the amplitude-damping channel and the phase-damping channel for the same parameter p: $S(\mathcal{E}_{\mathrm{ad}}) \geqslant S(\mathcal{E}_{\mathrm{pd}})$, which is illustrated in Fig. 1. We see that for the same parameter p, the amplitude-damping channel is more irreversible than the phase-damping channel.
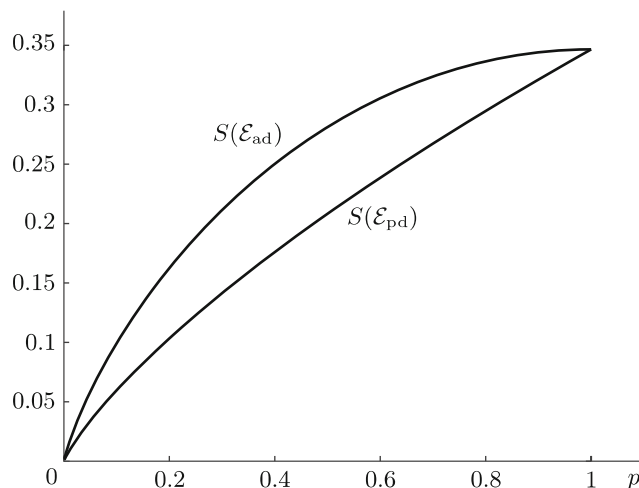

Fig. 1. Comparison between the degree of irreversibility of the amplitude-damping channel $\mathcal{E}_{\mathrm{ad}}$ and the phase-damping channel $\mathcal{E}_{\mathrm{pd}}$ for the parameter $p \in [0,1]$. We see that except in the two extreme cases ($p = 0$ or $p = 1$), the amplitude-damping channel $\mathcal{E}_{\mathrm{ad}}$ is more irreversible than the phase-damping channel $\mathcal{E}_{\mathrm{pd}}$.

7.9. Channel induced by weak measurements. We consider the channel

$$\mathcal{K}(\rho) = E_x \rho E_x + E_{1-x} \rho E_{1-x}, \qquad x \in [0, 1/2),$$

associated with the weak measurement $\{E_x, E_{1-x}\}$ with $E_x = \sqrt{1-x}\,\Pi_0 + \sqrt{x}\,\Pi_1$. Here, $\{\Pi_0, \Pi_1\}$ is a Lüders measurement in a d-dimensional system. In particular, as $x \to 0$, the weak measurement tends to the Lüders measurement $\{\Pi_0, \Pi_1\}$. The corresponding Jamiołkowski–Choi state is

$$J_{\mathcal{K}} = \frac{1}{d}\left(\sum_{1 \leqslant i,j \leqslant \operatorname{tr}\Pi_0} |ii\rangle\langle jj| + \sum_{\operatorname{tr}\Pi_0 + 1 \leqslant i,j \leqslant d} |ii\rangle\langle jj| + 2\sqrt{x(1-x)} \sum_{\substack{1 \leqslant i \leqslant \operatorname{tr}\Pi_0, \\ \operatorname{tr}\Pi_0 + 1 \leqslant j \leqslant d}} (|ii\rangle\langle jj| + |jj\rangle\langle ii|) \right)$$

and the irreversibility of $\mathcal{K}$ can be evaluated as

$$S(\mathcal{K}) = -\frac{1}{2}(\lambda_+ \ln \lambda_+ + \lambda_- \ln \lambda_-),$$

where

$$\lambda_\pm = \frac{1}{2}\left(1 \pm \left(1 - \frac{4(2x-1)^2}{d^2} \operatorname{tr}\Pi_0 \operatorname{tr}\Pi_1 \right)^{1/2} \right)$$

are the nonzero eigenvalues of $J_{\mathcal{K}}$. In particular, for a qubit weak measurement $\mathcal{E}_x = \mathcal{K}$ with $\operatorname{tr}\Pi_0 = \operatorname{tr}\Pi_1 = 1$, $d = 2$, we further obtain

$$S(\mathcal{E}_x) = -\frac{1}{2}\big(p_x \ln p_x + (1 - p_x)\ln(1 - p_x)\big)$$

with $p_x = 1/2 + \sqrt{x(1-x)}$.

7.10. Measurement-preparation channel. We consider the measurement-preparation channel

$$\mathcal{E}_{\mathrm{mp}}(\rho) = \sum_k \operatorname{tr}(\rho M_k)\tau_k$$

with τ_k quantum states and $M = \{M_k\}$ a quantum measurement (POVM) such that $\sum_k M_k = \mathbf{1}$. The corresponding Jamiołkowski–Choi state is

$$J_{\mathcal{E}_{\mathrm{mp}}} = \frac{1}{d}\sum_k M_k^{\mathrm{T}} \otimes \tau_k,$$

where T denotes the transposition of a matrix (of an operator relative to a basis). Consequently,

$$S(\mathcal{E}_{\mathrm{mp}}) = \frac{1}{2}S\left(\frac{1}{d}\sum_k M_k^{\mathrm{T}} \otimes \tau_k \right).$$

In particular, when $M = \Pi = \{\Pi_k : k = 1, 2, \ldots, m\}$ is a Lüders measurement, we have

$$S(\mathcal{E}_{\mathrm{mp}}) = \frac{1}{2}\ln d + \frac{1}{2d}\sum_k \operatorname{tr}(\Pi_k)S(\tau_k).$$

7.11. Teleportation channel. Let ρ^{ab} be a two-qubit state shared by Alice and Bob. The quantum teleportation via the resource state ρ^{ab} provides a way to transmit an unknown state γ from the sender Alice to the receiver Bob with fidelity better than the classical limit $2/3$ via a classical channel [19], [54]–[58]. It has been shown that the standard teleportation can be described by a generalized depolarizing channel as [58]

$$\mathcal{E}_{\mathrm{t}}(\gamma) = \sum_{k=0}^{3} p_k \sigma_k \gamma \sigma_k,$$

where $\sigma_0 = \mathbf{1}$, σ_k are the Pauli matrices, and $p_k = \text{tr}(\rho^{ab} M_k)$ with the Bell measurement operators

$$M_k = (\sigma_k \otimes \mathbf{1})|\Phi^+\rangle\langle\Phi^+|(\sigma_k \otimes \mathbf{1}), \qquad |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

For the resource state

$$\rho^{ab} = \frac{1-p}{4}\mathbf{1} \otimes \mathbf{1} + p|\Phi^+\rangle\langle\Phi^+|, \qquad 0 \leqslant p \leqslant 1,$$

the irreversibility of the teleportation channel $\mathcal{E}_t$ can be directly evaluated as

$$S(\mathcal{E}_t) = -\frac{1+3p}{8}\ln\frac{1+3p}{4} - \frac{3(1-p)}{8}\ln\frac{1-p}{4},$$

which is decreasing with the fraction parameter $p \in [0,1]$. This is consistent with intuition because larger p correspond to greater entanglement and hence to greater fidelity (less irreversibility).

The entanglement fidelity and the average fidelity of the teleportation channel $\mathcal{E}_t$ can be directly evaluated as

$$F(\mathcal{E}_t) = \frac{1+3p}{4}, \qquad \overline{F}(\mathcal{E}_t) = \frac{1+p}{2}.$$

Thus, the average fidelity of the teleportation protocol is greater than $2/3$ (the best possible fidelity when Alice and Bob communicate only through a classical channel) if and only if $p > 1/3$. For a two-qubit state ρ^{ab}, it has been shown that ρ^{ab} is entangled if and only if $p > 1/3$ [56], which implies that the entanglement of ρ^{ab} is a prerequisite for teleportation. In this case, $p > 1/3$ if and only if the irreversibility is $S(\mathcal{E}_t) < (\ln 12)/4$. This sheds some light on the teleportation protocol from the perspective of irreversibility.
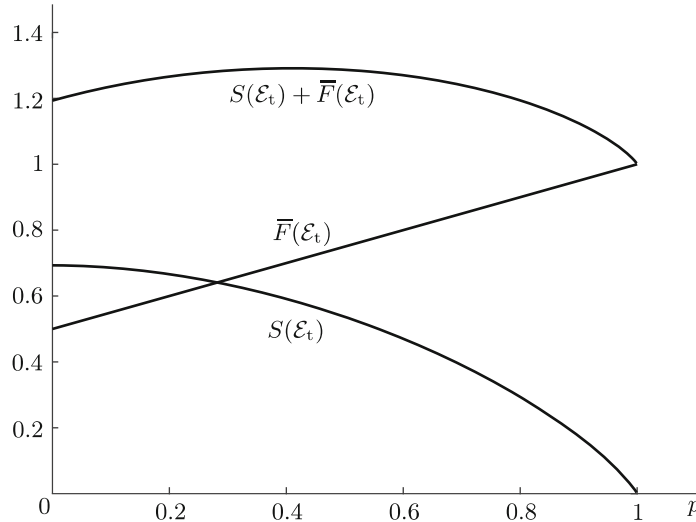

Fig. 2. Comparison between the irreversibility $S(\mathcal{E}_t)$ and the average fidelity $\overline{F}(\mathcal{E}_t)$ as functions of $p \in [0,1]$.

To visualize the difference between the irreversibility $S(\mathcal{E}_t)$ and the average fidelity $\overline{F}(\mathcal{E}_t)$ for the teleportation channel $\mathcal{E}_t$, we depict their behavior depending on the fraction parameter p in Fig. 2. To reflect the trade-off relation between them intuitively, we also depict the behavior of the sum $S(\mathcal{E}_t) + \overline{F}(\mathcal{E}_t)$, whence we find that the irreversibility $S(\mathcal{E}_t)$ and the fidelity $\overline{F}(\mathcal{E}_t)$ satisfy the trade-off relation

$$1 \leqslant S(\mathcal{E}_t) + \overline{F}(\mathcal{E}_t) \leqslant 1.3.$$

8. Summary

Motivated by intuitive and physical considerations, we have postulated some basic features for any reasonable measure of irreversibility, and have presented a concrete realization of the postulated requirements by quantifying the irreversibility of channels in terms of the entropy of the corresponding Jamiołkowski–Choi states. This is achieved by using the channel–state duality via the Jamiołkowski–Choi isomorphism between channels and states. We have further evaluated the irreversibility of several important channels, which highlights some basic features of channels from the perspective of irreversibility. We have also established several relations connecting irreversibility with entanglement fidelity, entropy exchange, nonunitality, and decorrelating power.

The measure of irreversibility has the apparent operational meaning as the entropy of the Jamiołkowski–Choi states of the channels, and thus may be used in characterizing information-theoretic aspects of quantum measurements and channels.

Irreversibility is a fundamental characteristic in information processing, and its quantitative characterization sheds light on the nature of channels and physical processes. It is desirable to find more applications of the measure of irreversibility, in particular, in the paradigm of quantum thermodynamics, where heat is usually related to irreversible processes.

Appendix: Proofs of Propositions 1–5

Below, we give the detailed proofs of Propositions 1–5. We further discuss an alternative measure of irreversibility in terms of the Tsallis entropy, which is easier to compute than the irreversibility based on the von Neumann entropy. We make a comparative study of this quantity and that introduced in the main text.

Appendix A: Proof of Proposition 1

1. By the properties of von Neumann entropy, we have $0 \leqslant S(J_{\mathcal{E}}) \leqslant \ln d^2$, $S(J_{\mathcal{E}}) = 0$ if and only if $J_{\mathcal{E}}$ is a pure state, and $S(J_{\mathcal{E}}) = \ln d^2$ if and only if $J_{\mathcal{E}} = \frac{1}{d^2}\mathbf{1} \otimes \mathbf{1}$ is a maximally mixed state on $H \otimes H$. Consequently,

$$0 \leqslant S(\mathcal{E}) = \frac{1}{2}S(J_{\mathcal{E}}) \leqslant \ln d.$$

It is clear that $S(\mathcal{E}) = 0$ if and only if $J_{\mathcal{E}}$ is a pure state, and $S(\mathcal{E}) = \ln d$ if and only if $J_{\mathcal{E}} = \mathbf{1} \otimes \mathbf{1}/d^2$ is a maximally mixed state on $H \otimes H$. Next, we show that $J_{\mathcal{E}}$ is a pure state if and only if $\mathcal{E}$ is a unitary channel, and $J_{\mathcal{E}} = \frac{1}{d^2}\mathbf{1} \otimes \mathbf{1}$ is equivalent to $\mathcal{E}$ being the completely depolarizing channel. For the first equivalence, we suppose that $\mathcal{E}(\rho) = U\rho U^\dagger$ is a unitary channel; then

$$J_{\mathcal{E}} = |\Phi_U^+\rangle\langle\Phi_U^+|,$$

where $|\Phi_U^+\rangle = \frac{1}{\sqrt{d}}\sum_i |i\rangle \otimes U|i\rangle$ is a pure state, with $S(\mathcal{E}) = 0$. Conversely, we suppose that

$$J_{\mathcal{E}} = \frac{1}{d}\sum_{i,j} |i\rangle\langle j| \otimes \mathcal{E}(|i\rangle\langle j|)$$

is a pure state; then $J_{\mathcal{E}}$ is a rank-1 operator, which makes $\mathcal{E}(|i\rangle\langle j|)$ an operator with rank not greater than 1 for any i and j. Because $\mathcal{E}(|i\rangle\langle i|)$ are quantum states, $\mathrm{rank}(\mathcal{E}(|i\rangle\langle i|)) = 1$ for any $i = 1, 2, \ldots, d$. If there is $\mathcal{E}(|i\rangle\langle j|) = 0$ for some $i \neq j$, we have $\mathrm{rank}(J_{\mathcal{E}}) \geqslant \mathrm{rank}(\mathcal{E}(|i\rangle\langle i|)) + \mathrm{rank}(\mathcal{E}(|j\rangle\langle j|)) = 2$, which contradicts the

rank condition $\text{rank}(J_{\mathcal{E}}) = 1$. Thus, $\mathcal{E}(|i\rangle\langle j|)$ is a rank-1 operator for any i and j. Combining $\text{rank}(J_{\mathcal{E}}) = 1$ and $\text{rank}(\mathcal{E}(|i\rangle\langle i|)) = 1$ for any i, we see that $\mathcal{E}(|i\rangle\langle j|)$ can be written as

$$\mathcal{E}(|i\rangle\langle j|) = |\phi_i\rangle\langle\phi_j|, \qquad i, j = 1, 2, \ldots, d,$$

where $\{|\phi_i\rangle : i = 1, 2, \ldots, d\}$ is a set of pure states. In what follows, we show that $\{|\phi_i\rangle : i = 1, 2, \ldots, d\}$ constitutes an orthonormal basis. By

$$\mathcal{E}(|i\rangle\langle i|) = \sum_k E_k|i\rangle\langle i|E_k^\dagger = |\phi_i\rangle\langle\phi_i|,$$

we have $E_k|i\rangle = c_{ki}|\phi_i\rangle$ for some complex numbers c_{ki}. Thus,

$$\mathcal{E}(|i\rangle\langle j|) = \left(\sum_k c_{ki}\bar{c}_{kj}\right)|\phi_i\rangle\langle\phi_j| = |\phi_i\rangle\langle\phi_j|,$$

which implies that $\sum_k c_{ki}\bar{c}_{kj} = 1$ for any i, j. The orthogonality of $\{|\phi_i\rangle : i = 1, 2, \ldots, d\}$ follows from

$$\langle i|j\rangle = \left\langle i\left|\sum_k E_k^\dagger E_k\right|j\right\rangle = \left(\sum_k \bar{c}_{ki}c_{kj}\right)\langle\phi_i|\phi_j\rangle = \langle\phi_i|\phi_j\rangle, \qquad i, j = 1, 2, \ldots, d.$$

Thus, $\mathcal{E}$ maps the orthonormal basis $\{|i\rangle : i = 1, 2, \ldots, d\}$ to the orthonormal basis $\{|\phi_i\rangle : i = 1, 2, \ldots, d\}$. Because $S(\mathcal{E})$ is independent of the choice of the orthonormal basis $\{|i\rangle : i = 1, 2, \ldots, d\}$, $\mathcal{E}$ maps any orthonormal basis to an orthonormal basis, which implies that $\mathcal{E}$ is a unitary channel.

To prove the second equivalence, we suppose that $\mathcal{E}$ is the completely depolarizing channel. Then

$$J_{\mathcal{E}} = \frac{1}{d}\sum_{i,j}|i\rangle\langle j| \otimes \mathcal{E}(|i\rangle\langle j|) = \frac{1}{d^2}\mathbf{1} \otimes \mathbf{1}$$

is the maximally mixed state on $H \otimes H$. Conversely, if $J_{\mathcal{E}} = \mathbf{1} \otimes \mathbf{1}/d^2$ is the maximally mixed state, then we conclude from

$$J_{\mathcal{E}} = \frac{1}{d}\sum_{i,j}|i\rangle\langle j| \otimes \mathcal{E}(|i\rangle\langle j|) = \frac{1}{d}\sum_i|i\rangle\langle i| \otimes \mathcal{E}(|i\rangle\langle i|) + \frac{1}{d}\sum_{i\neq j}|i\rangle\langle j| \otimes \mathcal{E}(|i\rangle\langle j|)$$

that $\mathcal{E}(|i\rangle\langle j|) = 0$ for any $i \neq j$ and $\mathcal{E}(|i\rangle\langle i|) = \mathbf{1}/d$ for any i. Therefore,

$$\mathcal{E}(\rho) = \mathcal{E}\left(\sum_{i,j}\langle i|\rho|j\rangle|i\rangle\langle j|\right) = \sum_{i,j}\langle i|\rho|j\rangle\mathcal{E}(|i\rangle\langle j|) = \frac{1}{d}\mathbf{1}$$

for any state ρ, i.e., $\mathcal{E}$ is the completely depolarizing channel.

2. Direct calculation shows that

$$J_{p_1\mathcal{E}_1 + p_2\mathcal{E}_2} = p_1 J_{\mathcal{E}_1} + p_2 J_{\mathcal{E}_2}.$$

By the concavity of von Neumann entropy, we have

$$S(p_1\mathcal{E}_1 + p_2\mathcal{E}_2) = \frac{1}{2}S(J_{p_1\mathcal{E}_1 + p_2\mathcal{E}_2}) = \frac{1}{2}S(p_1 J_{\mathcal{E}_1} + p_2 J_{\mathcal{E}_2}) \geqslant$$
$$\geqslant \frac{1}{2}\left(p_1 S(J_{\mathcal{E}_1}) + p_2 S(J_{\mathcal{E}_2})\right) = p_1 S(\mathcal{E}_1) + p_2 S(\mathcal{E}_2).$$

443

3. Let U and V be any unitary operators. Then

$$J_{\mathcal{E}_U \circ \mathcal{E} \circ \mathcal{E}_V} = \frac{1}{d}\sum_{i,j}|i\rangle\langle j| \otimes U\mathcal{E}(V|i\rangle\langle j|V^\dagger)U^\dagger =$$

$$= (V^\dagger \otimes U)\left(\frac{1}{d}\sum_{i,j}V|i\rangle\langle j|V^\dagger \otimes \mathcal{E}(V|i\rangle\langle j|V^\dagger)\right)(V^\dagger \otimes U)^\dagger.$$

Thus

$$S(\mathcal{E}_U \circ \mathcal{E} \circ \mathcal{E}_V) = \frac{1}{2}S(J_{\mathcal{E}_U \circ \mathcal{E} \circ \mathcal{E}_V}) = \frac{1}{2}S\left(\frac{1}{d}\sum_{i,j}V|i\rangle\langle j|V^\dagger \otimes \mathcal{E}(V|i\rangle\langle j|V^\dagger)\right) = \frac{1}{2}S(J_\mathcal{E}) = S(\mathcal{E})$$

whence item 3 follows.

4. Let H^a and H have the respective dimensions d_a and d, and orthonormal bases $\{|\mu\rangle\}$ and $\{|i\rangle\}$. For the channel $\mathcal{I}^a \otimes \mathcal{E}$ on the composite system Hilbert space $H^a \otimes H$, we have

$$d_a d J_{\mathcal{I}^a \otimes \mathcal{E}} = \sum_{\mu,\nu,i,j}|\mu\rangle\langle\nu| \otimes |i\rangle\langle j| \otimes |\mu\rangle\langle\nu| \otimes \mathcal{E}(|i\rangle\langle j|) =$$

$$= \sum_{\mu,\nu,i,j}(\mathbf{1}^a \otimes F \otimes \mathbf{1})(|\mu\rangle\langle\nu| \otimes |\mu\rangle\langle\nu| \otimes |i\rangle\langle j| \otimes \mathcal{E}(|i\rangle\langle j|))(\mathbf{1}^a \otimes F^\dagger \otimes \mathbf{1}) =$$

$$= d_a d(\mathbf{1}^a \otimes F \otimes \mathbf{1})(|\Phi_a^+\rangle\langle\Phi_a^+| \otimes J_\mathcal{E})(\mathbf{1}^a \otimes F^\dagger \otimes \mathbf{1}),$$

where $F = \sum_{\mu i}|i\rangle\langle\mu| \otimes |\mu\rangle\langle i|$ is the swap operator on the composite system Hilbert space $H^a \otimes H$ and $|\Phi_a^+\rangle = \frac{1}{\sqrt{d_a}}\sum_\mu |\mu\rangle \otimes |\mu\rangle$. Consequently,

$$S(\mathcal{I}^a \otimes \mathcal{E}) = \frac{1}{2}S(J_{\mathcal{I}^a \otimes \mathcal{E}}) = \frac{1}{2}S\big((\mathbf{1}^a \otimes F \otimes \mathbf{1})(|\Phi_a^+\rangle\langle\Phi_a^+| \otimes J_\mathcal{E})(\mathbf{1}^a \otimes F^\dagger \otimes \mathbf{1})\big) =$$

$$= \frac{1}{2}S(J_\mathcal{E} \otimes |\Phi_a^+\rangle\langle\Phi_a^+|) = \frac{1}{2}S(J_\mathcal{E}) = S(\mathcal{E}),$$

whence item 4 follows.

5. Let $\mathcal{F}$ be a unital channel on a d-dimensional system satisfying $\mathcal{F}(\mathbf{1}) = \mathbf{1}$. By the definition of Jamiołkowski–Choi states, we have $J_{\mathcal{F} \circ \mathcal{E}} = \mathcal{I} \otimes \mathcal{F}(J_\mathcal{E})$. It is obvious that $\mathcal{I} \otimes \mathcal{F}$ is also a unital channel. In view of the monotonicity of von Neumann entropy for a unital channel, we obtain

$$S(\mathcal{F} \circ \mathcal{E}) = \frac{1}{2}S(J_{\mathcal{F} \circ \mathcal{E}}) = \frac{1}{2}S(\mathcal{I} \otimes \mathcal{F}(J_\mathcal{E})) \geqslant \frac{1}{2}S(J_\mathcal{E}) = S(\mathcal{E}).$$

6. Let $\mathcal{E}^a$ and $\mathcal{E}^b$ be channels on systems a and b, with orthonormal bases $\{|\mu\rangle\}$ and $\{|i\rangle\}$. Because

$$J_{\mathcal{E}^a \otimes \mathcal{E}^b} = \frac{1}{d_a d_b}\sum_{\mu\nu ij}|\mu\rangle\langle\nu| \otimes |i\rangle\langle j| \otimes \mathcal{E}^a(|\mu\rangle\langle\nu|) \otimes \mathcal{E}^b(|i\rangle\langle j|),$$

we have

$$(\mathbf{1}^a \otimes F_{ab}^\dagger \otimes \mathbf{1}^b)J_{\mathcal{E}^a \otimes \mathcal{E}^b}(\mathbf{1}^a \otimes F_{ab} \otimes \mathbf{1}^b) =$$

$$= \frac{1}{d_a d_b}\sum_{\mu,\nu,i,j}|\mu\rangle\langle\nu| \otimes \mathcal{E}^a(|\mu\rangle\langle\nu|) \otimes |i\rangle\langle j| \otimes \mathcal{E}^b(|i\rangle\langle j|) = J_{\mathcal{E}^a} \otimes J_{\mathcal{E}^b},$$

where $F_{ab} = \sum_{\mu i}|\mu\rangle\langle i| \otimes |i\rangle\langle\mu|$ is the swap operator on the system Hilbert space $H^a \otimes H^b$. Consequently,

$$S(\mathcal{E}^a \otimes \mathcal{E}^b) = \frac{1}{2}S(J_{\mathcal{E}^a \otimes \mathcal{E}^b}) = \frac{1}{2}S\big((\mathbf{1}^a \otimes F_{ab}^\dagger \otimes \mathbf{1}^b)J_{\mathcal{E}^a \otimes \mathcal{E}^b}(\mathbf{1}^a \otimes F_{ab}^\dagger \otimes \mathbf{1}^b)\big) =$$

$$= \frac{1}{2}S(J_{\mathcal{E}^a} \otimes J_{\mathcal{E}^b}) = \frac{1}{2}S(J_{\mathcal{E}^a}) + \frac{1}{2}S(J_{\mathcal{E}^b}) = S(\mathcal{E}^a) + S(\mathcal{E}^b),$$

whence item 6 follows.

Appendix B: Proof of Proposition 2

For any random unitary channel $\mathcal{E}_{\mathrm{ru}}$, we have

$$J_{\mathcal{E}_{\mathrm{ru}}} = \sum_k p_k \left(\frac{1}{d} \sum_{i,j} |i\rangle\langle j| \otimes U_k |i\rangle\langle j| U_k^\dagger \right) = \sum_k p_k J_{\mathcal{E}_{U_k}},$$

whence

$$0 \leqslant S(J_{\mathcal{E}_{\mathrm{ru}}}) = S\left(\sum_k p_k J_{\mathcal{E}_{U_k}} \right) \leqslant H(\{p_k\}) + \sum_k p_k S(J_{\mathcal{E}_{U_k}}) = H(\{p_k\}),$$

which implies that

$$0 \leqslant S(\mathcal{E}_{\mathrm{ru}}) \leqslant \frac{1}{2} H(\{p_k\}).$$

By the properties of von Neumann entropy, $S(J_{\mathcal{E}_{\mathrm{ru}}}) = 0$ if and only if $\mathcal{E}_{\mathrm{ru}}$ is actually a unitary channel and $S(J_{\mathcal{E}_{\mathrm{ru}}}) = H(\{p_k\})$ if and only if the states $J_{\mathcal{E}_{U_k}}$ have support on orthogonal subspaces, i.e., $\mathrm{tr}(J_{\mathcal{E}_{U_k}} J_{\mathcal{E}_{U_l}}) = 0$ for any $k \neq l$. Direct manipulation of the definition of a Jamiołkowski–Choi state shows that

$$\mathrm{tr}(J_{\mathcal{E}_{U_k}} J_{\mathcal{E}_{U_l}}) = \frac{1}{d^2} |\mathrm{tr}\, U_k^\dagger U_l|^2.$$

Therefore, $S(J_{\mathcal{E}_{\mathrm{ru}}}) = H(\{p_k\})$ if and only if $\mathrm{tr}(U_k^\dagger U_l) = 0$ for any $k \neq l$.

Appendix C: Proof of Proposition 3

To prove Proposition 3, we first recall Pinsker's inequality, which states that [59]

$$S(\rho|\sigma) \geqslant 2T^2(\rho, \sigma) \tag{15}$$

for any states ρ and σ, where $S(\rho|\sigma) = \mathrm{tr}\,\rho(\ln\rho - \ln\sigma)$ is the quantum relative entropy, and

$$T(\rho, \sigma) = \frac{1}{2}\,\mathrm{tr}\,|\rho - \sigma|$$

is the trace distance between states ρ and σ with $|A| = \sqrt{A^\dagger A}$ for any operator A.

By Pinsker's inequality (15) and the triangle inequality for the trace distance, we have

$$S(J_{\mathcal{E}}|J_{\mathrm{cde}}) \geqslant 2T^2(J_{\mathcal{E}}, J_{\mathrm{cde}}) \geqslant 2\big(T(J_{\mathcal{E}}, J_{\mathcal{I}}) - T(J_{\mathcal{I}}, J_{\mathrm{cde}})\big)^2 = 2\left(1 - \frac{1}{d^2} - T(J_{\mathcal{E}}, J_{\mathcal{I}}) \right)^2,$$

where the equality holds by virtue of $T(J_{\mathcal{I}}, J_{\mathrm{cde}}) = 1 - 1/d^2$. By the well-known inequality relating fidelity and trace distance [11]

$$T(\rho, \sigma) \leqslant \sqrt{1 - F(\rho, \sigma)},$$

we further have

$$\sqrt{S(J_{\mathcal{E}}|J_{\mathrm{cde}})} \geqslant \sqrt{2}\left(1 - \frac{1}{d^2} - T(J_{\mathcal{E}}, J_{\mathcal{I}}) \right) \geqslant$$
$$\geqslant \sqrt{2}\left(1 - \frac{1}{d^2} - \sqrt{1 - F(J_{\mathcal{E}}, J_{\mathcal{I}})} \right) = \sqrt{2}\left(1 - \frac{1}{d^2} - \sqrt{1 - F(\mathcal{E})} \right),$$

whence the desired inequality follows.

Appendix D: Proof of Proposition 4

By Eqs. (9) and (12), we have

$$D(\mathcal{E}) = I(J_{\mathcal{I}}) - I(J_{\mathcal{E}}) = 2S(\mathbf{1}/d) - \big(S(\mathbf{1}/d) + S(\mathcal{E}(\mathbf{1}/d)) - S(J_{\mathcal{E}})\big) =$$
$$= \ln d - S(\mathcal{E}(\mathbf{1}/d)) + S(J_{\mathcal{E}}) = N(\mathcal{E}) + 2S(\mathcal{E}).$$

Appendix E: Proof of Proposition 5

1. By Eq. (13), we have $D(\mathcal{E}) \geqslant 0$ and the equality holds if and only if $S(J_{\mathcal{E}}) = 0$ and $S(\mathcal{E}(\mathbf{1}/d)|\mathbf{1}/d) = 0$, which implies that $\mathcal{E}$ is a unitary channel. For the upper bound $D(\mathcal{E}) \leqslant 2\ln d$, noting that

$$D(\mathcal{E}) = S(J_{\mathcal{E}}) + S(\mathcal{E}(\mathbf{1}/d)|\mathbf{1}/d) = S(J_{\mathcal{E}}) + S(\mathbf{1}/d) - S(\mathcal{E}(\mathbf{1}/d)) =$$
$$= 2S(\mathbf{1}/d) - S(J_{\mathcal{E}}|\mathbf{1}/d \otimes \mathcal{E}(\mathbf{1}/d)) \leqslant 2S(\mathbf{1}/d) = 2\ln d,$$

we have $D(\mathcal{E}) \leqslant 2\ln d$ and $D(\mathcal{E}) = 2\ln d$ if and only if $J_{\mathcal{E}} = \mathbf{1}/d \otimes \mathcal{E}(\mathbf{1}/d)$, which implies that $\mathcal{E}(\rho) = \mathcal{E}(\mathbf{1}/d)$ for any state ρ.

2. Direct calculations show that

$$J_{p_1\mathcal{E}_1 + p_2\mathcal{E}_2} = p_1 J_{\mathcal{E}_1} + p_2 J_{\mathcal{E}_2}.$$

By the joint convexity of relative entropy, we have

$$I(J_{p_1\mathcal{E}_1 + p_2\mathcal{E}_2}) = S(J_{p_1\mathcal{E}_1 + p_2\mathcal{E}_2}|\mathbf{1}/d \otimes (p_1\mathcal{E}_1 + p_2\mathcal{E}_2)(\mathbf{1}/d)) =$$
$$= S(p_1 J_{\mathcal{E}_1} + p_2 J_{\mathcal{E}_2}|p_1\mathbf{1}/d \otimes \mathcal{E}_1(\mathbf{1}/d) + p_2\mathbf{1}/d \otimes \mathcal{E}_2(\mathbf{1}/d)) \leqslant$$
$$\leqslant p_1 S(J_{\mathcal{E}_1}|\mathbf{1}/d \otimes \mathcal{E}_1(\mathbf{1}/d)) + p_2 S(J_{\mathcal{E}_2}|\mathbf{1}/d \otimes \mathcal{E}_2(\mathbf{1}/d)) =$$
$$= p_1 I(J_{\mathcal{E}_1}) + p_2 I(J_{\mathcal{E}_2}),$$

whence item 2 follows.

3. By the unitary invariance of von Neumann entropy, we have

$$I(J_{\mathcal{E}_U \circ \mathcal{E} \circ \mathcal{E}_V}) = S(\mathbf{1}/d) + S(\mathcal{E}_U \circ \mathcal{E} \circ \mathcal{E}_V(\mathbf{1}/d)) - S(\mathcal{I} \otimes \mathcal{E}_U \circ \mathcal{E} \circ \mathcal{E}_V(|\Phi^+\rangle\langle\Phi^+|)) =$$
$$= S(\mathbf{1}/d) + S(\mathcal{E}(\mathbf{1}/d)) - S(\mathcal{I} \otimes \mathcal{E}(|\Phi^+\rangle\langle\Phi^+|)) = I(J_{\mathcal{E}})$$

for any unitary operators U and V, which naturally implies the desired property.

4. Let H^a and H have respective dimensions d_a and d, and orthonormal bases $\{|\mu\rangle\}$ and $\{|i\rangle\}$. For the channel $\mathcal{I}^a \otimes \mathcal{E}$ on the composite system Hilbert space $H^a \otimes H$, we have

$$d_a d J_{\mathcal{I}^a \otimes \mathcal{E}} = \sum_{\mu\nu ij} |\mu\rangle\langle\nu| \otimes |i\rangle\langle j| \otimes |\mu\rangle\langle\nu| \otimes \mathcal{E}(|i\rangle\langle j|) =$$
$$= \sum_{\mu\nu ij} (\mathbf{1}^a \otimes F_{ab} \otimes \mathbf{1})(|\mu\rangle\langle\nu| \otimes |\mu\rangle\langle\nu| \otimes |i\rangle\langle j| \otimes \mathcal{E}(|i\rangle\langle j|))(\mathbf{1}^a \otimes F_{ab}^\dagger \otimes \mathbf{1}) =$$
$$= d_a d(\mathbf{1}^a \otimes F_{ab} \otimes \mathbf{1})(|\Phi_a^+\rangle\langle\Phi_a^+| \otimes J_{\mathcal{E}})(\mathbf{1}^a \otimes F_{ab}^\dagger \otimes \mathbf{1})$$

where $F_{ab} = \sum_{\mu i} |i\rangle\langle\mu| \otimes |\mu\rangle\langle i|$ is the swap operator on the composite system $H^a \otimes H$ and $|\Phi_a^+\rangle = \frac{1}{\sqrt{d_a}} \sum_\mu |\mu\rangle \otimes |\mu\rangle$. Consequently,

$$S(J_{\mathcal{I}^a \otimes \mathcal{E}}) = S\big((\mathbf{1}^a \otimes F_{ab} \otimes \mathbf{1})(|\Phi_a^+\rangle\langle\Phi_a^+| \otimes J_{\mathcal{E}})(\mathbf{1}^a \otimes F_{ab}^\dagger \otimes \mathbf{1})\big) =$$
$$= S(J_{\mathcal{E}} \otimes |\Phi_a^+\rangle\langle\Phi_a^+|) = S(J_{\mathcal{E}})$$

and

$$S(\mathbf{1}^a/d_a \otimes \mathcal{E}(\mathbf{1}/d)|\mathbf{1}^a/d_a \otimes \mathbf{1}/d) = S(\mathbf{1}^a/d_a) + S(\mathbf{1}/d) - S(\mathbf{1}^a/d_a \otimes \mathcal{E}(\mathbf{1}/d)) =$$
$$= S(\mathbf{1}^a/d_a) + S(\mathbf{1}/d) - S(\mathbf{1}^a/d_a) - S(\mathcal{E}(\mathbf{1}/d)) =$$
$$= S(\mathbf{1}/d) - S(\mathcal{E}(\mathbf{1}/d)) = S(\mathcal{E}(\mathbf{1}/d)|\mathbf{1}/d),$$

whence item 4 follows.

5. Let $\mathcal{F}$ be any channel on a system Hilbert space H. By the definition of Jamiołkowski–Choi states, we have $J_{\mathcal{F}\circ\mathcal{E}} = \mathcal{I} \otimes \mathcal{F}(J_{\mathcal{E}})$. In view of the monotonicity of relative entropy, item 5 follows from

$$I(J_{\mathcal{F}\circ\mathcal{E}}) = I(\mathcal{I} \otimes \mathcal{F}(J_{\mathcal{E}})) = S(\mathcal{I} \otimes \mathcal{F}(J_{\mathcal{E}})|\mathcal{I} \otimes \mathcal{F}(\mathbf{1}/d \otimes \mathcal{E}(\mathbf{1}/d))) \leqslant$$
$$\leqslant S(J_{\mathcal{E}}|\mathbf{1}/d \otimes \mathcal{E}(\mathbf{1}/d)) = I(J_{\mathcal{E}}).$$

6. Let $\mathcal{E}^a$ and $\mathcal{E}^b$ be channels on the systems a and b, with respective Hilbert spaces H^a and H^b, and orthonormal bases $\{|\mu\rangle\}$ and $\{|i\rangle\}$. Because

$$J_{\mathcal{E}^a \otimes \mathcal{E}^b} = \frac{1}{d_a d_b} \sum_{\mu\nu ij} |\mu\rangle\langle\nu| \otimes |i\rangle\langle j| \otimes \mathcal{E}^a(|\mu\rangle\langle\nu|) \otimes \mathcal{E}^b(|i\rangle\langle j|),$$

we have

$$(\mathbf{1}^a \otimes F^\dagger \otimes \mathbf{1}^b)J_{\mathcal{E}^a \otimes \mathcal{E}^b}(\mathbf{1}^a \otimes F \otimes \mathbf{1}^b) = \frac{1}{d_a d_b} \sum_{\mu\nu ij} |\mu\rangle\langle\nu| \otimes \mathcal{E}^a(|\mu\rangle\langle\nu|) \otimes |i\rangle\langle j| \otimes \mathcal{E}^b(|i\rangle\langle j|) = J_{\mathcal{E}^a} \otimes J_{\mathcal{E}^b},$$

where $F_{ab} = \sum_{\mu i} |\mu\rangle\langle i| \otimes |i\rangle\langle\mu|$ is the swap operator on the system Hilbert space $H^a \otimes H^b$. Consequently,

$$S(J_{\mathcal{E}^a \otimes \mathcal{E}^b}) = S\big((\mathbf{1}^a \otimes F_{ab} \otimes \mathbf{1}^b)J_{\mathcal{E}^a \otimes \mathcal{E}^b}(\mathbf{1}^a \otimes F_{ab}^\dagger \otimes \mathbf{1}^b)\big) = S(J_{\mathcal{E}^a} \otimes J_{\mathcal{E}^b}) = S(J_{\mathcal{E}^a}) + S(J_{\mathcal{E}^b}),$$

and

$$S(\mathcal{E}^a(\mathbf{1}^a/d_a) \otimes \mathcal{E}^b(\mathbf{1}^b/d_b)|\mathbf{1}^a/d_a \otimes \mathbf{1}^b/d_b) = S(\mathbf{1}^a/d_a \otimes \mathbf{1}^b/d_b) - S(\mathcal{E}^a(\mathbf{1}^a/d_a) \otimes \mathcal{E}^b(\mathbf{1}^b/d_b)) =$$
$$= S(\mathbf{1}^a/d_a) + S(\mathbf{1}^b/d_b) - S(\mathcal{E}^a(\mathbf{1}^a/d_a)) - S(\mathcal{E}^b(\mathbf{1}^b/d_b)) =$$
$$= S(\mathcal{E}^a(\mathbf{1}^a/d_a)|\mathbf{1}^a/d_a) + S(\mathcal{E}^b(\mathbf{1}^b/d_b|\mathbf{1}^b/d_b),$$

whence item 6 follows.

Appendix F: An alternative measure of irreversibility

We recall that the Tsallis r-entropy

$$S_r(\rho) = \frac{1 - \operatorname{tr}\rho^r}{r - 1}, \qquad r \in \mathbb{R},$$

is a simple and significant quantity characterizing the mixedness of a state ρ [60]. The case $r = 1$ is understood as the limit $r \to 1$ and actually corresponds to the von Neumann entropy. If we take ρ to be $J_{\mathcal{E}}$ of a channel $\mathcal{E}$, then we can regard

$$S_r(\mathcal{E}) = \frac{1}{2}S_r(J_{\mathcal{E}})$$

as a measure of irreversibility of the channel $\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$. For $r \in \mathbb{N}$, $S_r(\mathcal{E})$ has the explicit form

$$S_r(\mathcal{E}) = \frac{1}{2(r-1)} \left(1 - \frac{1}{d^r} \sum_{k_1, \ldots, k_r} \operatorname{tr}(E_{k_1}^\dagger E_{k_2}) \operatorname{tr}(E_{k_2}^\dagger E_{k_3}) \ldots \operatorname{tr}(E_{k_r}^\dagger E_{k_1}) \right).$$

In particular, the Tsallis 2-entropy $S_2(\rho) = 1 - \operatorname{tr} \rho^2$ is the linear entropy, and if we take ρ to be $J_\mathcal{E}$ of a channel $\mathcal{E}$, then we can regard

$$S_2(\mathcal{E}) = \frac{1}{2} S_2(J_\mathcal{E})$$

as a measure of irreversibility of the channel $\mathcal{E}$. It is interesting to note that this quantity can also be expressed as

$$S_2(\mathcal{E}) = \frac{1}{2} - \frac{1}{2d^2} \sum_k \|\mathcal{E}(X_k)\|^2$$

where $\{X_k \colon k = 1, \ldots, d^2\}$ is any orthonormal basis of the operator space $L(H)$ of all observables (Hermitian operators) on H with the Hilbert–Schmidt inner product $\langle A|B \rangle = \operatorname{tr} AB$. The quantity $S_2(\mathcal{E})$ satisfies the following properties, which parallel those of $S(\mathcal{E})$.

1. We have

   $$0 \leqslant S_2(\mathcal{E}) \leqslant \frac{1}{2} - \frac{1}{2d^2},$$

   and $S_2(\mathcal{E}) = 0$ if and only if $\mathcal{E}$ is a unitary channel, while $S_2(\mathcal{E})$ attains the maximum value $(d^2 - 1)/2d^2$ if and only if $\mathcal{E}$ is the completely depolarizing channel $\mathcal{E}_{\mathrm{cde}}(\rho) = \mathbf{1}/d$ for any state ρ.

2. $S_2(\mathcal{E})$ is concave in $\mathcal{E}$, i.e.,

   $$S_2(p_1 \mathcal{E}_1 + p_2 \mathcal{E}_2) \geqslant p_1 S_2(\mathcal{E}_1) + p_2 S_2(\mathcal{E}_2)$$

   for $p_1, p_2 \geqslant 0$, $p_1 + p_2 = 1$, and any channels $\mathcal{E}_1$ and $\mathcal{E}_2$.

3. $S_2(\cdot)$ is invariant under composition with unitary dynamics in the sense that

   $$S_2(\mathcal{E}_U \circ \mathcal{E}) = S_2(\mathcal{E} \circ \mathcal{E}_U) = S_2(\mathcal{E})$$

   for any unitary channel $\mathcal{E}_U(\rho) = U \rho U^\dagger$ with U being any unitary operator on the system Hilbert space.

4. $S_2(\cdot)$ is ancilla-independent in the sense that $S_2(\mathcal{I}^a \otimes \mathcal{E}) = S_2(\mathcal{E})$, where $\mathcal{I}^a$ is the identity channel on any ancilla system a.

5. We have

   $$S_2(\mathcal{E}^a \otimes \mathcal{E}^b) = S_2(\mathcal{E}^a) + S_2(\mathcal{E}^b) - S_2(\mathcal{E}^a) S_2(\mathcal{E}^b),$$

   where $\mathcal{E}^a$ and $\mathcal{E}^b$ are channels on systems a and b. This is a kind of nonextensivity of the Tsallis entropy.

6. $S_2(\cdot)$ is monotonic in the sense that

   $$S_2(\mathcal{F} \circ \mathcal{E}) \geqslant S_2(\mathcal{E})$$

   for any unital channel $\mathcal{F}$.

The measure of irreversibility $S_2(\mathcal{E})$ can be explicitly evaluated for various channels studied in Sec. 7. We list the results, together with those for $S(\mathcal{E})$, in Table 1.

Table 1. Comparison between the two irreversibility measures $S(\mathcal{E})$ and $S_2(\mathcal{E})$ for various channels defined in Sec. 7

Channel $\mathcal{E}$	$S(\mathcal{E}) = -\frac{1}{2}\operatorname{tr} J_{\mathcal{E}} \ln J_{\mathcal{E}}$	$S_2(\mathcal{E}) = \frac{1}{2}(1 - \operatorname{tr} J_{\mathcal{E}}^2)$
$\mathcal{E}_U$	0	0
$\mathcal{E}_{\mathrm{vN}}$	$\frac{1}{2}\ln d$	$\frac{1}{2}\left(1 - \frac{1}{d}\right)$
$\mathcal{E}_{\mathrm{sic}}$	$\frac{1}{2}\ln d + \frac{d-1}{2d}\ln(d+1)$	$\frac{1}{2} - \frac{1}{d(d+1)}$
$\mathcal{E}_{\mathrm{WH}}$	$\frac{1}{2}\ln\frac{d^2-d}{2}$	$\frac{d^2-d-2}{2d(d-1)}$
$\mathcal{E}_{\mathrm{cde}}$	$\ln d$	$\frac{1}{2}\left(1 - \frac{1}{d^2}\right)$
$\mathcal{E}_{\mathrm{cd}}$	$-\frac{1+\alpha}{4}\ln\frac{1+\alpha}{2} - \frac{1-\alpha}{4}\ln\frac{1-\alpha}{2}$	$\frac{1}{4}(1-\alpha^2)$
$\mathcal{E}_{\mathrm{ad}}$	$-\frac{p}{4}\ln\frac{p}{2} - \frac{2-p}{4}\ln\frac{2-p}{2}$	$\frac{1}{4}p(2-p)$
$\mathcal{E}_{\mathrm{pd}}$	$-\frac{1}{2}\big(p'\ln p' + (1-p')\ln(1-p')\big)$	$\frac{1}{4}p$
$\mathcal{E}_x$	$-\frac{1}{2}\big(p_x \ln p_x + (1-p_x)\ln(1-p_x)\big)$	$\frac{1}{4}(2x-1)^2$
$\mathcal{E}_{\mathrm{de}}$	$-\frac{1}{2}\big((1-3p)\ln(1-3p) + 3p\ln p\big)$	$3p(1-2p)$

Conflict of interest. The authors of this work declare that they have no conflicts of interest.

REFERENCES

1. E. Fermi, *Thermodynamics*, Dover, Mineola, NY (2012).

2. E. Schrödinger, *Statistical Thermodynamics*, Dover, Mineola, NY (1989).

3. L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics*, Vol. 5: *Statistical Physics*, Butterworth-Heinemann, Oxford (1980).

4. H. B. Callen, *Thermodynamics and an Introduction to Thermostatistics*, Wiley, New York (1991).

5. W. H. Zurek, "Pointer basis of quantum apparatus: Into what mixture does the wave packet collapse?," *Phys. Rev. D*, **24**, 1516–1525 (1981).

6. W. H. Zurek, "Decoherence and the transition from quantum to classical," *Physics Today*, **44**, 36–44 (1991).

7. E. Joos, H. D. Zeh, C. Kiefer, D. Giulini, J. Kupsch, I.-O. Stamatescu, *Decoherence and the Appearance of a Classical World in Quantum Theory*, Springer, Berlin (2003).

8. M. Schlosshauer, "Decoherence, the measurement problem, and interpretations of quantum mechanics," *Rev. Mod. Phys.*, **76**, 1267–1305 (2004).

9. M. Schlosshauer, *Decoherence and the Quantum-To-Classical Transition*, Springer, Berlin (2007).

10. H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems*, Oxford Univ. Press, Oxford (2007).

11. M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge Univ. Press, Cambridge (2010).

12. J. Watrous, *The Theory of Quantum Information*, Cambridge Univ. Press, Cambridge (2018).

13. A. S. Holevo, *Quantum Systems, Channels, Information: A Mathematical Introduction*, De Gruyter Studies in Mathematical Physics, Vol. 16, De Gruyter, Berlin, Boston (2019).

14. B. W. Schumacher, "Sending entanglement through noisy quantum channels," *Phys. Rev. A*, **54**, 2614–2628 (1996).

15. B. Schumacher and M. A. Nielsen, "Quantum data processing and error correction," *Phys. Rev. A*, **54**, 2629–2635 (1996).

16. M. A. Nielsen, "The entanglement fidelity and quantum error correction," arXiv: quant-ph/9606012.

17. H. Barnum, M. A. Nielsen, and B. W. Schumacher, "Information transmission through a noisy quantum channel," *Phys. Rev. A*, **57**, 4153–4175 (1998).

18. M. A. Nielsen, C. M. Caves, B. Schumacher, and H. Barnum, "Information-theoretic approach to quantum error correction and reversible measurement," *Proc. R. Soc. London Ser. A*, **454**, 277–304 (1998).

19. M. Horodecki, P. Horodecki, and R. Horodecki, "General teleportation channel, singlet fraction, and quasidistillation," *Phys. Rev. A*, **60**, 1888–1898 (1999).

20. M. A. Nielsen, "A simple formula for the average gate fidelity of a quantum dynamical operation," *Phys. Lett. A*, **303**, 249–252 (2002).

21. S. Luo, "Information conservation and entropy change in quantum measurements," *Phys. Rev. A*, **82**, 052103, 5 pp. (2010).

22. S. Luo and N. Li, "Decoherence and measurement-induced correlations," *Phys. Rev. A*, **84**, 052309, 8 pp. (2011).

23. H.-P. Breuer, E.-M. Laine, and J. Piilo, "Measure for the degree of non-Markovian behavior of quantum processes in open systems," *Phys. Rev. Lett.*, **103**, 210401, 4 pp. (2009).

24. Á. Rivas, S. F. Huelga, and M. B. Plenio, "Entanglement and non-Markovianity of quantum evolutions," *Phys. Rev. Lett.*, **105**, 050403, 4 pp. (2010).

25. S. Luo, S. Fu, and H. Song, "Quantifying non-Markovianity via correlations," *Phys. Rev. A*, **86**, 044101, 4 pp. (2012).

26. D. Chruściński and S. Maniscalco, "Degree of non-Markovianity of quantum evolution," *Phys. Rev. Lett.*, **112**, 120404, 5 pp. (2014).

27. H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, "Non-Markovian dynamics in open quantum systems," *Rev. Mod. Phys.*, **88**, 021002, 24 pp. (2016).

28. H. Spohn, "Entropy production for quantum dynamical semigroups," *J. Math. Phys.*, **19**, 1227–1230 (1978).

29. G. E. Crooks, "Entropy production fluctuation theorem and the nonequilibrium work relation for free energy differences," *Phys. Rev. E*, **60**, 2721–2726 (1999).

30. L. M. Martyushev and V. D. Seleznev, "Maximum entropy production principle in physics, chemistry and biology," *Phys. Rep.*, **426**, 1–45 (2006).

31. J. M. R. Parrondo, C. Van den Broeck, and R. Kawai, "Entropy production and the arrow of time," *New J. Phys.*, **11**, 073008, 14 pp. (2009).

32. S. Deffner and E. Lutz, "Nonequilibrium entropy production for open quantum systems," *Phys. Rev. Lett.*, **107**, 140404, 5 pp. (2011).

33. T. Sagawa and M. Ueda, "Role of mutual information in entropy production under information exchanges," *New J. Phys.*, **15**, 125012, 23 pp. (2013).

34. M. Lostaglio, K. Korzekwa, D. Jennings, and T. Rudolph, "Quantum coherence, time-translation symmetry, and thermodynamics," *Phys. Rev. X*, **5**, 021001, 11 pp. (2015), arXiv: 1410.4572.

35. H. Hossein-Nejad, E. J. O'Reilly, and A. Olaya-Castro, "Work, heat and entropy production in bipartite quantum systems," *New J. Phys.*, **17**, 075014, 11 pp. (2015).

36. M. Popovic, B. Vacchini, and S. Campbell, "Entropy production and correlations in a controlled non-Markovian setting," *Phys. Rev. A*, **98**, 012130, 8 pp. (2018).

37. J. P. Santos, L. C. Céleri, F. Brito, G. T. Landi, and M. Paternostro, "Spin-phase-space-entropy production," *Phys. Rev. A*, **97**, 052123, 10 pp. (2018).

38. K. Ptaszyǹski and M. Esposito, "Entropy production in open systems: The predominant role of intraenvironment correlations," *Phys. Rev. Lett.*, **123**, 200603, 6 pp. (2019).

39. P. Strasberg and A. Winter, "First and second law of quantum thermodynamics: A consistent derivation based on a microscopic definition of entropy," *PRX Quantum*, **2**, 030202, 26 pp. (2021).

40. G. T. Landi and M. Paternostro, "Irreversible entropy production: From classical to quantum," *Rev. Mod. Phys.*, **93**, 035008, 58 pp. (2021).

41. A. Belenchia, M. Paternostro, and G. T. Landi, "Informational steady states and conditional entropy production in continuously monitored systems: The case of Gaussian systems," *Phys. Rev. A*, **105**, 022213, 14 pp. (2022).

42. A. Jamiołkowski, "Linear transformations which preserve trace and positive semidefinitness of operators," *Rep. Math. Phys.*, **3**, 275–278 (1972).

43. M.-D. Choi, "Completely positive maps on complex matrices," *Linear Algebra Appl.*, **10**, 285–290 (1975).

44. M. Jiang, S. Luo, and S. Fu, "Channel-state duality," *Phys. Rev. A*, **87**, 022310, 8 pp. (2013).

45. K. Życzkowski and I. Bengtsson, "On duality between quantum maps and quantum states," *Open Syst. Inf. Dyn.*, **11**, 3-42 (2004).

46. Y. Sun, N. Li, and S. Luo, "Fidelity-disturbance-entropy tradeoff in quantum channels," *Phys. Rev. A*, **105**, 062458, 9 pp. (2022).

47. K. Banaszek, "Fidelity balance in quantum operations," *Phys. Rev. Lett.*, **86**, 1366–1369 (2001).

48. S. Luo, S. Fu, and N. Li, "Decorrelating capabilities of operations with application to decoherence," *Phys. Rev. A*, **82**, 052122, 12 pp. (2010).

49. G. Lüders, "Über die Zustandsänderung durch den Meßprozeß," *Ann. Physik*, **443**, 322–328 (1950).

50. J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, "Symmetric informationally complete quantum measurements," *J. Math. Phys*, **45**, 2171–2180 (2004).

51. R. F. Werner and A. S. Holevo, "Counterexample to an additivity conjecture for output purity of quantum channels," *J. Math. Phys.*, **43**, 4353–4357 (2002).

52. F. Buscemi, G. Chiribella, and G. M. D'Ariano, "Inverting quantum decoherence by classical feedback from the environment," *Phys. Rev. Lett.*, **95**, 090501, 4 pp. (2005).

53. K. Brádler, P. Hayden, D. Touchette, and M. M. Wilde, "Trade-off capacities of the quantum Hadamard channels," *Phys. Rev. A*, **81**, 062312, 23 pp. (2010).

54. C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, "Teleporting an unknown quantum state via dual classical and Einstein–Podolsky–Rosen channels," *Phys. Rev. Lett.*, **70**, 1895–1899 (1993).

55. S. Popescu, "Bell's inequalities versus teleportation: What is nonlocality?," *Phys. Rev. Lett.*, **72**, 797–799 (1994).

56. M. Horodecki and P. Horodecki, "Reduction criterion of separability and limits for a class of distillation protocols," *Phys. Rev. A*, **59**, 4206–4216 (1999).

57. K. Banaszek, "Optimal quantum teleportation with an arbitrary pure state," *Phys. Rev. A*, **62**, 024301, 4 pp. (2000).

58. G. Bowen and S. Bose, "Teleportation as a depolarizing quantum channel, relative entropy, and classical capacity," *Phys. Rev. Lett.*, **87**, 267901, 4 pp. (2001).

59. M. S. Pinsker, *Information and Information Stability of Random Variables and Processes*, Holden-Day, San Francisco, CA (1964).

60. C. Tsallis, "Possible generalization of Boltzmann–Gibbs statistics," *J. Stat. Phys.*, **52**, 479–487 (1988).