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A GENERALIZED CREWTHER RELATION AND THE V SCHEME: ANALYTIC RESULTS IN FOURTH-ORDER PERTURBATIVE QCD AND QED

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Using the analytic $\overline{\rm MS}$ scheme, three-loop contribution to the perturbative Coulomb-like part of the static color potential of a heavy quark–antiquark system, we obtain an analytic expression for the fourth-order β-function in the gauge-invariant effective V scheme in the case of the generic simple gauge group. We also present the Adler function of electron–positron annihilation into hadrons and the coefficient function of the Bjorken polarized sum rule in the V scheme up to a_s^4 terms. We demonstrate that at this level of the perturbation theory in this effective scheme, the generalized Crewther relation, which connects the flavor nonsinglet contributions to the Adler and Bjorken polarized sum rule functions, is satisfied. Starting from the a_s^2 order, it contains a conformal symmetry breaking term that factors into the conformal anomaly $\beta(a_s)/a_s$ and the polynomial in powers of a_s. We prove that this relation also holds in other gauge-invariant renormalization schemes. The obtained results allows revealing the difference between the V-scheme β-function in QED and the Gell-Mann–Low Ψ-function. This distinction arises due to the presence of the light-by-light type scattering corrections first appearing in the static potential at the three-loop level.

Keywords: Renormalization group, renormalization schemes, QCD and QED, conformal symmetry and its violation

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1. Introduction

It is well known that the binding energy of the quark–antiquark system in a color singlet state in QCD can be described by two terms, the perturbative Coulomb-like contribution at short distances and the essentially nonperturbative long-distance one, modeling the confinement description. Investigation of this phenomenon is actively underway by means of lattice calculations, where the linear dependence on distance r is predicted for the nonperturbative part of the static potential (see, e.g., [1]–[3] and the references therein). In its turn, a non-abelian analogue of the Coulomb potential of QED is determined in the framework of the perturbation theory (PT) and, for instance, it is the main component in studying spectroscopy of bound states such as heavy quarkonia in nonrelativistic QCD [4], [5].

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The static potential of the interaction of a heavy quark–antiquark pair is defined in general via the vacuum expectation value of the gauge-invariant Wilson loop, more precisely, in terms of the logarithm of the path-ordered Wilson loop over a closed rectangular contour divided by the interaction time T in the limit $T \to \infty$. The perturbative part of this potential is now available in analytic form in the $\overline{\text{MS}}$ renormalization scheme at the three-loop level. In the momentum representation, its is

$$V(\vec{q}^{\,2}) = -\frac{4\pi C_F \alpha_s(\vec{q}^{\,2})}{\vec{q}^{\,2}} \left[1 + a_1 a_s(\vec{q}^{\,2}) + a_2 a_s^2(\vec{q}^{\,2}) + \left(a_3 + \frac{\pi^2 C_A^3 L}{8}\right) a_s^3(\vec{q}^{\,2}) + \mathcal{O}(a_s^4)\right], \tag{1}$$

where α_s is the renormalized strong coupling constant in the $\overline{\text{MS}}$ scheme, $a_s = \alpha_s/\pi$, $L = \log(\mu^2/\vec{q}^{\,2})$, μ^2 is the scale $\overline{\text{MS}}$-scheme parameter of the dimensional regularization, and $\vec{q}^{\,2}$ is the square of the Euclidean three-dimensional momentum. The limit $T \to \infty$ formally leads to $q_0 \to 0$ and the squared Euclidean four-dimensional transferred momentum $Q^2 \to \vec{q}^{\,2}$. Thus, technically, we carry out the transition from the Euclidean four-dimensional space to its three-dimensional subspace. The renormalization group (RG) uncontrollable logarithmic term [4] arises in Eq. (1) due to the infrared (IR) divergences, which begin to manifest themselves in the static potential at the three-loop level. However, in the concrete applications of the effective nonrelativistic QCD, these IR-divergent terms cancel with certain ultraviolet (UV) divergent terms originating in the interaction of ultrasoft gluons with the heavy quark–antiquark bound states (see, e.g., [5], [6]). Because we consider regions of the intermediate and high energies only, these IR corrections do not affect the behavior of various physical quantities, and we therefore do not take them into account in our RG-oriented studies.

The analytic expression for the one-loop coefficient a_1 in Eq. (1) was calculated in [7], [8], while the two-loop a_2 is known from the calculations in [9], [10]. They are given by

$$a_1 = \frac{31}{36} C_A - \frac{5}{9} T_F n_f, \tag{2}$$

$$\begin{aligned} a_2 = {}& \left(\frac{4343}{2592} + \frac{\pi^2}{4} - \frac{\pi^4}{64} + \frac{11}{24}\zeta_3\right) C_A^2 - \left(\frac{899}{648} + \frac{7}{6}\zeta_3\right) C_A T_F n_f - \\ & - \left(\frac{55}{48} - \zeta_3\right) C_F T_F n_f + \left(\frac{5}{9} T_F n_f\right)^2, \end{aligned} \tag{3}$$

where n_f is the flavor number of active quarks and $\zeta_n = \sum_{k=1}^{\infty} k^{-n}$ is the Riemann zeta-function.

The eigenvalues C_F and C_A of the quadratic Casimir operator in the fundamental and adjoint representations of the generic simple gauge group are respectively defined as $(T^a T^a)_{ij} = C_F \delta_{ij}$ and $f^{acd} f^{bcd} = C_A \delta^{ab}$, where T^a are generators of the Lie algebra of the gauge group in the fundamental representation with the corresponding commutation relation $[T^a, T^b] = i f^{abc} T^c$. They are normalized as $\text{Tr}(T^a T^b) = T_F \delta^{ab}$ with the Dynkin index T_F. We note that in our study, we are primarily interested in the case of the $SU(N_c)$ color group with $C_A = N_c$ and $C_F = (N_c^2 - 1)/(2N_c)$, $T_F = 1/2$, and its particular case $N_c = 3$ of the $SU(3)$-group, relevant to physical QCD.

The three-loop contribution a_3 is a cubic polynomial in n_f:

$$a_3 = a_3^{(3)} n_f^3 + a_3^{(2)} n_f^2 + a_3^{(1)} n_f + a_3^{(0)}. \tag{4}$$

The leading terms in powers of n_f can be extracted from the renormalon-chain contributions to the Coulomb QED static potential (or from the representation of the QED invariant charge directly related to the photon vacuum polarization function [11]). The analytic expression for the quadratic n_f^2 coefficient was obtained in [12]. Because of technical difficulties, the contributions $a_3^{(1)}$ and $a_3^{(0)}$ were calculated analytically later in [13]. They turned out to be much more complicated than the coefficient $a_3^{(2)}$. Indeed, in addition to the expected appearance of the π^2, π^4, ζ_3, $\pi^2\zeta_3$, and ζ_5 terms (see, e.g., [14]), we also have contributions proportional to $\pi^2\log 2$, $\pi^4\log 2$ and, more significantly, the basic constants with the new greatest weight of transcendence six $w=6$, namely, π^6, ζ_3^2, $\pi^2\zeta_3\log 2$, $\pi^4\log^2 2$, and the ones that include more complicated functions, e.g., $\pi^2\alpha_4$ and s_6, where $\alpha_4 = \mathrm{Li}_4(1/2) + \log^4 2/4!$ with the polylogarithm function $\mathrm{Li}_n(x) = \sum_{k=1}^{\infty} x^k k^{-n}$ and $s_6 = \zeta_6 + \zeta_{-5,-1}$ with $\zeta_6 = \pi^6/945$ and multiple zeta value $\zeta_{-5,-1} = \sum_{k=1}^{\infty}\sum_{i=1}^{k-1}(-1)^{i+k}/ik^5$ (see Appendix A).

For the convenience of reader and for the purposes of the further discussion, it is useful to present all four coefficients in flavor expansion (4):

$$a_3^{(3)} = -\left(\frac{5}{9}\right)^3 T_F^3, \tag{5a}$$

$$a_3^{(2)} = \left(\frac{12541}{15552} + \frac{23}{12}\zeta_3 + \frac{\pi^4}{135}\right)C_A T_F^2 + \left(\frac{7001}{2592} - \frac{13}{6}\zeta_3\right)C_F T_F^2, \tag{5b}$$

$$\begin{aligned} a_3^{(1)} = &\Bigg[-\frac{58747}{31104} - \frac{89}{16}\zeta_3 + \frac{761}{161280}\pi^6 - \frac{3}{4}s_6 + \pi^4\left(-\frac{157}{3456} - \frac{5}{576}\log 2 + \frac{\log^2 2}{64}\right) + \\ &+ \pi^2\left(\frac{17}{1728} - \frac{19}{192}\zeta_3 - \frac{\log 2}{48} - \frac{7}{32}\zeta_3\log 2 - \frac{\alpha_4}{2}\right) + \\ &+ \frac{1091}{384}\zeta_5 + \frac{57}{128}\zeta_3^2\Bigg]C_A^2 T_F + \left(-\frac{71281}{10368} + \frac{33}{8}\zeta_3 + \frac{5}{4}\zeta_5\right)C_A C_F T_F + \\ &+ \left(\frac{143}{288} + \frac{37}{24}\zeta_3 - \frac{5}{2}\zeta_5\right)C_F^2 T_F + \Bigg[\frac{5}{96}\pi^6 + \pi^4\left(-\frac{23}{24} + \frac{\log 2}{6} - \frac{\log^2 2}{2}\right) + \\ &+ \pi^2\left(\frac{79}{36} - \frac{61}{12}\zeta_3 + \log 2 + \frac{21}{2}\zeta_3\log 2\right)\Bigg]\frac{d_F^{abcd}d_F^{abcd}}{N_A}, \end{aligned} \tag{5c}$$

$$\begin{aligned} a_3^{(0)} = &\Bigg[\frac{385645}{186624} + \frac{73}{24}\zeta_3 - \frac{4621}{193536}\pi^6 + \frac{9}{4}s_6 + \pi^4\left(\frac{1349}{17280} - \frac{5}{144}\log 2 - \frac{5}{72}\log^2 2\right) + \\ &+ \pi^2\left(-\frac{953}{3456} + \frac{175}{128}\zeta_3 - \frac{461}{288}\log 2 + \frac{217}{192}\zeta_3\log 2 + \frac{73}{24}\alpha_4\right) - \\ &- \frac{1927}{384}\zeta_5 - \frac{143}{128}\zeta_3^2\Bigg]C_A^3 + \Bigg[\frac{1511}{2880}\pi^6 + \pi^4\left(-\frac{39}{16} + \frac{35}{12}\log 2 + \frac{31}{12}\log^2 2\right) + \\ &+ \pi^2\left(\frac{929}{72} - \frac{827}{24}\zeta_3 - 74\alpha_4 + \frac{461}{6}\log 2 - \frac{217}{4}\zeta_3\log 2\right)\Bigg]\frac{d_F^{abcd}d_A^{abcd}}{N_A}, \end{aligned} \tag{5d}$$

Here, d_F^{abcd} and d_A^{abcd} are the rank-four totally symmetric higher-order group invariants, defined in the fundamental and adjoint representations, and N_A is the number of generators of the group. In the particular case of the $SU(N_c)$ gauge group, the aforementioned color structures are expressed through the number of colors N_c as

$$\frac{d_F^{abcd}d_A^{abcd}}{N_A} = \frac{N_c(N_c^2+6)}{48}, \qquad \frac{d_F^{abcd}d_F^{abcd}}{N_A} = \frac{N_c^4 - 6N_c^2 + 18}{96N_c^2},$$

where $N_A = N_c^2 - 1$.

Our further analysis is in part a continuation of [15], where we have investigated the requirements imposed on renormalization schemes leading to the factorization of the conformal symmetry breaking term $\Delta_{csb}(a_s)$ into the conformal anomaly $\beta(a_s)/a_s$ and the coupling-dependent polynomial $K(a_s) = \sum_{n\geq 1} K_n a_s^n$ in the generalized Crewther relation

$$D^{\mathrm{NS}}(a_s)C_{\mathrm{Bjp}}^{\mathrm{NS}}(a_s) = 1 + \Delta_{csb}(a_s) = 1 + \left(\frac{\beta(a_s)}{a_s}\right)K(a_s). \tag{6}$$

which involves two RG-invariant Euclidean quantities, namely, the flavor nonsinglet (NS) contributions to the Adler function $D(Q^2)$ and to the coefficient function $C_{\rm Bjp}(Q^2)$ of the Bjorken polarized sum rule. The first of them is characteristic of the e^+e^- annihilation into hadrons, whereas the second one enters a theoretical expression for the Bjorken sum rule of deep inelastic scattering (DIS) of the polarized charged leptons on nucleons. We note that $a_s = a_s(\mu^2 = Q^2)$ in Eq. (6).

In the normalization that we use, the unity in Eq. (6) corresponds to the original Crewther relation [16], derived in the Born approximation of the massless theory of strong interactions by using the operator product expansion (OPE) approach to the axial–vector–vector (AVV) triangle diagram in the conformal symmetry limit.

It was discovered in [17] that in the $\overline{\rm MS}$ scheme, starting with the a_s^2 terms of the PT, the Crewther relation is modified. In addition to unity, an extra contribution arises in (6) that turns out to be proportional to the RG β-function:

$$\mu^2 \frac{\partial a_s(\mu^2)}{\partial \mu^2} = \beta(a_s(\mu^2)) = -\sum_{i\geq 0} \beta_i a_s^{i+2}(\mu^2). \tag{7}$$

The renormalization procedure breaks the conformal symmetry of the massless QCD. In particular, this reflects the violation of the symmetry with respect to conformal transformations of the AVV function. The effect of this violation in Eq. (6) is described by the conformal symmetry breaking term $\Delta_{csb}(a_s)$, proportional to the factor $\beta(a_s)/a_s$ and containing the polynomial $K(a_s) = \sum_{n\geq 1} K_n a_s^n$ in a_s. This fact was discovered in the $\overline{\rm MS}$ scheme at the $\mathcal{O}(a_s^3)$ level in [17] and later confirmed at the $\mathcal{O}(a_s^4)$ level in [18]. Now it is customary to call this form of the generalized Crewther relation the Crewther–Broadhurst–Kataev (CBK) relation in the literature. It was intensively studied from different standpoints, e.g., in [19]–[22].

Recently, an analogue of the CBK relation was considered in the extended QCD model with an arbitrary number of fermion representations at the $\mathcal{O}(a_s^4)$ level in [23]. It was shown there that in this case, the CBK relation remains valid as well. This fact confirms the nonaccidental nature of the factorization of $\Delta_{csb}(a_s)$ at least at the $\mathcal{O}(a_s^4)$ order. Moreover, arguments presented in [24]–[26] indicate that the CBK relation must hold in the $\overline{\rm MS}$ scheme in QCD in all orders of the PT.

The natural question arises whether there are theoretical requirements on the choice of the ultraviolet subtraction schemes that ensure the realization of the fundamental property of the factorization of the conformal symmetry breaking term $\Delta_{csb}(a_s)$ in the CBK relation. The results in [15], [27], [28] demonstrate that this feature of CBK is implemented for a wide class of gauge-dependent momentum subtraction MOM-like schemes (e.g., the mMOM scheme [29]–[34]) in a linear covariant Landau gauge $\xi = 0$ at least at the $\mathcal{O}(a_s^4)$ level (and apparently in all PT orders). Therefore, the often prevailing opinion in the literature that the CBK relation is valid only for gauge-invariant MS-like schemes turns out to be incorrect.

Because it is not obvious that the CBK relation is also realized in some gauge-invariant schemes other than MS-like ones, we here study this issue with the example of the effective gauge-independent V scheme. In this scheme, the static potential of a heavy quark–antiquark pair has the Coulomb-like form and all higher-order corrections are absorbed into a redefinition of the effective charge with a corresponding change in the scale parameter. For this, and for the goals that are discussed later, we obtain analytic expressions for the β-function in the V scheme in the four-loop approximation and also for both the Adler and the coefficient function of the Bjorken polarized sum rule in the V scheme in the same order of the PT in the case of a generic simple gauge group. Further, we generalize the consideration of the factorization of $\Delta_{csb}(a_s)$ in the CBK relation to a wide class of gauge-invariant subtraction schemes. Similar problems are also investigated in the case of QED. In the end, we draw a number of conclusions on the relation between the β-function in the V scheme in QED and the Gell-Mann–Low Ψ-function, including a fixation of definite four-loop contributions to the static potential directly obtained in [35].

2. β-function in the V scheme

We turn to the effective gauge-invariant V scheme. It was first introduced in [9], [10] and was used in modeling the smooth transition of the QCD coupling constant through the thresholds of heavy-quark productions in the case where the mass corrections to the static potential are taken into account [36]. Other applications of the V scheme in the perturbative QCD studies can be found, e.g., in [2], [37]–[41].

We now use the analytic results on the static potential presented in the preceding section to refine the semianalytic form of the fourth-order expression for the RG β-function in the V scheme for a generic simple gauge group, obtained previously in [14] and applied to the analysis of theoretical QCD ambiguities for the e^+e^- annihilation into hadrons R-ratio at the $\mathcal{O}(a_s^4)$-level in the energy region below the manifestation of the left shoulder of the Z^0-peak.

Summarizing the aforesaid, we can succinctly describe the V scheme by an expression for the static heavy quark–antiquark potential in the Coulomb-like form,

$$V(\vec{q}^{\,2}) = -4\pi C_F \frac{\alpha_{s,V}(\vec{q}^{\,2})}{\vec{q}^{\,2}}\,, \tag{8}$$

where all higher-order PT corrections to $V(\vec{q}^{\,2})$ are absorbed into the effective coupling $\alpha_{s,V}(\vec{q}^{\,2})$ and, as was already stated, we neglect the contribution of the three-loop IR logarithmic term. In accordance with the technique of the effective charges (ECH) developed in [42]–[44], we define the effective V-scheme scale by means of the following relation, associated with its $\overline{\text{MS}}$-scheme counterpart:

$$\mu_V^2 = \mu^2 e^{a_1/\beta_0}, \tag{9}$$

where a_1 is given by Eq. (2) and β_0 is the first scheme-independent coefficient [45], [46] of the RG β-function (7). Further, fixing $\vec{q}^{\,2} = \mu_V^2$, we can finally obtain a link between couplings in the V and $\overline{\text{MS}}$ schemes:

$$\alpha_{s,V}(\mu_V^2) = \alpha_s(\mu_V^2)(1 + a_1 a_s(\mu_V^2) + a_2 a_s^2(\mu_V^2) + a_3 a_s^3(\mu_V^2) + \mathcal{O}(a_s^4)). \tag{10}$$

We then define the β-function in the V scheme

$$\beta^V(a_{s,V}) = \mu_V^2 \frac{\partial a_{s,V}}{\partial \mu_V^2} = -\sum_{i\geq 0} \beta_i^V a_{s,V}^{i+2}. \tag{11}$$

and its relation to the $\overline{\text{MS}}$-scheme $\beta(a_s)$-function:

$$\beta^V(a_{s,V}(a_s)) = \beta(a_s) \frac{\partial a_{s,V}(a_s)}{\partial a_s}\,. \tag{12}$$

The combination of Eqs. (10) and (12) yields the following relations between the β-functions coefficients in the V and $\overline{\text{MS}}$ schemes:

$$\beta_0^V = \beta_0, \qquad \beta_1^V = \beta_1, \tag{13a}$$

$$\beta_2^V = \beta_2 - a_1\beta_1 + (a_2 - a_1^2)\beta_0, \tag{13b}$$

$$\beta_3^V = \beta_3 - 2a_1\beta_2 + a_1^2\beta_1 + (2a_3 - 6a_1a_2 + 4a_1^3)\beta_0. \tag{13c}$$

and even in the higher PT orders with the still unknown correction a_4 to the static potential,

$$\beta_4^V = \beta_4 - 3a_1\beta_3 + (4a_1^2 - a_2)\beta_2 + (a_3 - 2a_1a_2)\beta_1 + (3a_4 - 12a_1a_3 - 5a_2^2 + 28a_1^2a_2 - 14a_1^4)\beta_0. \tag{13d}$$

These formulas reflect the transformation laws of the coefficients of the β-functions under the transition from one gauge-invariant renormalization scheme to another. The consequence of the use of the ECH approach is a scheme invariance of all coefficients of the effective β-functions within the gauge-independent MS-like schemes (see [47], [48] for details).

The first two coefficients of β^V coincide identically with their $\overline{\text{MS}}$-analogs (13a), calculated in [45], [46], [49]–[51]:

$$\beta_0^V = \frac{11}{12}C_A - \frac{1}{3}T_F n_f, \tag{14a}$$

$$\beta_1^V = \frac{17}{24}C_A^2 - \frac{5}{12}C_A T_F n_f - \frac{1}{4}C_F T_F n_f. \tag{14b}$$

The third and fourth terms β_2^V and β_3^V, Eqs. (13b) and (13c), are expressed in terms of the respective three- and four-loop coefficients of the $\overline{\text{MS}}$-scheme RG β-function, analytically computed in [52], [53] and [54], [55]. Using Eqs. (2), (3), and (13b), we can obtain the analytic three-loop coefficient β_2^V:

$$\begin{aligned}\beta_2^V = &\left(\frac{103}{96} + \frac{121}{288}\zeta_3 + \frac{11}{48}\pi^2 - \frac{11}{768}\pi^4\right)C_A^3 + \\ &+ \left(-\frac{445}{576} - \frac{11}{9}\zeta_3 - \frac{\pi^2}{12} + \frac{\pi^4}{192}\right)C_A^2 T_F n_f + \left(-\frac{343}{288} + \frac{11}{12}\zeta_3\right)C_A C_F T_F n_f + \\ &+ \frac{1}{32}C_F^2 T_F n_f + \left(\frac{1}{288} + \frac{7}{18}\zeta_3\right)C_A T_F^2 n_f^2 + \left(\frac{23}{72} - \frac{1}{3}\zeta_3\right)C_F T_F^2 n_f^2.\end{aligned} \tag{14c}$$

This result was originally derived in [10]. Unlike the $\overline{\text{MS}}$-scheme β_2-term, the coefficient β_2^V contains not only rational numbers but also transcendental ones, namely, the ζ_3, π^2, and π^4-contributions. They originate from the two-loop correction a_2 to the static potential in (3).

Using Eqs. (5a)–(5d) and (13c)), we find the four-loop coefficient β_3^V in analytic form:

$$\begin{aligned}\beta_3^V = &\Bigg[-\frac{3871}{2592} + \frac{1463}{432}\zeta_3 - \frac{21197}{2304}\zeta_5 - \frac{1573}{768}\zeta_3^2 + \frac{33}{8}s_6 - \frac{50831}{1161216}\pi^6 + \\ &+ \pi^4\left(\frac{45023}{207360} - \frac{55}{864}\log 2 - \frac{55}{432}\log^2 2\right) + \\ &+ \pi^2\left(-\frac{35035}{20736} + \frac{1925}{768}\zeta_3 + \frac{803}{144}\alpha_4 - \frac{5071}{1728}\log 2 + \frac{2387}{1152}\zeta_3\log 2\right)\Bigg]C_A^4 + \\ &+ \Bigg[\frac{731}{192} - \frac{13}{3}\zeta_3 + \frac{19709}{2304}\zeta_5 + \frac{1199}{768}\zeta_3^2 - \frac{23}{8}s_6 + \frac{10189}{414720}\pi^6 + \\ &+ \pi^4\left(-\frac{2419}{11520} + \frac{25}{3456}\log 2 + \frac{259}{3456}\log^2 2\right) + \\ &+ \pi^2\left(\frac{14477}{10368} - \frac{1259}{1152}\zeta_3 - \frac{53}{18}\alpha_4 + \frac{889}{864}\log 2 - \frac{665}{576}\zeta_3\log 2\right)\Bigg]C_A^3 T_F n_f + \\ &+ \left[-\frac{7645}{1152} + \frac{61}{24}\zeta_3 + \frac{55}{24}\zeta_5\right]C_A^2 C_F T_F n_f + \frac{23}{128}C_F^3 T_F n_f + \\ &+ \left[\frac{143}{576} + \frac{143}{48}\zeta_3 - \frac{55}{12}\zeta_5\right]C_A C_F^2 T_F n_f + \\ &+ \Bigg[-\frac{1171}{432} + \frac{89}{72}\zeta_3 - \frac{1091}{576}\zeta_5 - \frac{19}{64}\zeta_3^2 + \frac{1}{2}s_6 - \frac{761}{241920}\pi^6 + \\ &+ \pi^4\left(\frac{529}{8640} + \frac{5}{864}\log 2 - \frac{1}{96}\log^2 2\right) + \\ &+ \pi^2\left(-\frac{737}{2592} + \frac{19}{288}\zeta_3 + \frac{1}{3}\alpha_4 + \frac{1}{72}\log 2 + \frac{7}{48}\zeta_3\log 2\right)\Bigg]C_A^2 T_F^2 n_f^2 + \\ &+ \left[\frac{583}{144} - \frac{7}{4}\zeta_3 - \frac{5}{6}\zeta_5\right]C_A C_F T_F^2 n_f^2 + \left[-\frac{29}{288} - \frac{4}{3}\zeta_3 + \frac{5}{3}\zeta_5\right]C_F^2 T_F^2 n_f^2 +\end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{293}{648}+\frac{\zeta_3}{54}-\frac{2}{405}\pi^4\right]C_A T_F^3 n_f^3+\left[-\frac{1}{2}+\frac{\zeta_3}{3}\right]C_F T_F^3 n_f^3+\\
&+\left[-\frac{5}{144}+\frac{11}{12}\zeta_3\right]\frac{d_A^{abcd}d_A^{abcd}}{N_A}+\left[\frac{2}{9}-\frac{13}{6}\zeta_3\right]\frac{d_F^{abcd}d_A^{abcd}}{N_A}n_f+\\
&+\left[-\frac{1511}{4320}\pi^6+\pi^4\left(\frac{13}{8}-\frac{35}{18}\log 2-\frac{31}{18}\log^2 2\right)+\right.\\
&\left.+\pi^2\left(-\frac{929}{108}+\frac{827}{36}\zeta_3+\frac{148}{3}\alpha_4-\frac{461}{9}\log 2+\frac{217}{6}\zeta_3\log 2\right)\right]\frac{d_F^{abcd}d_A^{abcd}}{N_A}T_F n_f+\\
&+\left[\frac{16621}{17280}\pi^6+\pi^4\left(-\frac{143}{32}+\frac{385}{72}\log 2+\frac{341}{72}\log^2 2\right)+\right.\\
&\left.+\pi^2\left(\frac{10219}{432}-\frac{9097}{144}\zeta_3-\frac{407}{3}\alpha_4+\frac{5071}{36}\log 2-\frac{2387}{24}\zeta_3\log 2\right)\right]C_A\frac{d_F^{abcd}d_A^{abcd}}{N_A}+\\
&+\left[\frac{55}{576}\pi^6-\pi^4\left(\frac{253}{144}-\frac{11}{36}\log 2+\frac{11}{12}\log^2 2\right)+\right.\\
&\left.+\pi^2\left(\frac{869}{216}-\frac{671}{72}\zeta_3+\frac{11}{6}\log 2+\frac{77}{4}\zeta_3\log 2\right)\right]C_A\frac{d_F^{abcd}d_F^{abcd}}{N_A}n_f-\\
&-\left[\frac{11}{36}-\frac{2}{3}\zeta_3\right]\frac{d_F^{abcd}d_F^{abcd}}{N_A}n_f^2+\left[-\frac{5}{144}\pi^6+\pi^4\left(\frac{23}{36}-\frac{1}{9}\log 2+\frac{1}{3}\log^2 2\right)+\right.\\
&\left.+\pi^2\left(-\frac{79}{54}+\frac{61}{18}\zeta_3-\frac{2}{3}\log 2-7\zeta_3\log 2\right)\right]\frac{d_F^{abcd}d_F^{abcd}}{N_A}T_F n_f^2. \qquad (14d)
\end{aligned}
$$

The analytic result in (14d) improves the presentation of our previous semianalytic expression for β_3^V obtained in [14]. Indeed, the coefficient β_3^V presented there contained numerical uncertainties associated with the inability to calculate specific three-loop master integrals to the static potential with a high enough precision [12], [56], [57] to apply the PSLQ algorithm [58], [59] and restore their analytic expressions from the obtained numerical values. This problem was solved in [13] by means of a dimensional recurrence relation [60] and the convergence acceleration algorithm [61].

Unlike the β_3-coefficient in the $\overline{\text{MS}}$ scheme, which contains rational numbers and ζ_3-contributions only, the coefficient β_3^V is expressed through a much larger number of terms with higher transcendentalities initially appearing in the three-loop correction a_3 to the static potential. We also note that result (14d) includes four extra color structures originating from $2a_3\beta_0$-term in Eq. (13c) and not encountered in the representation of the β_3-coefficient, namely, the $C_A d_F^{abcd}d_A^{abcd}/N_A$, $C_A d_F^{abcd}d_F^{abcd}n_f/N_A$, $d_F^{abcd}d_A^{abcd}T_F n_f/N_A$, and $d_F^{abcd}d_F^{abcd}T_F n_f^2/N_A$ patterns. The term proportional to the $d_A^{abcd}d_A^{abcd}/N_A$ structure in (14d) follows from the $\overline{\text{MS}}$-scheme coefficient β_3. In the particular case of the $SU(N_c)$ gauge group, it is equal to $d_A^{abcd}d_A^{abcd}/N_A = N_c^2(N_c^2+36)/24$.

Taking the values $\alpha_4 \approx 0.5270972$ and $s_6 \approx 0.9874414$ into account, we arrive at the following numerical form of Eqs. (14a)–(14d) in the case of the $SU(3)$ color gauge group:

$$\beta_0^V = 2.75 - 0.1666667 n_f, \qquad (15a)$$

$$\beta_1^V = 6.375 - 0.7916667 n_f, \qquad (15b)$$

$$\beta_2^V = 66.00284 - 11.656347 n_f + 0.3261237 n_f^2, \qquad (15c)$$

$$\beta_3^V = 168.6484 - 50.59222 n_f + 2.761578 n_f^2 - 0.0190318 n_f^3. \qquad (15d)$$

Expressions (15c) and (15d) are to be compared with their $\overline{\text{MS}}$-counterparts [52]–[55]

$$\beta_2 = 22.32031 - 4.368924 n_f + 0.0940394 n_f^2, \qquad (16a)$$

$$\beta_3 = 114.2303 - 27.13394 n_f + 1.582379 n_f^2 + 0.0058567 n_f^2 \qquad (16b)$$

and with the mMOM ones in the Landau gauge [29], [30], [32]

$$\beta_2^{\mathrm{mMOM},\,\xi=0} = 47.50754 - 9.771667 n_f + 0.3028642 n_f^2, \tag{17a}$$
$$\beta_3^{\mathrm{mMOM},\,\xi=0} = 392.7385 - 95.40363 n_f + 6.349228 n_f^2 - 0.1073931 n_f^3. \tag{17b}$$

Naturally, the first two coefficients of the RG β-function coincide in the V, $\overline{\mathrm{MS}}$, and mMOM schemes with $\xi = 0$.

3. The Adler function, *R*-ratio, and the Bjorken polarized sum rule in the V scheme

3.1. The Adler function in the V scheme. It is known that the Adler function $D(Q^2)$ is a convenient ingredient for calculating the Minkowskian annihilation electron–positron cross section into hadrons with the help of a Källen–Lehmann-type dispersion relation (see, e.g., [62], [63]). This function is defined in the Euclidean domain with the Euclidean transferred momentum $Q^2 = -q^2$ and, importantly, is a renormalization-invariant quantity. Its two-, three-, and four-loop expressions in the $\overline{\mathrm{MS}}$ scheme were directly evaluated in [64]–[66], [67], [68], and [18], [69], [70].

In the massless limit, the function $D(Q^2)$ is decomposed into a sum of flavor nonsinglet (NS) and singlet (SI) components:

$$D(a_s) = d_R\biggl(\sum_f Q_f^2\biggr) D^{\mathrm{NS}}(a_s) + d_R\biggl(\sum_f Q_f\biggr)^2 D^{\mathrm{SI}}(a_s), \tag{18}$$

where Q_f is the electric charge of the active quark with a flavor f and d_R is the dimension of the quark representation of the Lie algebra of the considered generic simple gauge group. In the case of the $SU(N_c)$ color gauge group, $d_R = N_c$. The singlet (SI) flavor contribution $D^{\mathrm{SI}}(a_s)$ appears from the third-order a_s^3 due to the special diagrams of the light-by-light scattering type [67], [70].

To obtain an analytic four-loop expression for the NS Adler function $D_V^{\mathrm{NS}}(a_{s,V})$ in the V scheme, we use its explicit $\overline{\mathrm{MS}}$-scheme result at the same level, relation (10) between the corresponding couplings in two considered gauge-independent schemes, and take the RG-invariance of the flavor NS Adler function into account. Keeping the aforesaid in mind, we obtain the following results:

$$D_V^{\mathrm{NS}}(a_{s,V}) = 1 + \sum_{k\geq 1} d_{k,V}^{\mathrm{NS}} a_{s,V}^k, \tag{19a}$$
$$d_{1,V}^{\mathrm{NS}} = \frac{3}{4} C_F, \tag{19b}$$
$$d_{2,V}^{\mathrm{NS}} = -\frac{3}{32} C_F^2 + \biggl(\frac{307}{96} - \frac{11}{4}\zeta_3\biggr) C_F C_A + \biggl(-\frac{23}{24} + \zeta_3\biggr) C_F T_F n_f, \tag{19c}$$
$$\begin{aligned} d_{3,V}^{\mathrm{NS}} = {} & -\frac{69}{128} C_F^3 + \biggl(-\frac{175}{96} - \frac{143}{16}\zeta_3 + \frac{55}{4}\zeta_5\biggr) C_F^2 C_A + \\ & + \biggl(\frac{621}{32} - \frac{1403}{96}\zeta_3 - \frac{55}{24}\zeta_5 - \frac{3}{16}\pi^2 + \frac{3}{256}\pi^4\biggr) C_F C_A^2 + \\ & + \biggl(\frac{3}{2} - \zeta_3\biggr) C_F T_F^2 n_f^2 + \biggl(-\frac{375}{32} + \frac{205}{24}\zeta_3 + \frac{5}{6}\zeta_5\biggr) C_F C_A T_F n_f + \\ & + \biggl(\frac{29}{96} + 4\zeta_3 - 5\zeta_5\biggr) C_F^2 T_F n_f, \end{aligned} \tag{19d}$$

$$
\begin{aligned}
d_{4,V}^{\rm NS} = & \left(\frac{4157}{2048}+\frac{3}{8}\zeta_3\right)C_F^4+\left(-\frac{3335}{512}-\frac{139}{128}\zeta_3+\frac{2255}{32}\zeta_5-\frac{1155}{16}\zeta_7\right)C_F^3C_A+\\
&+\left(-\frac{498269}{18432}-\frac{17513}{192}\zeta_3+100\zeta_5+\frac{1155}{32}\zeta_7+\frac{3}{64}\pi^2-\frac{3}{1024}\pi^4\right)C_F^2C_A^2+\\
&+\left[\frac{668335}{4608}-\frac{101621}{1152}\zeta_3-\frac{89119}{1536}\zeta_5+\frac{34199}{1536}\zeta_3^2-\frac{385}{64}\zeta_7-\frac{27}{16}s_6+\frac{4621}{258048}\pi^6+\right.\\
&+\pi^4\left(\frac{1}{90}-\frac{11}{128}\zeta_3+\frac{5}{192}\log 2+\frac{5}{96}\log^2 2\right)+\\
&\left.+\pi^2\left(-\frac{4183}{4608}+\frac{179}{512}\zeta_3-\frac{73}{32}\alpha_4+\frac{461}{384}\log 2-\frac{217}{256}\zeta_3\log 2\right)\right]C_FC_A^3+\\
&+\left(\frac{287}{256}+\frac{17}{8}\zeta_3-\frac{235}{8}\zeta_5+\frac{105}{4}\zeta_7\right)C_F^3T_Fn_f+\\
&+\left(\frac{12277}{1152}+\frac{1117}{16}\zeta_3-\frac{145}{2}\zeta_5-\frac{11}{4}\zeta_3^2-\frac{105}{8}\zeta_7\right)C_F^2C_AT_Fn_f+\\
&+\left[-\frac{201725}{1536}+\frac{41071}{576}\zeta_3+\frac{87847}{1536}\zeta_5-\frac{20225}{1536}\zeta_3^2+\frac{35}{16}\zeta_7+\frac{9}{16}s_6-\frac{761}{215040}\pi^6+\right.\\
&+\pi^4\left(\frac{109}{4608}+\frac{\zeta_3}{32}+\frac{5}{768}\log 2-\frac{3}{256}\log^2 2\right)+\\
&\left.+\pi^2\left(\frac{367}{2304}-\frac{109}{256}\zeta_3+\frac{3}{8}\alpha_4+\frac{\log 2}{64}+\frac{21}{128}\zeta_3\log 2\right)\right]C_FC_A^2T_Fn_f+\\
&+\left(-\frac{125}{384}-\frac{281}{24}\zeta_3+\frac{25}{2}\zeta_5+\zeta_3^2\right)C_F^2T_F^2n_f^2+\\
&+\left(\frac{81103}{2304}-\frac{4859}{288}\zeta_3-\frac{35}{2}\zeta_5+\frac{11}{6}\zeta_3^2-\frac{\pi^4}{180}\right)C_FC_AT_F^2n_f^2-\\
&-\left(\frac{67}{24}-\frac{7}{6}\zeta_3-\frac{5}{3}\zeta_5\right)C_FT_F^3n_f^3+\left(\frac{3}{16}-\frac{\zeta_3}{4}-\frac{5}{4}\zeta_5\right)\frac{d_F^{abcd}d_A^{abcd}}{d_R}+\\
&+\left(-\frac{13}{16}-\zeta_3+\frac{5}{2}\zeta_5\right)\frac{d_F^{abcd}d_F^{abcd}}{d_R}n_f+\left[-\frac{1511}{3840}\pi^6+\pi^4\left(\frac{117}{64}-\frac{35}{16}\log 2-\frac{31}{16}\log^2 2\right)+\right.\\
&\left.+\pi^2\left(-\frac{929}{96}+\frac{827}{32}\zeta_3+\frac{111}{2}\alpha_4-\frac{461}{8}\log 2+\frac{651}{16}\zeta_3\log 2\right)\right]C_F\frac{d_F^{abcd}d_A^{abcd}}{N_A}+\\
&+\left[-\frac{5}{128}\pi^6+\pi^4\left(\frac{23}{32}-\frac{\log 2}{8}+\frac{3}{8}\log^2 2\right)+\right.\\
&\left.+\pi^2\left(-\frac{79}{48}+\frac{61}{16}\zeta_3-\frac{3}{4}\log 2-\frac{63}{8}\zeta_3\log 2\right)\right]C_F\frac{d_F^{abcd}d_F^{abcd}}{N_A}n_f. \qquad (19e)
\end{aligned}
$$

We make a few comments on the derived expressions. First, unlike the $\overline{\rm MS}$-scheme three-loop results, the coefficient $d_{3,V}^{\rm NS}$ in (19d) contains the complementary terms of the $C_FC_A^2$ contribution, which are proportional to π^2 and π^4. Second, the coefficient $d_{4,V}^{\rm NS}$ in (19e) includes all transcendental basic constants contained in β_3^V, plus the extra ζ_7-term with the greatest transcendence of weight $w=7$, originally appearing from d_4 in the $\overline{\rm MS}$ scheme [69]. It is also worth noting that in contrast to d_4, the analytic expression for $d_{4,V}^{\rm NS}$ has two additional color structures, namely, $C_Fd_F^{abcd}d_A^{abcd}/N_A$ and $C_Fd_F^{abcd}d_F^{abcd}n_f/N_A$, coming from the product of a_3 on d_1.

For $SU(3)$, these coefficients are numerically given by

$$d_{1,V}^{\rm NS}=1, \qquad (20a)$$

$$d_{2,V}^{\rm NS}=-0.597626+0.1624824n_f, \qquad (20b)$$

$$d_{3,V}^{\rm NS} = -7.21638 - 1.240217n_f + 0.0993144n_f^2, \tag{20c}$$
$$d_{4,V}^{\rm NS} = 19.9437 + 4.38696n_f - 1.114839n_f^2 + 0.0564909n_f^3. \tag{20d}$$

The SI contributions to the respective coefficients d_3 and d_4 of the Adler function were calculated in the $\overline{\rm MS}$ scheme in [67], [70]. Taking the renormalization invariance of the function $D(Q^2)$ into account, we can obtain the SI contributions to the coefficients $d_{3,V}$ and $d_{4,V}$,

$$D_V^{\rm SI}(a_{s,V}) = \sum_{k\geq 3} d_{k,V}^{\rm SI} a_{s,V}^k, \tag{21a}$$
$$d_{3,V}^{\rm SI} = d_3^{\rm SI} = \left(\frac{11}{192} - \frac{\zeta_3}{8}\right)\frac{d^{abc}d^{abc}}{d_R}, \tag{21b}$$
$$d_{4,V}^{\rm SI} = d_4^{\rm SI} - 3a_1 d_3^{\rm SI} = \left[\left(-\frac{13}{64} - \frac{\zeta_3}{4} + \frac{5}{8}\zeta_5\right)C_F + \right.$$
$$+\left(\frac{3211}{4608} - \frac{383}{384}\zeta_3 + \frac{45}{64}\zeta_5 - \frac{11}{32}\zeta_3^2\right)C_A +$$
$$\left. + \left(-\frac{47}{288} + \frac{19}{96}\zeta_3 - \frac{5}{16}\zeta_5 + \frac{\zeta_3^2}{8}\right)T_F n_f\right]\frac{d^{abc}d^{abc}}{d_R}, \tag{21c}$$

where d^{abc} is the symmetric color constant, which for the $SU(N_c)$ group satisfies the relation $d^{abc}d^{abc} = (N_c^2-4)(N_c^2-1)/N_c$.

Expressions (21b) and (21c) in numerical form are given by

$$d_{3,V}^{\rm SI} = -0.413179, \qquad d_{4,V}^{\rm SI} = -2.74010 - 0.152688n_f. \tag{22}$$

3.2. The $R(s)$-ratio in the V scheme. We move on to the case of the $R(s)$-ratio of the process of the electron–positron annihilation into hadrons. This quantity is directly measured in the Minkowski region of energies and is expressed through the cross section of this process,

$$R(s) = \frac{\sigma(e^+e^- \to \gamma^* \to \text{hadrons})}{\sigma_{\rm Born}(e^+e^- \to \gamma^* \to \mu^+\mu^-)} = d_R\left(\sum_f Q_f^2\right)R^{\rm NS}(a_s) + d_R\left(\sum_f Q_f\right)^2 R^{\rm SI}(a_s), \tag{23}$$

where $\sigma_{\rm Born}(e^+e^- \to \mu^+\mu^-) = 4\pi\alpha_{\rm EM}^2/3s$ is the Born massless normalization factor.

The Källen–Lehmann-type dispersion representation (see, e.g., [62], [63]), relating the Adler function to the $R(s)$-ratio, dictates the following analytic correspondence:

$$R(s) = D(s) - \frac{\pi^2}{3}d_1\beta_0^2 a_s^3 - \pi^2\left(d_2\beta_0^2 + \frac{5}{6}d_1\beta_1\beta_0\right)a_s^4 + \mathcal{O}(a_s^5). \tag{24}$$

The terms proportional to π^2 appear here as the effect of the analytic continuation from the Euclidean to Minkowskian domain.

Using Eq. (24) and the RG invariance of the R-ratio, we can conclude that the following relations hold between the coefficients of the NS and SI contributions to the R-ratio and the Adler function in the V scheme:

$$R_V^{\rm NS}(a_{s,V}) = 1 + \sum_{k\geq 1} r_{k,V}^{\rm NS} a_{s,V}^k, \qquad R_V^{\rm SI}(a_{s,V}) = \sum_{k\geq 3} r_{k,V}^{\rm SI} a_{s,V}^k, \tag{25a}$$
$$r_{1,V}^{\rm NS} = d_{1,V}^{\rm NS}, \qquad r_{2,V}^{\rm NS} = d_{2,V}^{\rm NS}, \tag{25b}$$
$$r_{3,V}^{\rm NS} = d_{3,V}^{\rm NS} - \frac{\pi^2}{3}d_1\beta_0^2, \qquad r_{3,V}^{\rm SI} = d_{3,V}^{\rm SI}, \tag{25c}$$
$$r_{4,V}^{\rm NS} = d_{4,V}^{\rm NS} - \pi^2\left(d_{2,V}^{\rm NS}\beta_0^2 + \frac{5}{6}d_1\beta_1\beta_0\right), \qquad r_{4,V}^{\rm SI} = d_{4,V}^{\rm SI}. \tag{25d}$$

Taking Eqs. (25b)–(25d) into account, we arrive at the complete numerical result for $R_V(s)$ in the V scheme in the physically relevant case of the $SU(3)$ group,

$$R_V(a_{s,V}) = 3\sum_f Q_f^2\biggl(1+\sum_{k\geq 1} r_{k,V}\, a_{s,V}^k\biggr), \tag{26a}$$

$$r_{1,V} = 1, \tag{26b}$$

$$r_{2,V} = -0.597626 + 0.1624824 n_f, \tag{26c}$$

$$r_{3,V} = -32.09600 + 1.775495 n_f + 0.0079291 n_f^2 - 0.413179\delta_f, \tag{26d}$$

$$\begin{aligned} r_{4,V} = {} & -79.6389 + 13.49715 n_f - 0.566196 n_f^2 + 0.0119455 n_f^3 + \\ & + (-2.74010 - 0.152688 n_f)\delta_f, \end{aligned} \tag{26e}$$

where terms with $\delta_f = (\sum_f Q_f)^2/(\sum_f Q_f^2)$ are the SI contributions. We note that the analogous V-scheme numerical expressions for the $R_V(s)$ coefficients were previously presented in [14] but with theoretical mean-square uncertainties following from the inaccuracies in the calculation of the a_3 correction to the static potential [12], [56], [57]. Naturally, these results are in full agreement with those given in (26b)–(26e). The discussed uncertainties are negligible and much smaller than the ones related to the determination of physical parameters such as $\alpha_s(M_Z^2)$ [71]. In [14], [28], [31], the interested reader can find the results of the study of the scheme- and scale-dependence of the $R(s)$-ratio at the in the cases $n_f = 4$ and $n_f = 5$.

3.3. The Bjorken polarized sum rule in the V scheme. One of the important physical quantities in the study of the DIS of polarized leptons on nucleons is the coefficient Bjorken function $C_{\mathrm{Bjp}}(Q^2)$. It is determined in the Euclidean region of energies and included in the Bjorken polarized sum rule (neglecting the $\mathcal{O}(1/Q^{2k})$ nonperturbative terms):

$$\int_0^1 \Bigl(g_1^{lp}(x,Q^2) - g_1^{ln}(x,Q^2)\Bigr)\,dx = \frac{1}{6}\left|\frac{g_A}{g_V}\right| C_{\mathrm{Bjp}}(Q^2), \tag{27}$$

Here, $g_1^{lp}(x,Q^2)$ and $g_1^{ln}(x,Q^2)$ are the structure functions of the DIS processes, which characterize the spin distribution of quarks and gluons inside nucleons, and g_A and g_V are the axial and vector neutron β-decay constants with $g_A/g_V = -1.2754 \pm 0.0013$ [71].

The coefficient Bjorken function is split in two components, the NS and SI ones:

$$C_{\mathrm{Bjp}}(a_s) = C_{\mathrm{Bjp}}^{\mathrm{NS}}(a_s) + d_R \sum_f Q_f C_{\mathrm{Bjp}}^{\mathrm{SI}}(a_s). \tag{28}$$

The one-, two-, three-, and four-loop results for the NS coefficient Bjorken function $C_{\mathrm{Bjp}}(a_s)$ in the $\overline{\mathrm{MS}}$ scheme were respectively obtained in [72], [73], [74], [18]. Unlike the $D^{\mathrm{SI}}(a_s)$-function, the SI part of $C_{\mathrm{Bjp}}(a_s)$ appears first at the $\mathcal{O}(a_s^4)$ level [75] and was calculated analytically in [76].

Using the explicit fourth-order approximation for the Bjorken function in the $\overline{\mathrm{MS}}$ scheme [18], [76], relation (10), and the RG-invariance of $C_{\mathrm{Bjp}}(a_s)$, we arrive at the following expressions for the NS and SI contributions to the coefficient Bjorken function in the V scheme:

$$C_{\mathrm{Bjp},V}^{\mathrm{NS}}(a_{s,V}) = 1 + \sum_{k\geq 1} c_{k,V}^{\mathrm{NS}} a_{s,V}^k, \qquad C_V^{\mathrm{SI}}(a_{s,V}) = \sum_{k\geq 4} c_{k,V}^{\mathrm{SI}} a_{s,V}^k, \tag{29a}$$

$$c_{1,V}^{\mathrm{NS}} = -\frac{3}{4} C_F, \tag{29b}$$

$$c_{2,V}^{\mathrm{NS}} = \frac{21}{32} C_F^2 - \frac{19}{24} C_F C_A + \frac{1}{12} C_F T_F n_f, \tag{29c}$$

$$
\begin{aligned}
c_{3,V}^{\rm NS} = & -\frac{3}{128}C_F^3 + \left(\frac{295}{288} - \frac{11}{12}\zeta_3\right)C_F^2 C_A + \left(\frac{73}{36} - \frac{\zeta_3}{8} - \frac{5}{6}\zeta_5\right)C_F C_A T_F n_f + \\
& + \left(-\frac{4231}{1152} + \frac{11}{32}\zeta_3 + \frac{55}{24}\zeta_5 + \frac{3}{16}\pi^2 - \frac{3}{256}\pi^4\right)C_F C_A^2 - \\
& - \frac{5}{24}C_F T_F^2 n_f^2 + \left(-\frac{13}{36} + \frac{\zeta_3}{3}\right)C_F^2 T_F n_f, \qquad \text{(29d)}
\end{aligned}
$$

$$
\begin{aligned}
c_{4,V}^{\rm NS} = & \left(-\frac{4823}{2048} - \frac{3}{8}\zeta_3\right)C_F^4 + \left(-\frac{857}{1152} - \frac{971}{96}\zeta_3 + \frac{1045}{48}\zeta_5\right)C_F^3 C_A + \\
& + \left(\frac{776809}{55296} + \frac{4921}{384}\zeta_3 - \frac{1375}{144}\zeta_5 - \frac{385}{16}\zeta_7 - \frac{21}{64}\pi^2 + \frac{21}{1024}\pi^4\right)C_F^2 C_A^2 + \\
& + \left[-\frac{247307}{13824} + \frac{4579}{1152}\zeta_3 + \frac{5557}{4608}\zeta_5 - \frac{3223}{1536}\zeta_3^2 + \frac{385}{64}\zeta_7 + \frac{27}{16}s_6 - \frac{4621}{258048}\pi^6 + \right. \\
& + \pi^4\left(\frac{2953}{46080} - \frac{5}{192}\log 2 - \frac{5}{96}\log^2 2\right) + \\
& \left. + \pi^2\left(-\frac{1361}{4608} + \frac{525}{512}\zeta_3 + \frac{73}{32}\alpha_4 - \frac{461}{384}\log 2 + \frac{217}{256}\zeta_3 \log 2\right)\right]C_F C_A^3 + \\
& + \left(\frac{317}{144} + \frac{109}{24}\zeta_3 - \frac{95}{12}\zeta_5\right)C_F^3 T_F n_f + \\
& + \left(-\frac{14177}{1728} - \frac{739}{144}\zeta_3 + \frac{205}{72}\zeta_5 + \frac{35}{4}\zeta_7\right)C_F^2 C_A T_F n_f + \\
& + \left[\frac{47693}{3456} - \frac{77}{18}\zeta_3 + \frac{851}{512}\zeta_5 + \frac{1921}{1536}\zeta_3^2 - \frac{35}{16}\zeta_7 - \frac{9}{16}s_6 + \frac{761}{215040}\pi^6 + \right. \\
& + \pi^4\left(-\frac{235}{4608} - \frac{5}{768}\log 2 + \frac{3}{256}\log^2 2\right) + \\
& \left. + \pi^2\left(\frac{641}{2304} - \frac{19}{256}\zeta_3 - \frac{3}{8}\alpha_4 - \frac{\log 2}{64} - \frac{21}{128}\zeta_3 \log 2\right)\right]C_F C_A^2 T_F n_f + \\
& + \left(\frac{1891}{3456} - \frac{\zeta_3}{36}\right)C_F^2 T_F^2 n_f^2 + \left(-\frac{8309}{3456} + \frac{9}{8}\zeta_3 - \frac{35}{36}\zeta_5 - \frac{\zeta_3^2}{6} + \frac{\pi^4}{180}\right)C_F C_A T_F^2 n_f^2 + \\
& + \frac{5}{72}C_F T_F^3 n_f^3 + \left(-\frac{3}{16} + \frac{\zeta_3}{4} + \frac{5}{4}\zeta_5\right)\frac{d_F^{abcd}d_A^{abcd}}{d_R} + \\
& + \left(\frac{13}{16} + \zeta_3 - \frac{5}{2}\zeta_5\right)\frac{d_F^{abcd}d_F^{abcd}}{d_R}n_f + \\
& + \left[\frac{1511}{3840}\pi^6 + \pi^4\left(-\frac{117}{64} + \frac{35}{16}\log 2 + \frac{31}{16}\log^2 2\right) + \right. \\
& \left. + \pi^2\left(\frac{929}{96} - \frac{827}{32}\zeta_3 - \frac{111}{2}\alpha_4 + \frac{461}{8}\log 2 - \frac{651}{16}\zeta_3 \log 2\right)\right]C_F\frac{d_F^{abcd}d_A^{abcd}}{N_A} + \\
& + \left[\frac{5}{128}\pi^6 - \pi^4\left(\frac{23}{32} - \frac{\log 2}{8} + \frac{3}{8}\log^2 2\right) + \right. \\
& \left. + \pi^2\left(\frac{79}{48} - \frac{61}{16}\zeta_3 + \frac{3}{4}\log 2 + \frac{63}{8}\zeta_3 \log 2\right)\right]C_F\frac{d_F^{abcd}d_F^{abcd}}{N_A}n_f, \qquad \text{(29e)}
\end{aligned}
$$

$$
c_{4,V}^{\rm SI} = c_4^{\rm SI} = \frac{1}{9}\beta_0 d^{abc}d^{abc}. \qquad \text{(29f)}
$$

Comparing Eqs. (19e) and (29e), we can see certain similarities between the analytic expressions for $d_{4,V}^{\rm NS}$ and $c_{4,V}^{\rm NS}$. For instance, the contributions proportional to the $d_F^{abcd}d_A^{abcd}/d_R$, $d_F^{abcd}d_F^{abcd}n_f/d_R$, $C_F d_F^{abcd}d_A^{abcd}/N_A$, and $C_F d_F^{abcd}d_F^{abcd}n_f/N_A$ color structures in their expressions are identical in absolute values, but opposite in sign. Moreover, the terms proportional to π^6, s_6, $\pi^4\log 2$, $\pi^4\log^2 2$, $\pi^2\alpha_4$, $\pi^2\log 2$, and $\pi^2\zeta_3\log 2$ share the same property. Therefore, all these color structures and transcendental constants cancel automatically in the sum of $d^{\rm NS}_{4,V}+c^{\rm NS}_{4,V}$. This fact is important in studying conditions of the factorization of the $\Delta_{csb}(a_s)$-term into $\beta(a_s)/a_s$ and $K(a_s)$ functions in the CBK relation in the V scheme (see a discussion below).

For the $SU(3)$ group, we obtain the following numerical form of the PT coefficients of the $C_{{\rm Bjp},V}$-function:

$$C_{{\rm Bjp},V}(a_{s,V}) = 1+\sum_{k\geq 1} c_{k,V}a^k_{s,V}, \tag{30a}$$

$$c_{1,V} = -1, \tag{30b}$$

$$c_{2,V} = -2+0.0555556n_f, \tag{30c}$$

$$c_{3,V} = -2.55978+2.062006n_f-0.0694444n_f^2, \tag{30d}$$

$$c_{4,V} = -122.1910+30.87144n_f-1.531353n_f^2+0.0115741n_f^3+ \\ +(12.22222-0.740741n_f)\eta_f, \tag{30e}$$

where $\eta_f=\sum_f Q_f$.

3.4. PT series for the Adler function, *R*-ratio, and the coefficient Bjorken function in the $\overline{\mathbf{MS}}$, V, and Landau mMOM schemes. For comparison of the behavior of the PT series for the Adler function and the coefficient Bjorken functions, we consider their expressions in the $\overline{\rm MS}$ [18], [64]–[70], [72]–[74], [76], V (20b)–(20d), (22), (30c)–(30e) [28], and mMOM schemes in the Landau gauge [15], [29], [31] for the $SU(3)$ color gauge group. Taking the results of the cited works into account, we can present them in Table 1.

The content of Table 1 indicates that the contributions to the NS coefficient Bjorken function in all three schemes have a sign-alternating structure in n_f. This property is also valid for the higher-order corrections to the Adler function in the $\overline{\rm MS}$ scheme. In other cases, this feature is violated starting from the a_s^3 term. This fact may be regarded as the argument in favor of the well-known statement that the renormalon-motivated large-β_0 approximation is more pronounced for quantities calculated from the PT in the $\overline{\rm MS}$ scheme in the Euclidean domain [77]. We note, however, that the PT expressions for the Adler function and the coefficient function of the Bjorken polarized sum rule in the $\overline{\rm MS}$ scheme contain contributions not only of the ultraviolet (UV) renormalons resulting in sign-alternating series but also of the infrared (IR) ones, which lead to sign-constant series (see, e.g., [77], [78] and the references therein). The possible irregularities in low orders may be caused by cancellations between IR and UV renormalons.

We also mention that the analytic expressions for $D(Q^2)$, $R(s)$, and $C_{\rm Bjp}(Q^2)$ at the $\mathcal{O}(a_s^4)$ level in the mMOM scheme with an arbitrary linear covariant gauge parameter can be found in [15], [27], [28]. The investigation of the behavior of the $R(s)$-ratio for the process $e^+e^-\to\gamma^*\to$ *hadrons* depending on the energy in the center of mass system and the study of its scheme dependence with the example of the three discussed renormalization schemes have been considered previously for $n_f=4,5$ in [14], [28].

4. The CBK relation in the V scheme

Having obtained analytic fourth-order approximations for the Adler function, the coefficient function of the Bjorken polarized sum rule, and the β-function in the V scheme, we can study CBK relation (6) in this scheme in detail. It is known that the CBK relation is implemented in the class of gauge-invariant MS-like schemes at least at the $\mathcal{O}(a_s^4)$ PT level [17], [18] (but apparently in all orders [24]–[26]). However, the following question remains open: does this relation hold in other gauge-independent schemes different from the MS-like ones? In this section, we investigate this issue with the example of the considered gauge-invariant V scheme.

Table 1. The PT series for the Adler function, the R-ratio, and the coefficient function of the Bjorken polarized sum rule in QCD in the $\overline{\text{MS}}$-, V, and Landau mMOM schemes. The overall factor $3\sum_f Q_f^2$ is omitted from the expressions for $D(Q^2)$ and $R(s)$. Here, $\delta_f = (\sum_f Q_f)^2/(\sum_f Q_f^2)$ and $\eta_f = \sum_f Q_f$

Scheme	**Adler function**
$\overline{\text{MS}}$	$1 + a_s + (1.9857 - 0.11529 n_f) a_s^2 +$ $+ (18.2427 - 4.2158 n_f + 0.0862 n_f^2 - 0.413 \delta_f) a_s^3 +$ $+ (135.792 - 34.440 n_f + 1.875 n_f^2 - 0.010 n_f^3 + \delta_f(-5.942 + 0.1916 n_f)) a_s^4$
V	$1 + a_{s,V} + (-0.5976 + 0.1625 n_f) a_{s,V}^2 +$ $+ (-7.2164 - 1.240 n_f + 0.0993 n_f^2 - 0.413 \delta_f) a_{s,V}^3 +$ $+ (19.944 + 4.387 n_f - 1.115 n_f^2 + 0.056 n_f^3 + \delta_f(-2.740 - 0.1527 n_f)) a_{s,V}^4$
mMOM $\xi = 0$	$1 + a_{s,M} + (-1.535 + 0.1625 n_f) a_{s,M}^2 +$ $+ (-0.6647 - 1.685 n_f + 0.0993 n_f^2 - 0.413 \delta_f) a_{s,M}^3 +$ $+ (-38.363 + 18.44 n_f - 1.71 n_f^2 + 0.056 n_f^3 + \delta_f(-1.578 - 0.1527 n_f)) a_{s,M}^4$
	***R*-ratio**
$\overline{\text{MS}}$	$1 + a_s + (1.9857 - 0.11529 n_f) a_s^2 +$ $+ (-6.6369 - 1.2000 n_f - 0.00518 n_f^2 - 0.413 \delta_f) a_s^3 +$ $+ (-156.608 + 18.7748 n_f - 0.7974 n_f^2 + 0.0215 n_f^3 + \delta_f(-5.942 + 0.1916 n_f)) a_s^4$
V	$1 + a_{s,V} + (-0.5976 + 0.1625 n_f) a_{s,V}^2 +$ $+ (-32.096 + 1.775 n_f + 0.0079 n_f^2 - 0.413 \delta_f) a_{s,V}^3 +$ $+ (-79.639 + 13.497 n_f - 0.566 n_f^2 + 0.0119 n_f^3 + \delta_f(-2.740 - 0.1527 n_f)) a_{s,V}^4$
mMOM $\xi = 0$	$1 + a_{s,M} + (-1.535 + 0.1625 n_f) a_{s,M}^2 +$ $+ (-25.544 + 1.331 n_f + 0.0079 n_f^2 - 0.413 \delta_f) a_{s,M}^3 +$ $+ (-67.981 + 19.068 n_f - 0.904 n_f^2 + 0.0114 n_f^3 + \delta_f(-1.578 - 0.1527 n_f)) a_{s,M}^4$
	Coefficient Bjorken function
$\overline{\text{MS}}$	$1 - a_s + (-4.5833 + 0.33333 n_f) a_s^2 + (-41.4399 + 7.6073 n_f - 0.1775 n_f^2) a_s^3 +$ $+ (-479.448 + 123.39 n_f - 7.69 n_f^2 + 0.104 n_f^3 + \eta_f(12.222 - 0.7407 n_f)) a_s^4$
V	$1 - a_{s,V} + (-2 + 0.05556 n_f) a_{s,V}^2 + (-2.5598 + 2.062 n_f - 0.0694 n_f^2) a_{s,V}^3 +$ $+ (-122.191 + 30.871 n_f - 1.531 n_f^2 + 0.011157 n_f^3 + \eta_f(12.222 - 0.7407 n_f)) a_{s,V}^4$
mMOM $\xi = 0$	$1 - a_{s,M} + (-1.0625 + 0.05556 n_f) a_{s,M}^2 + (-4.2409 + 2.097 n_f - 0.0694 n_f^2) a_{s,M}^3 +$ $+ (-66.891 + 17.790 n_f - 1.091 n_f^2 + 0.0120 n_f^3 + \eta_f(12.222 - 0.7407 n_f)) a_{s,M}^4$

In fact, our problem reduces to verifying the factorization of the conformal symmetry breaking term Δ_{csb} in the CBK relation in the V scheme:

$$D_{\rm V}^{\rm NS}(a_{s,{\rm V}})C_{{\rm Bjp},{\rm V}}^{\rm NS}(a_{s,{\rm V}}) = 1 + \frac{\beta^{\rm V}(a_{s,{\rm V}})}{a_{s,{\rm V}}}K^{\rm V}(a_{s,{\rm V}}). \tag{31}$$

Using the V-scheme analogues of Eqs. (2.3a)–(2.3d) in [15], which also follow from formula (31) we are testing, we can obtain that the first coefficient in the expansion

$$K^{\rm V}(a_{s,{\rm V}}) = \sum_{n\geq 1} K_n^{\rm V} a_{s,{\rm V}}^n \tag{32}$$

coincides with its $\overline{\rm MS}$-scheme analog, namely,

$$K_1^{\rm V} = \left(-\frac{21}{8} + 3\zeta_3\right)C_F. \tag{33}$$

Similarly, using the V-scheme results for a_s^3-corrections to the flavor NS Adler function n Eq. (19d), the coefficient function of the Bjorken polarized sum rule (29d) and the two-loop contribution to β^V (14b), we find the second term in expansion (32):

$$K_2^{\rm V} = \left(\frac{397}{96} + \frac{17}{2}\zeta_3 - 15\zeta_5\right)C_F^2 + \left(-\frac{1453}{96} + \frac{53}{4}\zeta_3\right)C_F C_A + \left(\frac{31}{8} - 3\zeta_3\right)C_F T_F n_f. \tag{34}$$

In expression (34), the analytic term proportional to C_F^2-factor is identical to its $\overline{\rm MS}$ [17] and mMOM scheme counterparts at $\xi = 0$ [15], [27], [28]. However, another abelian V-scheme contribution (34), containing a $C_F T_F n_f$-color structure, coincides with the Landau mMOM analogue [15], [27], [28], but not with the $\overline{\rm MS}$-scheme term [17]. The reason for this lies in the definition of the mMOM scheme [29], [30]. Indeed, the relation between the couplings $a_{s,M}$ in the mMOM scheme and a_s in the $\overline{\rm MS}$ one requires knowledge of the renormalization constants of the gluon and ghost fields only, but not of any vertex structures [29]. Taking the renormalization mMOM conditions into account, we can arrive at the relation [15], [29], [30], [32]

$$a_{s,M}(\mu^2) = \frac{a_s(\mu^2)}{(1+\Pi_A(a_s(\mu^2),\xi(\mu^2)))(1+\Pi_c(a_s(\mu^2),\xi(\mu^2)))^2}, \tag{35}$$

where Π_A and Π_c are respectively the $\overline{\rm MS}$-scheme gluon and ghosts self-energy functions. They were calculated with an explicit dependence on the gauge parameter ξ at the three-loop level in [79] and at the four-loop level in [32]. In the abelian limit of the $U(1)$-group, all gauge-dependent terms proportional to the eigenvalue C_A of the Casimir operator in the adjoint representation vanish and only the $C_F^i(T_F n_f)^j$-contributions with $C_F = 1$ and $T_F = 1$ remain, and therefore a $U(1)$-analogue of formula (35) has the form

$$a_M(\mu^2) = \frac{a(\mu^2)}{1+\Pi_{\rm QED}(a(\mu^2))} = a_{\rm MOM}(\mu^2) = a_{\rm inv}(\mu^2). \tag{36}$$

The left-hand side of formula (36) matches the definition of the RG-invariant and scheme-independent invariant charge in QED, governing the higher-order corrections to the Coulomb static potential in QED (except for the light-by-light type contributions that appear starting from the $\mathcal{O}(a^3)$ level and are not included in the photon vacuum polarization function in Eq. (36)). This fact makes the gauge-dependent mMOM and gauge-invariant V schemes akin in QCD. That is why the abelian C_F^2 and $C_F T_F n_f$ terms in Eq. (34) coincide in these two different schemes.

Using the a_s^4 approximations for the NS contributions to $D(Q^2)$ in (19e), to $C_{\rm Bjp}(Q^2)$ in (29e), and the three-loop expression for β^V in (14c), we obtain the third term in (32):

$$K_3^V = \left(\frac{2471}{768} + \frac{61}{8}\zeta_3 - \frac{715}{8}\zeta_5 + \frac{315}{4}\zeta_7\right)C_F^3 + \left(\frac{75143}{2304} + \frac{2059}{32}\zeta_3 - \frac{545}{6}\zeta_5 - \frac{105}{8}\zeta_7\right)C_F^2 C_A + \\ + \left(-\frac{1273}{144} - \frac{599}{24}\zeta_3 + \frac{75}{2}\zeta_5\right)C_F^2 T_F n_f + \left(-\frac{71389}{576} + \frac{15235}{192}\zeta_3 + \frac{2975}{48}\zeta_5 - \frac{187}{8}\zeta_3^2 + \frac{63}{32}\pi^2 - \\ - \frac{9}{4}\pi^2\zeta_3 - \frac{63}{512}\pi^4 + \frac{9}{64}\pi^4\zeta_3\right)C_F C_A^2 + \left(\frac{40931}{576} - \frac{1771}{48}\zeta_3 - \frac{125}{3}\zeta_5 + \frac{17}{2}\zeta_3^2\right)C_F C_A T_F n_f + \\ + \left(-\frac{49}{6} + \frac{7}{2}\zeta_3 + 5\zeta_5\right)C_F T_F^2 n_f^2. \tag{37}$$

As was expected, all abelian contributions in (37) are the same as in the mMOM scheme in Landau gauge [15] (unlike the $\overline{\rm MS}$-results [18], where only the C_F^3 term is equal to the V-scheme one). However, in contrast to both $\overline{\rm MS}$ and mMOM scheme cases, the V-scheme K_3^V coefficient contains extra π^2, $\pi^2\zeta_3$, π^4, and $\pi^4\zeta_3$ contributions to the $C_F C_A^2$ color structure. The other two non-abelian pieces, proportional to $C_F^2 C_A$ and $C_F C_A T_F n_f$, repeat the transcendental pattern of the $\overline{\rm MS}$ and Landau mMOM scheme results for the K_3 coefficient.

As we have noted, the light-by-light type scattering terms with the $d_F^{abcd}d_A^{abcd}/d_R$, $d_F^{abcd}d_F^{abcd}n_f/d_R$, $C_F d_F^{abcd}d_A^{abcd}/N_A$, and $C_F d_F^{abcd}d_F^{abcd}n_f/N_A$ color factors included in $d_{4,V}^{\rm NS}$ (19e) and $c_{4,V}^{\rm NS}$ (29e) cancel exactly in the expression for $d_{4,V}^{\rm NS} + c_{4,V}^{\rm NS}$, which is equal to

$$(d_{4,V}^{\rm NS} + c_{4,V}^{\rm NS})\Big|_{\rm l-b-l} = (d_4^{\rm NS} + c_4^{\rm NS})\Big|_{\rm l-b-l} - a_3(d_1^{\rm NS} + c_1^{\rm NS})\Big|_{\rm l-b-l}. \tag{38a}$$

Formula (38a) follows directly from the RG invariance of the $D(Q^2)$ and $C_{\rm Bjp}(Q^2)$ functions and from relation (10). We emphasize that the discussed cancellation is a consequence of conformal symmetry. Indeed, the equality $d_1^{\rm NS} + c_1^{\rm NS} = 0$ is the attribute of the CBK relation and arises from the nonrenormalizability of the AVV triangle graph at the $\mathcal{O}(\alpha_s)$ level. This feature was confirmed by direct calculations in [80]. Therefore, in this order, the application of the renormalization procedure does not lead to the appearance of conformal symmetry breaking term in the CBK relation. In its turn, in the conformal invariant limit when all coefficients β_k of the RG β-function vanish, the sum $d_4^{\rm NS} + c_4^{\rm NS}$ in the $\overline{\rm MS}$ scheme is expressed only through terms d_k and c_k with $1 \le k \le 3$ (see, e.g., Eq. (2.3d) in [15]), which do not contain the light-by-light contributions. This means that the equality $(d_4^{\rm NS} + c_4^{\rm NS})|_{\rm l-b-l} = 0$ is a consequence of conformal symmetry [19]. Based on these arguments, we conclude that the left-hand side of Eq. (38a) is identically zero:

$$(d_{4,V}^{\rm NS} + c_{4,V}^{\rm NS})|_{\rm l-b-l} = 0. \tag{38b}$$

This feature is extremely important for the validity of the CBK relation in the V scheme at the $\mathcal{O}(\alpha_s^4)$ level. Totally similarly, we can show that the terms originating from expression (4) for the a_3 coefficient and proportional to the π^6, s_6, $\pi^4\log 2$, $\pi^4\log^2 2$, $\pi^2\alpha_4$, $\pi^2\log 2$, and $\pi^2\zeta_3\log 2$ numbers cancel in the sum $d_{4,V}^{\rm NS} + c_{4,V}^{\rm NS}$. This fact is also a consequence of conformal symmetry.[1]

Thus, we have demonstrated that the factorization of Δ_{csb} in the CBK relation in the V scheme holds at least in the fourth PT order. The natural question arises: is this factorization property also true in other gauge-invariant renormalization schemes other than the V- or MS-like ones? We answer this question in the next section.

[1] Interesting consequences of conformal symmetry violation are briefly discussed in Appendix B.

5. The CBK relation in different gauge-invariant schemes

In this section, we continue developing the ideas proposed in [15]. The requirements for gauge-dependent schemes that ensure the CBK relation were investigated there. It turns out that if the CBK relation in QCD is valid in the $\overline{\text{MS}}$ scheme in all PT orders (the arguments to trust this assumption are adduced in [24]–[26]), then it also holds for a wide class of MOM-like schemes with a linear covariant Landau gauge in all orders. We now extend the ideas in [15] to a class of gauge-invariant renormalization schemes.

We conduct our study in the particular case of the V scheme. Without restricting the generality, by the V scheme, we can understand any other gauge-invariant scheme whose coupling constant is related to a_s in the $\overline{\text{MS}}$ scheme by a relation of type (10) with some coefficients a_k.

Taking the RG-invariance of the functions $D^{\text{NS}}(a_s)$ and $C^{\text{NS}}_{\text{Bjp}}(a_s)$ into account and using the explicit form of the conformal symmetry breaking term in the CBK relation, we obtain the equality

$$\frac{\beta(a_s)}{a_s}K(a_s) = \frac{\beta^V(a_{s,V}(a_s))}{a_{s,V}(a_s)}K^V(a_{s,V}(a_s)), \tag{39}$$

relating the respective polynomials $K(a_s)$ and $K^V(a_{s,V})$ in the $\overline{\text{MS}}$ and V (arbitrary gauge-invariant) schemes. Using the scheme independence of the first two coefficients of the RG β-function (14a), (14b) within a class of gauge-invariant schemes and substituting Eq. (10) in (39), we obtain the following relations between the V and $\overline{\text{MS}}$-scheme coefficients of the polynomial $K(a_s)$:

$$K_1^V = K_1, \tag{40}$$

$$K_2^V = K_2 - 2a_1K_1. \tag{41}$$

The $\overline{\text{MS}}$ coefficients K_1 and K_2 were first calculated in [17]. Using their explicit form, we find that formulas (40) and (41) are in full agreement with the results of direct calculation in Eqs. (33) and (34). Expression (41) is analogous to Eq. (4.3) in [15], which presents the relation between K_2 coefficients in the mMOM and $\overline{\text{MS}}$ scheme. However, as can be seen from Eq. (41) it no longer contains terms with the $1/\beta_0$ factor. This is a consequence of the scheme independence of the two-loop coefficient β_1 in a class of the gauge-invariant renormalization schemes like the V scheme. We recall that in gauge-dependent schemes such as mMOM, the two-loop coefficient of the β-function is already gauge dependent. Despite this fact, it was found in [15] that the CBK relation holds in the mMOM scheme in the $\mathcal{O}(a_s^3)$ approximation for three values of the gauge parameter only: $\xi = 0, -1, -3$. The mentioned $1/\beta_0$ term disappears at these values of ξ.

Carrying out similar steps in the next PT order, we obtain

$$K_3^V = K_3 - 3a_1K_2 + \left(\frac{\beta_2 - \beta_2^V - a_1\beta_1}{\beta_0} + 5a_1^2 - 2a_2\right)K_1. \tag{42}$$

Taking now Eq. (13b) into account, we derive the simplified form of Eq. (42) without the $1/\beta_0$ factor:

$$K_3^V = K_3 - 3a_1K_2 + (6a_1^2 - 3a_2)K_1. \tag{43}$$

The coefficient K_3 is known from the results in [18]. The application of formula (43) reproduces expression (37). We also note that the final structure of Eq. (43) is much simpler than the analogous one in Eq. (4.7) in [15] for the mMOM scheme in an arbitrary gauge. As was shown there, the CBK relation remains valid at the $\mathcal{O}(a_s^4)$ level in the mMOM scheme (and other QCD MOM-like schemes) only in the Landau gauge $\xi = 0$.

Further, proceeding similarly and using Eqs. (13b)–(13d), we straightforwardly obtain

$$K_4^V = K_4 - 4a_1K_3 + (10a_1^2 - 4a_2)K_2 + (20a_1a_2 - 20a_1^3 - 4a_3)K_1, \tag{44}$$

$$K_5^V = K_5 - 5a_1K_4 + (15a_1^2 - 5a_2)K_3 + (30a_1a_2 - 35a_1^3 - 5a_3)K_2 + \\ + (30a_1a_3 + 15a_2^2 - 105a_1^2a_2 + 70a_1^4 - 5a_4)K_1, \tag{45}$$

where K_4 and K_5 are still unknown $\overline{\text{MS}}$-scheme coefficients in Eq. (6) and a_4 is still an entirely unknown four-loop correction to the static potential in QCD, Eq. (1).

It is not difficult to obtain similar representations of the K_n^V terms in any PT order. We thus verify that if the CBK relation in QCD is valid in the $\overline{\text{MS}}$ scheme in all orders of the PT, then it is also true for arbitrary gauge-invariant renormalization scheme with "nonexotic" coefficients a_k included in the ratio of the couplings in this considered scheme and in the $\overline{\text{MS}}$ scheme (see an analogue of Eq. (10)). By the "nonexotic" coefficients, we understand those that are the polynomials in n_f with coefficients that are algebraic or transcendental numbers. For example, in the gauge-invariant 't Hooft scheme [81], [82], the coefficients a_k are not polynomials in n_f: they contain terms proportional to the $1/\beta_0$ factors. The β-function in this scheme contains two nonzero scheme-independent PT coefficients only and the rest are assumed to be zero by finite renormalization of charge. As shown in [83], the transition from the $\overline{\text{MS}}$ scheme to the 't Hooft scheme violates the property of factorization of the conformal symmetry breaking term Δ_{csb} in the CBK relation.

6. The QED case

We now consider the QED case with N charged leptons. The transition to the abelian $U(1)$ gauge group is performed by replacing $C_A = 0$, $C_F = 1$, $T_F = 1$, $d_A^{abcd} = 0$, $d_F^{abcd} = 1$, $d^{abc} = 1$, $N_A = 1$, $d_R = 1$, and $n_f = N$.

Using the analytic expression for the four-loop approximation of the β-function in the V scheme in the case of a generic simple gauge group, Eqs. (14a)–(14d), and taking the transition to $U(1)$ discussed above into account, we can obtain the QED analogue of the β-function in the V scheme

$$\beta_{\text{QED}}^V(a_V) = \frac{1}{3}Na_V^2 + \frac{1}{4}Na_V^3 + \left(-\frac{1}{32}N + \left(\frac{1}{3}\zeta_3 - \frac{23}{72}\right)N^2\right)a_V^4 + \\ + \left(-\frac{23}{128}N + \left(\frac{13}{32} + \frac{2}{3}\zeta_3 - \frac{5}{3}\zeta_5 + \frac{2}{3}\mathcal{C}\right)N^2 + \left(\frac{1}{2} - \frac{1}{3}\zeta_3\right)N^3\right)a_V^5, \tag{46}$$

where $a_V = \alpha_V/\pi$. The introduced constant $\mathcal{C}$ arises naturally in the definition of the static Coulomb potential in terms of the QED invariant charge:

$$V_{\text{QED}}(\vec{q}^{\,2}) = -\frac{4\pi}{\vec{q}^{\,2}} \frac{\alpha(\mu^2)}{1 + \Pi_{\text{QED}}(\vec{q}^{\,2}/\mu^2, \alpha)} \left(1 + N \cdot \mathcal{C}\left(\frac{\alpha}{\pi}\right)^3 + \cdots\right). \tag{47}$$

Here, $\mathcal{C}$ is the correction associated with the appearance of the light-by-light type scattering diagrams to the static potential [13], which do not occur in the photon vacuum polarization function Π_{QED} at this level [11]:

$$\mathcal{C} = \frac{5}{96}\pi^6 - \pi^4\left(\frac{23}{24} - \frac{\log 2}{6} + \frac{\log^2 2}{2}\right) + \pi^2\left(\frac{79}{36} - \frac{61}{12}\zeta_3 + \log 2 + \frac{21}{2}\zeta_3 \log 2\right) \approx -0.888062. \tag{48}$$

The numerical effect of the term $2\mathcal{C}N^2a_V^5/3$ in (46) is not negligible compared to the contribution of the remaining part, proportional to $N^2a_V^5$; on the contrary, it dominates it.

Expression (46) is to be compared with the QED result for the β-function in the MOM scheme identical to the Gell-Mann–Low Ψ-function [84]

$$\begin{aligned}\beta^{\rm MOM}_{\rm QED}(a_{\rm MOM}) = \Psi(a_{\rm MOM}) = \frac{1}{3}Na^2_{\rm MOM} + \frac{1}{4}Na^3_{\rm MOM} + \\ + \left(-\frac{1}{32}N + \left(\frac{1}{3}\zeta_3 - \frac{23}{72}\right)N^2\right)a^4_{\rm MOM} + \\ + \left(-\frac{23}{128}N + \left(\frac{13}{32} + \frac{2}{3}\zeta_3 - \frac{5}{3}\zeta_5\right)N^2 + \left(\frac{1}{2} - \frac{1}{3}\zeta_3\right)N^3\right)a^5_{\rm MOM},\end{aligned} \tag{49}$$

where $a_{\rm MOM} = \alpha_{\rm MOM}/\pi$ coincides with the QED invariant charge (36). We note that expression (49) can be also obtained, e.g., as a result of the transition to the $U(1)$ group for the β-function computed initially in the mMOM scheme with the generic simple gauge group [30], [32]. This fact directly follows from formulas (35) and (36).

We can see that at the three-loop level, $\beta^V_{\rm QED}$ in (46) completely coincides in form with the Gell-Mann–Low Ψ-function in (49). The difference between them starts to manifest itself only at the four-loop level due to the additional term $2\mathcal{C}N^2a^5_V/3$, related to the light-by-light type scattering effect in the perturbative expression for the static Coulomb potential (47):

$$\beta^V_{3,\rm QED} = \Psi_3 + \frac{2}{3}\mathcal{C}N^2. \tag{50}$$

The obtained result can be presented in the compact form

$$a_V = a_{\rm MOM} + \mathcal{C}Na^4_{\rm MOM} + \mathcal{O}(a^5_{\rm MOM}). \tag{51}$$

The arguments given in this section and in Sec. 4, allows us to conclude that in QCD, the V scheme has many properties similar to those of MOM-like schemes in the Landau gauge. Because the difference between the a_V and $a_{\rm MOM}$ couplings in QED (51) starts manifesting itself from the α^4-term only, the V scheme in QCD can arguably be interpreted as a gauge-independent scheme in which one can construct an analog of the gauge-invariant charge, namely, a gauge-invariant combination of the associated Green's functions. We recall that it is impossible to introduce this concept within the gauge-dependent MOM-like schemes in QCD (see, e.g., [14] for details).

Setting $N = 1$ in Eqs. (46) and (49), we arrive at their numerical form:

$$\beta^V_{\rm QED}(a_V) = 0.3333a^2_V + 0.25a^3_V + 0.0499a^4_V - 1.19301a^5_V, \tag{52}$$

$$\Psi(a_{\rm MOM}) = 0.3333a^2_{\rm MOM} + 0.25a^3_{\rm MOM} + 0.0499a^4_{\rm MOM} - 0.60096a^5_{\rm MOM}. \tag{53}$$

We observe that even at $N = 1$, the numerical effect of the light-by-light scattering contribution, which is typical for the V scheme, is rather sizable and is almost equal to twice four-loop correction to the Ψ-function.

We turn to the consideration of relations between higher-order corrections to the $\beta^V_{\rm QED}$ and Ψ-function. The dependence of these RG functions on the number N of the charged leptons is described by the following decompositions:

$$\beta^V_{\rm QED}(a_V) = \beta^{V(1)}_{{\rm QED},0}Na^2_V + \sum_{i\geq 1}\sum_{k=1}^{i}\beta^{V(k)}_{{\rm QED},i}N^k a^{i+2}_V, \tag{54}$$

$$\Psi(a_{\rm MOM}) = \Psi^{(1)}_0 Na^2_{\rm MOM} + \sum_{i\geq 1}\sum_{k=1}^{i}\Psi^{(k)}_i N^k a^{i+2}_{\rm MOM}. \tag{55}$$

As we have already seen, the coefficients $\beta^{V(k)}_{\mathrm{QED},i}$ and $\Psi^{(k)}_i$ differ only by corrections $\Delta\beta^{V(k)}_{\mathrm{QED},i}$ associated with the light-by-light scattering-type effects in the static potential:

$$\beta^{V(k)}_{\mathrm{QED},i} = \Psi^{(k)}_i + \Delta\beta^{V(k)}_{\mathrm{QED},i}. \tag{56}$$

This fact directly follows from the definition of the coupling constant a^V in the V scheme (a QED analog of Eq. (8) and (47)) and from relation (36). It is clear that the extra term $\Delta\beta^{V(k)}_{\mathrm{QED},i}$ appears only for the indices $\{i,k\} = \{i \geq 3,\, 2 \leq k \leq i-1\}$. In the cases where $\{i,k\} = \{i \geq 3,\, k = 1 \text{ or } k = i\}$, the coefficients of the β^V and Ψ-functions coincide. Indeed, we have already observed that at the four-loop level,

$$\beta^{V(1)}_{\mathrm{QED},3} = \Psi^{(1)}_3 = -\frac{23}{128}, \qquad \beta^{V(3)}_{\mathrm{QED},3} = \Psi^{(3)}_3 = \frac{1}{2} - \frac{1}{3}\zeta_3. \tag{57}$$

The RG β-function in the MOM scheme (the Gell-Mann–Low Ψ-function) was calculated in the fifth-loop approximation in QED in [85] for an arbitrary N (and in the $\overline{\mathrm{MS}}$ scheme as well):

$$\begin{aligned}\Psi_4 = {} & \left(\frac{4157}{6144} + \frac{1}{8}\zeta_3\right)N + \left(-\frac{251}{256} - \frac{23}{24}\zeta_3 - \frac{45}{8}\zeta_5 + \frac{35}{4}\zeta_7\right)N^2 + \\ & + \left(-\frac{3383}{3456} - \frac{205}{72}\zeta_3 + \frac{5}{2}\zeta_5 + \zeta_3^2\right)N^3 + \left(-\frac{67}{72} + \frac{7}{18}\zeta_3 + \frac{5}{9}\zeta_5\right)N^4.\end{aligned} \tag{58}$$

For instance, this result can be obtained as the $U(1)$ limit of the β-function computed at the five-loop level in the mMOM scheme with an arbitrary gauge parameter for the generic simple gauge group in [32].

Using formula (13d) and the expression for β_4 in the $\overline{\mathrm{MS}}$ scheme [85], we now find the five-loop coefficient of the β^V_{QED}-function,

$$\begin{aligned}\beta^V_{\mathrm{QED},4} = {} & \left(\frac{4157}{6144} + \frac{1}{8}\zeta_3\right)N + \left(-\frac{49}{48} - \frac{53}{96}\zeta_3 + \frac{65}{32}\zeta_5 + \frac{1}{4}\mathcal{C} + a_4^{(1)}\right)N^2 + \\ & + \left(-\frac{4255}{4608} + \frac{1013}{144}\zeta_3 - \frac{13}{96}\zeta_4 - \frac{215}{36}\zeta_5 - \frac{5}{3}\zeta_3^2 + \frac{20}{9}\mathcal{C} + a_4^{(2)}\right)N^3 + \\ & + \left(\frac{118907}{31104} - \frac{71}{24}\zeta_3 + a_4^{(3)}\right)N^4,\end{aligned} \tag{59}$$

where the constant $\mathcal{C}$ has been defined above and, similarly to with Eq. (4), we use the decomposition of the four-loop correction a_4 to the static Coulomb potential in QED in powers of N:

$$a_4 = \left(\frac{5}{9}\right)^4 N^3 + a_4^{(3)}N^3 + a_4^{(2)}N^2 + a_4^{(1)}N. \tag{60}$$

We note that the term $-13\zeta_4/96$ in the N^3-coefficient of $\beta^V_{\mathrm{QED},4}$ in (59) is not related to the light-by-light scattering effects but arises from the calculation of β_4 in the $\overline{\mathrm{MS}}$ scheme (see [85]–[87]).

As we have expected, the terms linear in N are the same in Eqs. (58) and (59):

$$\beta^{V(1)}_{\mathrm{QED},4} = \Psi^{(1)}_4 = \frac{4157}{6144} + \frac{1}{8}\zeta_3. \tag{61}$$

It was explained in [88] that the scheme independence of these linear terms in massless QED is a consequence of conformal symmetry.

Because the coefficients $\beta^{V(4)}_{\mathrm{QED},4}$ and $\Psi^{(4)}_4$ must also be the same, we can fix the contribution $a^{(3)}_4$ from matching Eqs. (58) and (59):

$$a_4^{(3)} = -\frac{147851}{31104} + \frac{241}{72}\zeta_3 + \frac{5}{9}\zeta_5, \tag{62}$$

$$\beta^{V(4)}_{\mathrm{QED},4} = \Psi^{(4)}_4 = -\frac{67}{72} + \frac{7}{18}\zeta_3 + \frac{5}{9}\zeta_5. \tag{63}$$

The four-loop expressions for $\beta^{V(2)}_{\mathrm{QED},4}$ and $\beta^{V(3)}_{\mathrm{QED},4}$ contain the contributions related to the light-by-light scattering-type effects in the static potential. They originate from the constant $\mathcal{C}$, occurring at the three-loop level, and the fourth-order corrections $a^{(1)}_4$ and $a^{(2)}_4$ (59). Based on the results in [13], we can conclude that the contributions of these effects are separated from the other ones by transcendent constants proportional to even powers of the π-number (see Eq. (48)). Without these still unknown terms, the corrections $a^{(1)}_4$ and $a^{(2)}_4$ are given by

$$a_4^{(1)}|_{\mathrm{no\,l-b-l}} = \frac{31}{768} - \frac{13}{32}\zeta_3 - \frac{245}{32}\zeta_5 + \frac{35}{4}\zeta_7, \tag{64}$$

$$a_4^{(2)}|_{\mathrm{no\,l-b-l}} = -\frac{767}{13824} - \frac{1423}{144}\zeta_3 + \frac{13}{96}\zeta_4 + \frac{305}{36}\zeta_5 + \frac{8}{3}\zeta_3^2. \tag{65}$$

These expressions directly follow from equating $\Psi^{(2)}_4$ to $\beta^{V(2)}_{\mathrm{QED},4}$ and $\Psi^{(3)}_4$ to $\beta^{V(3)}_{\mathrm{QED},4}$ in the approximation where the light-by-light scattering effects in the static potential are discarded. The following relations then hold:

$$\beta^{V(2)}_{\mathrm{QED},4} = \Psi^{(2)}_4 + \Delta\beta^{V(2)}_{\mathrm{QED},4}, \qquad \Delta\beta^{V(2)}_{\mathrm{QED},4} = a_4^{(1)}|_{\mathrm{l-b-l}} + \frac{1}{4}\mathcal{C}, \tag{66}$$

$$\beta^{V(3)}_{\mathrm{QED},4} = \Psi^{(3)}_4 + \Delta\beta^{V(3)}_{\mathrm{QED},4}, \qquad \Delta\beta^{V(3)}_{\mathrm{QED},4} = a_4^{(2)}|_{\mathrm{l-b-l}} + \frac{20}{9}\mathcal{C}. \tag{67}$$

Formulas (62), (64), and (65) can generalized without significant obstructions to the case of the generic simple gauge group and then become more transparent:

$$a_4^{(3)}|_{\mathrm{abelian}} = \left(-\frac{147851}{31104} + \frac{241}{72}\zeta_3 + \frac{5}{9}\zeta_5\right) C_F T_F^3, \tag{68}$$

$$\begin{aligned} a_4^{(2)}|_{\mathrm{abelian,\,no\,l-b-l}} = &\left(\frac{13025}{13824} - \frac{403}{36}\zeta_3 - \frac{11}{96}\zeta_4 + \frac{175}{18}\zeta_5 + 2\zeta_3^2\right) C_F^2 T_F^2 + \\ &+ \left(-\frac{431}{432} + \frac{21}{16}\zeta_3 + \frac{1}{4}\zeta_4 - \frac{5}{4}\zeta_5 + \frac{2}{3}\zeta_3^2\right)\frac{d_F^{abcd}d_F^{abcd}}{N_A}, \end{aligned} \tag{69}$$

$$a_4^{(1)}|_{\mathrm{abelian,\,no\,l-b-l}} = \left(\frac{31}{768} - \frac{13}{32}\zeta_3 - \frac{245}{32}\zeta_5 + \frac{35}{4}\zeta_7\right) C_F^3 T_F. \tag{70}$$

The $d_F^{abcd}d_F^{abcd}$ contribution to $a^{(2)}_4$ in (69) originates from the $d_F^{abcd}d_F^{abcd}$ contributions to the coefficients β_3 and β_4. This fact can be directly observed from (13d), where the abelian terms proportional to C_F and $d_F^{abcd}d_F^{abcd}$, can be fixed from considering β_4^{mMOM} [32] (whose the abelian contributions in the Landau gauge are equal to those in β_4^V without taking the light-by-light scattering-type corrections to the static potential into account) and from analytic results for β_3 [54], [55] and β_4 [86], [87]. We emphasize that expressions (68)–(70) are in full agreement with the analogous results in Eq. (14.4) in [35].

7. Conclusions

In this work, we obtain an explicit analytic form of the RG β-function in the gauge-invariant V scheme at the four-loop level in the case of a generic simple gauge group. Using the renormalization invariance of the Adler function for the of $e^+e^- \to \gamma^* \to$ *hadrons*, the $R_{e^+e^-}(s)$-ratio and the coefficient function of the Bjorken polarized sum rule of DIS of polarized charged leptons on nucleons, we also obtain their PT expressions in the V scheme up to α_s^4-corrections. We compare the derived V-scheme results with the $\overline{\text{MS}}$ and mMOM counterparts in the Landau gauge. In the cases of the Adler function and $R_{e^+e^-}(s)$-ratio in the V and mMOM schemes, the nonregular behavior of the perturbative corrections in their decomposition in powers of n_f is observed in higher orders. Taking the obtained V-scheme results into account, we demonstrate explicitly that the CBK relation remains valid in this effective scheme at the $\mathcal{O}(\alpha_s^4)$ level. Furthermore, we prove our hypothesis that the factorization of the conformal symmetry breaking term $\Delta_{csb}(a_s)$ of the CBK relation into $\beta(a_s)/a_s$ and a polynomial $K(a_s)$ holds in any gauge-invariant scheme at least in the fourth PT order. The chosen gauge-invariant scheme should only lead to the "nonexotic" coefficients in the relation between couplings defined in the $\overline{\text{MS}}$ scheme and in the considered one, i.e., these coefficients should be polynomials in n_f. Moreover, it turns out that if the CBK relation in QCD is valid in the $\overline{\text{MS}}$ scheme in all PT orders, then it also holds for the discussed gauge-invariant class of the renormalization schemes in all orders. We show that in QED, the coefficients of the β-function in the V scheme coincide with the analogous ones in the MOM scheme at the three-loop level. Starting from the fourth PT order, their N^2-coefficients begin to differ by a correction associated with the manifestation of the effects of the light-by-light scattering in the static potential. The other terms proportional to N and N^3 remain the same. In even higher orders, this tendency persists, i.e., two N-dependent terms in the coefficients of the perturbative expansions of the β^V_{QED} and Ψ-functions always coincide, and the remaining ones differ by a correction related to the light-by-light scattering in the static potential. Based on these findings, we predict several contributions to the four-loop correction to the static potential in the case of the generic simple gauge group, corroborating recent independent results in [35].

Appendix A

We consider the question related to the integral representation of multiple zeta values. In general, these functions are defined as

$$\zeta_{m_1,\ldots,m_k} = \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1-1}\cdots\sum_{i_k=1}^{i_{k-1}-1}\prod_{j=1}^{k}\frac{\text{sign}(m_j)^{i_j}}{i_j^{|m_j|}}. \tag{71}$$

They were studied in detail in a number of works on the subject (see, e.g., [89]–[91], [61]). We use the Hurwitz–Lerch zeta function $\Phi(z, s, q)$

$$\Phi(z, s, q) = \sum_{k=0}^{\infty}\frac{z^k}{(k+q)^s} \tag{72}$$

and its integral representation

$$\Phi(z, s, q) = \frac{1}{\Gamma(s)}\int_0^1 \frac{x^{q-1}(-\log x)^{s-1}}{1-zx}\,dx, \tag{73}$$

which is valid for $\text{Re}(q) > 0$, $\text{Re}(s) > 0$ and $z \in [-1; 1)$ or $\text{Re}(s) > 1$ and $z = 1$.

Then, for the constant $\zeta_{-5,-1}$ with transcendence of weight 6, appearing in calculating the three-loop correction to the static potential [13], we can write [28]

$$\zeta_{-5,-1} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^5} \sum_{i=1}^{k-1} \frac{(-1)^i}{i} = \frac{15}{16}\zeta_5 \log 2 - \sum_{k=1}^{\infty} \frac{\Phi(-1,1,k)}{k^5} =$$
$$= \frac{15}{16}\zeta_5 \log 2 - \int_0^1 \frac{dx}{x(x+1)} \sum_{k=1}^{\infty} \frac{x^k}{k^5} = \frac{15}{16}\zeta_5 \log 2 - \zeta_6 + \int_0^1 dx \frac{\mathrm{Li}_5(x)}{x+1}. \tag{74}$$

Therefore, the constant $s_6 = \zeta_6 + \zeta_{-5,-1}$ can be represented in the form [90]

$$s_6 = \frac{15}{16}\zeta_5 \log 2 + \int_0^1 dx \frac{\mathrm{Li}_5(x)}{x+1} \approx 0.9874414. \tag{75}$$

Similarly, we can obtain an integral representations for multiple zeta values with specific arguments arising in the intermediate calculations in [13]:

$$\zeta_{5,2} = \zeta_5\zeta_2 - \zeta_7 + \int_0^1 dx \frac{\mathrm{Li}_5(x)\log x}{1-x} \approx 0.0385751, \tag{76}$$
$$\zeta_{-5,2} = -\frac{15}{16}\zeta_5\zeta_2 + \frac{63}{64}\zeta_7 + \int_0^1 dx \frac{\mathrm{Li}_5(-x)\log x}{1-x} \approx 0.0271089. \tag{77}$$

For instance, the function $\zeta_{5,3}$ occurs in the computation of the $\overline{\mathrm{MS}}$-scheme β-function of the $O(N)$-symmetric ϕ^4 theory in the six-loop approximation [92] (in the notations in that paper, $\zeta_{3,5}$):

$$\zeta_{5,3} = \zeta_3\zeta_5 - \zeta_8 - \frac{1}{2}\int_0^1 dx \frac{\mathrm{Li}_5(x)\log^2(x)}{1-x} \approx 0.0377077. \tag{78}$$

Appendix B

It is interesting to note some common features of the CBK relation and the action sum rule [93]–[97] (in lattice QCD, it is also known as the Michael sum rule). Indeed, both of them contain a conformal anomaly term, reflecting the effect of conformal symmetry violation. However, the second relation can be directly used in the nonperturbative region as well.

We recall that the conformal anomaly in the trace of the energy–momentum tensor of a massless $SU(N_c)$ gauge theory in the Euclidean domain has the form [98]–[100]

$$T_{\mu\mu}(x) = \frac{\beta(a_s)}{2a_s} F^a_{\mu\nu}(x)F^a_{\mu\nu}(x) = 2\frac{\beta(a_s)}{a_s}\mathcal{L}(x), \tag{79}$$

where $\mathcal{L}(x)$ is the gluon gauge part of the *Euclidean* Lagrangian density of the $SU(N_c)$ theory, expressed trough the Euclidean chromoelectric and chromomagnetic fields

$$\mathcal{L}(x) = \frac{1}{4}F^a_{\mu\nu}(x)F^a_{\mu\nu}(x) = \frac{1}{2}(\vec{E}(x)^2 + \vec{B}(x)^2). \tag{80}$$

We note that owing to a change in the metric signature, the square of the Euclidean electric field has an opposite sing to its Minkowskian counterpart, while signs of the squares of the Euclidean and Minkowskian magnetic fields coincide. The action sum rule relates a certain combination of the static potential to the Euclidean chromoelectric and chromomagnetic condensates and the β-function [93]–[97],

$$\widetilde{V}(r) + r\,\frac{\partial \widetilde{V}(r)}{\partial r} = \frac{\beta(a_s)}{a_s}\left\langle \int d^3x\,(\vec{E}(x)^2 + \vec{B}(x)^2)\right\rangle_r, \tag{81}$$

where $\widetilde{V}(r)$ is the static potential in coordinate space including the confining and nonconfining components and $\langle\cdot\rangle_r$ is the vacuum expectation value in the presence of a static quark–antiquark pair spaced apart from each other at a distance r excluding the analogous contribution without these field sources.

It would be interesting to study the possible relation of the action sum rule and the CBK relation based on the first principles of quantum field theory.

Conflicts of interest. The authors declare no conflicts of interest.

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