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ON A CLASS OF QUADRATIC CONSERVATION LAWS FOR NEWTON EQUATIONS IN EUCLIDEAN SPACE

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We discuss quadratic conservation laws for the Newton equations and the corresponding second-order Killing tensors in Euclidean space. In this case, the complete set of integrals of motion consists of polynomials of the second, fourth, sixth, and so on degrees in momenta, which can be constructed using the Lax matrix related to the hierarchy of the multicomponent nonlinear Schrödinger equation.

Keywords: Killing tensors, integrable systems, symmetric spaces

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1. Introduction

Studying the problem of the existence of integrals of motion is closely related to the choice of the functional class where the first integrals are to be found. Polynomial, rational, algebraic, and nonalgebraic integrals of motion are typically selected [1]–[10].

In this paper, we discuss the known integrable systems that describe the motion in Euclidean space and have at least two conservation laws quadratic in velocities, with the other integrals of motion being higher-degree polynomials in velocities (momenta).

Most often, quadratic integrals of motion appear in studying Hamiltonian systems that are integrable by separation of variables. Let A and B be a pair of nondegenerate symmetric second-order tensors in Euclidean space $\mathbb{R}^n$ that generate the pair of quadratic polynomials in momenta,

$$T_A = \sum_{i,j} A^{ij} p_i p_j, \qquad T_B = \sum_{i,j} B^{ij} p_i p_j.$$

These polynomials are in involution, $\{T_A, T_B\} = 0$, with respect to the standard Poisson bracket on the cotangent bundle $T^*\mathbb{R}^n$ if the Schouten bracket of the tensors A and B vanishes,

$$[\![A, B]\!] = 0.$$

The Schouten bracket allows passing to the geometric description of the dynamical system, i.e, replacing the equation $\{T_A, T_B\} = 0$ in phase space with the equation $[\![A, B]\!] = 0$ in configuration space and then using the geometric properties of the configuration space to study the properties of the dynamical system.

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For instance, if the spectral problem

$$(A - \lambda B)\psi = 0 \tag{1.1}$$

has n different real eigenvalues and the corresponding eigenvectors are normal, i.e., the orthogonal complements to any eigenvector form an integrable distribution [11], [12], then the tensors A and B generate an n-dimensional linear space of the second-order tensor fields that are in involution and have common eigenvectors [13]–[15].

This allows constructing n independent second-degree polynomials in momenta in the cotangent bundle $T^*\mathbb{R}^n$,

$$T_1 = \sum_{i,j} A^{ij} p_i p_j, \quad T_2 = \sum_{i,j} B^{ij} p_i p_j, \quad T_3 = \sum_{i,j} K_3^{ij} p_i p_j, \quad \ldots, \quad T_n = \sum_{i,j} K_n^{ij} p_i p_j,$$

which are in involution

$$\{T_i, T_j\} = 0$$

with respect to the canonical Poisson brackets

$$\{q_i, q_j\} = 0, \qquad \{p_i, p_j\} = 0, \qquad \{q_i, p_j\} = \delta_{ij}, \qquad i, j = 1, \ldots, n. \tag{1.2}$$

Adding the appropriate potentials

$$H_1 = T_1 + V_1(q_1, \ldots, q_n), \quad H_2 = T_2 + V_2(q_1, \ldots, q_n), \quad \ldots, \quad H_n = T_n + V_n(q_1, \ldots, q_n),$$

we obtain an n-dimensional space of first integrals [13] (also see [16], [17]). Historical details and a more complete list of appropriate references can be found in [18], [19].

In other words, two tensors A and B define equivalent metrics with common geodesics if and only if they can be reduced to some special normal forms [13], [15], [20]. It turns out that the corresponding geodesic flow is integrable and has a complete set of the integrals of motion that are quadratic in momenta. We then consider the problem of adding nontrivial potentials to the Hamiltonian and preserving the integrability properties in a given functional class of integrals of motion quadratic in momenta [6], [7], [16], [17], [21].

To obtain something new, we propose to abandon the common scheme of first considering the integrable geodesic flows and consequently adding appropriate potentials to the obtained integrals of motion. In fact, if we begin with motion in the potential field, we can find a number of new examples of second-order tensors A and B that also define integrable and superintegrable systems in Euclidean space [22]–[25]. The corresponding spectral problem (1.1) does not necessarily have a set of different real eigenvalues and normal eigenvectors, i.e., the Hamilton–Jacobi equation does not admit the separation of variables in any of the known curvilinear orthogonal coordinate systems in Euclidean space.

We study the properties of the second-order tensors A and B in Euclidean space $\mathbb{R}^n$, which correspond to the quadratic conservation laws appearing in the study of integrable Hamiltonian systems associated with the hierarchy of the multicomponent nonlinear Schrödinger equations [26]–[29]. The appropriate spectral problem (1.1) does not then have the required set of different real eigenvalues and normal eigenvectors, which does not prevent the Liouville integrability, however.

Although a number of explicit expressions for the Hamiltonians

$$H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + V(q_1, \ldots, q_n),$$

corresponding to Hermitian symmetric spaces of types A.III, BD.I, C.I, and D.III in Cartan's classification are reproduced in various textbooks (see, e.g., [30]–[33]), the corresponding integrals of motion that are polynomial in momenta have not been studied. In this paper, we partially correct this defect.

1.1. Integrable systems and symmetric spaces. A simply connected symmetric space is a homogeneous space G/K, where G is a Lie group and K is its subgroup, which is the isotropy group of the symmetric space.

For any Hermitian symmetric space G/K, there is an element in the Cartan subalgebra $\mathcal{A}$ of the Lie algebra $\mathfrak{g}$ of the group G such that

- the Cartan automorphism $\sigma = \operatorname{ad}\mathcal{A}$ defines the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \quad \text{where} \quad \mathfrak{k} = \{X \in \mathfrak{g},\, \sigma(X) = X\}, \quad \mathfrak{m} = \{X \in \mathfrak{g},\, \sigma(X) = -X\},$$

  i.e.,

$$\mathfrak{k} = \{X \in \mathfrak{g},\, [X, \mathcal{A}] = 0\};$$

- the root system of $\mathfrak{g}$ decomposes into the subsets

$$\Delta = \Delta_0 \cup \Delta_+ \cup \Delta_-,$$

  where

$$\Delta_0 = \{\alpha \in \Delta,\, \alpha(\mathcal{A}) = 0\}, \qquad \Delta_\pm = \{\alpha \in \Delta,\, \alpha(\mathcal{A}) = \pm a\},$$

  $a > 0$ is a constant whose value is determined by the type of the chosen Hermitian symmetric space;

- consequently,

$$[\mathcal{A}, e_\alpha] = \pm a e_\alpha, \qquad \alpha \in \Delta_\pm, \qquad \mathfrak{m} = \operatorname{span}\{e_{\pm\alpha},\, \alpha \in \Delta_+\};$$

- consequently,

$$[e_\alpha, e_\beta] = 0, \qquad \alpha, \beta \in \Delta_+, \qquad \alpha, \beta \in \Delta_-.$$

Using the Killing formula, which in our case has the standard form

$$\langle X, Y \rangle = b\ \operatorname{tr}(X \cdot Y), \qquad b \in \mathbb{R},$$

we define the metric

$$\mathrm{g}^{\alpha,\beta} = \langle e_\alpha, e_\beta \rangle,$$

for which b plays the role of a constant Gaussian curvature, and the Riemann curvature tensor with the components

$$\mathcal{R}_{\alpha,\beta,\gamma,\delta} = \langle [e_\alpha, e_\beta], [e_\gamma, e_\delta] \rangle.$$

The exact definitions and all necessary details and references can be found in textbook [34]. More precisely, the tensors g and $\mathcal{R}$ have the properties of the Riemann metric and the Riemann tensor [26], which allows identifying g with the metric in Euclidean space and using $\mathcal{R}$ for constructing the potential in Euclidean space.

Using the Cartan involution σ, we can construct the decomposition of the twisted affine algebra $\mathcal{L}(\mathfrak{g}, \sigma) = \mathcal{L}_+ + \mathcal{L}_-$ into two subalgebras, construct the classical r-matrix corresponding to this decomposition, and describe the orbits of $\mathcal{L}_-$ passing through the point $\mathcal{A}\lambda^2$ in the dual space $\mathcal{L}_-^*$ [26], [28], [29].

The shift of the orbit by an arbitrary element of the Cartan subalgebra Λ, which plays a key role in our calculations, generates the Lax matrix

$$L(\lambda) = \lambda^2 \mathcal{A} + \lambda \sum_\alpha q^\alpha (e_\alpha - e_{-\alpha}) - \frac{1}{a} \sum_\alpha \mathrm{g}^{\alpha,-\alpha} p_\alpha (e_\alpha + e_{-\alpha}) + \frac{1}{a} \sum_{\alpha,\beta} q_\alpha q_\beta [e_\alpha, e_{-\beta}] + \Lambda. \tag{1.3}$$

We use the notation from [28], with q_α being Cartesian coordinates in Euclidean space and p_α the corresponding momenta denoted by q_i and p_i hereafter, for which the Poisson bracket has form (1.2). The sum in (1.3) ranges all coordinates α and β from the subset Δ_+.

The corresponding Hamiltonian

$$H = \frac{1}{4}\operatorname{tr} L^2(\lambda)\Big|_{\lambda=0} = \frac{1}{2}\sum_{\alpha} \mathrm{g}^{\alpha,-\alpha} p_\alpha^2 - \frac{1}{4}\sum_{\alpha,\beta,\gamma,\delta} \mathcal{R}_{-\alpha,\beta,\gamma,-\delta} q^\alpha q^\beta q^\gamma q^\delta + \frac{1}{2}\sum_\alpha \omega_\alpha (q^\alpha)^2 \tag{1.4}$$

depends on the arbitrary "frequencies" ω_α, which are functions of the matrix elements of Λ. The Newton equations of motion in Euclidean space corresponding to this Hamiltonian are

$$\ddot{q}^\alpha = \sum_{\beta,\gamma,\delta} \mathcal{R}^\alpha_{\beta,\gamma,-\delta} q^\beta q^\gamma q_\delta - \omega_\alpha q_\alpha, \qquad \alpha,\beta,\gamma,\delta = 1,\dots,N. \tag{1.5}$$

According to [28], Lax matrix (1.3) also generates several quadratic integrals of motion commuting with the Hamiltonian H in (1.4),

$$H_i^{(2)} = \sum_{j,k} K_i^{ik} p_j p_k + U_i.$$

In this case, $A = \mathrm{g}$ is the standard metric in Euclidean space, and $B = K$ is the Killing tensor satisfying the Killing equation

$$\nabla_i K^{jk} + \nabla_j K^{ki} + \nabla_k K^{ij} = 0, \tag{1.6}$$

where ∇ is the Levi-Civita connection for the metric g.

Proposition 1. *In the general case, there is a fourth-order integral of motion in momenta independent of the quadratic integrals of motion* $H_i^{(2)}$,

$$G = \operatorname{tr} L^4(\lambda)\Big|_{\lambda=0} = \sum_{\alpha,\beta,\gamma,\delta} \mathcal{R}_{-\alpha,\beta,\gamma,-\delta} p^\alpha p^\beta p^\gamma p^\delta + \sum_{\alpha,\beta} S^{\alpha,\beta}(q) p_\alpha p_\beta + W(q), \tag{1.7}$$

whose principal part is determined by the tensor $\mathcal{R}$.

In the particular case of an anharmonic oscillator or the Garnier system, this fourth-degree polynomial can be expressed in terms of quadratic integrals of motion [28]. We do not write the corresponding second-order tensor $S^{\alpha,\beta}$ and potential W explicitly because they are not related to the main goal of this paper.

If all the parameters $\omega_\alpha = 0$, i.e., $\Lambda = 0$, the Hamiltonian H in (1.4) commutes with the family of noncommutative linear integrals of motion associated with various combinations of rotations. In this case, spectral invariants of the Lax matrix generate a family of commuting integrals of motion, whose number is insufficient for the Liouville integrability in the general case, as well as in the case of the complete Toda chain [35]. This means that other tensor invariants associated with the Lax matrix must also be used to prove the integrability.

If the parameters $\omega_\alpha \neq 0$, the Lax matrix $L(\lambda)$ in (1.3) generates the required number of integrals of motion, which are polynomials of the second, fourth, sixth, and so on degrees in momenta. The principal parts of these polynomials

$$H_i^{(2\ell)} = \sum_{j,k,\dots,m}^{2\ell} K_i^{j,k,\dots,m} p_j p_k \dots p_m + \cdots, \qquad \ell = 1,2,\dots, \tag{1.8}$$

define the Killing tensors of valence 2ℓ in the Euclidean space $\mathbb{R}^n$,

$$[\![g, K_i]\!] = 0.$$

In this paper, using the Haantjes torsion [11], we consider several second-order Killing tensors and prove that spectral problem (1.1) does not have the required number of real simple eigenvalues and normal eigenvectors.

1.2. Second-order Killing tensors in Euclidean space. Killing vector fields are generators of local symmetries of the metric in configuration space. For instance, the standard basis of shifts and rotations in Euclidean space

$$X_i = \partial_i, \qquad X_{i,j} = q_i X_j - q_j X_i, \tag{1.9}$$

where $\partial_i = \frac{\partial}{\partial q_i}$, allows describing various symmetries of a physical system. By the Noether theorem, these symmetries correspond to conservation laws that are linear in velocities and associated with space–time coordinate transformations. For instance, the integrals of motion for the physical systems invariant under rotations are linear combinations of the components of the angular momentum tensor J,

$$X_{ij} \mapsto J_{ij} = q_i p_j - q_j p_i.$$

If the integral of motion is the square of either the angular momentum or spin, then Killing vectors do not suffice for the description, and we have to use Killing tensors.

Killing tensors of rank m are associated with the existence of polynomial integrals of motion that have the mth degree in velocities. Because the coordinate transformation in space–time is not associated with Killing tensors of rank $m > 1$, they are usually identified with the so-called hidden symmetries [21]. To construct higher-order tensors, the Weyl theory of tensor products is typically used. For instance, a second-order Killing tensor in Euclidean space has the general form

$$K = \sum_{i,j} a_{ij} X_i \circ X_j + \sum_{i,j,k} b_{ijk} X_i \circ X_{j,k} + \sum_{i,j,k,m} c_{ijkm} X_{i,j} \circ X_{k,m}, \tag{1.10}$$

where a_{ij}, b_{ijk}, and c_{ijm} are arbitrary parameters, and $\circ$ denotes the symmetric product of Killing vector fields.

The dimension of the vector space of the m-valence Killing tensors in the n-dimensional Euclidean space is given by the Delong–Takeuchi–Thompson formula

$$d = \frac{1}{n}\binom{n+m}{m+1}\binom{n+m-1}{m} = \frac{1}{n}\binom{n+2}{3}\binom{n+1}{2} = \frac{n(n+2)(n+1)^2}{12}.$$

In our case of second-order tensors, to find the total number of independent parameters a_{ij}, b_{ijk}, and c_{ijm} involved in definition (1.10) of the general solution of Killing equation (1.6), we set $m = 2$.

Because we do not consider the geodesic flows and pass to the motion in a potential field, all second-order Killing tensors associated with the Hamiltonian $H = T + V$ in (1.4) can be found by directly solving the equation

$$d(KdV) = 0, \tag{1.11}$$

which states that the 1-form KdV is exact. We recall that Eq. (1.11) on configuration space can be obtained from the equation

$$\left\{ \sum_{i,j}^{n} \mathrm{g}^{ij} p_i p_j + V(q), \sum_{i,j}^{n} K^{ij} p_i p_j + U(q) \right\} = 0$$

on phase space, which is responsible for the involution of integrals of motion with respect to Poisson bracket (1.2). Here, V is a function on $\mathbb{R}^n$, and KdV denotes the Killing 1-form with the components $\mathrm{g}_{ij} K^{jk} \partial_k V$, with g_{ij} being the inverse tensor to g^{ij}.

Substituting the general solution of the Killing equation, K in (1.10), and the potential

$$V = \frac{1}{4} \sum_{\alpha,\beta,\gamma,\delta} \mathcal{R}_{-\alpha,\beta,\gamma,-\delta} q^\alpha q^\beta q^\gamma q^\delta - \frac{1}{2} \sum_{\alpha} \omega_\alpha (q^\alpha)^2 \tag{1.12}$$

in Eq. (1.11), we obtain a linear system of equations for the coefficients a_{ij}, b_{ijk}, and c_{ijkm}. Solving this system of equations with modern computer software, we obtain the sought Killing tensors [36]. For brevity in what follows, by a Killing tensor, we mean tensor fields of types (2,0), (1,1), and (0,2), because the metric tensor g can be used to change the tensor field type.

To study the properties of the obtained solutions of Eq. (1.11), we can use the following criteria. A tensor field K of type (1,1) has simple eigenvalues if

$$D = \det \begin{pmatrix} S_0 & S_1 & \cdots & S_{n-1} \\ S_1 & S_2 & \cdots & S_n \\ \vdots & \vdots & \ddots & \vdots \\ S_{n-1} & S_n & \cdots & S_{2n-2} \end{pmatrix} \neq 0, \qquad S_m = \mathrm{tr}(K^m). \tag{1.13}$$

This is a consequence of the Sylvester theorem on the discriminant D of an algebraic equation, applied to the characteristic equation $\det(A - \lambda B) = 0$ for the symmetric tensor K.

The Killing tensor K with simple eigenvalues has normal eigenvectors if and only if the Nijenhuis conditions are satisfied [12]:

$$\mathcal{N}^m_{[jk}\mathrm{g}_{i]m} = 0, \qquad \mathcal{N}^m_{[jk}K_{i]m} = 0, \qquad \mathcal{N}^m_{[jk}K_{i]\ell}K^\ell_m = 0. \tag{1.14}$$

The square brackets appearing in this equation denote antisymmetrization over the three indexes i, j, k, and N is the Nijenhuis tensor or the Nijenhuis torsion tensor K,

$$\mathcal{N}_K(X,Y) = K^2[X,Y] - K[KX,Y] - K[X,KY] + [KX,KY].$$

In terms of the local coordinates $q = (q_1, \ldots, q_n)$, the elements of the antisymmetric tensor field $\mathcal{N}_K$ of type (1,2) are given by

$$\mathcal{N}^i_{jk} = \sum_{m=1}^n \frac{\partial K^i_k}{\partial q_m}K^m_j - \frac{\partial K^i_j}{\partial q_m}K^m_k + \left(\frac{\partial K^m_j}{\partial q_k} - \frac{\partial K^m_k}{\partial q_j}\right)K^i_m.$$

As a criterion of normality of the eigenvectors of the Killing tensor K with respect to the metric g, instead of Nijenhuis conditions (1.14), we can use the condition that the Haantjes tensor, or Haantjes torsion, is zero [11]:

$$\mathcal{H}_K(X,Y) = K^2\mathcal{N}(X,Y) - K\mathcal{N}(KX,Y) - K\mathcal{N}(X,KY) + \mathcal{N}(KX,KY).$$

In terms of the local coordinates $q = (q_1, \ldots, q_n)$, the condition $\mathcal{H}_K(X,Y) = 0$ is a system of equations of the form

$$\mathcal{H}^i_{jk} = \sum_{m,\ell=1}^n K^i_m K^m_\ell \mathcal{N}^\ell_{jk} + \mathcal{N}^i_{m\ell}K^m_j K^\ell_k - K^i_m(\mathcal{N}^m_{\ell k}K^\ell_j + \mathcal{N}^m_{j\ell}K^\ell_k) = 0. \tag{1.15}$$

These are the fourth-order equations with respect to elements of the Killing tensor K, while the Nijenhuis equations are of the second, third, and fourth order with respect to elements of K, which can be considered successively (see discussions in [18], [19]).

The Nijenhuis and Haantjes tensors determine the deformation of the structures of nonassociative and alternated algebras in the tangent bundle TQ of a manifold Q [37]. The Nijenhuis and Haantjes tensors are therefore used in many problems of mathematical physics, but the basic applications of these tensors are related just to their triviality conditions $\mathcal{N}_K(X,Y) = 0$ and $\mathcal{H}_K(X,Y) = 0$ (see [38]). Hence, virtually nothing is known on tensors that do not satisfy the Nijenhuis conditions.

Below. we prove that the 2-valence Killing tensors K in (1.8) associated with the Hamiltonian H in (1.4) have a nonzero Haantjes torsion $\mathcal{H}(K) \neq 0$. Thus, constructing a sufficient number of independent commuting integrals of motion, which is necessary for Liouville integrability, is an open question if we are restricted by classical Euclidean geometry, without using the Lax matrices.

2. Symmetric spaces of type A.III

We consider equation of motion (1.5) in the Euclidean space $\mathbb{R}^{mn}$ and the corresponding Hamiltonian H in (1.4) associated with the Riemann pair

$$SU(m+n)/S(U(m)\times U(n)),\qquad 1<m\le n,\quad n+m\ge 4.$$

We use the representation of $su(m+n)$ by $(m+n)\times(m+n)$ matrices with the block-matrix structure [39], [26]–[28] associated with the Cartesian decomposition

$$\mathfrak{g}\equiv\mathfrak{k}\oplus\mathfrak{m},\qquad \mathfrak{k}=s(u(m)\oplus u(n)),$$

where the subalgebra $\mathfrak{k}$ consists of block-diagonal matrices of the form

$$\mathfrak{k}\simeq\begin{pmatrix}u(m)&0\\0&u(n)\end{pmatrix}.$$

In this case, the elements of the complement subspace $\mathfrak{m}$ are

$$X\in\mathfrak{m}\quad\to\quad X=\sum_{\alpha\in\Delta_+}(X^\alpha e_\alpha+X^{-\alpha}e_{-\alpha}),$$

where the Weyl generators corresponding to the subset Δ_+ of the root system are realized as the the lower-triangular matrices

$$e_\alpha=E_{ij},\qquad i<j,\ i>m,\ j<n,$$

with the only nonzero element at the intersection of the ith row and jth column.

We use a normalization slightly different from the one in [26]–[28], [39], and therefore represent the appropriate Lax matrix (1.3) in explicit matrix form as

$$L(\lambda)=\begin{pmatrix}-2\lambda^2I_m+QQ^{\mathrm T}+a&0\\0&2\lambda^2I_n-Q^{\mathrm T}Q+b\end{pmatrix}+\begin{pmatrix}0&P-2\mathrm{i}\lambda Q\\P^{\mathrm T}+2\mathrm{i}\lambda Q^{\mathrm T}&0\end{pmatrix},\tag{2.1}$$

where I_m and I_n are the identity $(m\times m)$ and $(n\times n)$ matrices, a and b are diagonal matrices depending on m real numbers a_k and n real numbers b_i,

$$a=\mathrm{diag}_m(a_1,\dots,a_m),\qquad b=\mathrm{diag}_n(b_1,\dots,b_n),\qquad a_i,b_i\in\mathbb{R},$$

the superscript T denotes transposition, and $\mathrm{i}=\sqrt{-1}$ is the imaginary unit.

The matrices Q and P are $m\times n$-matrices depending linearly on the Cartesian coordinates q_i and momenta p_i,

$$Q_{ij}=q_{(i-1)n+j},\qquad P_{ij}=p_{(i-1)n+j},\qquad i=1,\dots,m,\quad j=1,\dots,n,$$

i.e.,

$$Q=\begin{pmatrix}q_1&q_2&\cdots&q_n\\q_{n+1}&q_{n+2}&\cdots&q_{2n}\\\vdots&\vdots&\ddots&\vdots\\q_{n(m-1)+1}&q_{n(m-1)+2}&\cdots&q_{mn}\end{pmatrix},\qquad P=\begin{pmatrix}p_1&p_2&\cdots&p_n\\p_{n+1}&p_{n+2}&\cdots&p_{2n}\\\vdots&\vdots&\ddots&\vdots\\p_{n(m-1)+1}&p_{n(m-1)+2}&\cdots&p_{mn}\end{pmatrix}.\tag{2.2}$$

For the considered A.III type symmetric space, Hamiltonian (1.4) has the form

$$H=\frac{1}{4}\operatorname{tr}L^2\Big|_{\lambda=0}-\frac{1}{4}\sum_{j=1}^{m}a_j^2-\frac{1}{4}\sum_{i=1}^{n}b_i^2=\frac{1}{2}\sum_{i=1}^{n}p_i^2+\frac{1}{2}\sum_{j=0}^{m-1}\biggl(\sum_{i=1}^{n}q_{jn+i}^2\biggr)^2+$$
$$+\sum_{k,j=0;k>j}^{m-1}\biggl(\sum_{i=1}^{n}q_{jn+i}q_{kn+i}\biggr)^2+\frac{1}{2}\sum_{j=0}^{m-1}a_{j+1}\biggl(\sum_{i=1}^{n}q_{jn+i}^2\biggr)-\frac{1}{2}\sum_{i=1}^{n}b_i\biggl(\sum_{j=0}^{m-1}q_{jn+i}^2\biggr). \tag{2.3}$$

When $a_i\neq 0$ and $b_i\neq 0$, there are two bases in the space of the integrals of motion obtained from the characteristic polynomial for the Lax matrix

$$\tau(z,\lambda)=\det(z\,I-L(\lambda)),$$

which are associated with representations of the respective algebras $so(m)$ and $so(n)$. Because

$$\{\tau(x,\lambda),\tau(y,\mu)\}=0,$$

all these integrals of motion are in involution with respect to Poisson brackets (1.2).

2.1. The first basis in the space of integrals of motion. We consider the residues of the function

$$\Delta_1(z,\lambda)=\frac{\tau(z,\lambda)}{\prod_{i=1}^{m}(z-a_i+2\lambda^2)} \tag{2.4}$$

with respect to the variable z at the m points $z=a_i-2\lambda^2$:

$$\operatorname{Res}\Delta_1(z,\lambda)\Big|_{z=a_i-2\lambda^2}=\sum_{k=0}^{n-1}\lambda^{2k}h_i^{(2(n-k))},\qquad i=1,\dots,m.$$

Because $m\le n$, the coefficients $h^{(2(n-k))}$ are polynomials of degree not greater than $2m$ in momenta.

Proposition 2. *For the integrable systems associated with symmetric Hermitian* A.III*-type spaces, there is a basis of mn independent integrals of motion, which include*

- *m second-degree polynomials in momenta $h_1^{(2)},\dots,h_m^{(2)}$,*
- *m fourth-degree polynomials in momenta $h_1^{(4)},\dots,h_m^{(4)}$,*
- *m sixth-degree polynomials in momenta $h_1^{(6)},\dots,h_m^{(6)}$,*
- ....
- *m $2m$th-degree polynomials in momenta $h_1^{(2m)},\dots,h_m^{(2m)}$*

and $m(n-m)$ other $2m$th-degree polynomials in momenta.

The quadratic integrals of motion have the form

$$h_i^{(2)}=\sum_{k\neq i}^{m}\frac{M_{ik}^2}{a_i-a_k}+t_i(p)+v_i(q), \tag{2.5}$$

where the functions

$$M_{ik}=\sum^{n}J_{j\ell},\qquad J_{j\ell}=q_jp_\ell-q_\ell p_j,$$

provide a realization of the elements of the Lie algebra $so^*(m)$ *as linear combinations of* n *rotations* $X_{j,\ell}$ *in* (1.9) *in the configuration space* $\mathbb{R}^{mn}$. *The functions* $t_i(p)$ *correspond to the sequence of* n *shifts* X_ℓ *in* (1.9),

$$t_i(p) = \sum^n p_\ell^2,$$

and the potentials $v_i(q)$ *in* (2.5) *are fourth-degree polynomials in the coordinates* q_j.

The corresponding Killing tensors do not satisfy Nijenhuis conditions (1.4), *and their Haantjes torsion is nonvanishing.*

2.2. The second basis in the space of integrals of motion . We consider the residues of the function

$$\Delta_2(z,\lambda) = \frac{\tau(z,\lambda)}{\prod_{i=1}^n (z - b_i - 2\lambda^2)} \tag{2.6}$$

with respect to z at the n points $z = b_i + 2\lambda^2$:

$$\operatorname{Res} \Delta_2(z,\lambda)\big|_{z=b_i+2\lambda^2} = \sum_{k=0}^{m-1} \lambda^{2k} H_i^{(2(m-k))}, \qquad i = 1,\dots,n.$$

The coefficients $H_i^{(2(m-k))}$ at the different powers of λ are polynomial integrals of motion of degree not greater than $2m$.

Proposition 3. *For the integrable systems associated with symmetric Hermitian spaces of type* A.III, *there is a basis of* mn *independent integrals of motion, which include*

- n *second-degree polynomials in momenta* $H_1^{(2)},\dots,H_n^{(2)}$,
- n *fourth-degree polynomials in momenta* $H_1^{(4)},\dots,H_n^{(4)}$,
- n *sixth-degree polynomials in momenta* $H_1^{(6)},\dots,H_n^{(6)}$,
- ....
- n $2m$*th-degree polynomials in momenta* $H_1^{(2m)},\dots,H_n^{(2m)}$.

The quadratic integrals of motion have the form

$$H_i^{(2)} = \sum_{k\neq i}^n \frac{N_{ik}^2}{b_i - b_k} + T_i(p) + U_i(q), \tag{2.7}$$

where the functions

$$N_{ik} = \sum^m J_{j\ell}, \qquad J_{j\ell} = q_j p_\ell - q_\ell p_j,$$

provide a realization of the elements of the Lie algebra $so^*(n)$ *as linear combination of* m *rotations* $X_{i,\ell}$ *in* (1.9) *in the configuration space* $\mathbb{R}^{mn}$. *The functions* $T_i(p)$ *correspond to the sequence of* m *shifts* X_ℓ *along the coordinate axes* (1.9),

$$T_i(p) = \sum_\ell^m p_\ell^2,$$

and the potentials $U_i(q)$ *in* (2.7) *are fourth-degree polynomials in the coordinates.*

The corresponding Killing tensors do not satisfy Nijenhuis conditions (1.14), *and their Haantjes torsion is nonvanishing.*

Thus, there are $m+n-1$ independent quadratic integrals of motion

$$h_1^{(2)} + \dots + h_m^{(2)} = 2H = H_1^{(2)} + \dots + H_n^{(2)},$$

associated with linear combinations of rotations in configuration space that correspond to representations of the $so^*(m)$ and $so^*(n)$ algebras.

Proposition 4. *Equations of motion* (1.5) *defined by the Hamiltonian H in* (2.3) *have only the above $n+m-1$ independent quadratic conservation laws in involution.*

To prove this proposition, we have to estimate the dimension of the solution space of Eq. (1.11) for the potentials occurring in Hamiltonian (2.3).

2.3. Example: $so(m+n)$ with $m=n=2$. We consider motion in the 4-dimensional Euclidean space $\mathbb{R}^4$ when the quadratic conservation laws involve left and right isoclinic double rotations (Clifford shifts), which are the basic objects in classical four-dimensional Euclidean space geometry and in the theory of Clifford algebras [40]–[42].

The 4×4 Lax matrix can be written explicitly in this example:

$$L(\lambda) = \begin{pmatrix} q_1^2+q_2^2+a_1-2\lambda^2 & q_1q_3+q_2q_4 & p_1-2i\lambda q_1 & p_2-2i\lambda q_2 \\ q_1q_3+q_2q_4 & q_3^2+q_4^2+a_2-2\lambda^2 & p_3-2i\lambda q_3 & p_4-2i\lambda q_4 \\ p_1-2i\lambda q_1 & p_3-2i\lambda q_3 & b_1-q_1^2-q_3^2+2\lambda^2 & -q_1q_2-q_3q_4 \\ p_2-2i\lambda q_2 & p_4-2i\lambda q_4 & -q_1q_2-q_3q_4 & b_2-q_2^2-q_4^2+2\lambda^2 \end{pmatrix}. \tag{2.8}$$

The corresponding Hamiltonian H in Eq. (2.3) is then given by

$$\begin{aligned} H = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{p_3^2}{2} + \frac{p_4^2}{2} + \frac{1}{2}(q_1^2+q_2^2)^2 + \frac{1}{2}(q_3^2+q_4^2)^2 + (q_1q_3+q_2q_4)^2 + \\ + \frac{a_1-b_1}{2}q_1^2 + \frac{a_1-b_2}{2}q_2^2 + \frac{a_2-b_1}{2}q_3^2 + \frac{a_2-b_2}{2}q_4^2. \end{aligned} \tag{2.9}$$

The spectral curve of the Lax matrix $L(\lambda)$ in (2.8) is not a hyperelliptic curve of genus $g=5$: it is defined by the characteristic equation

$$\mathcal{C}\colon \quad \det(zI - L(\lambda)) = 0.$$

The first basis in the space of integrals of motion. Both residues of function (2.4),

$$\Delta(z,\lambda) = \frac{\det(zI-L(\lambda))}{(z-a_1+2\lambda^2)(z-a_2+2\lambda^2)}$$

with respect to the variable z at the points $z = a_{1,2} - 2\lambda^2$ are second-degree polynomials in λ,

$$\operatorname{Res}\big|_{z=a_i-2\lambda^2} \Delta(z,\lambda) = 4\lambda^2 f_i + g_i, \qquad i = 1,2,$$

where the coefficients $f_{1,2}$ and $g_{1,2}$ are the respective second- and fourth-degree polynomials in momenta.

Calculating the residue at infinity,

$$\operatorname{Res}\big|_{z=\infty} \Delta(z,\lambda) = -4\lambda^2(f_1+f_2) - (g_1+g_2),$$

allows finding relations between the coefficients $f_{1,2}$ and $g_{1,2}$,

$$f_1 + f_2 = 2H, \qquad g_1 + g_2 = \tilde{f}_3,$$

where $\tilde{f}_3$ is a second-degree polynomial in momenta that is independent of $f_{1,2}$.

We write the quadratic integrals of motion as

$$f_1 = -\frac{M_{12}^2}{a_1 - a_2} + p_1^2 + p_2^2 + v_1, \qquad f_2 = \frac{M_{12}^2}{a_1 - a_2} + p_3^2 + p_4^2 + v_2, \tag{2.10}$$

where

$$v_1 = (q_1^2 + q_2^2 + q_3^2 + a_1 - b_1)q_1^2 + (q_1^2 + q_2^2 + q_4^2 + a_1 - b_2)q_2^2 + 2q_1q_2q_3q_4,$$
$$v_2 = (q_1^2 + q_3^2 + q_4^2 + a_2 - b_1)q_3^2 + (q_2^2 + q_3^2 + q_4^2 + a_2 - b_2)q_4^2 + 2q_1q_2q_3q_4,$$

and M_{12} is the function associated with the double rotation in $\mathbb{R}^4$,

$$M_{12} = J_{1,3} + J_{2,4} = (q_1p_3 - q_3p_1) + (q_2p_4 - q_4p_2).$$

This function commutes with those terms in the definition of the coefficients $f_{1,2}$ in (2.10) that are associated with shifts,

$$\{M_{12}, p_1^2 + p_2^2\} = \{M_{12}, p_3^2 + p_4^2\} = 0,$$

and with the function describing the second independent double rotation in $\mathbb{R}^4$,

$$N_{12} = J_{1,2} + J_{3,4} = (q_1p_2 - q_2p_1) + (q_3p_4 - p_3q_4).$$

Therefore,

$$\{M_{12}, N_{12}\} = 0.$$

This function enters the following combination of the integrals of motion:

$$\begin{aligned} f_3 &= (b_1 + b_2)H - g_1 - g_2 - a_1f_1 - a_2f_2 = \\ &= N_{12}^2 - \frac{1}{2}(b_1 - b_2)((q_1^2 + q_2^2 + q_3^2 + q_4^2)(q_1^2 - q_2^2 + q_3^2 - q_4^2) + \\ &\quad + (q_1^2 - q_2^2)a_1 + (q_3^2 - q_4^2)a_2 - (q_1^2 + q_3^2)b_1 + (q_2^2 + q_4^2)b_2). \end{aligned}$$

For $b_1 = b_2$, the linear integral of motion N_{12} is a function of the integrals of motion $f_{1,2}$ and $g_{1,2}$ forming the first basis in the space of the integrals of motion.

The second basis in the space of integrals of motion. Both residues of function (2.6)

$$\Delta(z, \lambda) = \frac{\det(zI - L(\lambda))}{(z - b_1 - 2\lambda^2)(z - b_2 - 2\lambda^2)}$$

with respect to the variable z at the points $z = b_{1,2} + 2\lambda^2$ are second-degree polynomials in λ,

$$\mathrm{Res}\big|_{z=b_i+2\lambda^2} \Delta(z, \lambda) = -4\lambda^2 F_i + G_i, \qquad i = 1, 2,$$

where the coefficients $F_{1,2}$ and $G_{1,2}$ are the respective second- and fourth-degree polynomials in momenta.

Calculating the residue at infinity

$$\mathrm{Res}\big|_{z=\infty}\Delta(z,\lambda)=8\lambda^2H-(G_1+G_2)$$

allows finding relations between the coefficients $F_{1,2}$ and $G_{1,2}$,

$$F_1+F_2=2H,\qquad G_1+G_2=\widetilde{F}_3,$$

where $\widetilde{F}_3$ is a second-degree polynomial in momenta that is independent of $F_{1,2}$.

We write the quadratic integrals of motion as

$$F_1=\frac{N_{12}^2}{b_1-b_2}+p_1^2+p_3^2+V_1,\qquad F_2=-\frac{N_{12}^2}{b_1-b_2}+p_2^2+p_4^2+V_2,$$

where

$$\begin{aligned}V_1&=(q_1^2+q_2^2+q_3^2+a_1-b_1)q_1^2+(q_1^2+q_3^2+q_4^2+a_2-b_1)q_3^2+2q_1q_2q_3q_4,\\ V_2&=(q_1^2+q_2^2+q_4^2+a_1-b_2)q_2^2+(q_2^2+q_3^2+q_4^2+a_2-b_2)q_4^2+2q_1q_2q_3q_4.\end{aligned}$$

Here, N_{12} is the function associated with the double rotation in $\mathbb{R}^4$:

$$N_{12}=J_{1,2}+J_{3,4}=(q_1p_2-q_2p_1)+(q_3p_4-p_3q_4).$$

It commutes with those terms in the definition of $F_{1,2}$ that are responsible for shifts,

$$\{N_{12},p_1^2+p_3^2\}=\{N_{12},p_2^2+p_4^2\}=0,$$

and with the function responsible for the second double rotation

$$M_{12}=J_{1,3}+J_{2,4}=(q_1p_3-q_3p_1)+(q_2p_4-p_2q_4),$$

which appears in the definition of the quadratic integrals of motion $f_{1,2}$ in (2.10) from the first basis.

This function also appears in a combination of the integrals of motion

$$\begin{aligned}F_3&=G_1+G_2-b_1F_1-b_2F_2-(a_1+a_2)H=\\ &=M_{12}^2+\frac{1}{2}(a_1-a_2)(p_3^2+p_4^2-p_1^2-p_2^2+(q_1^2-q_3^2)b_1+(q_2^2-q_4^2)b_2-\\ &\quad-(q_1^2+q_2^2)a_1+(q_3^2+q_4^2)a_2-(q_1^2+q_2^2)^2+(q_3^2+q_4^2)^2).\end{aligned}$$

For $a_1=a_2$, the linear integral of motion N_{13} is a function of the integrals $F_{1,2}$ and $G_{1,2}$, which determine the second basis in the space of integrals of motion.

Thus, we have presented six polynomials f_1,f_2,f_3 and F_1,F_2,F_3 of the second degree in momenta, among which only $m+n-1=3$ polynomials are functionally independent. Direct calculation shows that the corresponding Killing tensors of valence 2 have a nonzero Haantjes torsion.

We can verify that the polynomial $G=\operatorname{tr}L^4(\lambda=0)$ of the fourth degree in momenta, Eq. (1.7), cannot be expressed in terms of these second-degree polynomials in momenta.

2.4. Example: $so(m+n)$ with $m = 2$ and $n = 3$. In this example, the 5×5 Lax matrix is given by

$$L(\lambda)=\begin{pmatrix} q_1^2+q_2^2+q_3^2+a_1-2\lambda^2 & q_1q_4+q_2q_5+q_3q_6 & p_1-2i\lambda q_1 & p_2-2i\lambda q_2 & p_3-2i\lambda q_3\\ q_1q_4+q_2q_5+q_3q_6 & q_4^2+q_5^2+q_6^2+a_2-2\lambda^2 & p_4-2i\lambda q_4 & p_5-2i\lambda q_5 & p_6-2i\lambda q_6\\ p_1+2i\lambda q_1 & p_4+2i\lambda q_4 & b_1-q_1^2-q_4^2+2\lambda^2 & -q_1q_2-q_4q_5 & -q_1q_3-q_4q_6\\ p_2+2i\lambda q_2 & p_5+2i\lambda q_5 & -q_1q_2-q_4q_5 & b_2-q_2^2-q_5^2+2\lambda^2 & -q_2q_3-q_5q_6\\ p_3+2i\lambda q_3 & p_6+2i\lambda q_6 & -q_1q_3-q_4q_6 & -q_2q_3-q_5q_6 & b_3-q_3^2-q_6^2+2\lambda^2 \end{pmatrix},$$

and the Hamiltonian H in Eqs. (1.4) and (2.3) has the form

$$\begin{aligned} H &= \frac{1}{2}\sum_{i=1}^{6} p_i^2 + \frac{(q_1^2+q_2^2+q_3^2)^2}{2} + \frac{(q_4^2+q_5^2+q_6^2)^2}{2} + (q_1q_4+q_2q_5+q_3q_6)^2 - \\ &- \frac{q_1^2+q_4^2}{2}b_1 - \frac{q_2^2+q_5^2}{2}b_2 - \frac{q_3^2+q_6^2}{2}b_3 + \frac{q_1^2+q_2^2+q_3^2}{2}a_1 + \frac{q_4^2+q_5^2+q_6^2}{2}a_2. \end{aligned} \tag{2.11}$$

At $a_i = 0$ and $b_k = 0$, this Hamiltonian is invariant under four rotations (1.9) of the configuration space $\mathbb{R}^6$,

$$\begin{aligned} Y_1 &= (q_1\partial_4 - q_1\partial_4) + (q_2\partial_5 - q_5\partial_2) + (q_3\partial_6 - q_6\partial_3),\\ Y_2 &= (q_1\partial_2 - q_2\partial_1) + (q_4\partial_5 - q_5\partial_4),\\ Y_3 &= (q_1\partial_3 - q_3\partial_1) + (q_4\partial_6 - q_6\partial_4),\\ Y_4 &= (q_2\partial_3 - q_3\partial_2) + (q_5\partial_6 - q_6\partial_5), \end{aligned} \tag{2.12}$$

and hence the Lie derivative along these vector fields is zero:

$$\mathcal{L}_{Y_j} H = 0, \quad j = 1,\dots,4, \quad \text{at} \quad a_i = 0,\ b_k = 0.$$

The presence of these four symmetries leads to the existence of four integrals of motion that are linear in momenta, and some of them do not commute with each other.

The equation of the spectral curve for the 5×5 Lax matrix contains five commuting functions H, F_1, F_2 and G_1, G_2,

$$\begin{aligned} \tau(z,\lambda) &= z^5 - 2\lambda^2 z^4 - 2(4\lambda^4 + H)z^3 + (16\lambda^6 + 4H\lambda^2 + F_1)z^2 + (16\lambda^8 + \\ &+ 8H\lambda^4 - 4F_2^2\lambda^2 + G_1)z - 32\lambda^{10} - 16H\lambda^6 + (8F_2^2 - 4F_1)\lambda^4 - 2G_1\lambda^2 + G_2, \end{aligned}$$

where the integrals of motion quadratic in momenta

$$F_1 = M_{12}^2 - N_{12}^2 - N_{13}^2 - N_{23}^2, \qquad F_2 = M_{12}^2$$

are associated with the symmetries Y_k in Eq. (2.12) (see the explicit expressions for M_{12} in (2.14) and for N_{ij} in (2.15) below). The functions $G_{1,2}$ are fourth-degree polynomials in momenta, which are functionally independent of H, F_1, and F_2.

As a missing sixth independent integral of motion, we can take any linear integral of motion N_{ij}, which nevertheless is not formally generated by spectral invariants of the 5×5 Lax matrix. Thus, to prove the integrability in the framework of the classical r-matrix method, we have to find the required sixth integral of motion using other tensor invariants of the Lax matrix, as this was for the complete Toda chain [35].

In the general case, for $a_i \neq 0$ and $b_k \neq 0$, the terms added to potential (2.11) are not invariant under rotations Y_i in (2.12). Nevertheless, the spectral curve of the Lax matrix $L(\lambda)$ is not a sixth-order hyperelliptic curve; this allows immediately obtaining six independent polynomial integrals of motion in involution that no longer belong to the class of second-degree polynomials in momenta.

The first basis in the space of integrals of motion. We find the residues of the function $\Delta(z,\lambda)$ in (2.4),

$$\Delta(z,\lambda) = \frac{\det(zI - L(\lambda))}{(z - a_1 + 2\lambda^2)(z - a_2 + 2\lambda^2)}$$

with respect to the variable z at two points $z = a_{1,2} - 2\lambda^2$:

$$\mathrm{Res}\big|_{z=a_i-2\lambda^2}\Delta(z,\lambda) = -16\lambda^4 f_i + \lambda^2 g_i + w_i, \qquad i = 1, 2.$$

The coefficients $f_{1,2}$ are second-degree polynomials in momenta, while $g_{1,2}$ and $w_{1,2}$ are fourth-degree polynomials in momenta.

We write the quadratic integrals of motion as

$$f_1 = -\frac{M_{12}^2}{b_1 - b_2} + p_1^2 + p_2^2 + p_3^2 + v_1, \qquad f_2 = \frac{M_{12}^2}{b_1 - b_2} + p_4^2 + p_5^2 + p_6^2 + v_2, \tag{2.13}$$

where

$$\begin{aligned} v_1 &= (q_1^2 + q_2^2 + q_3^2 + q_4^2 + a_1 - b_1)q_1^2 + (q_1^2 + q_2^2 + q_3^2 + q_5^2 + a_1 - b_2)q_2^2 + \\ &\quad + (q_1^2 + q_2^2 + q_3^2 + q_6^2 + a_1 - b_3)q_3^2 + 2q_1q_2q_4q_5 + 2q_1q_3q_4q_6 + 2q_2q_3q_5q_6, \\ v_2 &= (q_1^2 + q_4^2 + q_5^2 + q_6^2 + a_2 - b_1)q_4^2 + (q_2^2 + q_4^2 + q_5^2 + q_6^2 + a_2 - b_2)q_5^2 + \\ &\quad + (q_3^2 + q_4^2 + q_5^2 + q_6^2 + a_2 - b_3)q_6^2 + 2q_1q_2q_4q_5 + 2q_1q_3q_4q_6 + 2q_2q_3q_5q_6. \end{aligned}$$

Because the residue at infinity is

$$\mathrm{Res}\big|_{z=\infty}\Delta(z,\lambda) = 32\lambda^4 H - \lambda^2(g_1 + g_2) - (w_1 + w_2), \qquad f_1 + f_2 - 2H = 0,$$

the sum of these integrals of motion is equal to twice the Hamiltonian.

The function M_{12} is associated with the triple rotation of the configuration space $\mathbb{R}^6$, because $n = 3$:

$$M_{12} = J_{14} + J_{25} + J_{36} = (q_1p_4 - p_4q_1) + (q_2p_5 - p_2q_5) + (q_3p_6 - p_3q_6). \tag{2.14}$$

Various combinations of the basis elements $f_{1,2}$, $g_{1,2}$, and $w_{1,2}$ are also associated with various double rotations in $\mathbb{R}^6$. For instance, the second-degree polynomial in momenta

$$f_3 = 2(b_1 + b_2 + b_3)H + \frac{g_1 + g_2}{4} - 2a_1f_1 - 2a_2f_2$$

is equal to

$$f_3 = N_{12}^2 + N_{13}^2 + N_{23}^2 + (p_1^2 + p_4^2)b_1 + (p_2^2 + p_5^2)b_2 + (p_3^2 + p_6^2)b_3 + v_3,$$

where

$$\begin{aligned} N_{12} &= J_{12} + J_{45} = (q_1p_2 - p_1q_2) + (q_4p_5 - p_4q_5), \\ N_{13} &= J_{13} + J_{46} = (q_1p_3 - p_1q_3) + (q_4p_6 - p_4q_6), \\ N_{23} &= J_{23} + J_{56} = (q_2p_3 - p_2q_3) + (q_5p_6 - p_5q_6) \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} v_3 = {} & (q_1^4 + q_1^2q_2^2 + q_1^2q_3^2 + 2q_1^2q_4^2 + 2q_1q_2q_4q_5 + 2q_1q_3q_4q_6 + q_4^4 + q_4^2q_5^2 + q_4^2q_6^2 + a_1q_1^2 + \\ & + a_2q_4^2)b_1 + (q_1^2q_2^2 + 2q_1q_2q_4q_5 + q_2^4 + q_2^2q_3^2 + 2q_2^2q_5^2 + 2q_2q_3q_5q_6 + q_4^2q_5^2 + q_5^4 + \\ & + q_5^2q_6^2 + a_1q_2^2 + a_2q_5^2)b_2 + (q_1^2q_3^2 + 2q_1q_3q_4q_6 + q_2^2q_3^2 + 2q_2q_3q_5q_6 + q_3^4 + 2q_3^2q_6^2 + \\ & + q_4^2q_6^2 + q_5^2q_6^2 + q_6^4 + a_1q_3^2 + a_2q_6^2)b_3 - (q_1^2 + q_4^2)b_1^2 - (q_2^2 + q_5^2)b_2^2 - (q_3^2 + q_6^2)b_3^2. \end{aligned}$$

The second basis in the space of integrals of motion. We calculate the residues of the function $\Delta(z,\lambda)$ in (2.6)

$$\Delta(z,\lambda)=\frac{\det(zI-L(\lambda))}{(z-b_1-2\lambda^2)(z-b_2-2\lambda^2)(z-b_3-2\lambda^2)}$$

with respect to the variable z at three points $z=b_i+2\lambda^2$:

$$\mathrm{Res}\big|_{z=b_i+2\lambda^2}\Delta(z,\lambda)=4\lambda^2F_i+G_i,\qquad i=1,2,3.$$

The six coefficients F_i and G_i are, respectively, second- and fourth-degree polynomials in momenta.

The residue at infinity is

$$\mathrm{Res}\big|_{z=\infty}\Delta(z,\lambda)=8\lambda^2H-(G_1+G_2+G_3),\qquad 2H+F_1+F_2+F_3=0.$$

The quadratic integrals of motion are determined by double rotations and double shifts (2.7) of the configuration space:

$$\begin{aligned}F_1&=-\frac{N_{12}^2}{b_1-b_2}-\frac{N_{13}^2}{b_1-b_3}-p_1^2-p_4^2-(q_1^2+q_2^2+q_3^2+2q_4^2+a_1-b_1)q_1^2-\\&\quad-(q_4^2+q_5^2+q_6^2+a_2-b_1)q_4^2-2(q_2q_5+q_3q_6)q_1q_4,\\F_2&=-\frac{N_{21}^2}{b_2-b_1}-\frac{N_{23}^2}{b_2-b_3}-p_2^2-p_5^2-(q_1^2+q_2^2+q_3^2+2q_5^2+a_1-b_2)q_2^2-\\&\quad-(q_4^2+q_5^2+q_6^2+a_2-b_2)q_5^2-2(q_1q_4+q_3q_6)q_2q_5,\\F_3&=-\frac{N_{31}^2}{b_3-b_1}-\frac{N_{32}^2}{b_3-b_2}-p_3^2-p_5^2-(q_1^2+q_2^2+q_3^2-2q_6^2+a_1-b_3)q_3^2-\\&\quad-(q_4^2+q_5^2+q_6^2+a_2-b_3)q_6^2-2(q_1q_4+q_2q_5)q_3q_6.\end{aligned}$$

The functions $N_{ij}=-N_{ji}$ in (2.15) can be considered a realization of the elements of the Lie algebra $so^*(3)$ via double rotations of the configuration space $\mathbb{R}^6$, because

$$\{N_{12},N_{13}\}=N_{23},\qquad\{N_{13},N_{23}\}=N_{12},\qquad\{N_{23},N_{12}\}=N_{13}.$$

The highest-order term of the second-degree polynomials in momenta, which is independent of F_1,F_2 and F_3,

$$\begin{aligned}F_4&=G_1+G_2+G_3-b_1F_1-b_2F_2-b_3F_3-(a_1+a_2)H=\\&=M_{12}^2-\frac{a_1-a_2}{2}(p_1^2+p_2^2+p_3^2-p_4^2-p_5^2-p_6^2+V_4)\end{aligned}$$

contains the function M_{12} in (2.14), which is associated with a triple rotation in $\mathbb{R}^6$. The corresponding potential is given by

$$\begin{aligned}V_4&=(q_1^2+q_2^2+q_3^2+q_4^2+q_5^2+q_6^2)(q_1^2+q_2^2+q_3^2-q_4^2-q_5^2-q_6^2)+\\&\quad+(q_1^2+q_2^2+q_3^2)a_1-(q_4^2+q_5^2+q_6^2)a_2-(q_1^2-q_4^2)b_1-(q_2^2-q_5^2)b_2-(q_3^2-q_6^2)b_3.\end{aligned}$$

For $a_1=a_2$, the linear integral of motion M_{12} is a function of the basis elements F_k and G_k in the space of integrals of motion.

Thus, in the case $m=2$ and $n=3$, we have presented seven integrals of motion that are quadratic in momenta, f_1,f_2,f_3 and F_1,F_2,F_3,F_4. Direct calculation shows that the corresponding Killing tensors of valence 2 have a nontrivial Haantjes torsion. Among those integrals, only those with $m+n-1=4$ are functionally independent.

As before, we can verify that the fourth-degree polynomial in momenta $G=\mathrm{tr}\,L^4(\lambda=0)$, Eq. (1.7), cannot be expressed in terms of these second-degree polynomials in momenta.

2.5. Example: $so(m+n)$ with $m = n = 3$. We write the 6×6 Lax matrix explicitly,

$$L(\lambda) = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}, \tag{2.16}$$

where the 3×3 blocks are

$$L_{11} = \begin{pmatrix} -2\lambda^2+q_1^2+q_2^2+q_3^2+a_1 & q_1q_4+q_2q_5+q_3q_6 & q_1q_7+q_2q_8+q_3q_9 \\ q_1q_4+q_2q_5+q_3q_6 & -2\lambda^2+q_4^2+q_5^2+q_6^2+a_2 & q_4q_7+q_5q_8+q_6q_9 \\ q_1q_7+q_2q_8+q_3q_9 & q_4q_7+q_5q_8+q_6q_9 & -2\lambda^2+q_7^2+q_8^2+q_9^2+a_3 \end{pmatrix},$$

$$L_{22} = \begin{pmatrix} 2\lambda^2-q_1^2-q_4^2-q_7^2+b_1 & -q_1q_2-q_4q_5-q_7q_8 & -q_1q_3-q_4q_6-q_7q_9 \\ -q_1q_2-q_4q_5-q_7q_8 & 2\lambda^2-q_2^2-q_5^2-q_8^2+b_2 & -q_2q_3-q_5q_6-q_8q_9 \\ -q_1q_3-q_4q_6-q_7q_9 & -q_2q_3-q_5q_6-q_8q_9 & 2\lambda^2-q_3^2-q_6^2-q_9^2+b_3 \end{pmatrix},$$

$$L_{12} = \begin{pmatrix} p_1-2i\lambda q_1 & p_2-2i\lambda q_2 & p_3-2i\lambda q_3 \\ p_4-2i\lambda q_4 & p_5-2i\lambda q_5 & p_6-2i\lambda q_6 \\ p_7-2i\lambda q_7 & p_8-2i\lambda q_8 & p_9-2i\lambda q_9 \end{pmatrix}, \qquad L_{21} = \begin{pmatrix} p_1+2i\lambda q_1 & p_4+2i\lambda q_4 & p_7+2i\lambda q_7 \\ p_2+2i\lambda q_2 & p_5+2i\lambda q_5 & p_8+2i\lambda q_8 \\ p_3+2i\lambda q_3 & p_6+2i\lambda q_6 & p_9+2i\lambda q_9 \end{pmatrix}.$$

The Hamiltonian H in Eq. (2.3) is given by

$$\begin{aligned} H = {} & \frac{1}{2}\sum_{i=1}^{9} p_i^2 + \frac{(q_1^2+q_2^2+q_3^2)^2}{2} + \frac{(q_4^2+q_5^2+q_6^2)^2}{2} + \frac{(q_7^2+q_8^2+q_9^2)^2}{2} + \\ & + (q_1q_4+q_2q_5+q_3q_6)^2 + (q_1q_7+q_2q_8+q_3q_9)^2 + (q_4q_7+q_5q_8+q_6q_9)^2 - \\ & - \frac{q_1^2+q_4^2+q_7^2}{2}b_1 - \frac{q_2^2+q_5^2+q_8^2}{2}b_2 - \frac{q_3^2+q_6^2+q_9^2}{2}b_3 + \\ & + \frac{q_1^2+q_2^2+q_3^2}{2}a_1 + \frac{q_4^2+q_5^2+q_6^2}{2}a_2 + \frac{q_7^2+q_8^2+q_9^2}{2}a_3. \end{aligned}$$

The first basis in the space of integrals of motion. The residues of function (2.4)

$$\Delta(z,\lambda) = \frac{\det(zI - L(\lambda))}{(z-a_1+2\lambda^2)(z-a_2+2\lambda^2)(z-a_3+2\lambda^2)}$$

with respect to the variable z determine a fourth-degree polynomials in the variable λ,

$$\begin{aligned} &\operatorname{Res}\big|_{z=a_i+2\lambda^2}\Delta(z,\lambda) = 16\lambda^4 f_i + \lambda^2 g_i + s_i, \qquad i=1,2,3. \\ &\operatorname{Res}\big|_{z=\infty}\Delta(z,\lambda) = 32H\lambda^4 - (g_1+g_2+g_3)\lambda^2 - (s_1+s_2+s_3). \end{aligned}$$

The nine coefficients f_i, g_i, and s_i are second-, fourth-, and sixth-degree polynomials in momenta.

We write the quadratic integrals of motion as

$$\begin{aligned} f_1 = {} & \frac{M_{12}^2}{a_1-a_2} + \frac{M_{13}^2}{a_1-a_3} - p_1^2 - p_2^2 - p_3^2 - (2{q_2}^2 + 2{q_3}^2 + {q_4}^2 + {q_7}^2 + a_1 - b_1){q_1}^2 - \\ & - (2{q_3}^2 - {q_5}^2 - {q_8}^2 - a_1 + b_2){q_2}^2 - ({q_3}^2 + {q_6}^2 + {q_9}^2 + a_1 - b_3){q_3}^2 - \\ & - 2q_2q_3(q_5q_6+q_8q_9) - 2q_1q_2(q_4q_5+q_7q_8) - 2q_1q_3(q_4q_6+q_7q_9) - {q_1}^4 - {q_2}^4, \\ f_2 = {} & \frac{M_{21}^2}{a_2-a_1} + \frac{M_{23}^2}{a_2-a_3} - p_4^2 - p_5^2 - p_6^2 - ({q_1}^2 + 2{q_5}^2 + 2{q_6}^2 + {q_7}^2 + a_2 - b_1){q_4}^2 - \\ & - ({q_2}^2 + 2{q_6}^2 + {q_8}^2 + a_2 - b_2){q_5}^2 - (q_3^2 + q_6^2 + q_9^2 + a_2 - b_3)q_6^2 - \\ & - q_5q_4(2q_1q_2 + 2q_7q_8) - 2q_6q_4(q_1q_3+q_7q_9) - 2q_5q_6(q_2q_3+q_8q_9) - q_4^4 - q_5^4, \end{aligned}$$

$$f_3 = \frac{M_{31}^2}{a_3 - a_1} + \frac{M_{32}^2}{a_3 - a_2} - p_7^2 - p_8^2 - p_9^2 - (q_1^2 + q_4^2 + 2q_8^2 + 2q_9^2 + a_3 - b_1)q_7^2 - \\ - (q_2^2 + q_5^2 + 2q_9^2 + a_3 - b_2)q_8^2 - (q_3^2 + q_6^2 + q_9^2 + a_3 - b_3)q_9^2 - \\ - 2q_7q_8(q_1q_2 + q_4q_5) - 2q_7q_9(q_1q_3 + q_4q_6) - 2q_8q_9(q_2q_3 + q_5q_6) - q_7^4 - q_8^4.$$

The functions M_{ij} occurring in the definition of the quadratic integrals of motion are

$$\begin{aligned} M_{12} &= J_{14} + J_{25} + J_{36} = (q_1p_4 - p_1q_4) + (q_2p_5 - p_2q_5) + (q_3p_6 - p_3q_6), \\ M_{13} &= J_{17} + J_{28} + J_{39} = (q_1p_7 - p_1q_7) + (q_2p_8 - p_2q_8) + (q_3p_9 - p_3q_9), \\ M_{23} &= J_{47} + J_{58} + J_{69} = (q_4p_7 - p_4q_7) + (q_5p_8 - p_5q_8) + (q_6p_9 - p_6q_9). \end{aligned} \tag{2.17}$$

The combination of the basis integrals of motion

$$f_4 = \frac{g_1 + g_2 + g_2}{4} + 2a_1f_1 + 2a_2f_2 + 2a_3f_3$$

is also a quadratic polynomial in momenta, and its definition

$$f_4 = -\left(\sum_{j=1}^{n} b_j\right)\left(\sum_{i=1}^{nm} p_i^2\right) + \sum_{j=1}^{n} b_j \left(\sum_{i=0}^{m-1} p_{j+im}^2\right) + N_{12}^2 + N_{23}^2 + N_{31}^2 + u_4(q)$$

contains the functions N_{ij}:

$$\begin{aligned} N_{12} &= J_{12} + J_{45} + J_{78} = (q_1p_2 - p_1q_2) + (q_4p_5 - p_4q_5) + (q_7p_8 - p_7q_8), \\ N_{13} &= J_{13} + J_{46} + J_{79} = (q_1p_3 - p_1q_3) + (q_4p_6 - p_4q_6) + (q_7p_9 - p_7q_9), \\ N_{23} &= J_{23} + J_{56} + J_{89} = (q_2p_3 - p_2q_3) + (q_5p_6 - p_5q_6) + (q_8p_9 - p_8q_9). \end{aligned} \tag{2.18}$$

The functions M_{ij} in (2.17) and N_{ij} in (2.18) are associated with two independent realizations of elements of the Lie algebra $so^*(3)$ via triple rotations in $\mathbb{R}^9$. The corresponding Poisson brackets are

$$\begin{aligned} &\{M_{12}, M_{13}\} = M_{23}, \qquad \{M_{13}, M_{23}\} = M_{12}, \qquad \{M_{23}, M_{12}\} = M_{13}, \\ &\{N_{12}, N_{13}\} = N_{23}, \qquad \{N_{13}, N_{23}\} = N_{12}, \qquad \{N_{23}, N_{12}\} = N_{13}, \\ &\{N_{ij}, M_{kl}\} = 0. \end{aligned}$$

The second basis in the space of integrals of motion. The residues of function (2.6)

$$\Delta(z, \lambda) = \frac{\det(zI - L(\lambda))}{(z - b_1 - 2\lambda^2)(z - b_2 - 2\lambda^2)(z - b_3 - 2\lambda^2)}$$

with respect to the variable z are

$$\begin{aligned} &\mathrm{Res}\big|_{z=b_i+2\lambda^2} \Delta(z, \lambda) = 16\lambda^4 F_i + \lambda^2 G_i + S_i, \qquad i = 1, 2, 3, \\ &\mathrm{Res}\big|_{z=\infty} \Delta(z, \lambda) = 32H\lambda^4 - (G_1 + G_2 + G_3)\lambda^2 - (S_1 + S_2 + S_3). \end{aligned}$$

The nine coefficients F_i, G_i and S_i are respectively second-, fourth-, and sixth-degree polynomials in momenta.

We write the quadratic integrals of motion as

$$\begin{aligned}
F_1 = &-\frac{N_{12}^2}{b_1-b_2}-\frac{N_{13}^2}{b_1-b_3}-p_1^2-p_4^2-p_7^2-(q_1^2+q_2^2+q_3^2+q_4^2+q_7^2+a_1-b_1)q_1^2-\\
&-(q_1^2+q_4^2+q_5^2+q_6^2+q_7^2+a_2-b_1)q_4^2-(q_1^2+q_4^2+q_7^2+q_8^2+q_9^2+a_3-b_1)q_7^2-\\
&-2q_1q_4(q_2q_5+q_3q_6)-2q_1q_7(q_2q_8+q_3q_9)-2q_4q_7(q_5q_8+q_6q_9),\\
F_2 = &-\frac{N_{21}^2}{b_2-b_1}-\frac{N_{23}^2}{b_2-b_3}-p_2^2-p_5^2-p_8^2-(q_1^2+q_2^2+q_3^2+q_5^2+q_8^2+a_1-b_2)q_2^2-\\
&-(q_2^2+q_4^2+q_5^2+q_6^2+q_8^2+a_2-b_2)q_5^2-(q_2^2+q_5^2+q_7^2+q_8^2+q_9^2+a_3-b_2)q_8^2-\\
&-2q_2q_5(q_1q_4+q_3q_6)-2q_2q_8(q_1q_7+q_3q_9)-2q_5q_8(q_4q_7+q_6q_9),\\
F_3 = &-\frac{N_{31}^2}{b_3-b_1}-\frac{N_{32}^2}{b_3-b_2}-p_3^2-p_6^2-p_9^2-(q_1^2+q_2^2+q_3^2+q_6^2+q_9^2+a_1-b_3)q_3^2-\\
&-(q_3^2+q_4^2+q_5^2+q_6^2+q_9^2+a_2-b_3)q_6^2-(q_3^2+q_6^2+q_7^2+q_8^2+q_9^2+a_3-b_3)q_9^2-\\
&-2q_3q_6(q_1q_4+q_2q_5)-2q_3q_9(q_1q_7+q_2q_8)-2q_6q_9(q_4q_7+q_5q_8).
\end{aligned}$$

The functions N_{kl} occurring in these definitions are given in (2.18).

The combination of the basis integrals of motion

$$F_4 = \frac{1}{8}(G_1+G_2+G_3)-b_1F_1-b_2F_2-b_3F_3$$

is also a second-degree polynomial in momenta and is independent of F_1, F_2, and F_3:

$$\begin{aligned}
F_4 = \frac{1}{2}\bigg(\sum_{j=1}^{m}a_j\bigg)\bigg(\sum_{i=1}^{n}p_i^2\bigg)-\frac{1}{2}\sum_{j=0}^{m-1}a_{j+1}\bigg(\sum_{i=1}^{n}p_{jn+i}^2\bigg)+\\
+\frac{M_{12}^2}{2}+\frac{M_{13}^2}{2}+\frac{M_{23}^2}{2}+U_4(q).
\end{aligned}$$

Thus, in the case $m=n=3$, we have presented eight integrals of motion quadratic in momenta, f_1, f_2, f_3, f_4 and F_1, F_2, F_3, F_4, for which the corresponding Killing tensors of valence 2 have a nontrivial Haantjes torsion. Among them, only the $m+n-1=5$ integrals of motion are functionally independent.

As before, we can verify that the fourth-degree polynomial in momenta $G=\operatorname{tr}L^4(\lambda=0)$, Eq. (1.7), cannot be expressed in terms of these second-degree polynomials in momenta.

3. Symmetric spaces of type C.I

The group $Sp(n)$ is associated with the root space C_n, and its matrix representation can be realized using symplectic and unitary $2n\times 2n$ matrices. Because

$$\frac{Sp(n)}{U(n)}\subset\frac{SU(2n)}{S(U(n)\times U(n))},$$

we can obtain the required Lax matrices by means of Lax matrices (2.1) that have already been used.

Roughly speaking, by imposing conditions on the Cartesian coordinates in definition (2.1) with $m=n$, we can make the $n\times n$ matrices Q and P in (2.2) symmetric, and then divide the nondiagonal elements of P by 2 and impose the appropriate restrictions on the parameters a_i and b_i. Following [28], we then obtain a nonstandard constant metric in Euclidean space.

Below, we write these Lax matrices for $n=2$ and $n=3$ and discuss the appropriate quadratic integrals of motion.

3.1. Example: $sp(n)$ with $n = 2$.

In this case, the 4×4 Lax matrix is given by

$$L(\lambda) = \begin{pmatrix} -2\lambda^2+q_1^2+q_2^2+a_1 & q_1q_2+q_2q_3 & p_1-2\mathrm{i}\lambda q_1 & \frac{p_2}{2}-2\mathrm{i}\lambda q_2 \\ q_1q_2+q_2q_3 & -2\lambda^2+q_2^2+q_3^2+a_2 & \frac{p_2}{2}-2\mathrm{i}\lambda q_2 & p_3-2\mathrm{i}\lambda q_3 \\ p_1+2\mathrm{i}\lambda q_1 & \frac{p_2}{2}+2\mathrm{i}\lambda q_2 & 2\lambda^2-q_1^2-q_2^2+b_1 & -q_1q_2-q_2q_3 \\ \frac{p_2}{2}+2\mathrm{i}\lambda q_2 & p_3+2\mathrm{i}\lambda q_3 & -q_1q_2-q_2q_3 & 2\lambda^2-q_2^2-q_3^2+b_2 \end{pmatrix}, \tag{3.1}$$

where

$$a_i = -b_i. \tag{3.2}$$

The Hamiltonian has the form

$$H = T + V = \frac{p_1^2}{2} + \frac{p_2^2}{4} + \frac{p_3^2}{2} + \frac{(q_1^2 + 2q_2^2 + q_3^2)^2}{2} - (q_1q_3 - q_2^2)^2 - b_1(q_1^2 + q_2^2) - b_2(q_2^2 + q_3^2). \tag{3.3}$$

It can be easily verified that this Hamiltonian corresponds to case (13c) in [28].

After the canonical change of variables $p_2 \to \sqrt{2}p_2$, $q_2 \to q_2/\sqrt{2}$, we obtain the standard metric $\mathrm{g} = \mathrm{diag}(1, 1, 1)$ in Euclidean space and an integrable fourth-degree potential

$$V = \frac{1}{2}(q_1^2 + q_2^2 + q_3^2)^2 - \frac{(2q_1q_3 - q_2^2)^2}{4}, \tag{3.4}$$

where we set $b_i = a_i = 0$ for brevity. This potential is absent in the classification of integrable fourth-degree potentials in [43] based on the Singlin–Joshid method, because the authors of that paper restricted themselves to considering a particular ansatz for potentials in the form

$$\widetilde{V} = q_1^4 + aq_1^2q_2^2 + bq_1^2q_3^2 + cq_2^4 + dq_2^2q_3^2 + eq_3^4, \qquad a, b, c, d, e \in \mathbb{R},$$

while potential (3.4) includes the term $q_1q_3q_2^2$ that is linear in q_1 and q_3.

Basis in the space of integrals of motion. The residues of the functions

$$\Delta(z, \lambda) = \frac{\det(Iz - L(\lambda))}{(z + 2\lambda^2 - a_1)(z + 2\lambda^2 - a_2)}, \qquad \Delta(z, \lambda) = \frac{\det(Iz - L(\lambda))}{(z - 2\lambda^2 - b_1)(z - 2\lambda^2 - b_2)}$$

coincide with each other up to a sign and the substitution $a_1 - a_2 = -(b_1 - b_2)$, which correspond to condition (3.2) imposed on the parameters.

Calculating these residues

$$\mathrm{Res}\big|_{z=b_i+2\lambda^2}\Delta(z, \lambda) = -4\lambda^2F_i + G_i, \qquad i = 1, 2,$$
$$\mathrm{Res}\big|_{z=\infty}\Delta(z, \lambda) = 8\lambda^2H - (G_1 + G_2),$$

we find the conditions for the coefficients

$$F_1 + F_2 - 2H = 0, \qquad G_1 + G_2 + 2(b_1 + b_2)H = 0.$$

Thus, there are two second-degree polynomials in momenta $F_{1,2}$ and one forth-degree polynomial in momenta $G_{1,2}$ or G, Eq. (1.7), which are independent of each other and therefore form a basis in the space of integrals of motion.

In [28], the authors claim that because the three integrals of motion F_1, F_2, and $G_1 + G_2$ are quadratic polynomials in momenta, there is a point (i.e., coordinate) canonical transformation that allows separating variables in the Hamilton–Jacobi equation. Obviously, the authors just did not recognize that these quadratic integrals of motion are functionally dependent, and therefore their statement on the separation of variables is incorrect.

We write the quadratic integrals of motion as

$$F_1 = p_1^2 + \frac{p_2^2}{4} + \frac{M_{12}^2}{b_1 - b_2} + (q_1^2 + 2q_2^2 + a_1 - b_1)q_1^2 + (q_1^2 + q_2^2 + q_3^2 + 2q_1q_3 - b_1 - b_2)q_2^2,$$
$$F_2 = p_3^2 + \frac{p_2^2}{4} + \frac{M_{12}^2}{b_2 - b_1} + (2q_2^2 + q_3^2 + a_2 - b_2)q_3^2 + (q_1^2 + q_2^2 + q_3^2 + 2q_1q_3 - b_2 - b_1)q_2^2,$$

where

$$M_{12} = \frac{1}{2}(q_1p_2 - 2q_2p_1 - q_3p_2 + 2q_2p_3).$$

For $b_1 = b_2$, we have a linear integral of motion M_{12} associated with a double rotation in $\mathbb{R}^3$. After reduction with respect to the corresponding symmetry, we obtain a quadratic–linear Hamiltonian H commuting with the fourth-degree integral G in (1.7) and describing integrable motion on the plane $\mathbb{R}^2$ in presence of a magnetic field.

Proposition 5. *The general solution K, Eq.* (1.10), *of Killing equation* (1.6) *depends on* 20 *parameters in the case of the three-dimensional Euclidean space $\mathbb{R}^3$. Using modern software, we can directly prove that there are only two independent solutions of Eq.* (1.11) *for the potential V in* (1.12) *entering Hamiltonian* (3.3), *and they are associated with the integrals of motion $F_{1,2}$.*

Moreover, substituting the Killing tensors corresponding to the integrals of motion $F_{1,2}$ and the unknown function $V(q_1, q_2, q_3)$ in Eq. (1.11), we obtain an integrable potential that is more general than the potential in (3.3). In fact, we consider the quadratic integrals of motion F_1, F_2, and $2H = F_1 + F_2$,

$$H = p_1^2 + p_2^2 + p_3^2 + V(q),$$

and

$$F_1 = 2p_1^2 + p_2^2 + \frac{(q_1p_2 - p_1q_2 + q_2p_3 - p_2q_3)^2}{b_1 - b_2} + U_1(q),$$
$$F_2 = 2p_3^2 + p_2^2 + \frac{(q_1p_2 - p_1q_2 + q_2p_3 - p_2q_3)^2}{b_2 - b_1} + U_2(q).$$

Proposition 6. *The general solution V_g of Eq.* (1.11) *in this case is given by*

$$\begin{aligned} V_g = c_1\bigg(q_1^4 + 2q_1^2q_2^2 + 2q_1q_2^2q_3 + \frac{q_2^4}{2} + 2q_2^2q_3^2 + q_3^4 - 2(q_1^2 + q_2^2)(b_1 - b_2)\bigg) + \\ + c_2(2q_1^3 + 3q_2^2(q_1 + q_3) + 2q_3^3 - 2(b_1 - b_2)q_1) + c_3(q_1^2 + q_2^2 + q_3^2) + \\ + c_4(q_1 + q_3) + \frac{c_5}{q_2^2}, \qquad c_i \in \mathbb{R}, \end{aligned} \tag{3.5}$$

or

$$\begin{aligned} V_g = \sum_{\alpha,\beta,\gamma,\delta} \mathcal{R}_{-\alpha,\beta,\gamma,-\delta}(c_1q^\alpha q^\beta q^\gamma q^\delta + c_2\zeta^\alpha q^\beta q^\gamma q^\delta + c_3\zeta^\alpha\zeta^\beta q^\gamma q^\delta + c_4\zeta^\alpha\zeta^\beta\zeta^\gamma q^\delta) - \\ - 2c_1(q_1^2 + q_2^2)(b_1 - b_2) - 2c_2(b_1 - b_2)q_1 + \frac{c_5}{q_2^2}, \end{aligned}$$

where $\zeta = (1, 0, 1)$ is a constant vector.

We do not write the corresponding integral of motion (1.7).

Setting $c_1 = 0$ in all integrals of motion in what follows, we obtain a three-dimensional integrable analogue of the Hénon–Heiles system with a fourth-degree integral in momenta. According to [39], this integrable system with a cubic potential is associated with the third stationary flow of the vector Korteweg–de Vries equation.

3.2. Example: $sp(n)$ with $n=3$.

Lax matrix (2.16) is given by

$$L(\lambda)=\begin{pmatrix}\widehat{L}_{11} & \widehat{L}_{12}\\ \widehat{L}_{21} & \widehat{L}_{22}\end{pmatrix},$$

where the 3×3 blocks are

$$\widehat{L}_{11}=\begin{pmatrix}-2\lambda^2+q_1^2+q_2^2+q_3^2+a_1 & q_1q_2+q_2q_4+q_3q_5 & q_1q_3+q_2q_5+q_3q_6\\ q_1q_2+q_2q_4+q_3q_5 & -2\lambda^2+q_2^2+q_4^2+q_5^2+b_1-b_2+a_1 & q_2q_3+q_4q_5+q_5q_6\\ q_1q_3+q_2q_5+q_3q_6 & q_2q_3+q_4q_5+q_5q_6 & -2\lambda^2+q_3^2+q_5^2+q_6^2+b_1-b_3+a_1\end{pmatrix},$$

$$\widehat{L}_{22}=\begin{pmatrix}2\lambda^2-q_1^2-q_2^2-q_3^2+b_1 & -q_1q_2-q_2q_4-q_3q_5 & -q_1q_3-q_2q_5-q_3q_6\\ -q_1q_2-q_2q_4-q_3q_5 & 2\lambda^2-q_2^2-q_4^2-q_5^2+b_2 & -q_2q_3-q_4q_5-q_5q_6\\ -q_1q_3-q_2q_5-q_3q_6 & -q_2q_3-q_4q_5-q_5q_6 & 2\lambda^2-q_3^2-q_5^2-q_6^2+b_3\end{pmatrix},$$

$$\widehat{L}_{12}=\begin{pmatrix}p_1-2\mathrm{i}\lambda q_1 & \frac{p_2}{2}-2\mathrm{i}\lambda q_2 & \frac{p_3}{2}-2\mathrm{i}\lambda q_3\\ \frac{p_2}{2}-2\mathrm{i}\lambda q_2 & p_4-2\mathrm{i}\lambda q_4 & \frac{p_5}{2}-2\mathrm{i}\lambda q_5\\ \frac{p_3}{2}-2\mathrm{i}\lambda q_3 & \frac{p_5}{2}-2\mathrm{i}\lambda q_5 & p_6-2\mathrm{i}\lambda q_6\end{pmatrix},\qquad \widehat{L}_{21}=\begin{pmatrix}p_1+2\mathrm{i}\lambda q_1 & \frac{p_2}{2}+2\mathrm{i}\lambda q_2 & \frac{p_3}{2}+2\mathrm{i}\lambda q_3\\ \frac{p_2}{2}+2\mathrm{i}\lambda q_2 & p_4+2\mathrm{i}\lambda q_4 & \frac{p_5}{2}+2\mathrm{i}\lambda q_5\\ \frac{p_3}{2}+2\mathrm{i}\lambda q_3 & \frac{p_5}{2}+2\mathrm{i}\lambda q_5 & p_6+2\mathrm{i}\lambda q_6\end{pmatrix}.$$

In this case, we have to impose the following restrictions on the parameters in the original Lax matrix (2.16), which have been arbitrary up to now:

$$a_i=-b_i.$$

As before, we define a basis in the space of integrals of motion using residues of the function

$$\Delta=\frac{\det(Iz-L(\lambda))}{(z-2\lambda^2-b_1)(z-2\lambda^2-b_2)(z-2\lambda^2-b_3)}$$

with respect to the variable z at the finite points $z=b_i+2\lambda^2$

$$\mathrm{Res}\big|_{z=b_i+2\lambda^2}\Delta(z,\lambda)=-16\lambda^4F_i+\lambda^2G_i+S_i.$$

We restrict ourself to the consideration of quadratic integrals of motion

$$\begin{aligned}F_1&=\frac{M_{12}^2}{b_1-b_2}+\frac{M_{13}^2}{b_1-b_3}+T_1+V_1, &\qquad T_1&=p_1^2+\frac{p_2^2}{4}+\frac{p_3^2}{4},\\ F_2&=\frac{M_{21}^2}{b_2-b_1}+\frac{M_{23}^2}{b_2-b_3}+T_2+V_2, &\qquad T_2&=\frac{p_2^2}{4}+p_4^2+\frac{p_5^2}{4},\\ F_3&=\frac{M_{31}^2}{b_3-b_1}+\frac{M_{32}^2}{b_3-b_2}+T_3+V_3, &\qquad T_3&=\frac{p_3^2}{4}+\frac{p_5^2}{4}+p_6^2,\end{aligned}$$

whose definition involves functions associated with triple rotations in configuration space, which after a suitable canonical transformation reduces the metric to the standard unit metric:

$$\begin{aligned}M_{12}&=-M_{21}=\frac{1}{2}(q_1p_2-2p_1q_2+2q_2p_4-p_2q_4+q_3p_5-p_3q_5),\\ M_{13}&=-M_{31}=\frac{1}{2}(q_1p_3-2p_1q_3+q_2p_5-p_2q_5+2q_3p_6-p_3q_6),\\ M_{23}&=-M_{32}=\frac{1}{2}(q_2p_3-p_2q_3+q_4p_5-2p_4q_5+2q_5p_6-p_5q_6).\end{aligned}$$

For brevity, we omit explicit expressions for the potentials V_k.

The residue at infinity generates a relation among the quadratic integrals,

$$F_1+F_2+F_3-2H=0$$

and relations among other integrals of motion:

$$\frac{1}{4}(G_1+G_2+G_3)+b_1F_1+b_2F_2+b_3F_3+2(b_1+b_2+b_3)H=0$$

and

$$S_1+S_2+S_3-\frac{1}{4}(b_1G_1+b_2G_2+b_3G_3)-(b_1^2-b_2b_3)F_1-(b_2^2-b_1b_3)F_2-(b_3^2-b_1b_2)F_3=0.$$

Of the nine dependent integrals of motion F_i, G_i and S_i constructed in this way, we have to select six independent integrals. For instance, we can take three quadratic integrals of motion, two fourth-degree integrals of motion, and one sixth-degree integral in momenta.

4. Symmetric spaces of type D.III

Because

$$\frac{SO(2n)}{U(n)}\subset\frac{SU(2n)}{S(U(n)\times U(n))},$$

we can use reduction for constructing integrable systems corresponding to the symmetric spaces associated with the D_n root space.

In fact, we take Lax matrix (2.1) with $m=n$ and, imposing conditions on the Cartesian coordinates and the corresponding momenta, we make the $n\times n$ matrices Q and P in (2.2) antisymmetric, and then impose the appropriate restrictions on the parameters a_i and b_i in (2.1).

4.1. Example: $so(2n)$ with $n=2$. After the reduction, the 8×8 Lax matrix (2.1) remains a block matrix

$$L(\lambda)=\begin{pmatrix}\bar L_{11} & \bar L_{12}\\ \bar L_{21} & \bar L_{22}\end{pmatrix},$$

where the two diagonal blocks are the symmetric matrices

$$\bar L_{11}=\begin{pmatrix} q_1^2+q_2^2+q_3^2+a_1-2\lambda^2 & q_2q_4+q_3q_5 & -q_1q_4+q_3q_6 & -q_1q_5-q_2q_6\\ q_2q_4+q_3q_5 & q_1^2+q_4^2+q_5^2+a_2-2\lambda^2 & q_1q_2+q_5q_6 & q_1q_3-q_4q_6\\ -q_1q_4+q_3q_6 & q_1q_2+q_5q_6 & q_2^2+q_4^2+q_6^2+a_3-2\lambda^2 & q_2q_3+q_4q_5\\ -q_1q_5-q_2q_6 & q_1q_3-q_4q_6 & q_2q_3+q_4q_5 & q_3^2+q_5^2+q_6^2+a_4-2\lambda^2\end{pmatrix},$$

$$\bar L_{22}=\begin{pmatrix} 2\lambda^2-q_1^2-q_2^2-q_3^2+b_1 & -q_2q_4-q_3q_5 & q_1q_4-q_3q_6 & q_1q_5+q_2q_6\\ -q_2q_4-q_3q_5 & 2\lambda^2-q_1^2-q_4^2-q_5^2+b_2 & -q_1q_2-q_5q_6 & -q_1q_3+q_4q_6\\ q_1q_4-q_3q_6 & -q_1q_2-q_5q_6 & 2\lambda^2-q_2^2-q_4^2-q_6^2+b_3 & -q_2q_3-q_4q_5\\ q_1q_5+q_2q_6 & -q_1q_3+q_4q_6 & -q_2q_3-q_4q_5 & 2\lambda^2-q_3^2-q_5^2-q_6^2+b_4\end{pmatrix},$$

and the off-diagonal blocks are antisymmetric matrices of the form

$$\bar L_{12}=\begin{pmatrix} 0 & p_1-2i\lambda q_1 & p_2-2i\lambda q_2 & p_3-2i\lambda q_3\\ -p_1+2i\lambda q_1 & 0 & p_4-2i\lambda q_4 & p_5-2i\lambda q_5\\ -p_2+2i\lambda q_2 & -p_4+2i\lambda q_4 & 0 & p_6-2i\lambda q_6\\ -p_3+2i\lambda q_3 & -p_5+2i\lambda q_5 & -p_6+2i\lambda q_6 & 0\end{pmatrix},$$

$$\bar L_{21}=\begin{pmatrix} 0 & -p_1-2i\lambda q_1 & -p_2-2i\lambda q_2 & -p_3-2i\lambda q_3\\ p_1+2i\lambda q_1 & 0 & -p_4-2i\lambda q_4 & -p_5-2i\lambda q_5\\ p_2+2i\lambda q_2 & p_4+2i\lambda q_4 & 0 & -p_6-2i\lambda q_6\\ p_3+2i\lambda q_3 & p_5+2i\lambda q_5 & p_6+2i\lambda q_6 & 0\end{pmatrix}.$$

In this case, the parameters (arbitrary up to now) have to satisfy the restrictions

$$a_2-a_1=b_1-b_2,\qquad a_3-a_1=b_1-b_3,\qquad a_4-a_1=b_1-b_4.$$

Four residues of the function

$$\Delta=\frac{\det(Iz-L(\lambda))}{(z-2\lambda^2-b_1)(z-2\lambda^2-b_2)(z-2\lambda^2-b_3)(z-2\lambda^2-b_4)}$$

with respect to the variable z at the points $z = b_i + 2\lambda^2$ are the sixth-degree polynomials in λ:

$$\operatorname{Res} t\big|_{z=b_i+2\lambda^2} \Delta(z,\lambda) = -64\lambda^6 F_i + \lambda^4 G_i + \lambda^2 S_i + W_i.$$

The coefficients F_i, G_i, S_i, and W_i are second-, fourth-, sixth-, and eighth-degree polynomials in momenta. As a result, we have 16 dependent integrals of motion and the residue at infinity yields various relations among these polynomials, for instance,

$$F_1 + F_2 + F_3 + F_4 - 2H = 0.$$

We present only the highest-order part of the quadratic integrals of motion and omit explicit expressions for the appropriate potentials V_k:

$$\begin{aligned}
F_1 &= \frac{M_{12}^2}{b_1 - b_2} + \frac{M_{13}^2}{b_1 - b_2} + \frac{M_{14}^2}{b_1 - b_4} + T_1 + V_1,\\
F_2 &= \frac{M_{21}^2}{b_2 - b_1} + \frac{M_{23}^2}{b_2 - b_3} + \frac{M_{24}^2}{b_2 - b_4} + T_2 + V_2,\\
F_3 &= \frac{M_{31}^2}{b_3 - b_1} + \frac{M_{32}^2}{b_3 - b_2} + \frac{M_{34}^2}{b_3 - b_4} + T_3 + V_3,\\
F_4 &= \frac{M_{41}^2}{b_4 - b_1} + \frac{M_{42}^2}{b_4 - b_2} + \frac{M_{43}^2}{b_4 - b_3} + T_4 + V_4.
\end{aligned}$$

The functions appearing in the definition of the integrals of motion,

$$\begin{aligned}
M_{12} &= (q_2p_4 - p_2q_4) + (q_3p_5 - p_3q_5), & M_{13} &= (q_1p_4 - p_1q_4) + (q_6p_3 - p_6q_3),\\
M_{14} &= (q_1p_5 - p_1q_5) + (q_2p_6 - p_2q_6), & M_{23} &= (q_1p_2 - p_1q_2) + (q_5p_6 - p_5q_6),\\
M_{24} &= (q_1p_3 - p_1q_3) + (q_6p_4 - p_6q_4), & M_{34} &= (q_2p_3 - p_2q_3) + (q_4p_5 - p_4q_5)
\end{aligned}$$

are associated with double rotations of the configuration space $\mathbb{R}^6$, while the functions

$$\begin{aligned}
T_1 &= p_1^2 + p_2^2 + p_3^2, & T_2 &= p_1^2 + p_4^2 + p_5^2,\\
T_3 &= p_2^2 + p_4^2 + p_6^2, & T_4 &= p_3^2 + p_5^2 + p_6^2
\end{aligned}$$

are determined by the sequences of three shifts along the coordinate axes. Direct calculation shows that the Haantjes torsion of the corresponding Killing tensors is not zero.

Of the sixteen dependent integrals of motion F_i, G_i, S_i, and W_i, we have to select six independent integrals of motion, four of which can be quadratic polynomials in momenta.

5. Symmetric spaces of type BD.I

The symmetric space

$$\frac{SO(m+n)}{SO(m) \times SO(n)}$$

is Hermitian only if $m = 2$, because $so(m) + so(n)$ has no center in the general case. At $m = 2$, the $so(2)$ subalgebra is the center, and, depending on whether n is even or odd, this symmetric space is associated with the $B_{(n+1)/2}$ or $D_{(n+2)/2}$ root system.

5.1. Example: $SO(2) \times SO(4) \simeq S(U(1) \times U(3))$. Using the Weyl–Cartan basis from [26], we construct 6×6 Lax matrix (1.3) in the form

$$L(\lambda) = \begin{pmatrix} \widetilde{L}_{11} & \widetilde{L}_{12} \\ \widetilde{L}_{21} & \widetilde{L}_{22} \end{pmatrix},$$

where

$$\widetilde{L}_{11} = \begin{pmatrix} -2\lambda^2+2q_1^2+2q_2^2+2q_3^2+2q_4^2+a_1 & p_1-2\mathrm{i}\lambda q_1 & p_2-2\mathrm{i}\lambda q_2 \\ p_1+2\mathrm{i}\lambda q_1 & -2q_1^2+2q_3^2+a_2 & -2q_1q_2+2q_3q_4 \\ p_2+2\mathrm{i}\lambda q_2 & -2q_1q_2+2q_3q_4 & -2q_2^2+2q_4^2+a_3 \end{pmatrix},$$

$$\widetilde{L}_{22} = \begin{pmatrix} 2\lambda^2-2q_1^2-2q_2^2-2q_3^2-2q_4^2+b_1 & -p_1-2\mathrm{i}\lambda q_1 & -p_2-2\mathrm{i}\lambda q_2 \\ -p_1+2\mathrm{i}\lambda q_1 & 2q_1^2-2q_3^2+b_2 & 2q_1q_2-2q_3q_4 \\ -p_2+2\mathrm{i}\lambda q_2 & 2q_1q_2-2q_3q_4 & 2q_2^2-2q_4^2+b_3 \end{pmatrix},$$

$$\widetilde{L}_{12} = \begin{pmatrix} 0 & p_3-2\mathrm{i}\lambda q_3 & p_4-2\mathrm{i}\lambda q_4 \\ -p_3+2\mathrm{i}\lambda q_3 & 0 & -2q_1q_4+2q_2q_3 \\ -p_4+2\mathrm{i}\lambda q_4 & 2q_1q_4-2q_2q_3 & 0 \end{pmatrix},$$

$$\widetilde{L}_{21} = \begin{pmatrix} 0 & -p_3-2\mathrm{i}\lambda q_3 & -p_4-2\mathrm{i}\lambda q_4 \\ p_3+2\mathrm{i}\lambda q_3 & 0 & 2q_1q_4-2q_2q_3 \\ p_4+2\mathrm{i}\lambda q_4 & -2q_1q_4+2q_2q_3 & 0 \end{pmatrix}.$$

The parameters involved in the Lax matrix satisfy the relations

$$a_2 = a_1 + b_1 - b_2, \qquad a_3 = a_1 + b_1 - b_3.$$

In this case, the Hamiltonian H in (2.3) has the form

$$H = p_1^2 + p_2^2 + p_3^2 + p_4^2 + 4(q_1^2 + q_2^2 + q_3^2 + q_4^2)^2 - 8(q_1q_3 + q_2q_4)^2 + \\ + 2(b_2 - b_1)q_1^2 + 2(b_3 - b_1)q_2^2 + 2(a_1 - b_2)q_3^2 + 2(a_1 - b_3)q_4^2.$$

This Hamiltonian coincides with Hamiltonian (2.9) up to a scaling transformation and the canonical transformation $q_i \to -q_i$ and $p_i \to -p_i$ of one of the coordinates and momenta (see a discussion of the isomorphism of root systems and the corresponding integrable systems in [26]).

5.2. Example: $so(2n+1)$ with $n = 2$. We consider the realization of elements of the Lie algebra $so(2n+1)$ by $(2n+1) \times (2n+1)$ matrices X satisfying the relation

$$X + SX^{\mathrm{T}}S^{-1} = 0, \qquad S = \sum_{k=1}^{2n+1} (-1)^{k+1} E_{k,2n+2-k},$$

where E_{ij} are matrices with the only nonzero element at the intersection of the ith row and jth column [34].

In this case, the Cartan evolution is associated with the element $\mathcal{A} = E_{1,1} - E_{2n+1,2n+1}$ of the Cartan subalgebra, while Lax matrix (1.3) has the block structure

$$L(\lambda) = \begin{pmatrix} 2\lambda^2 & \vec{x}^{\mathrm{T}} & 0 \\ \vec{y} & 0 & s \cdot \vec{x} \\ 0 & \vec{y}^{\mathrm{T}} \cdot s & -2\lambda^2 \end{pmatrix} + C + \Lambda,$$

where the central zero block in the first term has the size $(2n-1) \times (2n-1)$, the elements of the vector columns x and y are

$$\vec{x}_i = p_i - 2\mathrm{i}q_i, \qquad \vec{y}_i = p_i + 2\mathrm{i}q_i, \qquad i = 1, \ldots, 2n-1,$$

and s is the $(2n-1)\times(2n-1)$ matrix

$$s=\sum_{k=1}^{2n-1}(-1)^k E_{k,2n-k}.$$

The matrix C corresponds to the term $\sum_{\alpha,\beta} q_\alpha q_\beta [e_\alpha, e_{-\beta}]$ in definition (1.3), and Λ is a numerical matrix satisfying the relation

$$\Lambda + S\Lambda^{\mathrm{T}} S^{-1}=0$$

and defining the shift of the orbit as follows from [29].

The first nontrivial example appears at $n=2$ [26]. For the corresponding symmetric space, we write an explicit expression for the 5×5 Lax matrix (1.3):

$$L(\lambda)=\begin{pmatrix} 2\lambda^2 & p_1-2i\lambda q_1 & p_2-2i\lambda q_2 & p_3-2i\lambda q_3 & 0 \\ p_1+2i\lambda q_1 & 0 & 0 & 0 & -p_3+2i\lambda q_3 \\ p_2+2i\lambda q_2 & 0 & 0 & 0 & p_2-2i\lambda q_2 \\ p_3+2i\lambda q_3 & 0 & 0 & 0 & -p_1+2i\lambda q_1 \\ 0 & -p_3-2i\lambda q_3 & p_2+2i\lambda q_2 & -p_1-2i\lambda q_1 & -2\lambda^2 \end{pmatrix} + C + \Lambda,$$

where

$$C=2\begin{pmatrix} -q_1^2-q_2^2-q_3^2 & 0 & 0 & 0 & 0 \\ 0 & q_1^2-q_3^2 & (q_1+q_3)q_2 & 0 & 0 \\ 0 & (q_1+q_3)q_2 & 0 & (q_1+q_3)q_2 & 0 \\ 0 & 0 & (q_1+q_3)q_2 & -q_1^2+q_3^2 & 0 \\ 0 & 0 & 0 & 0 & q_1^2+q_2^2+q_3^2 \end{pmatrix}, \quad \Lambda = 2\begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & a_3 & 0 & 0 \\ 0 & a_3 & 0 & a_3 & 0 \\ 0 & 0 & a_3 & -a_2 & 0 \\ 0 & 0 & 0 & 0 & -a_1 \end{pmatrix}.$$

This example is interesting because the matrix Λ in this representation is not necessarily diagonal.

Hamiltonian (1.4) is given by

$$H=\frac{1}{4}\operatorname{tr} L^2\big|_{\lambda=0} -2a_1^2-2a_2^2-4a_3^2 = p_1^2+p_2^2+p_3^2+4(q_1^2+q_2^2+q_3^2)^2 - \\ -2(2q_1q_3-q_2^2)^2-4(a_1-a_2)q_1^2-4(a_1+a_2)q_3^2-4q_2(a_1q_2-2a_3(q_1+q_3)).$$

The quadratic integral of motion is

$$F=(q_1p_2-p_1q_2+q_2p_3-q_3p_2)^2-(p_1+p_3)(a_2(p_1-p_3)+2a_3p_2)+U,$$

where

$$U=4(q_1+q_3)(a_2(q_1-q_3)+2a_3q_2)(a_1-q_1^2-q_2^2-q_3^2)-4(a_2^2+a_3^2)(q_1^2+q_3^2)- \\ -8q_2(q_1-q_3)a_2a_3-8(q_1q_3+q_2^2)a_3^2,$$

determines the second-order Killing tensor with a nonzero Haantjes torsion.

The characteristic equation for this Lax matrix is

$$z^5-2(2\lambda^4+4a_1\lambda^2+2a_1^2+2a_2^2+4a_3^2+H)z^3+ \\ +[16(a_2^2+2a_3^2)\lambda^4+8(F+4a_1(a_2^2+2a_3^2))\lambda^2+G/2-H^2]z=0.$$

The highest-order term in the fourth-degree polynomial in momenta G that appears in this equation is determined by the curvature tensor $\mathcal{R}$:

$$G=-\frac{1}{4}\sum_{\alpha,\beta,\gamma,\delta}\mathcal{R}_{-\alpha,\beta,\gamma,-\delta}q^\alpha q^\beta q^\gamma q^\delta+\cdots= \\ =4(p_1^2+p_2^2+p_3^2)^2-2(2p_1p_3-p_2^2)^2+\cdots.$$

At $a_i=0$, we have the Hamiltonian in (3.3), (3.4) up to a canonical transformation,

$$H=\frac{1}{4}\operatorname{tr} L^2\big|_{\lambda=0}=p_1^2+p_2^2+p_3^2+4(q_1^2+q_2^2+q_3^2)^2-2(2q_1q_3-q_2^2)^2,$$

which follows from the isomorphism of the root systems (see [26], [34]). At $a_i\neq 0$, the terms missed in [28] appear in the potential.

5.3. Example: $so(2n+1)$ with $n=3$. In this case, 7×7 Lax matrix (1.3), where we temporarily set $\Lambda=0$, is given by

$$L(\lambda)=\begin{pmatrix} 2\lambda^2 & p_1-2\mathrm{i}\lambda q_1 & p_2-2\mathrm{i}\lambda q_2 & p_3-2\mathrm{i}\lambda q_3 & p_4-2\mathrm{i}\lambda q_4 & p_5-2\mathrm{i}\lambda q_5 & 0 \\ p_1+2\mathrm{i}\lambda q_1 & 0 & 0 & 0 & 0 & 0 & -p_5+2\mathrm{i}\lambda q_5 \\ p_2+2\mathrm{i}\lambda q_2 & 0 & 0 & 0 & 0 & 0 & p_4-2\mathrm{i}\lambda q_4 \\ p_3+2\mathrm{i}\lambda q_3 & 0 & 0 & 0 & 0 & 0 & -p_3+2\mathrm{i}\lambda q_3 \\ p_4+2\mathrm{i}\lambda q_4 & 0 & 0 & 0 & 0 & 0 & p_2-2\mathrm{i}\lambda q_2 \\ p_5+2\mathrm{i}\lambda q_5 & 0 & 0 & 0 & 0 & 0 & -p_1+2\mathrm{i}\lambda q_1 \\ 0 & -p_5-2\mathrm{i}\lambda q_5 & p_4+2\mathrm{i}\lambda q_4 & -p_3-2\mathrm{i}\lambda q_3 & p_2+2\mathrm{i}\lambda q_2 & -p_1-2\mathrm{i}\lambda q_1 & -2\lambda^2 \end{pmatrix}+$$

$$+2\begin{pmatrix} -\sum_{k=1}^5 q_k^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_1^2-q_5^2 & q_1q_2+q_4q_5 & q_3(q_1-q_5) & q_1q_4+q_2q_5 & 0 & 0 \\ 0 & q_1q_2+q_4q_5 & q_2^2-q_4^2 & q_3(q_2+q_4) & 0 & q_1q_4+q_2q_5 & 0 \\ 0 & q_3(q_1-q_5) & q_3(q_2+q_4) & 0 & q_3(q_2+q_4) & -q_3(q_1-q_5) & 0 \\ 0 & q_1q_4+q_2q_5 & 0 & q_3(q_2+q_4) & -q_2^2+q_4^2 & q_1q_2+q_4q_5 & 0 \\ 0 & 0 & q_1q_4+q_2q_5 & -q_3(q_1-q_5) & q_1q_2+q_4q_5 & -q_1^2+q_5^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sum_{k=1}^5 q_k^2 \end{pmatrix}. \tag{5.1}$$

The corresponding Hamiltonian

$$H=\frac{1}{4}\operatorname{tr}L^2\big|_{\lambda=0}=\sum_{k=1}^5 p_k^2+4\left(\sum_{k=1}^5 q_k^2\right)^2-2(2q_1q_5-2q_2q_4+q_3^2)^2 \tag{5.2}$$

commutes with the four integrals of motion

$$R_1=(q_1p_2-p_1q_2)+(q_4p_5-p_4q_5),\qquad R_2=(q_2p_3-p_2q_3)+(q_3p_4-p_3q_4),$$
$$R_3=(q_1p_3-p_1q_3)+(q_5p_3-p_5q_3),\qquad R_4=(q_1p_4-p_1q_4)+(q_2p_5-p_2q_5)$$

such that

$$\{R_1,R_2\}=-R_3,\qquad \{R_1,R_3\}=R_2,\qquad \{R_1,R_4\}=0,$$

$$\{R_4,R_2\}=R_3,\qquad \{R_4,R_3\}=-R_2,\qquad \{R_2,R_3\}=R_4-R_1.$$

As before, the existence of these integrals of motion is associated with the invariance of the Hamiltonian under rotations of the configuration space $\mathbb{R}^5$.

The characteristic equation for the Lax matrix

$$\det(z\cdot I-L(\lambda))=z^7-2(2\lambda^4+H)z^5+(8F_1\lambda^2+G_1)z^3-4G_2z=0$$

includes four independent integrals of motion in involution, H, G_1, and

$$F_1=R_1^2+R_2^2+R_3^2+R_4^2,\qquad G_2=(R_1+R_4)^2[(R_1-R_4)^2+2(R_2^2+R_3^2)].$$

Using the Hamiltonian and the fourth-degree polynomial G_1, we can obtain the integral of motion G in (1.7), whose principal part is determined by the curvature tensor $\mathcal{R}$ in (1.4), (2.9):

$$G=2G_1+2H^2=-\frac{1}{4}\sum_{\alpha,\beta,\gamma,\delta}\mathcal{R}_{-\alpha,\beta,\gamma,-\delta}p^\alpha p^\beta p^\gamma p^\delta+\cdots=$$
$$=-4(p_1^2+p_2^2+p_3^3+p_4^2+p_5^2)^2+2(2p_1p_5-2p_2p_4+p_3^2)^2+\cdots.$$

This integral of motion is independent of the Hamiltonian H and linear integrals of motion R_k.

Because $\{R_k, G_1\} = 0$, we have a completely integrable system with five independent integrals of motion in involution, for instance,

$$R_1,\, R_4,\, R_2^2 + R_3^2, H,\, G_1.$$

Nevertheless, the spectral invariants of Lax matrix (5.1) generate only four integrals of motion, similarly to the complete Toda chain [35].

Adding a constant nondiagonal matrix to $L(\lambda)$ in (5.1) ,

$$\Lambda = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & a_4 & 0 & 0 & 0 \\ 0 & 0 & a_4 & 0 & a_4 & 0 & 0 \\ 0 & 0 & 0 & a_4 & -a_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_1 \end{pmatrix},$$

we add the terms quadratic in coordinates to the original Hamiltonian (5.2):

$$\begin{aligned} H &= \frac{1}{4}\operatorname{tr} L^2\big|_{\lambda=0} - 2a_1^2 - 2a_2^2 - 2a_3^2 - 4a_4^2 = \\ &= \sum_{k=1}^{5} p_k^2 + 4\left(\sum_{k=1}^{5} q_k^2\right)^2 - 2(2q_1q_5 - 2q_2q_4 + q_3^2)^2 + (a_1 - a_2)q_1^2 + \\ &\quad + (a_1 - a_3)q_2^2 + q_3(a_1q_3 - 2a_4q_2 - 2a_4q_4) + (a_1 + a_3)q_4^2 + (a_2 + a_1)q_5^2. \end{aligned}$$

The characteristic equation for the Lax matrix

$$\begin{gathered} z^7 - 4(\lambda^4 - 2\lambda^2 a_1 + a_1^2 + a_2^2 + a_3^2 + 2a_4^2 + H/2)z^5 + (16(a_2^2 + a_3^2 + 2a_4^2)\lambda^4 + \\ + F_1\lambda^2 + G_1)z^3 - [64a_2^2(a_3^2 + 2a_4^2)\lambda^4 + F_2\lambda^2 + G_2]z = 0 \end{gathered}$$

then contains a sufficient number of commuting and independent integrals of motion to ensure the integrability by the Liouville theorem. There are three polynomials H, F_1, and F_2 of the second degree in momenta and two polynomials G_1 and G_2 of the fourth degree.

Thus, we can say that the constant term Λ in Lax matrix (1.3) allows removing degeneration in a certain sense and obtaining the complete set of integrals of motion that suffices to prove the integrability of the system by the Liouville theorem.

6. Conclusions

The problem of the existence of quadratic integrals of motion for Hamiltonians in the natural form

$$H = \sum_{i,j} \mathrm{g}^{ij} p_i p_j + V(q)$$

has been under discussion for a long time, starting with the papers by Jacobi, Levi-Civita, and Darboux, and until now. Most of the classic and contemporary papers study the problem of the existence of integrable geodesic flows at $V(q) = 0$ or the problem of the equivalent metrics first. After that, they describe the class of potentials $V(q) \neq 0$ that can be added to a given geodesic flow and preserve the integrability property.

It turns out that abandoning this common such that strategy allows constructing quadratic conservation laws for a sufficiently wide class of Hamiltonians in the natural form, describing motion in Euclidean space. Some examples were constructed in [22]–[25] by directly solving both the Killing equation (1.6) and the equation for potential (1.11).

In this paper, the quadratic conservation laws for Newton equations (1.5) are constructed using the known Lax representation [26], [28], [29]. The corresponding Killing tensors are associated with special linear combinations of basis rotations about the coordinate axes (these combinations form a representation of the rotation subalgebra) and with a sequences of shifts along these axes. For instance, to construct the integrals of motion in four-dimensional Euclidean space, we use the right and left isoclinic rotations (Clifford shifts), which are classical objects in Euclidean geometry and the theory of Clifford algebra.

Open problems include a rigorous mathematical definition of this class of Killing tensors and a construction of the corresponding integrals of motion of highest degrees in momenta in the framework of standard Euclidean, Riemannian, and pseudo-Riemannian geometries, i.e., without using the Lax matrices.

Conflicts of interest. The authors declare no conflicts of interest.

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