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ON AN ALTERNATIVE STRATIFICATION OF KNOTS

E. N. Lanina,*† A. V. Popolitov,*†‡ and N. S. Tselousov*†

We introduce an alternative stratification of knots: by the size of the lattice on which a knot can be first met. Using this classification, we find the fraction of unknots and knots with more than 10 *minimal crossings inside different lattices and answer the question of which knots can be realized inside* 3×3 *and* 5×5 *lattices. In accordance with previous research, the fraction of unknots decreases exponentially with the growth of the lattice size. Our computational results are consistent with theoretical estimates for the number of knots with a fixed crossing number inside lattices of a given size.*

Keywords: knot theory, knots classification, Jones polynomial, lattice knot

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1. Introduction

The study of knot representations is a central theme in knot theory. We are especially interested in the so-called universal representations, which contain all knots. These include the minimum braid representations, Morse link representations, arc representations, and many others. However, for example, arborescent knots are not universal, because an arbitrary knot cannot be represented in arborescent form.

Different knot representations are useful for different purposes. For example, the braid representation allows calculating quantum knot invariants using the Reshetikhin–Turaev algorithm [1]–[14]. The HOMFLY polynomials for arborescent knots are easier to calculate in terms of modular transformation matrices S and T and their conjugates [15]–[19].

In this paper, we investigate classical knot theory questions with the use of the so-called lattice representation. A lattice representation is defined on a square lattice of size $(2n+1) \times (2n+1)$. Each node of the lattice is equipped with one of two crossings, denoted as $+$ and $-$. It has been proved that each knot has at least one lattice diagram [20], and hence this representation is universal. Moreover, each lattice knot in a fixed-size lattice can also be realized inside any bigger lattice. We can therefore introduce the knot representation by lattice diagrams (also see Sec. 2).

*Moscow Institute of Physics and Technology (National Research University), Dolgoprudny, Moscow Region, Russia, e-mails: lanina.en@phystech.edu (corresponding author), tselousov.ns@phystech.edu.

†National Research Center “Kurchatov Institute,” Moscow, Russia.

‡Kharkevich Institute for Information Transmission Problems, Russian Academy of Sciences, Moscow, Russia, e-mail: popolit@gmail.com.

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The set of all distinct knots $\mathbb{K}$ is infinite and to deal with this infinity, the set is usually stratified (split) into finite parts $\mathbb{K}_n$, $n = 1, 2, \ldots$ according to the value of some simple knot invariant. Commonly used is the splitting by the crossing number, i.e., the minimal number of crossings among all the knot representations for a given knot. Knots with small crossing numbers are collected in the celebrated Rolfsen table (see, e.g., [21]).

After a stratification of the knot set $\mathbb{K}$ is chosen, one can ask how this splitting relates to some other splitting of the knot set into different classes; for instance, what is the asymptotic fraction of different knot classes (with respect to the latter splitting) in the $\mathbb{K}_n$ stratum as $n \to \infty$?

Indeed, following Thurston's famous theorem, all knots are divided into three types: torus, satellite, and hyperbolic. For knots classification by a crossing number n, it was proved that hyperbolic knots do not dominate as $n \to \infty$, and it was argued that satellite knots do dominate [22]. However, this result is rather unexpected because based on numerical evidence for small n it had long been believed that hyperbolic knots dominate for infinitely large crossing numbers due to a well-known conjecture (see [23], p.119). It was also obtained that the number of prime knots grows exponentially with n [24]. Given an alternative stratification of knots, one can ask whether these counterintuitive statements are stable/sensitive with respect to changing the stratification. In particular, which type of knots dominates if we split knots by the minimal size of their respective lattice diagram? What is the distribution of knots coming from a fixed-size lattice? In this short note, we give some theoretical bounds on the number of different types of knots in a fixed-size lattice (see Sec. 4) and present some numerical computation results to answer these questions (see Sec. 5).

The results in this paper are as follows. First, we derive the upper and lower bounds for the number of knots with fixed crossing numbers that can be realized inside a fixed-size lattice (Sec. 4). Second, we confirm our bounds by numerical computations of knots that can be obtained from 3×3 and 5×5 lattice diagrams (Sec. 5.1). In addition, we calculate the number of unknots and knots with crossing numbers greater than 10 inside bigger lattices (Sec. 5.2).

Lattice diagrams are also interesting because they connect knot theory and statistical models. The relation between knot theory and statistical mechanics was first noted by Jones [25]. In that paper, a connection between the Jones polynomials and the Potts model was established. Later, Jones developed a method to compute the HOMFLY polynomials using vertex models. This method was also used by Turaev for the Kauffman polynomials [26]. The connection between knot theory and statistical mechanics was formalized and further extended by Jones [27] with the use of spin models. The connection between exactly solvable statistical models and knot theory is interesting because it can lead to progress both in proving mathematical theorems in knot theory (for example, regarding the dominance of satellite knots [22]) and in the search for new knot invariants and simpler methods for computing well-known invariants. We hope to move forward in this direction using the well developed theory of integrable models of statistical physics [28], [29].

2. Basic theorems

In this section, we introduce some basic facts about lattice diagrams to be used in what follows. Detailed explanations can be found in recent paper [20].

Definition 1. A lattice diagram L_n is a closed immersed curve that has $2n+1$ horizontal and $2n+1$ vertical line segments and $(2n+1)^2$ crossing points.

In [20], lattice diagrams are called potholder curves or potholders. Knots carried by potholder curves were studied by Grosberg and Nechaev [30], [31], who calculated the number of unknots carried by such curves via a connection to the Potts model of statistical mechanics. A lattice knot is obtained from a lattice diagram by resolving each crossing to one of the two types, here denoted by $+$ and $-$. Examples of lattice knots coming from L_1, L_2 and L_3 are shown in Fig. 1.

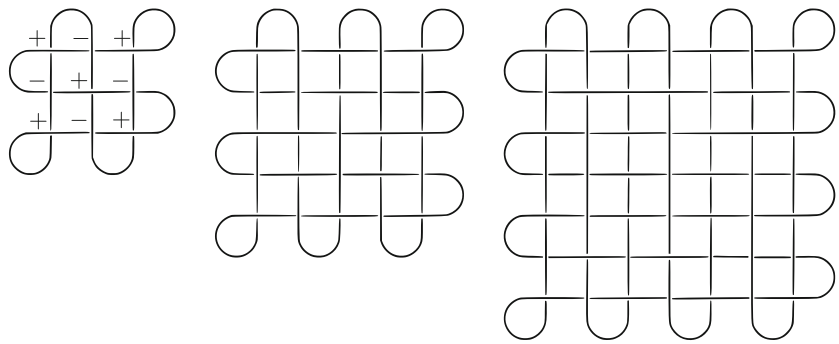


Fig. 1. Examples of lattice knots: 3×3, 5×5, and 7×7 lattice knots.

Theorem 1. *Every knot is carried by lattice knots.*

Theorem 2. *All lattice knots coming from L_n also come from L_{n+1}.*

From these two theorems, we conclude that there is another classification of knots. Namely, we can sort knots by the minimal size of lattice diagrams from which they are produced.

3. How can knots be distinguished?

The main question that we address here is which types of knots come from the lattice diagram L_n? To answer it, we discuss how lattice knots can be distinguished.

3.1. Reidemeister moves. The simplest but laborious way is to use the Reidemeister theorem.

Theorem 3. *Two knots $\mathcal{K}_1$ and $\mathcal{K}_2$ are equivalent if there exists a sequence of Reidemeister moves* (*Fig.* 2) *that transforms one projection into the other.*

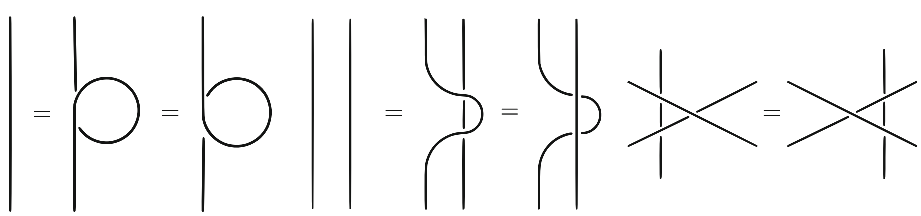


Fig. 2. Reidemeister moves.

Given a lattice knot, this theorem allows obtaining the same knots but inside bigger lattices. We use this approach to present theoretical bounds (Sec. 4).

3.2. The Jones polynomial. Knots can be distinguished in practice with the help of peculiar special functions, *knot invariants*.

Definition 2. A knot invariant $I(\mathcal{K})$ is a function of a knot $\mathcal{K}$ that takes the same value on equivalent knots:

$$\mathcal{K}_1 = \mathcal{K}_2 \quad \Longrightarrow \quad I(\mathcal{K}_1) = I(\mathcal{K}_2). \tag{3.1}$$

In this paper, we use the following property of knot invariants:

$$\mathcal{K}_1 \neq \mathcal{K}_2 \quad \Longrightarrow \quad I(\mathcal{K}_1) \neq I(\mathcal{K}_2). \tag{3.2}$$

Many known knot invariants of different types are known in the literature; we are interested in polynomial ones because they can be effectively computed with the help of special computer programs. One of the most famous polynomial knot invariants is the Jones polynomial $J^{\mathcal{K}}$,

$$J^{\mathcal{K}}\colon \mathcal{K} \to \mathbb{Z}[q, q^{-1}]. \tag{3.3}$$

Given a knot $\mathcal{K}$, it returns a Laurent polynomial $J^{\mathcal{K}}(q)$. We give examples of Jones polynomials for several knots with small crossing numbers:

$$\begin{aligned} J^{0_1} &= 1, \\ J^{3_1} &= q^2 + q^6 - q^8, \\ J^{4_1} &= 1 - q^2 - q^{-2} + q^4 + q^{-4}. \end{aligned} \tag{3.4}$$

We consider the *reduced* or *normalized* Jones polynomial whose value for the unknot is 1 rather than $q + q^{-1}$. The Jones polynomial is a very good tool to distinguish knots with small crossing numbers. It distinguishes almost all knots in the Rolfsen table through 9 crossings. The first example of knots that cannot be distinguished by the Jones polynomial is

$$J^{5_1}(q) = J^{10_{132}}(q). \tag{3.5}$$

The Jones polynomial has a number of particular properties.

- at the point $q = 1$, the Jones polynomial of any knot reduces to unity:

$$J^{\mathcal{K}}(1) = 1; \tag{3.6}$$

- the Jones polynomial of the mirror image of a knot $\mathcal{K}^{\mathrm{mir}}$ is expressed by the following formula in terms of the Jones polynomial of the original knot $\mathcal{K}$:

$$J^{\mathcal{K}^{\mathrm{mir}}}(q) = J^{\mathcal{K}}(q^{-1}); \tag{3.7}$$

- the Jones polynomial of a composite knot $\mathcal{K}_1 \# \mathcal{K}_2$ is factored:

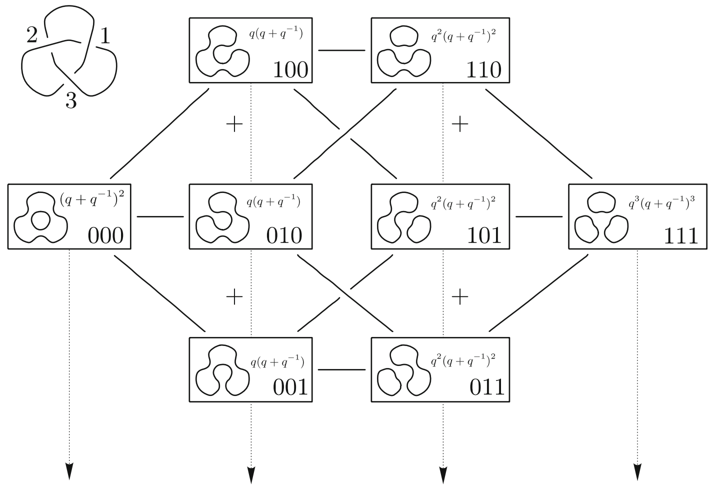
$$J^{\mathcal{K}_1 \# \mathcal{K}_2}(q) = J^{\mathcal{K}_1}(q) \cdot J^{\mathcal{K}_2}(q); \tag{3.8}$$

- the Jones polynomial of a disjoint union of two knots $\mathcal{K}_1 \bigsqcup \mathcal{K}_2$ is

$$J^{\mathcal{K}_1 \bigsqcup \mathcal{K}_2}(q) = (q + q^{-1}) \cdot J^{\mathcal{K}_1}(q) \cdot J^{\mathcal{K}_2}(q). \tag{3.9}$$

We use properties (3.8) and (3.7) in our computations.

The Jones polynomial has several equivalent definitions and approaches to its computation. In our computations, we use the *state sum formula* or the *Khovanov algorithm* [32], [33], which is related to the rapidly growing field of the knot homology theory. We briefly describe this algorithm to demonstrate its usefulness for distinguishing lattice knots.



$$(q+q^{-1})^2 \quad - \quad 3q(q+q^{-1}) \quad + \quad 3q^2(q+q^{-1})^2 \quad - \quad q^3(q+q^{-1})^3 =$$

$$= q^{-2}+1+q^2-q^6 \xrightarrow[((n_+,n_-)=(3,0))]{(-1)^n - q^{n_+}-2n_-} q+q^3+q^5-q^9 \xrightarrow{(q+q^{-1})^{-1}} J^{3_1} = q^2+q^6-q^8$$

Fig. 3. Khovanov's algorithm for calculating the Jones polynomial for the trefoil [33].

Given a knot diagram, we write all knot smoothings (each intersection can be replaced with either a 0- or a 1-smoothing); they then become vertices of the so-called Khovanov cube. Each knot smoothing correspond to a summand $(-1)^r q^r (q+q^{-1})^k$, where r is the number of 1-smoothings and k is the number of components of the smoothing. We then add the contributions of each knot smoothings and multiply the resulting polynomial by $(-1)^{n_-} q^{n_+-2n_-}(q+q^{-1})^{-1}$, where n_- is the number of $-$ crossings and n_+ is the number of $+$ crossings.

This algorithm becomes clear in the example of the trefoil (Fig. 3).

Remark 1. Khovanov cubes are the same for all lattice knots coming from a lattice diagram of a fixed size.

This fact allows us to effectively calculate the Jones polynomials for lattice knots. Namely, we need to construct the Khovanov cube for a fixed-size lattice only once, and then, in order to obtain the Jones polynomials for all lattice knots inside a fixed-size lattice, we start the Khovanov algorithm for each vertex of the cube. This is exactly how our computer program in Sec. 5.1 works.

4. Upper and lower bounds

For any $n > k$, the types of knots realized on a lattice of size $(2k+1)\times(2k+1)$ can also be obtained on a lattice of size $(2n+1)\times(2n+1)$. Thus, we introduce another enumeration of knots analogous to the Rolfsen table: each knot is enumerated by two numbers, m and l_m, where m corresponds to the minimal lattice size $(2m+1)\times(2m+1)$ on which the knot can be met and l_m just enumerates distinct knots that can be represented as lattice knots with a fixed minimal m. For example, for the trivial knot, we have $m=0$ and $l_0=1$, and for the trefoil, we have $m=1$ and $l_1=1$.

We want to know which knot dominates (i.e., has more representations) on a fixed-size lattice. It was proved recently in [22] that hyperbolic knots with a crossing number $\leq n$ do not dominate as $n\to\infty$ for the standard stratification of knots, and it was argued that satellite knots dominate instead. Does this picture change if knots are classified by the lattices size?

In this section, we evaluate how many different knots can be met inside a fixed-size lattice. Namely, we establish the connection between the introduced stratification by lattice diagrams and by the Rolfsen table. In other words, we answer the question of how many knots n_m given in the Rolfsen table come from a lattice diagram L_k.

To evaluate the numbers of different lattice knots, we split the problem into two parts.

1. First, in Sec. 4.1, we evaluate how many times the $(2k+1)\times(2k+1)$ lattice can be found inside the $(2n+1)\times(2n+1)$ lattice for $n > k$;
2. Second, in Sec. 4.2, we define which knots classified by the Rolfsen table can be found in the $(2k+1)\times(2k+1)$ lattice.

This allows us to eliminate the trivial combinatorial contribution and count only distinct knots.

4.1. Lattice embeddings. To find the number of ways to embed a fixed lattice $(2k+1)\times(2k+1)$ into a lattice of size $(2n+1)\times(2n+1)$, we "untie" the bigger lattice to a lattice of a smaller size.

A loop that intersects m threads can be pulled out in 2^m ways such that the intersections of each individual thread are of the same type. Examples are given in Fig. 4.

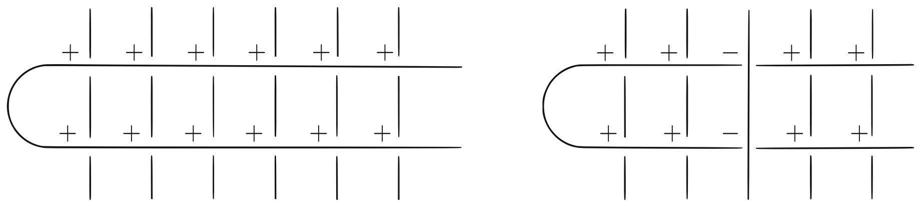


Fig. 4. Examples of pulling loops out.

Thus, to obtain the $(2k+1)\times(2k+1)$ lattice from the $(2n+1)\times(2n+1)$ lattice, we follow three steps.

1. Pull out k horizontal and k vertical loops that belong to the $(2k+1)\times(2k+1)$ lattice. We can do this in $2^{2(n-k)\cdot 2k}$ ways.
2. Then, pull out $n-k$ vertical loops in $2^{2(n-k+1)(n-k)}$ ways. We give examples in Fig. 5.

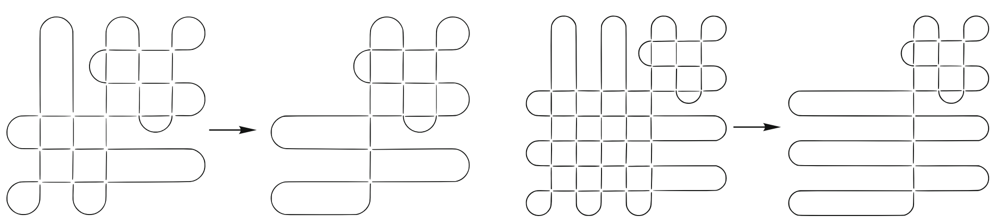


Fig. 5. Pulling out vertical loops. The smaller 3×3 lattice is placed in the upper right corner; its intersections are left empty because they can be chosen arbitrarily to form any lattice knot. The other intersections are to be determined so as to "untie" the bigger lattice to the 3×3 lattice.

3. At the last step, we obtain a trivial loop, and hence the corresponding intersections can be arbitrary. This yields another $2^{2(n-k)}$ options.

Actually, after step 1, we can untangle the remaining part of the lattice starting with the $n-k$ horizontal loops. This is possible if $n-k > 1$. We emphasize that the variants obtained at the previous stages must be excluded.

2′. Pull out $n-k$ horizontal loops. This way, we obtain $2^{(n-k+2)(n-k)}$ more options.

3′. Finally, pull out $n-k$ vertical loops, in $2^{2(n-k)}$ ways.

We note that the position of the $(2k+1)\times(2k+1)$ lattice inside the $(2n+1)\times(2n+1)$ lattice can be chosen in $(n-k+1)\times(n-k+1)$ ways, and the number of the above variants stays the same even if the $(2k+1)\times(2k+1)$ lattice does not lie on the boundary of the $(2n+1)\times(2n+1)$ lattice (for example, see Fig. 6). It is crucial that the $(2k+1)\times(2k+1)$ lattice be untiable to a lower-size lattice because otherwise we would overcount the options to transform a bigger lattice knot into a smaller one.

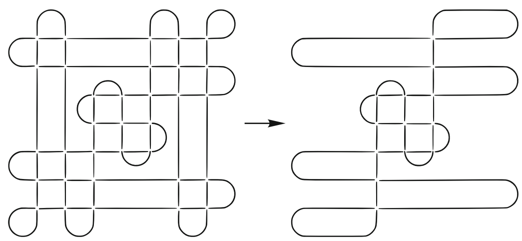


Fig. 6. A smaller lattice knot in the middle of a bigger lattice.

In this analysis, we do not consider the cases where the resultant smaller lattice knot is split into several parts separated in the bigger lattice. We then have the following lemma.

Lemma 1. *In total, there are more than*

$$(\sigma_k^n)_{\min} = (n-k+1)^2[2^{2(n-k)(n+k+2)} + 2^{(n-k)(n+3k+4)}(1-\delta_{n-k,1}-\delta_{n-k,0})]. \tag{4.1}$$

lattices of size $(2k+1)\times(2k+1)$ *inside the* $(2n+1)\times(2n+1)$ *lattice.*

Remark 2. We emphasize that in this analysis, we systematically exclude repeated knots.

4.2. Knots inside lattices. To find how many knots with a fixed crossing number can be embedded into a lattice of size $(2n+1)\times(2n+1)$, we need to find which crossing numbers are allowed in a lattice with a fixed size.

It can be shown that there are knots with an odd crossing number $\leq (2+1)^2-2$ in the $(2k+1)\times(2k+1)$ lattice. Namely, we can easily find the biggest knot, which is the endless knot with the crossing number $(2k+1)^2-2$. To obtain other knots, we pulls out loops of the endless knot, leaving the remaining crossings alternating. An example is given in Fig. 7 (each step results in removing two crossings, leaving an alternating remaining knot).

We note that there is no need to obtain other knots with smaller crossing numbers due to the theorem stating that all knots realized in a smaller lattice can also be realized in a bigger one.

Thus, because we just need to embed an untiable lattice of size $(2k+1)\times(2k+1)$ into the $(2n+1)\times(2n+1)$ lattice, we have the following lemma.

Lemma 2. *The number* σ_k^n *of knots with the crossing numbers*

$$(2k-1)^2, (2k-1)^2+2, \ldots, (2k+1)^2-4, (2k+1)^2-2$$

in the $(2n+1)\times(2n+1)$ *lattice can be estimated from below by the number* $(\sigma_k^n)_{\min}$ *of lattice embeddings in* (4.1) (*for* $k=1$, *the knot with the crossing number* 1 *is the unknot and it must be discarded*).

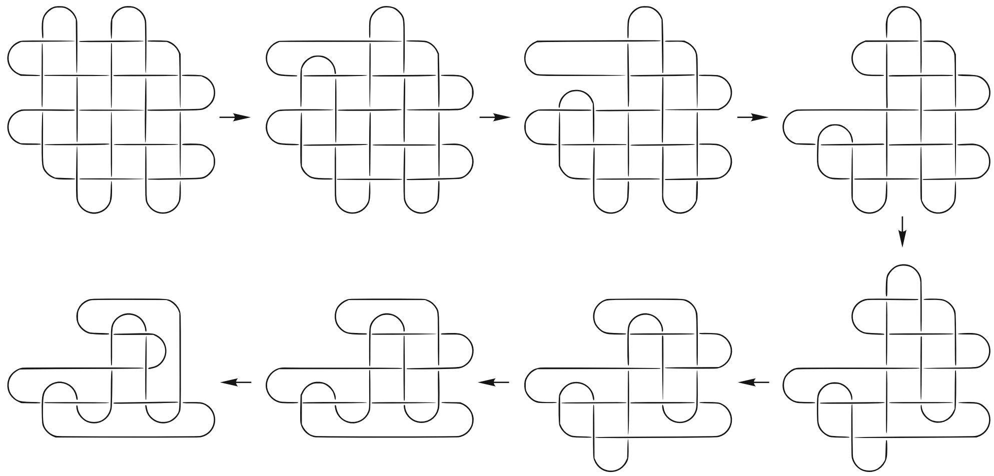


Fig. 7. Knots with crossing numbers 23, 21, 19, 17, 15, 13, 11, and 9 inside the 5×5 lattice.

We now obtain an upper bound. As we have established, the number of knots with odd crossing numbers from $(2k-1)^2$ to $(2k+1)^2-2$ on a $(2n+1)\times(2n+1)$ lattice satisfies the estimate $\sigma_k^n \geqslant (\sigma_k^n)_{\min}$, where $(\sigma_k^n)_{\min}$ is defined in (4.1). The number of different types of knots inside an untiable $(2k+1)\times(2k+1)$ lattice is at least $8k$ for $k>1$, 6 for $k=1$, and 1 for $k=0$, and hence their total is

$$\begin{aligned}
&\Sigma_k^n \geqslant (\Sigma_k^n)_{\min} = 8k(n-k+1)^2[2^{2(n-k)(n+k+2)} + 2^{(n-k)(n+3k+4)}(1-\delta_{n-k,1}-\delta_{n-k,0})], \quad k>1,\\
&\Sigma_1^n \geqslant (\Sigma_1^n)_{\min} = 6n^2[2^{2(n-1)(n+3)} + 2^{(n-1)(n+7)}(1-\delta_{n,2}-\delta_{n,1})],\\
&\Sigma_0^n \geqslant (\Sigma_0^n)_{\min} = (\sigma_0^n)_{\min} = (n+1)^2[2^{2n(n+2)} + 2^{n(n+4)}(1-\delta_{n,1}-\delta_{n,0})].
\end{aligned} \tag{4.2}$$

We note that in the case $k>0$, we take mirror knots into account, and hence the number of knots with a fixed crossing number doubles.

The $(2n+1)\times(2n+1)$ lattice contains $2^{(2n+1)^2}$ knots because each intersection can be resolved in any of two possible ways. Therefore, to obtain an upper bound for the number of knots with a fixed crossing number, we need to subtract the minimal number of knots with all other crossing numbers from the total number of $2^{(2n+1)^2}$ knots. Then

$$\sigma_k^n \leqslant 2^{(2n+1)^2} - \sum_{i=0}^{n}(\Sigma_i^n)_{\min} + (\sigma_k^n)_{\min}. \tag{4.3}$$

To summarize, we have arrived as the following theorem.

Theorem 4. *The number σ_k^n of knots with*

$$(2k-1)^2, (2k-1)^2+2, \dots, (2k+1)^2-4, (2k+1)^2-2$$

crossing numbers in the $(2n+1)\times(2n+1)$ lattice satisfies the constraints

$$(\sigma_k^n)_{\min} \leqslant \sigma_k^n \leqslant 2^{(2n+1)^2} - \sum_{i=0}^{n}(\Sigma_i^n)_{\min} + (\sigma_k^n)_{\min}, \tag{4.4}$$

where $(\sigma_k^n)_{\min}$ and $(\Sigma_i^n)_{\min}$ are defined in (4.1) and (4.2) (for $k=1$, the knot with crossing number 1 is the unknot and it must be discarded). We emphasize that in this analysis, we differentiate between knot projections and their mirror projections.

Remark 3. According to our analysis, if there are knots with even crossing numbers $(2k-1)^2+1$, $(2k-1)^2+3, \ldots, (2k+1)^2-3$ (for $k=1$, the knot with crossing number 2 is the unknot and it must be discarded), their number also admit the above bounds (4.4).

We give some examples:

$$\begin{aligned} &256 \leqslant \sigma_0^1 \leqslant 506, \\ &1 \leqslant \sigma_1^1 \leqslant 252, \\ &626\,688 \leqslant \sigma_0^2 \leqslant 2^{25} - 6144, \\ &1024 \leqslant \sigma_1^2 \leqslant 2^{25} - 631\,824, \\ &1 \leqslant \sigma_2^2 \leqslant 2^{25} - 632\,847. \end{aligned} \tag{4.5}$$

Here, σ_0^1 is the number of unknots inside the 3×3 lattice and σ_1^1 is the number of other types of knots inside the 3×3 lattice. Inside the 5×5 lattice, σ_0^2 is the number of unknots, σ_1^2 is the number of knots with crossing numbers 3, 5, 7 and 4, 6 (if they appear), and σ_2^2 is the number of knots with crossing numbers $2n+1$, $n = 4, \ldots, 11$, and $2n$, $n = 5, \ldots, 11$ (if they appear). It also follows from the estimates in (4.4) that unknots dominate for any n.

5. Numerical computations

To check our estimates (4.4), we compute the Jones polynomials for small lattice knots. The Jones polynomials are convenient tools to distinguish small knots. The first example of knots with the identical Jones polynomials is $J^{5_1}(q) = J^{10_{132}}(q)$, and such identities become essential for knots with 10 and more crossings.

5.1. Knots inside 3×3 and 5×5 lattices. We have written a computer program that calculates the Jones polynomial of lattice knots by using the state sum formula (see Sec. 3.2).

First, we found the knot types in the 3×3 lattice and their numbers. The program returns 10 different Jones polynomials. Comparing them with the Jones polynomials for knots with crossing numbers less than 7 (with the use of knot atlas [34]) and using property (3.7), we obtain

$$\begin{aligned} &J^{0_1}(q) = 1, \\ &J^{3_1}(q) = J^{3_1^{\mathrm{mir}}}(q^{-1}) = q^2 + q^6 - q^8, \\ &J^{4_1}(q) = 1 - q^2 - q^{-2} + q^4 + q^{-4}, \\ &J^{5_2}(q) = J^{5_2^{\mathrm{mir}}}(q^{-1}) = q^2 - q^4 + 2q^6 - q^8 + q^{10} - q^{12}, \\ &J^{6_1}(q) = J^{6_1^{\mathrm{mir}}}(q^{-1}) = q^{-4} - q^{-2} + 2 - 2q^2 + q^4 - q^6 + q^8, \\ &J^{7_4}(q) = J^{7_4^{\mathrm{mir}}}(q^{-1}) = q^2 - 2q^4 + 3q^6 - 2q^8 + 3q^{10} - 2q^{12} + q^{14} - q^{16}. \end{aligned} \tag{5.1}$$

The results are given in Table 1. We note that the numbers of knots agree with our estimates (4.5). There are $2^9 = 512$ knots in the 3×3 lattice. The fraction of unknots is 0.609. Torus knots dominate, their fraction being 0.766.

There are many more different knots inside the 5×5 lattice. We have computed 13 829 different Jones polynomials (up to the replacement $q \to q^{-1}$), and hence there are not less then 13 829 different knots inside the 5×5 lattice. In Table 2, we write types and numbers of knots with crossing numbers less than or equal to 9. We note that the numbers of knots agree with our estimates (4.5).

Table 1. The number of knots in the 3×3 lattice. In the "type" row, H denotes hyperbolic knots and T torus knots. If the number of knots is split into a sum of two equal numbers, the corresponding knot is not amphichiral, and hence the knot and its mirror image differ

knot	0_1	3_1	4_1	5_2	6_1	7_4
type	T	T	H	H	H	H
number	312	40+40	48	24+24	8+8	4+4

Table 2. The number of knots in the 5×5 lattice with no more than 9 crossings. For brevity, we do not distinguish between knots and their mirror images

knot	0_1	3_1	4_1	5_1	5_2	6_1	6_2	6_3
type	T	T	H	T	H	H	H	H
number	5 063 616	2 785 728	1 896 896	525 360	2 327 776	1 253 216	719 496	381 312
knot	7_1	7_2	7_3	7_4	7_5	7_6	7_7	8_1
type	T	H	H	H	H	H	H	H
number	26 560	508 000	350 784	519 312	426 256	475 232	281 280	250 656
knot	8_2	8_3	8_4	8_5	8_6	8_7	8_8	8_9
type	H	H	H	H	H	H	H	H
number	29 824	130 208	171 648	26 880	194 816	42 944	249 840	81 536
knot	8_{10}	8_{11}	8_{12}	8_{13}	8_{14}	8_{15}	8_{16}	8_{17}
type	H	H	H	H	H	H	H	H
number	288 704	49 536	261 184	130 144	332 752	27 456	37 312	29 440
knot	8_{18}	8_{19}	8_{20}	8_{21}	9_1	9_2	9_3	9_4
type	H	H	T	H	T	H	H	H
number	13 392	31 680	69 712	48 064	0	89 568	16 192	21 888
knot	9_5	9_6	9_7	9_8	9_9	9_{10}	9_{11}	9_{12}
type	H	H	H	H	H	H	H	H
number	179 072	13 632	57 088	101 104	27 968	65 392	16 000	169 152
knot	9_{13}	9_{14}	9_{15}	9_{16}	9_{17}	9_{18}	9_{19}	9_{20}
type	H	H	H	H	H	H	H	H
number	157 568	117 584	111 936	14 112	8 128	153 072	142 864	35 008
knot	9_{21}	9_{22}	9_{23}	9_{24}	9_{25}	9_{26}	9_{27}	9_{28}
type	H	H	H	H	H	H	H	H
number	165 568	44 352	98 208	28 800	25 344	32 896	40 128	15 552
knot	9_{29}	9_{30}	9_{31}	9_{32}	9_{33}	9_{34}	9_{35}	9_{36}
type	H	H	H	H	H	H	H	H
number	3 680	42 160	17 312	32 256	16 320	14 432	9 504	39 104
knot	9_{37}	9_{38}	9_{39}	9_{40}	9_{41}	9_{42}	9_{43}	9_{44}
type	H	H	H	H	H	H	H	H
number	19 200	3 792	16 304	336	5 728	44 928	57 312	64 992
knot	9_{45}	9_{46}	9_{47}	9_{48}	9_{49}			
type	H	H	H	H	H			
number	50 464	35 712	12 608	15 592	7 264			

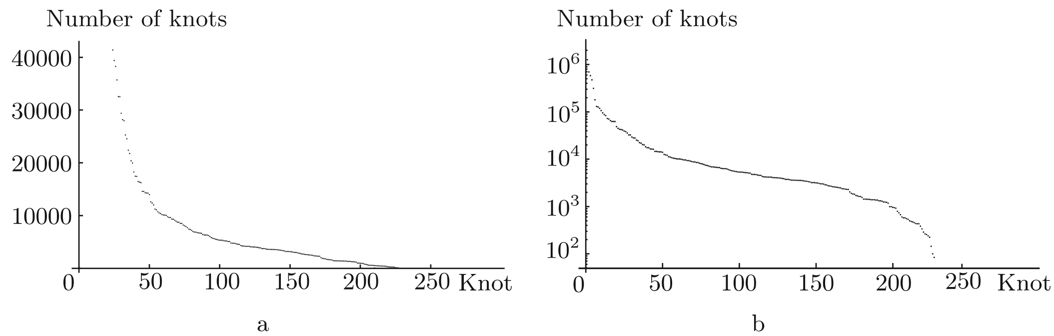


Fig. 8. The number of knots with up to 10 crossings in the 5×5 lattice, in (a) ordinary and (b) logarithmic scales. The horizontal axis just enumerates the knots.

We could identify only knots with up to 12 crossings because the known databases [21], [34] only contain the Jones polynomials for knots with the crossing number not greater than 12. We cannot therefore find the fraction of hyperbolic and satellite knots. The fraction of unknots is 0.151, and they dominate. The numbers of knots with crossing number not greater than 9 are shown in Fig. 8.

5.2. The fraction of unknots. In the foregoing, we presented results of extensive numeric experiments on 3×3 and 5×5 lattices. To move to bigger lattices, we need to impose two constraints, otherwise the numerical computations required become overwhelming even for modern computers.

- Limit the scope: switch from detecting all knots to just detecting the unknots and the knots outside the Rolfsen table (i.e., with the minimal crossing number greater than 10). As mentioned above, this is done with help of evaluating the Jones polynomial, and hence, due to unavoidable collisions, the result will be an estimate from above. This optimization drastically reduces memory requirements because we do not keep track of all different knots.
- Reduce precision: instead of faithfully iterating over all knots on a given lattice, generate a (reasonably) large number of random knots on this lattice (by a Monte Carlo type method). From this sampling, we obtain an estimate of the fraction of unknots, rather than the accurate total number. With this optimization, we can set the computation time to any (reasonable) desired value.

With these optimizations, we obtain estimates for the number of unknots and of knots with more than 10 crossings on a given lattice (see Table 3).

Table 3. The number of unknots and knots with the minimal crossing number greater than 10

Lattice size	Fraction of unknots	Fraction of knots with > 10 crossings
1×1	1.0	0.0
3×3	0.614	0.0
5×5	0.134	0.272
7×7	0.015	0.872
9×9	0.0001	0.9972

We see that

1. these numbers agree (modulo the inaccuracies introduced by random sampling and Jones polynomial collisions) with theoretical estimates (4.4) and with the full-iteration results (Tables 1 and 2);

2. the fraction of unknots seems to be exponentially decreasing, confirming the (asymptotic mean-field) estimates of Grosberg and Nechaev [30], [31];

3. the number of non-Rolfsen knots (those with the minimal crossing number greater than 10) in a rectangular grid is rapidly (exponentially) approaching 1. This means that the computational task of finding a knot with a small number of crossings on a large grid is exponentially complex. This also agrees with our theoretical estimates (4.4), which predict exponential growth in the number of knots with large crossing numbers. The implications of this will be explored in future research.

6. Conclusions and Discussion

In this paper, we have answered classical questions of knot theory regarding the introduced stratification by lattice knots. We have obtained the following results.

1. We obtained estimates for the numbers of knots with fixed a crossing number inside the $(2n+1) \times (2n+1)$ lattice for all n (Section 4).

2. We classified (in accordance with the Rolfsen table) all knots in the 3×3 lattice (Table 1) and knots with up to 12 crossings that can be distinguished by the Jones polynomial in the 5×5 lattice (only knots with no more than 9 crossings are given in Table 2), Sec. 5.1.

3. We found the numbers of unknots for five lattices (Sec. 5.2). Their fraction decreases exponentially with as the lattice size increases.

4. We found the numbers of knots with > 10 minimal crossings for five lattices (Sec. 5.2). Their fraction increases exponentially as the lattice size increases.

To find which type of knots dominates for large lattices, we need to learn to effectively calculate knot invariants that distinguish between hyperbolic, torus, and satellite knots.

With the increase in the lattice size, more knots appear that cannot be distinguished by the Jones polynomial. Moreover, for the Jones polynomial, it is not even proved whether it distinguishes the unknots. A more effective method is to use the Khovanov polynomial [33], [32], which is a categorification of the Jones polynomial, because there is a theorem stating that the Khovanov polynomial detects the unknot [35]. Moreover, it distinguishes more knots. However, it is much more difficult to calculate, and the development of a simple method for its calculation for lattice knots is a separate problem.

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Conflicts of interest. The authors declare no conflicts of interest.

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