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APPLICATION OF THE TRIGONAL CURVE TO A HIERARCHY OF GENERALIZED TODA LATTICES

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Starting from the zero-curvature equation and Lenard recurrence relations, we derive a hierarchy of generalized Toda lattices. The trigonal curve is introduced through the Lax pair characteristic polynomial for the discrete hierarchy, from which a Dubrovin-type equation is established. Then the asymptotic behavior of the Baker–Akhiezer function and the meromorphic function is analyzed, and the divisors of the two functions are also discussed. Moreover, the Abel map is defined and the corresponding flows are straightened out on the Jacobian variety, such that the final algebro-geometric solutions of the hierarchy are calculated in terms of the Riemann theta function.

Keywords: discrete matrix spectral problem, generalized Toda lattices, trigonal curve, algebro-geometric solutions

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1. Introduction

The Toda lattice [1], [2]

$$\frac{\partial^2 y}{\partial t^2} = \exp(y^- - y) - \exp(y - y^+), \qquad y = y(n,t), \quad (n,t) \in \mathbb{Z} \times \mathbb{R}, \tag{1.1}$$

is an absolutely integrable equation with exponential interaction that was discovered in the course of seeking a system with rigorous periodic solutions; its exponential interaction was used to explain the nonergodic character of the famous Fermi–Pasta–Ulam problem [3]. It has abundant mathematical structures and is regarded as a model of physical phenomena, with the well-known equations such as the nonlinear Schrödinger (NLS) and Korteweg–de Vries equations being closely related to it or deduced from it by appropriate limit procedures [4], [5]. In addition, it can describe the motion of a chain of particles with nearest-neighbor interaction in constructing different mathematical models; the Toda lattice model of DNA is also a typical representative in biology [6].

It is worth mentioning that by the variable transformation $\varpi = -\exp(y - y^+)$ and $x = y_t$, the Toda lattice can be rewritten in the form

$$\varpi_t = \varpi(x - x^+), \qquad x_t = \varpi - \varpi^-. \tag{1.2}$$

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With the increase in scholars' attention to the Toda lattice, a variety of important methods were applied to it and numerous results have been achieved since it was proposed [7]–[12]. As one of the most effective research tools, the algebro-geometric methods were extensively applied to the Toda lattices. With the development of the finite-gap integration method in the works of Novikov, Matveev, Its, and others, the mathematical theory of the algebro-geometric method has been systematically developed since the early 1970s [13], [14]. The solutions can not only describe the integrability properties of the equations but also reveal the internal structure of the solutions for soliton equations [15]–[18].

The algebro-geometric solutions for numerous soliton equations related to a 2×2 matrix spectral problem have been obtained using the theta functions of hyperelliptic curves in a series of studies [19]–[21]. However, the studies of algebro-geometric solutions of 3rd-order soliton equations are very few. In the course of studying the algebro-geometric solutions of the 3rd-order soliton equations, the most classical findings originates in the Boussinesq equation, whose 3rd-order differential operators were studied in terms of the reduction theory of Riemann theta functions [22]; finite-gap solutions of the NLS equation were also confirmed smoothly by means of a special algorithm [23]. In 1999, based on the algebro-geometric method, Dickson, Gesztesy, and Unterkofler proposed a unified framework that yields all algebro-geometric solutions of the entire Boussinesq hierarchy related to a 3rd-order differential operator [24]. Based on the framework proposed previously, a systematic method for introducing a trigonal curve was developed with the help of the characteristic polynomial of the Lax matrix associated with the higher-order matrix spectral problem, from which the algebro-geometric method was successfully generalized to yield algebro-geometric solutions of the continuous hierarchies related to 3×3 matrices [25], [26]. Then the algebro-geometric method was further extended to 3rd-order discrete hierarchies [27], [28].

In this paper, we introduce the trigonal curve to define the Baker–Akhiezer function Ξ and the corresponding meromorphic function Θ. The soliton equations can then be separated into solvable Dubrovin-type ordinary differential equations. Based on the above step, the characteristics of the functions can be further analyzed. With the systematic algebro-geometric theory as support, we discuss the application of the algebro-geometric methods to the discrete hierarchy of a 3rd-order generalized Toda lattice

$$\begin{aligned} &q_t = 2rq^{--}q^- - \frac{s^+q}{s}, \qquad && r_t = \frac{s}{s^-} - \frac{s^+}{s} + 2(qv^+ - q^-v), \\ &s_t = -qsv - rs, && v_t = \frac{qs}{s^-} - 2rv + 2v^-, \end{aligned} \tag{1.3}$$

which becomes is the Toda lattice mentioned above (1.2) if $q = 0$, $r = \varpi^+$, $x = s/s^-$, and $v = 0$. The Hamiltonian system for (1.3) was constructed in [29].

The paper is organized as follows. In Sec. 2, the difference operators K_n and J_n are deduced in accordance with the Lenard recurrence relations and hierarchy (1.3) is then derived from the zero-curvature equation. In Sec. 3, the trigonal curve $\mathcal{K}_{l-1}$ is defined for the characteristic polynomial of the Lax pair for hierarchy (1.3), whence the functions Ξ and Θ can be defined. In Sec. 4, in the stationary case, we analyze the characteristics of the functions and introduce the Abel differentials; the potentials of the Lax pair are then expressed in terms of the Riemann theta function. In Sec. 5, we apply the analysis in last two sections to the time-dependent case and separate hierarchy (1.3) into solvable Dubrovin-type ordinary differential equations. Then we straighten out the flows and obtain the Riemann theta representation. On the whole, the algebro-geometric solutions of hierarchy (1.3) are obtained and we rewrite the Riemann theta representation of the potentials for low genera. We summarize and conclude in Sec. 6.

2. The hierarchy of a generalized Toda lattice

We suppose that q, r, s, and v satisfy the following conditions: in the stationary case,

$$q(n,\,\cdot\,), r(n,\,\cdot\,), s(n,\,\cdot\,), v(n,\,\cdot\,) \in C^1(\mathbb{R}),$$

and in the time-dependent case,

$$q(\,\cdot\,,t), r(\,\cdot\,,t), s(\,\cdot\,,t), v(\,\cdot\,,t) \in \mathbb{C}^{\mathbb{Z}}, \qquad t \in \mathbb{R},$$

where $\mathbb{C}^{\mathbb{Z}}$ is the set of all complex-valued functions of a variable in $\mathbb{Z}$.

On the complex-valued sequence $\hbar = \{\hbar(n)\}_{n\in\mathbb{Z}}$, we define the shift operators $E^{\pm}$ as

$$(E^{\pm}\hbar)(n) = \hbar(n \pm 1), \qquad n \in \mathbb{Z},$$

and write $\hbar^{\pm} = E^{\pm}\hbar$ with $\hbar \in \mathbb{C}^{\mathbb{Z}}$.

We consider the discrete 3×3 matrix spectral problem [29]

$$E\Xi = U\Xi, \qquad \Xi = \begin{pmatrix} \Xi_1 \\ \Xi_2 \\ \Xi_3 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & q & 0 \\ v & \lambda + r & s \\ 0 & -1/s & 0 \end{pmatrix}, \tag{2.1}$$

where q, r, s, and v are potentials and λ is a constant. The Lenard recurrence relations are

$$\begin{aligned} K_n\tilde{g}_j &= J_n\tilde{g}_{j+1}, \qquad \tilde{g}_j = (\tilde{a}_j, \tilde{b}_j, \tilde{c}_j, \tilde{d}_j)^{\mathrm{T}}, \\ K_n\bar{g}_j &= J_n\bar{g}_{j+1}, \qquad \bar{g}_j = (\bar{a}_j, \bar{b}_j, \bar{c}_j, \bar{d}_j)^{\mathrm{T}} \end{aligned} \tag{2.2}$$

We then introduce the starting points

$$\tilde{g}_0 = (1, 0, -1, 0)^{\mathrm{T}}, \qquad \bar{g}_0 = (-1, 0, 2, 0)^{\mathrm{T}}, \tag{2.3}$$

and define two difference operators J_n and K_n as

$$J_n = \begin{pmatrix} 0 & E & 0 & 0 \\ 0 & 0 & E-1 & 0 \\ 0 & 0 & 0 & s^2E \\ -\dfrac{1}{q}(E-1) & -\dfrac{v}{q}E & 0 & 0 \end{pmatrix},$$

$$K_n = \begin{pmatrix} -qE & K_{12} & q & \dfrac{1}{s}Eqs^2E \\ -qE\dfrac{1}{q}(E-1) & v - qE\dfrac{v}{q}E & -r(E-1) & s - \dfrac{1}{s}Es^2E \\ -s & vsE & -sE-s & K_{34} \\ K_{41} & K_{42} & -vE & -sE^{-1}vE \end{pmatrix},$$

where

$$\begin{aligned} K_{12} &= 1 + \frac{1}{s}EsE - rE, \qquad K_{34} = vqs^2E - rs^2E, \\ K_{41} &= (r-E)\frac{1}{q}(E-1) + \frac{s(E^{-1}-1)}{s^-q^-} + v, \\ K_{42} &= -E\frac{v}{q} + \frac{rv}{q}E - \frac{sE^{-1}vE}{s^-q^-}. \end{aligned} \tag{2.4}$$

Hence, $\tilde{g}_j$ and $\bar{g}_j$ can be found using the operators K_n and J_n; the first two members are given by

$$\tilde{g}_1 = \left(1, -2q^-, 0, \frac{1}{s^-}\right)^{\mathrm{T}}, \qquad \bar{g}_1 = \left(1, 3q^-, 1, -\frac{3}{s^-}\right)^{\mathrm{T}}. \tag{2.5}$$

To deduce the hierarchy related to spectral problem (2.1), we introduce the stationary zero-curvature equation

$$(E\Gamma)U - U\Gamma = 0, \qquad \Gamma = (\Gamma_{ij})_{3\times 3} = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{32} & -\Gamma_{11}-\Gamma_{22} \end{pmatrix}, \tag{2.6}$$

which is equivalent to

$$\begin{aligned}
&E\Gamma_{11} + vE\Gamma_{12} - \Gamma_{11} - q\Gamma_{21} = 0, \\
&qE\Gamma_{11} + (r+\lambda)E\Gamma_{12} - \frac{1}{s}E\Gamma_{13} - \Gamma_{12} - q\Gamma_{22} = 0, \\
&sE\Gamma_{12} - \Gamma_{13} - q\Gamma_{23} = 0, \\
&E\Gamma_{21} + vE\Gamma_{22} - v\Gamma_{11} - (r+\lambda)\Gamma_{21} - s\Gamma_{31} = 0, \\
&qE\Gamma_{21} + (r+\lambda)E\Gamma_{22} - \frac{1}{s}E\Gamma_{23} - v\Gamma_{12} - (r+\lambda)\Gamma_{22} - s\Gamma_{32} = 0, \\
&sE\Gamma_{22} - v\Gamma_{13} - (r+\lambda)\Gamma_{23} + s(\Gamma_{11}+\Gamma_{22}) = 0, \\
&E\Gamma_{31} + vE\Gamma_{32} + \frac{1}{s}\Gamma_{21} = 0, \\
&qE\Gamma_{31} + (r+\lambda)E\Gamma_{32} + \frac{1}{s}E(\Gamma_{11}+\Gamma_{22}) + \frac{1}{s}\Gamma_{22} = 0, \\
&sE\Gamma_{32} + \frac{1}{s}\Gamma_{23} = 0.
\end{aligned} \tag{2.7}$$

where each element $\Gamma_{ij} = \Gamma_{ij}(a,b,c,d)$ is a Laurent expansion in λ,

$$\begin{aligned}
&\Gamma_{11} = a, && \Gamma_{12} = b, \quad && \Gamma_{13} = sEb + qs^2Ed, \\
&\Gamma_{21} = \frac{1}{q}(E-1)a + \frac{v}{q}Eb, && \Gamma_{22} = c, \quad && \Gamma_{23} = -s^2Ed, \\
&\Gamma_{31} = \frac{(E^{-1}-1)}{s^-q^-}a - \frac{E^{-1}vE}{s^-q^-}b - E^{-1}vEd, && \Gamma_{32} = d, \quad && -\Gamma_{11}-\Gamma_{11} = -a-c,
\end{aligned} \tag{2.8}$$

with

$$a = \sum_{j\geqslant 0} a_j\lambda^{-j}, \qquad b = \sum_{j\geqslant 0} b_j\lambda^{-j}, \qquad c = \sum_{j\geqslant 0} c_j\lambda^{-j}, \qquad d = \sum_{j\geqslant 0} d_j\lambda^{-j}. \tag{2.9}$$

We can show by direct calculation that Eqs. (2.6) and (2.7) imply the Lenard equation

$$K_nG = \lambda J_nG, \qquad G = (a,b,c,d)^{\mathrm{T}}. \tag{2.10}$$

We substitute (2.8) in (2.9) and compare the powers of λ to deduce the recurrence relations

$$K_nG_j = \lambda J_nG_{j+1}, \quad J_nG_0 = 0, \qquad j \geqslant 0, \tag{2.11}$$

where $G_j = (a_j,b_j,c_j,d_j)^{\mathrm{T}}$. It is evident that $\ker J_n = \{\alpha_0\tilde{g}_0 + \beta_0\bar{g}_0 \mid \alpha_0,\beta_0 \in \mathbb{R}\}$ and G_j has the expansion

$$G_j = \alpha_0\tilde{g}_j + \beta_0\bar{g}_j + \cdots + \alpha_j\tilde{g}_0 + \beta_j\bar{g}_0, \qquad j \geqslant 0, \tag{2.12}$$

where α_j and β_j are constants.

Assuming that Ξ satisfies the discrete matrix spectral problem (2.1), we have

$$\Xi_{t_m} = \widehat{\Gamma}^{(m)}\Xi, \qquad \widehat{\Gamma}^{(m)} = (\widehat{\Gamma}^{(m)}_{ij})_{3\times 3} = \begin{pmatrix} \widehat{\Gamma}^{(m)}_{11} & \widehat{\Gamma}^{(m)}_{12} & \widehat{\Gamma}^{(m)}_{13} \\ \widehat{\Gamma}^{(m)}_{21} & \widehat{\Gamma}^{(m)}_{22} & \widehat{\Gamma}^{(m)}_{23} \\ \widehat{\Gamma}^{(m)}_{31} & \widehat{\Gamma}^{(m)}_{32} & -\widehat{\Gamma}^{(m)}_{11} - \widehat{\Gamma}^{(m)}_{22} \end{pmatrix}, \tag{2.13}$$

where $\widehat{\Gamma}^{(m)}_{ij} = \widehat{\Gamma}_{ij}(\hat{a}^{(m)}, \hat{b}^{(m)}, \hat{c}^{(m)}, \hat{d}^{(m)})$ and

$$\begin{aligned} \hat{a}^{(m)} &= \sum_{j=0}^{m} \hat{a}_j \lambda^{m-j}, \qquad \hat{b}^{(m)} = \sum_{j=0}^{m} \hat{b}_j \lambda^{m-j}, \\ \hat{c}^{(m)} &= \sum_{j=0}^{m} \hat{c}_j \lambda^{m-j}, \qquad \hat{d}^{(m)} = \sum_{j=0}^{m} \hat{d}_j \lambda^{m-j}. \end{aligned} \tag{2.14}$$

Similarly, the elements $\hat{a}_j$, $\hat{b}_j$, $\hat{c}_j$, and $\hat{d}_j$ can be determined as

$$\widehat{G}_j = \hat{\alpha}_0 \tilde{g}_j + \hat{\beta}_0 \bar{g}_j + \cdots + \hat{\alpha}_j \tilde{g}_0 + \hat{\beta}_j \bar{g}_0, \qquad j \geqslant 0, \tag{2.15}$$

where $\widehat{G}_j$ are also solutions of (2.10). We note, importantly, that $\hat{\alpha}_j$, $\hat{\beta}_j$ and α_j, β_j in (2.12) are absolute of each other. The zero-curvature equation $U_{t_m} = (E\widehat{\Gamma}^m)U - U\widehat{\Gamma}^m$ is generated by the compatibility condition of Eqs. (2.1) and (2.12), which is equivalent to discrete hierarchy (1.3),

$$(q_{t_m}, r_{t_m}, s_{t_m}, v_{t_m})^{\mathrm{T}} = \widehat{\mathcal{I}}_m, \qquad m \geqslant 0, \tag{2.16}$$

and the vector can be represented as

$$\widehat{\mathcal{I}}_j = K_n \widehat{G}_j = J_n \widehat{G}_{j+1}, \qquad \widehat{\mathcal{I}}_j = \mathcal{I}(q, r, s, v, \underline{\hat{\alpha}}^{(j)}, \underline{\hat{\beta}}^{(j)}),$$

where $\underline{\hat{\alpha}}^{(j)} = (\hat{\alpha}_0, \ldots, \hat{\alpha}_j)$, and $\underline{\hat{\beta}}^{(j)} = (\hat{\beta}_0, \ldots, \hat{\beta}_j)$ for $j \geqslant 0$.

The first nontrivial member of hierarchy (2.16) is given by as

$$\widehat{\mathcal{I}}_0 = K_n \widehat{G}_0 = K_n(\alpha_0 \tilde{g}_0 + \beta_0 \bar{g}_0),$$

whence, with $\alpha_0 = 1$ and $\beta_0 = 0$, we have

$$q_{t_0} = -2q, \qquad r_{t_0} = 0, \qquad s_{t_0} = s, \qquad v_{t_0} = 2v. \tag{2.17}$$

Similarly, for $j = 2$ and $\widehat{\mathcal{I}}_1 = K_n(\alpha_0 \tilde{g}_1 + \alpha_1 \tilde{g}_0 + \beta_0 \bar{g}_1 + \beta_1 \bar{g}_0)$ with $\alpha_0 = 1$, $\beta_0 = 1$, and $t_0 = t$, we obtain the hierarchy that we study in what follows:

$$\begin{aligned} q_t &= 2rq^{--}q^- - \frac{s^+ q}{s}, \qquad r_t = \frac{s}{s^-} - \frac{s^+}{s} + 2(qv^+ - q^- v), \\ s_t &= -qsv - rs, \qquad v_t = \frac{qs}{s^-} - 2rv + 2v^-. \end{aligned} \tag{2.18}$$

If $q = 0$, $r = \varpi^+$, $x = s/s^-$, and $v = 0$, Eqs. (2.18) become the Toda lattice (1.2).

3. The stationary meromorphic function

We consider hierarchy (1.3) in the stationary case $\mathcal{I}_p = \mathcal{I}(q,r,s,v;\underline{\alpha}^{(p)},\ \underline{\beta}^{(p)}) = 0$, $\underline{\alpha}^{(p)} = (\alpha_0 \dots \alpha_p)$, and $\underline{\beta}^{(p)} = (\beta_0 \dots \beta_p)$. It is then equivalent to the stationary zero-curvature equation

$$(E\Gamma^{(p)})U - U\Gamma^{(p)} = 0, \qquad \Gamma^{(p)} = (\lambda^p \Gamma)_+ = (\Gamma^{(p)}_{ij})_{3\times 3}, \tag{3.1}$$

with $\Gamma^{(p)}_{ij} = \Gamma_{ij}(a^{(p)}, b^{(p)}, c^{(p)}, d^{(p)})$,

$$a^{(p)} = \sum_{j=0}^{p} a_j \lambda^{p-j}, \quad b^{(p)} = \sum_{j=0}^{p} b_j \lambda^{p-j}, \quad c^{(p)} = \sum_{j=0}^{p} c_j \lambda^{p-j}, \quad d^{(p)} = \sum_{j=0}^{p} d_j \lambda^{p-j}. \tag{3.2}$$

Direct calculation indicates that the characteristic polynomial $\mathcal{F}_l(\lambda, f) = \det(fI - \Gamma^{(p)})$ of $\Gamma^{(p)}$ also satisfies zero-curvature equation (3.1) and is a constant independent of n. It has the expansion

$$\det(fI - \Gamma^{(p)}) = f^3 - f^2 X_l(\lambda) + f Y_l(\lambda) - Z_l(\lambda), \tag{3.3}$$

where $X_l(\lambda), Y_l(\lambda)$ and $Z_l(\lambda)$ are constant-coefficient polynomials in λ,

$$\begin{aligned}
X_l(\lambda) &= t_r\Gamma^{(p)} = \Gamma^{(p)}_{11} + \Gamma^{(p)}_{22} + (-\Gamma^{(p)}_{11} - \Gamma^{(p)}_{22}) = 0,\\
Y_l(\lambda) &= \begin{vmatrix} \Gamma^{(p)}_{11} & \Gamma^{(p)}_{12} \\ \Gamma^{(p)}_{21} & \Gamma^{(p)}_{22} \end{vmatrix} + \begin{vmatrix} \Gamma^{(p)}_{11} & \Gamma^{(p)}_{13} \\ \Gamma^{(p)}_{31} & -\Gamma^{(p)}_{11} - \Gamma^{(p)}_{22} \end{vmatrix} + \begin{vmatrix} \Gamma^{(p)}_{22} & \Gamma^{(p)}_{23} \\ \Gamma^{(p)}_{32} & -\Gamma^{(p)}_{11} - \Gamma^{(p)}_{22} \end{vmatrix} = \\
&= (-\alpha_0^2 + \alpha_0\beta_0 - 3\beta_0^2)\lambda^{2p} + \cdots,\\
Z_l(\lambda) &= \det \Gamma^{(p)}_n = \begin{vmatrix} \Gamma^{(p)}_{11} & \Gamma^{(p)}_{12} & \Gamma^{(p)}_{13} \\ \Gamma^{(p)}_{21} & \Gamma^{(p)}_{22} & \Gamma^{(p)}_{23} \\ \Gamma^{(p)}_{31} & \Gamma^{(p)}_{32} & -\Gamma^{(p)}_{11} - \Gamma^{(p)}_{22} \end{vmatrix} = \\
&= ({\alpha_0}^2\beta_0 - 3\alpha_0{\beta_0}^2 + 2{\beta_0}^3)\lambda^{3p} + \cdots.
\end{aligned} \tag{3.4}$$

Then the trigonal curve $\mathcal{F}_l(\lambda, f) = 0$ whose degree is $l = 3p$ for $\alpha_0\beta_0 \neq 0$ can be introduced as

$$\mathcal{K}_{l-1}\colon\ \mathcal{F}_l(\lambda, f) = f^3 - f^2 X_l(\lambda) + f Y_l(\lambda) - Z_l(\lambda) = 0. \tag{3.5}$$

Under the condition $l = 3p$, it is obvious that the trigonal curve $\mathcal{K}_{l-1}$ can be compactified by adding different infinite points $u_{\infty'}$ and $u_{\infty''}$ based on (3.2) and (3.4), where we choose $u_{\infty''}$ as a double branch point. We still use $\mathcal{K}_{l-1}$ to denote the compactified curve. The discriminant of (3.5) is

$$\Delta(\lambda) = 27{Z_l}^2 - 18X_lS_lZ_l + 4{Y_l}^3 - {X_l}^2{Y_l}^2 + 4{X_l}^3 Z_l. \tag{3.6}$$

By the Riemann–Hurwitz formula, we can obtain that the arithmetic genus of $\mathcal{K}_{l-1}$ is $l-1$. Therefore, $\mathcal{K}_{l-1}$ turns into a three-sheet Riemann surface of genus $l-1$ if the curve is irreducible and

$$\left(\frac{\partial \mathcal{F}_l(\lambda, f)}{\partial \lambda}, \frac{\partial \mathcal{F}_l(\lambda, f)}{\partial f}\right)\Bigg|_{(\lambda,f)=(\lambda_0,f_0)} \neq 0$$

for any $u_0 = (\lambda_0, f_0) \in \mathcal{K}_{l-1}$.

We introduce the stationary Baker–Akhiezer function Ξ as

$$\begin{aligned}
&E\Xi(u,n,n_0) = U(q(n),r(n),s(n),v(n);\lambda(u))\Xi(u,n,n_0),\\
&\Gamma^{(p)}(q(n),r(n),s(n),v(n);\lambda(u))\Xi(u,n,n_0) = f(u)\Xi(u,n,n_0),\\
&\Xi_1(u,n_0,n_0) = 1, \qquad u=(\lambda,f)\in\mathcal{K}_{l-1}\backslash\{u_{\infty'},u_{\infty''}\}, \quad n,n_0\in\mathbb{Z}.
\end{aligned} \tag{3.7}$$

Based on the function Ξ, the meromorphic function Θ on $\mathcal{K}_{l-1}$ is defined as

$$\Theta(u,n) = \frac{\Xi_2(u,n,n_0)}{\Xi_1(u,n,n_0)}, \qquad u\in\mathcal{K}_{l-1}, \quad n\in\mathbb{Z}, \tag{3.8}$$

whence we have

$$\Xi_1(u,n,n_0) = \begin{cases} \prod\limits_{n'=n_0}^{n-1}(1+q(n)\Theta(u,n')), & n\geqslant n_0+1,\\ 1, & n=n_0,\\ \prod\limits_{n'=n_0}^{n-1}(1+q(n)\Theta(u,n'))^{-1}, & n\leqslant n_0-1. \end{cases} \tag{3.9}$$

The meromorphic function $\Theta(u,n)$ obtained in accordance with (3.7) and (3.8) is

$$\begin{aligned}
\Theta(u,n) &= \frac{y\Gamma_{23}^{(p)}+A_l(\lambda,n)}{f\Gamma_{13}^{(p)}+B_l(\lambda,n)} = \frac{E_{l-1}(\lambda,n)}{f^2\Gamma_{23}^{(p)}-fA_l(\lambda,n)+C_l(\lambda,n)} =\\
&= \frac{f^2\Gamma_{13}^{(p)}-fB_l(\lambda,n)+D_l(\lambda,n)}{F_{l-1}(\lambda,n)},
\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}
&A_l = \Gamma_{13}^{(p)}\Gamma_{21}^{(p)}-\Gamma_{11}^{(p)}\Gamma_{23}^{(p)}, \qquad B_l = \Gamma_{12}^{(p)}\Gamma_{23}^{(p)}-\Gamma_{13}^{(p)}\Gamma_{22}^{(p)},\\
&C_l = \Gamma_{21}^{(p)}(\Gamma_{13}^{(p)}\Gamma_{22}^{(p)}-\Gamma_{12}^{(p)}\Gamma_{23}^{(p)})-\Gamma_{23}^{(p)}(\Gamma_{11}^{(p)}\Gamma_{22}^{(p)}+(\Gamma_{22}^{(p)})^2+\Gamma_{23}^{(p)}\Gamma_{32}^{(p)}),\\
&D_l = \Gamma_{12}^{(p)}(\Gamma_{11}^{(p)}\Gamma_{23}^{(p)}-\Gamma_{13}^{(p)}\Gamma_{21}^{(p)})-\Gamma_{13}^{(p)}((\Gamma_{11}^{(p)})^2+\Gamma_{11}^{(p)}\Gamma_{22}^{(p)}+\Gamma_{13}^{(p)}\Gamma_{31}^{(p)}),\\
&E_{l-1} = (\Gamma_{23}^{(p)})^2\Gamma_{31}^{(p)}+\Gamma_{21}^{(p)}\Gamma_{23}^{(p)}(2\Gamma_{11}^{(p)}+\Gamma_{22}^{(p)})-(\Gamma_{21}^{(p)})^2\Gamma_{13}^{(p)},\\
&F_{l-1} = (\Gamma_{13}^{(p)})^2\Gamma_{32}^{(p)}+\Gamma_{12}^{(p)}\Gamma_{13}^{(p)}(2\Gamma_{22}^{(p)}+\Gamma_{11}^{(p)})-(\Gamma_{12}^{(p)})^2\Gamma_{23}^{(p)}.
\end{aligned} \tag{3.11}$$

Moreover, we introduce two other elements

$$\begin{aligned}
&G_l = \Gamma_{13}^{(p)}\Gamma_{32}^{(p)}+\Gamma_{12}^{(p)}(\Gamma_{11}^{(p)}+\Gamma_{22}^{(p)}),\\
&H_l = \Gamma_{12}^{(p)}(\Gamma_{11}^{(p)}\Gamma_{22}^{(p)}-\Gamma_{12}^{(p)}\Gamma_{21}^{(p)})+\Gamma_{13}^{(p)}(\Gamma_{11}^{(p)}\Gamma_{32}^{(p)}-\Gamma_{12}^{(p)}\Gamma_{31}^{(p)}).
\end{aligned} \tag{3.12}$$

Obviously, we can find various relations among polynomials (3.11), (3.12) and X_l, Y_l, Z_l. We list some of them:

$$\begin{aligned}
&F_{l-1} = \Gamma_{13}^{(p)}G_l-\Gamma_{12}^{(p)}B_l,\\
&B_lE_{l-1} = (\Gamma_{23}^{(p)})^2Z_l+A_lC_l, \qquad A_lF_{l-1} = (\Gamma_{13}^{(p)})^2Z_l+B_lD_l,\\
&\Gamma_{13}^{(p)}E_{l-1} = \Gamma_{23}^{(p)}C_l-(\Gamma_{23}^{(p)})^2Y_l-A_l^2, \qquad \Gamma_{23}^{(p)}F_{l-1} = \Gamma_{13}^{(p)}B_l-(\Gamma_{13}^{(p)})^2Y_l-B_l^2,\\
&E_{l-1}^- = -F_{l-1}, \qquad G_l = A_l^-, \qquad H_l = C_l^-.
\end{aligned} \tag{3.13}$$

Using (3.1), (3.2), (3.11), and (3.13), we find that F_{l-1} and E_{l-1} are polynomials of degree $l-1$ and can therefore be represented as

$$\begin{aligned} F_{l-1}(\lambda,n) &= F_{l-1,0}\prod_{j=1}^{l-1}(\lambda-\mu_j(n)),\\ E_{l-1}(\lambda,n) &= -F_{l-1,0}\prod_{j=1}^{l-1}(\lambda-\mu_j^+(n)). \end{aligned} \tag{3.14}$$

On the trigonal curve $\mathcal{K}_{l-1}$, we define $\{\tilde{\mu}_j(n)\}_{j=\overline{1,l-1}}$ and $\{\tilde{\mu}_j^+(n)\}_{j=\overline{1,l-1}}$ as

$$\begin{aligned} \tilde{\mu}_j(n) &= \big(\mu_j(n), y(\hat{\mu}_j(n))\big) = \left(\mu_j(n) - \frac{B_l(\lambda,n)}{\Gamma_{32}^{(p)}(\mu_j(n))}\right),\\ \tilde{\mu}_j^+(n) &= \big(\mu_j^+(n), y(\hat{\mu}_j^+(n))\big) = \left(\mu_j^+(n) - \frac{A_l(\lambda,n)}{\Gamma_{32}^{(p)}(\mu_j^+(n))}\right). \end{aligned} \tag{3.15}$$

For convenience, we let u, u^*, and u^{**} denote points on each of the three different sheets of the Riemann surface $\mathcal{K}_{l-1}$ and suppose that $f_i(\lambda)$ $(i=1,2,3)$ are three roots of $\mathcal{F}_l(\lambda,f)=0$:

$$\big(f-f_1(\lambda)\big)\big(f-f_2(\lambda)\big)\big(f-f_3(\lambda)\big) = f^3 - f^2X_l + fY_l - Z_l = f^3 + fY_l - Z_l = 0. \tag{3.16}$$

Then the three points $(\lambda,f_1(\lambda))$, $(\lambda,f_2(\lambda))$, and $(\lambda,f_3(\lambda))$ are also on the Riemann surface $\mathcal{K}_{l-1}$. Let $\{u,u^*,u^{**}\} = \{(\lambda,f_i(\lambda)),\ i=1,2,3\}$ be any one of the three points. From (3.16), the following system can easily be obtained:

$$\begin{aligned} &f_1+f_2+f_3 = X_l = 0, \qquad f_1f_2+f_2f_3+f_3f_1 = Y_l, \qquad f_1f_2f_3 = Z_l,\\ &f_1^2+f_2^2+f_3^2 = -2Y_l, \qquad f_1^3+f_2^3+f_3^3 = 3Z_l, \qquad f_1^2f_2^2+f_1^2f_3^2+f_2^2f_3^2 = X_l^2,\\ &(f_1+f_2)f_3^2+(f_2+f_3)f_1^2+(f_1+f_3)f_2^2 = -3Z_l. \end{aligned}$$

The function $\Theta(u,n)$ then satisfies the relations

$$\begin{aligned} &\Theta(u,n)\Theta(u^*,n)\Theta(u^{**},n) = -\frac{E_{l-1}(\lambda,n)}{F_{l-1}(\lambda,n)},\\ &\Theta(u,n)+\Theta(u^*,n)+\Theta(u^{**},n) = \frac{3D_l(\lambda,n)-2\Gamma_{32}^{(p)}Y_l(\lambda)}{F_{l-1}(\lambda,n)},\\ &\frac{1}{\Theta(u,n)}+\frac{1}{\Theta(u^*,n)}+\frac{1}{\Theta(u^{**},n)} = \frac{3C_l(\lambda,n)-2\Gamma_{12}^{(p)}(\lambda,n)Y_l(\lambda)}{E_{l-1}(\lambda,n)}. \end{aligned} \tag{3.17}$$

4. Algebro-geometric solutions of the stationary hierarchy

We analyze the asymptotic behavior of the functions $\Theta(u,n)$ and $\Xi(u,n)$, and then introduce the Abel differential and the Riemann theta function. As a result, we obtain algebro-geometric solutions in the stationary case, whereby the potentials q, r, s, and v can be expressed as in terms of the Riemann theta function.

First, it follows by direct calculation that $\Theta(u,n)$ satisfies the Riccati-type equation

$$\begin{aligned} q^-(n)q(n)\Theta^+(u,n)\Theta(u,n)\Theta^-(u,n) &= \left(v(n)q^-(n) - \frac{s(n)}{s^-(n)}\right)\Theta^-(u,n) + \\ &+ (\lambda+r(n))\Theta(u,n) - \Theta^+(u,n) + v(n) + (\lambda+r(n))q^-(n)\Theta^-(u,n)\Theta(u,n) - \\ &- q(n)\Theta(u,n)\Theta^+(u,n) - q^-(n)\Theta^-(u,n)\Theta^+(u,n). \end{aligned} \tag{4.1}$$

Introducing the local coordinate $\varsigma = \lambda^{-1}$ near $u_{\infty'}$ and comparing the powers of ς, we have the formula

$$\Theta = \sum_{j=1}^{\infty} \delta_j \varsigma^j, \qquad u \to u_{\infty'},$$

with

$$\begin{aligned} &\delta_1 = -v, \qquad \delta_2 = v^+ - \frac{s}{s^-} v^- + rv, \\ &\delta_3 = v(r + r^+ + 1 - qv + r^2) + v^+(2 + r - qv - 2q^- v^- - r^+) - v^{++}. \end{aligned} \tag{4.2}$$

As at the preceding step, we introduce the local coordinate $\lambda = \eta^{-2}$ near $u_{\infty''}$ and compare the powers of η, which yields

$$\Theta = \sum_{j=0}^{\infty} \kappa_j \eta^j, \qquad u \to u_{\infty''},$$

with

$$\kappa_0 = 1, \qquad \kappa_1 = -q^- - v, \qquad \kappa_2 = 1 - r + q^{-2} + q^- v + q^- q^{--} + \frac{s}{s^-}. \tag{4.3}$$

The divisors [16] of the meromorphic function are

$$(\Theta(u,n)) = \mathcal{D}_{u_{\infty'}, \tilde{\mu}_1^+(n), \ldots, \tilde{\mu}_{l-1}^+(n)}(u) - \mathcal{D}_{u_{\infty''}, \tilde{\mu}_1^+(n), \ldots, \tilde{\mu}_{l-1}^+(n)}(u), \tag{4.4}$$

whence it follows that $\Theta(u,n)$ has l zeros, $u_{\infty'}$, $\tilde{\mu}_1^+(n), \ldots, \tilde{\mu}_{l-1}^+(n)$, and l poles $u_{\infty''}$, $\tilde{\mu}_1(n), \ldots, \tilde{\mu}_{l-1}(n)$. Besides, according to (3.8), (4.2) and (4.3), we have

$$\Xi_1(u,n,n_0) \underset{\varsigma \to 0}{=} \Upsilon(n,n_0) \varsigma^{n-n_0} (1 + O(\varsigma)), \qquad u \to u_{\infty'}, \quad \varsigma = \lambda^{-1}, \tag{4.5}$$

where

$$\Upsilon(n,n_0) = \begin{cases} \prod_{n'=n_0}^{n-1} -v(n'), & n \geqslant n_0 + 1, \\ 1, & n = n_0, \\ \prod_{n'=n_0}^{n_0-1} (-v(n'))^{-1}, & n \leqslant n_0 - 1, \end{cases}$$

and

$$\Xi_1(u,n,n_0) \underset{\eta \to 0}{=} \eta^{n-n_0}(1 + O(\eta)), \qquad u \to u_{\infty''}, \quad \eta = \lambda^{1/2}. \tag{4.6}$$

The divisors of the Baker–Akhiezer function $\Xi_1(u,n,n_0)$ are

$$(\Xi_1(u,n,n_0)) = \mathcal{D}_{\tilde{\mu}_1(n), \ldots, \tilde{\mu}_{l-1}(n)} - \mathcal{D}_{\tilde{\mu}_1(n_0), \ldots, \tilde{\mu}_{l-1}(n_0)} + (n - n_0)(\mathcal{D}_{u_{\infty'}} - \mathcal{D}_{u_{\infty''}}). \tag{4.7}$$

The Riemann surface $\mathcal{K}_{l-1}$ has a canonical basis of cycles $\mathbf{w}_1, \ldots, \mathbf{w}_{l-1}$ and $\mathbf{o}_1, \ldots, \mathbf{o}_{l-1}$ whose intersection numbers are

$$\mathbf{w}_j \circ \mathbf{o}_\sigma = 0, \quad \mathbf{w}_j \circ \mathbf{w}_\sigma = 0, \quad \mathbf{o}_j \circ \mathbf{o}_\sigma = 0, \qquad j, \sigma = 1, \ldots, l-1. \tag{4.8}$$

On $\mathcal{K}_{l-1}$, we define

$$\tilde{\omega}_h(u) = \frac{1}{3f^2 + Y_l} = \begin{cases} \lambda^{h-1} d\lambda, & 1 \leqslant h \leqslant 2p - 1, \\ f \lambda^{h-2p-2}, & 2p \leqslant h \leqslant l - 1, \end{cases} \tag{4.9}$$

and set

$$\mathbb{O}_{ij} = \int_{\mathbf{w}_j} \tilde{\omega}_i, \qquad \mathbb{P}_{ij} = \int_{\mathbf{o}_j} \tilde{\omega}_i,$$

where the matrices $\mathbb{O}$ and $\mathbb{P}$ are invertible. Now, we introduce new matrices $\mathbb{Q}$ and τ such that $\mathbb{Q} = \mathbb{O}^{-1}$ and $\tau = \mathbb{O}^{-1}\mathbb{P}$. It is easy to see that τ is symmetric ($\tau_{ij} = \tau_{ji}$) and its imaginary part is positive definite ($\operatorname{Im}\tau > 0$).

Transforming $\tilde{\omega}_h$ into the new basis ω_j,

$$\omega_j = \sum_{h=1}^{l-1} \mathbb{Q}_{jh}\tilde{\omega}_h, \qquad j = 1, \ldots, l-1, \tag{4.10}$$

we have

$$\int_{\mathbf{w}_\sigma} \omega_j = \sum_{h=1}^{l-1} \mathbb{Q}_{jh} \int_{\mathbf{w}_i} \tilde{\omega}_h = \sum_{h=1}^{l-1} \mathbb{Q}_{jh}\mathbb{P}_{h\sigma} = \gamma_{j\sigma},$$
$$\int_{\mathbf{o}_\sigma} \omega_j = \sum_{h=1}^{l-1} \mathbb{Q}_{jh} \int_{\mathbf{o}_i} \tilde{\omega}_h = \sum_{h=1}^{l-1} \mathbb{Q}_{jh}\mathbb{P}_{h\sigma} = \tau_{j\sigma}.$$

We define the third kind holomorphic differential on $\mathcal{K}_{l-1}\backslash\{Q', Q''\}$ as $\omega^{(3)}_{Q',Q''}$. It has poles at Q_k with the residues $(-1)^{k+1}$, $k = 1, 2$. In particular,

$$\int_{\mathbf{w}_j} \omega^{(3)}_{Q',Q''} = 0, \qquad \int_{\mathbf{o}_j} \omega^{(3)}_{Q',Q''} = 2\pi i \int_{Q''}^{Q'} \omega_j, \qquad j = 1, \ldots, l-1.$$

For $\omega^{(3)}_{u_{\infty'},u_{\infty''}}$, we have

$$\begin{aligned} \omega^{(3)}_{u_{\infty'},u_{\infty''}} &\underset{\varsigma\to 0}{=} (\varsigma^{-1} + \omega_0^{\infty'}\varsigma^0 + O(\varsigma))\, d\varsigma, \qquad && u \to u_{\infty'}, \quad \varsigma = \lambda^{-1}, \\ \omega^{(3)}_{u_{\infty'},u_{\infty''}} &\underset{\eta\to 0}{=} (-\eta^{-1} + \omega_0^{\infty''}\eta^0 + O(\eta))\, d\eta, \qquad && u \to u_{\infty''}, \quad \eta = \lambda^{-1/2}, \end{aligned} \tag{4.11}$$

whence

$$\begin{aligned} \int_{Q_0}^{u} \omega^{(3)}_{u_{\infty'},u_{\infty''}} &\underset{\varsigma\to 0}{=} \ln\varsigma + \ell_1(Q_0) + \omega_0^{\infty'}\varsigma + O(\varsigma^2), \qquad && u \to u_{\infty'}, \\ \int_{Q_0}^{u} \omega^{(3)}_{u_{\infty'},u_{\infty''}} &\underset{\eta\to 0}{=} -\ln\eta + \ell_2(Q_0) + \omega_0^{\infty''}\eta + O(\eta^2), \qquad && u \to u_{\infty''}, \end{aligned} \tag{4.12}$$

where Q_0 is a variable base point on $\mathcal{K}_{l-1}\{u_{\infty'}, u_{\infty''}\}$, and $\ell_1(Q_0)$, $\ell_2(Q_0)$, $\omega_0^{\infty'}$, and $\omega_0^{\infty''}$ are constants.

Let $\mathcal{T}_{l-1}$ be the period lattice $\{\underline{z} \in \mathbb{C}^{l-1} | \underline{z} = \underline{\mathcal{F}} + \underline{\mathcal{H}}\tau, \underline{\mathcal{F}}, \underline{\mathcal{H}} \in \mathbb{Z}^{l-1}\}$. On $\mathcal{K}_{l-1}$, we regard $\mathcal{J}_{l-1} = \mathbb{C}^{l-1}/\mathcal{T}_{l-1}$ as the Jacobian variety. We can then introduce the Abel map $\underline{\mathcal{A}}\colon \mathcal{K}_{l-1} \to \mathcal{J}_{l-1}$,

$$\underline{\mathcal{A}}(u) = \big(\mathcal{A}_1(u), \ldots, \mathcal{A}_{l-1}(u)\big) = \bigg(\int_{Q_0}^{u} \omega_1, \ldots, \int_{Q_0}^{u} \omega_{l-1}\bigg) \pmod{\mathcal{T}_{l-1}}.$$

We define the divisors group $\operatorname{Div}(\mathcal{K}_{l-1})$ and continue the above equation to it by linearity:

$$\underline{\mathcal{A}}\Big(\sum h_\sigma u_\sigma\Big) = \sum h_\sigma \underline{\mathcal{A}}(u_\sigma).$$

We consider the nonspecial divisor $\mathcal{D}_{\underline{\tilde{\mu}}(n)} = \sum_{\sigma=1}^{l-1} \tilde{\mu}_\sigma(n)$ and define

$$\underline{\rho}(n) = \underline{\mathcal{A}}\left(\sum_{\sigma=1}^{l-1} \tilde{\mu}_\sigma(n) \right) = \sum_{\sigma=1}^{l-1} \underline{\mathcal{A}}(\tilde{\mu}_\sigma(n)) = \sum_{\sigma=1}^{l-1} \int_{Q_0}^{\tilde{\mu}_\sigma(n)} \underline{\omega}, \tag{4.13}$$

where $\underline{\rho} = (\rho_1(n), \ldots, \rho_{l-1}(n))$ and $\underline{\omega} = (\omega_1, \ldots, \omega_{l-1})$. We define the Riemann theta function $\theta(\underline{z})$ on $\mathcal{K}_{l-1}$ as

$$\begin{aligned} &\theta(\underline{z}) = \sum_{\underline{\mathcal{F}} \in \mathbb{Z}^{l-1}} \exp\{2\pi i \langle \underline{\mathcal{F}}, \underline{z} \rangle + \pi i \langle \underline{\mathcal{F}}, \underline{\mathcal{F}\tau} \rangle\}, \qquad \underline{z} = (z_1, \ldots, z_{l-1}) \in \mathbb{C}^{l-1}, \\ &\langle \underline{\mathcal{F}}, \underline{z} \rangle = \sum_{j=1}^{l-1} \mathcal{F}_j z_j, \qquad \langle \underline{\mathcal{F}}, \underline{\mathcal{F}\tau} \rangle = \sum_{j,\sigma=1}^{l-1} \tau_{j\sigma} \mathcal{F}_j \mathcal{F}_\sigma. \end{aligned} \tag{4.14}$$

We then introduce the function

$$\begin{aligned} &\theta(\underline{z}(u, \underline{\tilde{\mu}}(n))) = \theta(\underline{\Lambda} - \mathcal{A}(u) + \underline{\rho}(n)), \\ &u \in \mathcal{K}_{l-1}, \qquad \underline{\tilde{\mu}}(n) = \{\tilde{\mu}_1(n), \ldots, \tilde{\mu}_{l-1}(n)\} \in \sigma^{l-1}\mathcal{K}_{l-1}, \end{aligned} \tag{4.15}$$

where $\sigma^{l-1}\mathcal{K}_{l-1}$ is the $(l-1)$th symmetric power of $\mathcal{K}_{l-1}$, and the expression of the vector Λ depending on the base point Q_0 is

$$\Lambda_j = \frac{1}{2}(1 + \tau_{jj}) - \sum_{\substack{\sigma=1, \\ \sigma \neq j}}^{l-1} \int_{\mathbf{w}_\sigma} \omega_\sigma \int_{Q_0}^{u} \omega_j, \qquad j = 1, \ldots, l-1.$$

Theorem 1. *Let $u = (\lambda, f) \in \mathcal{K}_{l-1} \backslash \{u_{\infty'}, u_{\infty''}\}$, $(n, n_0) \in \mathbb{Z}^2$, and $\mathcal{D}_{\underline{\tilde{\mu}}(n)}$ be a nonspecial divisor. Then*

$$\begin{aligned} \Theta(u, n) &= \frac{\theta(\underline{z}(u_{\infty''}, \underline{\tilde{\mu}}(n)))\theta(\underline{z}(u, \underline{\tilde{\mu}}^+(n)))}{\theta(\underline{z}(u_{\infty''}, \underline{\tilde{\mu}}^+(n)))\theta(\underline{z}(u, \underline{\tilde{\mu}}(n)))} \exp\left(\int_{Q_0}^{u} \omega^{(3)}_{u_{\infty'}, u_{\infty''}} - \ell_2(Q_0) \right), \\ \Xi_1(u, n, n_0) &= \frac{\theta(\underline{z}(u_{\infty''}, \underline{\tilde{\mu}}(n_0)))\theta(\underline{z}(u, \underline{\tilde{\mu}}(n)))}{\theta(\underline{z}(u_{\infty''}, \underline{\tilde{\mu}}(n)))\theta(\underline{z}(u, \underline{\tilde{\mu}}(n_0)))} \times \\ &\quad \times \exp\left((n - n_0) \left(\int_{Q_0}^{u} \omega^{(3)}_{u_{\infty'}, \infty''} - \ell_2(Q_0) \right) \right). \end{aligned} \tag{4.16}$$

The divisor $\mathcal{D}_{\underline{\tilde{\mu}}}$ can be linearized as follows under the Abel map:

$$\underline{\rho}(n) = \underline{\rho}(n_0) + (n - n_0)\big(\underline{\mathcal{A}}(u_{\infty''}) - \underline{\mathcal{A}}(u_{\infty'}) \big). \tag{4.17}$$

Proof. Using the Abel theorem, we can obtain (4.17) from (4.7) and deduce the equations by (4.10),

$$\begin{aligned} &\exp\left(\int_{Q_0}^{u} \omega^{(3)}_{u_{\infty'}, u_{\infty''}} - \ell_2(Q_0) \right) \underset{\varsigma \to 0}{=} \varsigma \exp(\ell_1(Q_0) - \ell_2(Q_0) + O(\varsigma^2)), \quad u \to u_{\infty'}, \\ &\exp\left(\int_{Q_0}^{u} \omega^{(3)}_{u_{\infty'}, u_{\infty''}} - \ell_2(Q_0) \right) \underset{\eta \to 0}{=} \eta^{-1} + O(1), \qquad u \to u_{\infty''}. \end{aligned} \tag{4.18}$$

Letting ϕ denote the right-hand side of (4.16), we find that ϕ and Θ have the same zeros and poles. According to the Riemann–Roch theorem and Eqs. (4.3) and (4.18), we have

$$\frac{\phi}{\Theta} \underset{\eta \to 0}{=} \frac{(1 + O(\eta))(\eta^{-1} + O(1))}{\eta^{-1} + O(1)} = 1 + O(\eta), \qquad u \to u_{\infty''}.$$

Hence, the Riemann theta representation of $\Theta(u, n)$ can be proved and the representation of $\Xi_1(u, n, n_0)$ can also be proved by (3.8) and the representation of Θ.

Theorem 2. *Let $n \in \mathbb{Z}$, $\mathcal{D}_{\underline{\tilde{\mu}}(n)}$ be a nonspecial divisor. Then the potentials q, r, s, and v can be expressed in terms of the Riemann theta function as*

$$\begin{aligned}
q(n) = & -\omega_0^{\infty''} - \sum_{j=1}^{l-1}\left(\frac{1}{3\beta_0}\mathbb{Q}_{j,l-1} + \frac{1}{\alpha_0\beta_0}\mathbb{Q}_{j,2p-1}\right)\frac{\partial}{\partial_{zj}}\ln\frac{\theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}^{++}(n)))}{\theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}^{+}(n)))} - \\
& -\frac{\theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}(n)))\theta(\underline{z}(u_{\infty'},\underline{\mu}^{++}(n)))}{\theta(\underline{z}(u_{\infty''},\underline{\tilde{\mu}}^{++}(n)))\theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}^{+}(n)))}\exp(\ell_1(Q_0)-\ell_2(Q_0)),\\
r(n) = & -\omega_0^{\infty'} + \sum_{j=1}^{l-1}\left(\frac{2}{3\alpha_0}\mathbb{Q}_{j,l-1} - \frac{1}{\alpha^2}\mathbb{Q}_{j,2p-1}\right)\frac{\partial}{\partial_{zj}}\ln\frac{\theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}^{+}(n)))}{\theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}(n)))},\\
\frac{s(n)}{s^-(n)} = & \ \omega_0^{\infty'} - \sum_{j=1}^{l-1}\left(\frac{2}{3\alpha_0}\mathbb{Q}_{j,l-1} - \frac{1}{\alpha^2}\mathbb{Q}_{j,2p-1}\right)\frac{\partial}{\partial_{zj}}\ln\frac{\theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}^{+}(n)))}{\theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}(n)))},\\
v(n) = & -\frac{\theta(\underline{z}(u_{\infty''},\underline{\mu}(n)))\theta(\underline{z}(u_{\infty'},\underline{\mu}^{+}(n)))}{\theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}(n)))^2}\exp(\ell_1(Q_0)-\ell_2(Q_0)).
\end{aligned} \tag{4.19}$$

Proof. According to the Abel theorem and (4.6), we have

$$\underline{\rho}^+(n) + \underline{\mathcal{A}}(u_{\infty'}) = \underline{\rho}(n) + \underline{\mathcal{A}}(u_{\infty''}),$$

so

$$\theta(\underline{z}(u_{\infty''},\underline{\tilde{\mu}}^{+}(n))) = \theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}(n))),$$

whence using $f = \Gamma_{11}^{(p)} + \Gamma_{12}^{(p)} + \Gamma_{13}^{(p)}/\Theta^{-1}$ we have

$$f \underset{\substack{\varsigma\to 0,\\ \eta\to 0}}{=} \begin{cases} \varsigma^{-p-1}[\alpha_0 - \alpha_1\varsigma^2 + O(\varsigma^3)], & u \to u_{\infty'},\ \varsigma = \lambda^{-1},\\ \eta^{-2p-1}[\beta_0 - \beta_1\eta^2 + O(\eta^3)], & u \to u_{\infty''},\ \eta = \lambda^{-1/2}. \end{cases}$$

Using (4.9) and (4.10), we deduce the equality

$$\omega_j = \sum_{h=1}^{l-1}\mathbb{Q}_{ij}\hat{\omega}_h = \sum_{h=1}^{2p-1}\mathbb{Q}_{ij}\frac{\lambda^{h-1}d\lambda}{3f^2+Y_l} + \sum_{h=2p}^{l-1}\mathbb{Q}_{ij}\frac{f\lambda^{h-2p-2}d\lambda}{3f^2+Y_l}, \qquad j = 1,\ldots,l-1.$$

The expression of ω_j can then be obtained by direct calculation:

$$\omega_j \underset{\substack{\varsigma\to 0,\\ \eta\to 0}}{=} \begin{cases} \dfrac{2}{3\alpha_0}\mathbb{Q}_{j,l-1} - \dfrac{1}{\alpha_0{}^2}\mathbb{Q}_{j,2p-1} + O(\varsigma)d\varsigma, & u \to u_{\infty'},\\ -\dfrac{1}{3\beta_0}\mathbb{Q}_{j,l-1} - \dfrac{1}{\alpha_0\beta_0}\mathbb{Q}_{j,2p-1} + O(\eta)d\eta, & u \to u_{\infty''}. \end{cases}$$

With the Riemann theta representation of $\Theta(u,n)$ in (4.16), we have

$$\begin{aligned}
&\frac{\theta(\underline{z}(u,\underline{\tilde{\mu}}^{+}(n)))}{\theta(\underline{z}(u,\underline{\tilde{\mu}}(n)))} = \frac{\theta(\underline{\Delta} - \underline{\mathcal{A}}(u) + \underline{\rho}^+(n))}{\theta(\underline{\Delta} - \underline{\mathcal{A}}(u) + \underline{\rho}(n))} = \frac{\theta(\underline{\Delta} - \underline{\mathcal{A}}(u_{\infty'}) + \underline{\rho}^+(n) + \int_u^{u_{\infty'}}\underline{\omega})}{\theta(\underline{\Delta} - \underline{\mathcal{A}}(u_{\infty'}) + \underline{\rho}(n) + \int_u^{u_{\infty'}}\underline{\omega})} \underset{\varsigma\to 0}{=}\\
&\underset{\varsigma\to 0}{=} \frac{\theta(\ldots,\Lambda_j - \mathcal{A}_j(u_{\infty'}) + \rho_j^+(n) - \left(\frac{2}{3\alpha_0}\mathbb{Q}_{j,l-1} - \frac{1}{\alpha_0{}^2}\mathbb{Q}_{j,2p-1}\right)\varsigma + O(\varsigma^2),\ldots)}{\theta(\cdots,\Lambda_j - \mathcal{A}_j(u_{\infty'}) + \rho_j(n) - \left(\frac{2}{3\alpha_0}\mathbb{Q}_{j,l-1} - \frac{1}{\alpha_0{}^2}\mathbb{Q}_{j,2p-1}\right)\varsigma + O(\varsigma^2),\ldots)} \underset{\varsigma\to 0}{=}\\
&\underset{\varsigma\to 0}{=} \frac{\theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}^{+}(n))) - \sum_{j=1}^{l-1}\left(\frac{2}{3\alpha_0}\mathbb{Q}_{j,l-1} - \frac{1}{\alpha_0{}^2}\mathbb{Q}_{j,2p-1}\right)\frac{\partial}{\partial z_j}\ln\theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}(n)))\varsigma + O(\varsigma^2)}{\theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}(n))) - \sum_{j=1}^{l-1}\left(\frac{2}{3\alpha_0}\mathbb{Q}_{j,l-1} - \frac{1}{\alpha_0{}^2}\mathbb{Q}_{j,2p-1}\right)\frac{\partial}{\partial z_j}\ln\theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}^{+}(n)))\varsigma + O(\varsigma^2)} \underset{\varsigma\to 0}{=}\\
&\underset{\varsigma\to 0}{=} \frac{\theta'_+}{\theta'}\left(1 - \sum_{j=1}^{l-1}\left(\frac{2}{3\alpha_0}\mathbb{Q}_{j,l-1} - \frac{1}{\alpha_0{}^2}\mathbb{Q}_{j,2p-1}\right)\frac{\partial}{\partial z_j}\ln\frac{\theta'_+}{\theta'}\varsigma + O(\varsigma^2)\right),
\end{aligned}$$

where $u \to u_{\infty'}$ and $\theta' = \theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}(n)))$, $\theta'_+ = \theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}^{+}(n)))$.

Similarly to the previous steps, we obtain

$$\frac{\theta(\underline{z}(u,\underline{\tilde{\mu}}^+(n)))}{\theta(\underline{z}(u,\underline{\tilde{\mu}}(n)))}=\frac{\theta''_+}{\theta''}\Big(1-\sum_{j=1}^{l-1}(-\frac{1}{3\beta_0}\mathbb{Q}_{j,l-1}-\frac{1}{\alpha_0\beta_0}\mathbb{Q}_{j,2p-1})\frac{\partial}{\partial z_j}\ln\frac{\theta''_+}{\theta''}\eta+O(\eta^2)\Big),$$

where $\eta\to 0$, $\theta''=\theta(\underline{z}(u_{\infty''},\underline{\tilde{\mu}}(n)))$, and $\theta''_+=\theta(\underline{z}(u_{\infty''},\underline{\tilde{\mu}}^+(n)))$. Hence, we have the following formulas as $\eta\to 0$ and $\varsigma\to 0$:

$$\Theta(u,n)\underset{\substack{\varsigma\to 0,\\ \eta\to 0}}{=}\begin{cases}\Big(\varsigma+\Big(\omega_0^{\infty'}-\sum_{j=1}^{l-1}\Big(\frac{2}{3\alpha_0}\mathbb{Q}_{j,l-1}-\frac{1}{\alpha_0{}^2}\mathbb{Q}_{j,2p-1}\Big)\frac{\partial}{\partial z_j}\ln\frac{\theta'_+}{\theta'}\Big)\varsigma^2+O(\varsigma^3)\Big)\\ \quad\times\frac{\theta''\theta'_+}{\theta''_+\theta'}\exp(\ell_1(Q_0)-\ell_2(Q_0)), & u\to u_{\infty'},\\ \eta^{-1}+\omega_0^{\infty''}+\sum_{j=1}^{l-1}(\frac{1}{3\beta_0}\mathbb{Q}_{j,l-1}+\frac{1}{\alpha_0\beta_0}\mathbb{Q}_{j,2p-1})\frac{\partial}{\partial z_j}\ln\frac{\theta''_+}{\theta''}+O(\eta),\\ & u\to u_{\infty''}.\end{cases}$$

In addition, in accordance with (4.2) and (4.3), we have

$$\Theta(u,n)\underset{\substack{\varsigma\to 0,\\ \eta\to 0}}{=}\begin{cases}-v\varsigma+\Big(v^+-\frac{s}{s^-}v^-+rv\Big)\varsigma^2+O(\varsigma^3), & u\to u_{\infty'},\\ \eta^{-1}-q^-+v+O(\eta), & u\to u_{\infty''}.\end{cases}$$

Formulas (4.19) are thus proved.

We let the **o**-periods of $\omega^{(3)}_{u_{\infty'},u_{\infty''}}$ be denoted as

$$\underline{\mathcal{A}}^{(3)}=(\mathcal{A}^{(3)}_1,\ldots,\mathcal{A}^{(3)}_{l-1}),\qquad \mathcal{A}^{(3)}_\sigma=\frac{1}{2\pi i}\int_{\mathbf{o}_\sigma}\omega^{(3)}_{u_{\infty'},u_{\infty''}},\quad \sigma=1,\ldots,l-1. \tag{4.20}$$

Combining (4.11), (4.17), (4.19), and (4.20) shows that the Riemann theta representation for $q(n)$, $r(n)$, $s(n)$, and $v(n)$ has a remarkable linearity in $n\in\mathbb{Z}\times\mathbb{R}$. As a matter of fact, Eqs. (4.19) can be rewritten as

$$\begin{aligned}
q(n)&=-\,\omega_0^{\infty''}+\sum_{j=1}^{l-1}\mathbb{K}^{(\infty'')}_{j,0}\frac{\partial}{\partial_{z_j}}\ln\frac{\theta(\underline{\mathcal{B}}_2+\underline{\mathcal{A}}^{(3)}n)}{\theta(\underline{\mathcal{B}}_1+\underline{\mathcal{A}}^{(3)}n)}-\frac{\theta(\underline{\mathcal{B}}_0+\underline{\mathcal{A}}^{(3)}n)\theta(\underline{\mathcal{B}}_2+\underline{\mathcal{A}}^{(3)}n)}{\theta(\underline{\mathcal{B}}'_2+\underline{\mathcal{A}}^{(3)}n)\theta(\underline{\mathcal{B}}_1+\underline{\mathcal{A}}^{(3)}n)}\exp(\ell_1(Q_0)-\ell_2(Q_0)),\\
r(n)&=-\,\omega_0^{\infty'}+\sum_{j=1}^{l-1}\mathbb{K}^{(\infty')}_{j,0}\frac{\partial}{\partial_{z_j}}\ln\frac{\theta(\underline{\mathcal{B}}_1+\underline{\mathcal{A}}^{(3)}n)}{\theta(\underline{\mathcal{B}}_0+\underline{\mathcal{A}}^{(3)}n)},\\
\frac{s(n)}{s^-(n)}&=\omega_0^{\infty'}-\sum_{j=1}^{l-1}\mathbb{K}^{(\infty')}_{j,0}\frac{\partial}{\partial_{z_j}}\ln\frac{\theta(\underline{\mathcal{B}}_1+\underline{\mathcal{A}}^{(3)}n)}{\theta(\underline{\mathcal{B}}_0+\underline{\mathcal{A}}^{(3)}n)},\\
v(n)&=-\frac{\theta(\underline{\mathcal{B}}'_0+\underline{\mathcal{A}}^{(3)}n)\theta(\underline{\mathcal{B}}_1+\underline{\mathcal{A}}^{(3)}n)}{\theta(\underline{\mathcal{B}}_0+\underline{\mathcal{A}}^{(3)}n)^2}\exp(\ell_1(Q_0)-\ell_2(Q_0)).
\end{aligned}$$

where

$$\begin{aligned}
&\underline{\mathcal{B}}_0=\underline{\mathcal{S}}-\underline{\mathcal{A}}^{(3)},\qquad \underline{\mathcal{B}}'_0=\underline{\mathcal{S}}'-\underline{\mathcal{A}}^{(3)},\qquad \underline{\mathcal{B}}_1=\underline{\mathcal{S}}+\underline{\mathcal{A}}^{(3)},\\
&\underline{\mathcal{B}}'_1=\underline{\mathcal{S}}'+\underline{\mathcal{A}}^{(3)},\qquad \underline{\mathcal{B}}_2=\underline{\mathcal{S}}+2\underline{\mathcal{A}}^{(3)},\qquad \underline{\mathcal{B}}'_2=\underline{\mathcal{S}}'+2\underline{\mathcal{A}}^{(3)},\\
&\mathbb{K}^{(\infty')}_{j,0}=\frac{2}{3\alpha_0}\mathbb{Q}_{j,l-1}-\frac{1}{\alpha^2}\mathbb{Q}_{j,2p-1},\qquad \mathbb{K}^{(\infty'')}_{j,0}=-\frac{1}{3\beta_0}\mathbb{Q}_{j,l-1}-\frac{1}{\alpha_0\beta_0}\mathbb{Q}_{j,2p-1},\\
&\underline{\mathcal{S}}=\underline{\Lambda}-\underline{\mathbb{O}}(u_{\infty'})+\underline{\rho}(n_0)-\underline{\mathcal{A}}^{(3)}n_0,\qquad \underline{\mathcal{S}}'=\underline{\Lambda}-\underline{\mathbb{O}}(u_{\infty''})+\underline{\rho}(n_0)-\underline{\mathcal{A}}^{(3)}n_0.
\end{aligned}$$

5. Algebro-geometric solutions of the hierarchy in the time-dependent case

In this section, we discuss the algebro-geometric solutions of (1.3) in the time-dependent case. We first define the time-dependent Baker–Akhiezer function

$$
\begin{aligned}
&E\Xi(u,n,n_0,t_m,t_{0m}) = U(q(n,t_m),r(n,t_m),s(n,t_m),v(n,t_m);\lambda(u))\Xi(u,n,n_0,t_m,t_{0m}),\\
&\Xi_{tm}(u,n,n_0,t_m,t_{0m}) = \widehat{\Gamma}^m U(q(n,t_m),r(n,t_m),s(n,t_m),v(n,t_m);\lambda(u))\Xi(u,n,n_0,t_m,t_{0m}),\\
&\Gamma^{(p)}(q(n,t_m),r(n,t_m),s(n,t_m),v(n,t_m);\lambda(u))\Xi(u,n,n_0,t_m,t_{0m}) = f(u)\Xi(u,n,n_0,t_m,t_{0m}),\\
&\Xi_1(u,n_0,n_0,t_{0m},t_{0m}) = 1,\\
&u\in\mathcal{K}_{l-1}\backslash\{u_{\infty'},u_{\infty''}\},\quad (n,t_m),(n_0,t_{0m})\in\mathbb{Z}\times\mathbb{R}.
\end{aligned}
\tag{5.1}
$$

From the compatibility condition for Eqs. (5.1), we have

$$
U_{t_m} - (E\widehat{\Gamma}^m)U + U\widehat{\Gamma}^m = 0,\qquad (E\Gamma^{(p)})U - U\Gamma^{(p)} = 0,\qquad \Gamma^{(p)}_{t_m} - [\widehat{\Gamma}^m,\Gamma^{(p)}] = 0. \tag{5.2}
$$

Direct calculation shows that $\mathcal{F}_l(\lambda,f) = \det(fI-\Gamma^{(p)})$ satisfies the stationary zero-curvature equation. The Lax pair $\Gamma^{(p)}$ characteristic polynomial is a constant independent of n and t_m, and we have

$$
\det(fI-\Gamma^{(p)}) = f^3 - f^2X_l(\lambda) + fY_l(\lambda) - Z_l(\lambda).
$$

Then the trigonal curve $\mathcal{K}_{l-1}$ is naturally defined in the time-dependent case as

$$
\mathcal{K}_{l-1}\colon\ f_l(\lambda,f) = f^3 - f^2X_l(\lambda) + fY_l(\lambda) - Z_l(\lambda).
$$

The meromorphic function $\Theta(u,n,t_m)$ on $\mathcal{K}_{l-1}$ is defined as

$$
\Theta(u,n,t_m) = \frac{\Xi_2(u,n,n_0,t_m,t_{0m})}{\Xi_1(u,n,n_0,t_m,t_{0m})},\qquad u\in\mathcal{K}_{l-1},\quad (n,t_m)\in\mathbb{Z}\times\mathbb{R}, \tag{5.3}
$$

whence we have

$$
\Xi_1(u,n,n_0,t_0,t_{0m}) = \begin{cases}
\displaystyle\prod_{n'=n_0}^{n-1}(1+q(n,t_m)\Theta(u,n',t_m)), & n\geqslant n_0+1,\\
1, & n=n_0,\\
\displaystyle\prod_{n'=n_0}^{n-1}(1+q(n,t_m)\Theta(u,n',t_m))^{-1}, & n\leqslant n_0-1.
\end{cases}
$$

From (5.3), we have

$$
\begin{aligned}
\Theta(u,n,t_m) &= \frac{f\Gamma^m_{23}(\lambda,n,t_m)+A_l(\lambda,n,t_m)}{f\Gamma^m_{13}(\lambda,n,t_m)+B_l(\lambda,n,t_m)} =\\
&= \frac{E_{l-1}(\lambda,n,t_m)}{f^2\Gamma^m_{23}(\lambda,n,t_m)-fA_l(\lambda,n,t_m)+C_l(\lambda,n,t_m)} =\\
&= \frac{f^2\Gamma^m_{13}(\lambda,n,t_m)-fB_l(\lambda,n,t_m)+D_l(\lambda,n,t_m)}{F_{l-1}(\lambda,n,t_m)},
\end{aligned}
\tag{5.4}
$$

where $u = (\lambda, f)$ and the elements such as $A_l(\lambda, n, t_m)$ are defined the same as in the stationary case. Similarly,

$$F_{l-1}(\lambda, n, t_m) = F_{l-1,0} \prod_{j=1}^{l-1} (\lambda - \mu_j(n, t_m)),$$
$$E_{l-1}(\lambda, n, t_m) = -F_{l-1,0} \prod_{j=1}^{l-1} (\lambda - \mu_j^+(n, t_m)).$$

We give the expressions for $\{\tilde{\mu}_j(n, t_m)\}_{j=1,\dots,l-1} \subset \mathcal{K}_{l-1}$ and $\{\tilde{\mu}_j^+(n, t_m)\}_{j=1,\dots,l-1} \subset \mathcal{K}_{l-1}$ in the form

$$\begin{aligned} \tilde{\mu}_j(n, t_m) &= \big(\mu_j(n, t_m), f(\hat{\mu}_j(n, t_m))\big) = \\ &= \left(\mu_j(n, t_m) - \frac{B_l(\mu_j(n, t_m), n, t_m)}{\Gamma_{32}^m(\mu_j(n, t_m), n, t_m)}\right), \\ \tilde{\mu}_j^+(n, t_m) &= \big(\mu_j^+(n, t_m), f(\hat{\mu}_j^+(n, t_m))\big) = \\ &= \left(\mu_j^+(n, t_m) - \frac{A_l(\mu_j(n, t_m), n, t_m)}{\Gamma_{32}^m(\mu_j^+(n, t_m), n, t_m)}\right), \qquad (n, t_m) \in \mathbb{Z} \times \mathbb{R}. \end{aligned} \tag{5.5}$$

From (5.4), the divisor of $\Theta(u, n, t_m)$ can be expressed as

$$(\Theta(u, n, t_m)) = \mathcal{D}_{u_{\infty'}, \tilde{\mu}_1^+(n, t_m), \dots, \tilde{\mu}_{l-1}^+(n, t_m)}(u) - \mathcal{D}_{u_{\infty''}, \tilde{\mu}_1^+(n, t_m), \dots, \tilde{\mu}_{l-1}^+(n, t_m)}(u), \tag{5.6}$$

and hence $\Theta(u, n, t_m)$ still has l zeros, $u_{\infty'}, \tilde{\mu}_1^+(n, t_m), \dots, \tilde{\mu}_{l-1}^+(n, t_m)$, and l poles $u_{\infty''}, \tilde{\mu}_1(n, t_m), \dots, \tilde{\mu}_{l-1}(n, t_m)$.

By the same calculation, it is clear that $\Theta(u, n, t_m)$ satisfies the Riccati-type equation

$$\begin{aligned} &q^-(n, t_m) q(n, t_m) \Theta^+(u, n, t_m) \Theta(u, n, t_m) \Theta^-(u, n, t_m) = \\ &= \left(v(n, t_m) q^-(n, t_m) - \frac{s(n, t_m)}{s^-(n, t_m)}\right) \Theta^-(u, n, t_m) + \\ &\quad + (\lambda + r(n, t_m)) \Theta(u, n, t_m) - \Theta^+(u, n, t_m) + v(n, t_m) + \\ &\quad + (\lambda + r(n, t_m)) q^-(n, t_m) \Theta^-(u, n, t_m) \Theta(u, n, t_m) - \\ &\quad - q(n, t_m) \Theta(u, n, t_m) \Theta^+(u, n, t_m) - q^-(n, t_m) \Theta^-(u, n, t_m) \Theta^+(u, n, t_m). \end{aligned} \tag{5.7}$$

Also similarly to the preceding subsection, it can be shown that the function $\Theta(u, n, t_m)$ satisfies the system of equations

$$\begin{aligned} &\Theta(u, n, t_m) \Theta(u^*, n, t_m) \Theta(u^{**}, n, t_m) = -\frac{E_{l-1}(\lambda, n, t_m)}{F_{l-1}(\lambda, n, t_m)}, \\ &\Theta(u, n, t_m) + \Theta(u^*, n, t_m) + \Theta(u^{**}, n, t_m) = \frac{3D_l(\lambda, n, t_m) - 2\Gamma_{32}^m(\lambda, n, t_m) Y_l(\lambda, n, t_m)}{F_{l-1}(\lambda, n, t_m)}, \\ &\frac{1}{\Theta(u, n, t_m)} + \frac{1}{\Theta(u^*, n, t_m)} + \frac{1}{\Theta(u^{**}, n, t_m)} = \frac{3C_l(\lambda, n, t_m) - 2\Gamma_{12}^m(\lambda, n, t_m)(\lambda, n, t_m) Y_l(\lambda, n, t_m)}{E_{l-1}(\lambda, n, t_m)}. \end{aligned} \tag{5.8}$$

Differentiating the meromorphic function with respect to t_m, we have

$$\begin{aligned} \Theta_{t_m} = \left(\frac{\Xi_1^+}{\Xi_1}\right)_{t_m} &= \frac{\Xi_{1,t_m}^+ \Xi_1 - \Xi_1^+ \Xi_{1,t_m}}{\Xi_1^2} = \frac{\Xi_1^+}{\Xi_1}\left(\frac{\Xi_{1,t_m}^+}{\Xi_1^+} - \frac{\Xi_{1,t_m}}{\Xi_1}\right) = \\ &= \Theta \Delta \frac{\Xi_{1,t_m}}{\Xi_1} = \Theta \Delta \left(\widehat{\Gamma}_{11}^m + \widehat{\Gamma}_{12}^m \Theta + \widehat{\Gamma}_{13}^m \frac{1}{\Theta^-}\right), \end{aligned}$$

whence

$$\frac{\Theta(u, n, t_m)_{t_m}}{\Theta(u, n, t_m)} = \Delta \left(\widehat{\Gamma}_{11}^m(\lambda, n, t_m) + \widehat{\Gamma}_{12}^m(\lambda, n, t_m) \Theta + \widehat{\Gamma}_{13}^m(\lambda, n, t_m) \frac{1}{\Theta^-(u, n, t_m)}\right), \tag{5.9}$$

where Δ is the difference operator and $\Delta = E - 1$.

The dynamics of $\mu_j(n,t_m)$ of $F_{l-1}(\lambda,n,t_m)$ can be described by Dubrovin-type equations in accordance with the following lemma.

Lemma 1. *Let $(n,t_m)\in\mathbb{Z}\times\mathbb{R}$. The zeros $\{\mu_j(n,t_m)\}_{j=\overline{1,l-1}}$ of $F_{l-1}(\lambda,n,t_m)$ satisfy the equations*

$$\begin{aligned}\mu_{j,t_m}(n,t_m) = [\widehat{\Gamma}^m_{12}(\mu_j(n,t_m),n,t_m)\Gamma^{(p)}_{13}(\mu_j(n,t_m),n,t_m) -\\ - \widehat{\Gamma}^m_{13}(\mu_j(n,t_m),n,t_m)\Gamma^{(p)}_{12}(\mu_j(n,t_m),n,t_m)]\frac{3f^2(\hat{\mu}_j(n,t_m))+Y_l(\mu_j(n,t_m))}{F_{l-1,0}\prod_{\substack{\sigma=1,\\ \sigma\neq j}}^{l-1}(\mu_j(n,t_m)-\mu_\sigma(n,t_m))},\end{aligned} \tag{5.10}$$

where $1\leqslant j\leqslant l-1$.

Proof. From (3.10), (3.11), and (5.2), we have

$$\begin{aligned}F_{l-1,t_m}(\lambda,n,t_m) &= \big((\Gamma^{(p)}_{13})^2\Gamma^{(p)}_{32}+\Gamma^{(p)}_{12}\Gamma^{(p)}_{13}(\Gamma^{(p)}_{22}-\Gamma^{(p)}_{33})-(\Gamma^{(p)}_{12})^2\Gamma^{(p)}_{23}\big)_{t_m} =\\ &= 3\widehat{\Gamma}^m_{11}F_{l-1}+3(\widehat{\Gamma}^m_{12}A_l-\widehat{\Gamma}^m_{13}G_l)-2(\widehat{\Gamma}^m_{12}\Gamma^{(p)}_{13}-\widehat{\Gamma}^m_{13}\Gamma^{(p)}_{12})Y_l =\\ &= 3\widehat{\Gamma}^m_{11}F_{l-1}+3\widehat{\Gamma}^m_{12}(\Gamma^{(p)}_{23}G_l-\Gamma^{(p)}_{22}B_l)-3\widehat{\Gamma}^m_{13}(\Gamma^{(p)}_{32}B_l-\Gamma^{(p)}_{33}G_l)+\\ &\quad+(\widehat{\Gamma}^m_{12}\Gamma^{(p)}_{13}-\widehat{\Gamma}^m_{13}\Gamma^{(p)}_{12})Y_l.\end{aligned}$$

With (3.12) and (5.5), we then have

$$\frac{B_l}{\Gamma^{(p)}_{13}}\bigg|_{\lambda=\mu_j(n,t_m)} = \frac{G_l}{\Gamma^{(p)}_{12}}\bigg|_{\lambda=\mu_j(n,t_m)} = -f(\tilde{\mu}_j(n,t_m)),$$

whence

$$\begin{aligned}&\widehat{\Gamma}^m_{12}(\Gamma^{(p)}_{23}G_l-\Gamma^{(p)}_{22}B_l)|_{\lambda=\mu_j(n,t_m)} = f^2(\hat{\mu}_j(n,t_m))\widehat{\Gamma}^m_{12}\Gamma^{(p)}_{13}|_{\lambda=\mu_j(n,t_m)},\\ &\widehat{\Gamma}^m_{13}(\Gamma^{(p)}_{32}B_l-\Gamma^{(p)}_{33}G_l)|_{\lambda=\mu_j(n,t_m)} = f^2(\hat{\mu}_j(n,t_m))\widehat{\Gamma}^m_{13}\Gamma^{(p)}_{12}|_{\lambda=\mu_j(n,t_m)},\\ &(\widehat{\Gamma}^m_{12}B_l-\widehat{\Gamma}^m_{13}G_l)|_{\lambda=\mu_j(n,t_m)} = -f(\hat{\mu}_j(n,t_m))(\widehat{\Gamma}^m_{12}\Gamma^{(p)}_{13}-\widehat{\Gamma}^m_{13}\Gamma^{(p)}_{12})\big|_{\lambda=\mu_j(n,t_m)}.\end{aligned}$$

Therefore,

$$\begin{aligned}F_{l-1,t_m}(\lambda,n,t_m)|_{\lambda=\mu_j(n,t_m)} &= -\mu_{j,t_m}(n,t_m)F_{l-1,0}\prod_{\substack{\sigma=1,\\ \sigma\neq j}}^{l-1}(\mu_j(n,t_m)-\mu_\sigma(n,t_m)) =\\ &= \big(3f^2(\hat{\mu}_j(n,t_m))+Y_l(\mu_j(n,t_m))\big)\big(\widehat{\Gamma}^m_{12}\Gamma^{(p)}_{13}-\widehat{\Gamma}^m_{13}\Gamma^{(p)}_{12}\big)\big|_{\lambda=\mu_j(n,t_m)},\end{aligned}$$

and Eq. (5.10) is thus proved.

Moreover, in accordance with (5.1), we have

$$\begin{aligned}\Xi_1(u,n,n_0,t_m,t_{0m}) = \exp\bigg(\int_{t_{0m}}^{t_m}\Big(\widehat{\Gamma}^m_{11}(\lambda,n,t_m)+\widehat{\Gamma}^m_{12}(\lambda,n,t_m)\Theta+\widehat{\Gamma}^m_{13}(\lambda,n,t_m)\frac{1}{\Theta^-(u,n,t_m)}\Big)dt'\bigg)\times\\ \times\begin{cases}\prod\limits_{n'=n_0}^{n-1}(1+q(n,t_m)\Theta(u,n',t_m)), & n\geqslant n_0+1,\\ 1, & n=n_0,\\ \prod\limits_{n'=n}^{n_0-1}(1+q(n,t_m)\Theta(u,n',t_m))^{-1}, & n\leqslant n_0-1,\end{cases}\end{aligned} \tag{5.11}$$

and

$$\Xi_1(u,n,n_0,t_m,t_{0m}) = \Xi_1(u,n,n_0,t_m,t_m)\Xi_1(u,n_0,n_0,t_m,t_{0m}), \tag{5.12}$$

where $u=(\lambda,f)\in\mathcal{K}_{l-1}\backslash\{u_{\infty'},u_{\infty''}\}$, and $(n,t_m),(n_0,t_{0m})\in\mathbb{Z}\times\mathbb{R}$. Using the function in (5.11), we define $\Pi_m(u,n,t_m)$ as

$$\Pi_m(u,n,t_m) = \widehat{\Gamma}^m_{11}(\lambda,n,t_m) + \widehat{\Gamma}^m_{12}(\lambda,n,t_m)\Theta(u,n,t_m) + \widehat{\Gamma}^m_{13}\frac{1}{\Theta^-(u,n,t_m)},$$

whence

$$\widetilde{\Pi}^{(k)}_m(u,n,t_m) = \widetilde{\widehat{\Gamma}}^{(m,k)}_{11}(\lambda,n,t_m) + \widetilde{\widehat{\Gamma}}^{(m,k)}_{12}(\lambda,n,t_m)\Theta(u,n,t_m) + \widetilde{\widehat{\Gamma}}^{(m,k)}_{13}\frac{1}{\Theta^-(u,n,t_m)}, \tag{5.13}$$

where

$$\widetilde{\widehat{\Gamma}}^{(m,1)}_{1j} = \tilde{\Gamma}^m_{1j}|_{\hat{\alpha}_0=1,\hat{\beta}_1=0}, \qquad \widetilde{\widehat{\Gamma}}^{(m,2)}_{1j} = \tilde{\Gamma}^m_{1j}|_{\hat{\alpha}_0=1,\hat{\beta}_1=0},$$

and $\hat{\alpha}_1=\cdots=\hat{\alpha}_m=\hat{\beta}_1=\cdots=\hat{\beta}_m=0$. Hence,

$$\Pi_m(u,n,t_m) = \sum_{h=0}^{m}\hat{\alpha}_{m-h}\widetilde{\Pi}^{(1)}_h(u,n,t_m) + \sum_{h=0}^{m}\hat{\beta}_{m-h}\widetilde{\Pi}^{(2)}_h(u,n,t_m). \tag{5.14}$$

Lemma 2. *Let $(n,t_m)\in\mathbb{Z}\times\mathbb{R}$, and let $\varsigma=\lambda^{-1}$ and $\eta=\lambda^{1/2}$ be local coordinates near $u_{\infty'}$ and $u_{\infty''}$. Then*

$$\begin{aligned}
\widetilde{\Pi}^{(1)}_m(u,n,t_m) &\underset{\substack{\varsigma\to0,\\ \eta\to0}}{=} \begin{cases} \varsigma^{-(m+1)} + O(\varsigma), & u\to u_{\infty'},\\ -\eta^{-(m+1)} - O(\eta), & u\to u_{\infty''},\end{cases}\\
\widetilde{\Pi}^{(2)}_m(u,n,t_m) &\underset{\substack{\varsigma\to0,\\ \eta\to0}}{=} \begin{cases} -\bar{b}_m(n,t_m) - \bar{b}^+_{m+1}(n,t_m)\bar{d}^+_{m+1}(n,t_m) + O(\varsigma), & u\to u_{\infty'},\\ 2\dfrac{\bar{b}_{m+1}(n,t_m)}{\bar{b}^{++}_{m+1}(n,t_m)} + \dfrac{1}{3}\bar{d}^+_{m+1}(n,t_m) + O(\eta), & u\to u_{\infty''}.\end{cases}
\end{aligned} \tag{5.15}$$

Proof. We set $\tilde{\alpha}^m = \hat{\alpha}^m|_{\hat{\alpha}_0=1,\hat{\beta}_0=0}$. From (5.1) and (5.13), we then have

$$\begin{aligned}
\widetilde{\Pi}^{(1)}_m(u,n,t_m) &= \widetilde{\widehat{\Gamma}}^{(m,1)}_{11}(\lambda,n,t_m) + \widetilde{\widehat{\Gamma}}^{(m,1)}_{12}(\lambda,n,t_m)\Theta(u,n,t_m) + \frac{\widetilde{\widehat{\Gamma}}^{(m,1)}_{13}(\lambda,n,t_m)}{\Theta^-(u,n,t_m)} =\\
&= \tilde{\alpha}^m + \left(\Theta(u,n,t_m) + s(n,t_m)E\frac{1}{\Theta^-(u,n,t_m)}\right)\tilde{b}^m +\\
&\quad + q(n,t_m)s(n,t_m)^2E\tilde{d}^m\frac{1}{\Theta^-(u,n,t_m)}.
\end{aligned}$$

Using (4.2) and (4.3), we have the following result as $m=0$:

$$\widetilde{\Pi}^{(1)}_0(u,n,t_m) \underset{\substack{\varsigma\to0,\\ \eta\to0}}{=} \begin{cases} \varsigma^{-1} + O(\varsigma), & u\to u_{\infty'},\\ -\eta^{-1} - O(\eta), & u\to u_{\infty''},\end{cases}$$

We now suppose that

$$\widetilde{\Pi}^{(1)}_m(u,n,t_m) \underset{\substack{\varsigma\to0,\\ \eta\to0}}{=} \begin{cases} \varsigma^{-(m+1)} + \sum_{j=0}^{\infty}\delta_j(n,t_m)\varsigma^j, & u\to u_{\infty'},\\ -\eta^{-(m+1)} - \sum_{j=0}^{\infty}\kappa_j(n,t_m)\eta^j, & u\to u_{\infty''},\end{cases}$$

where the coefficients of $\{\delta_j(n,t_m)\}_{j\in\mathbb{Q}_0}$ and $\{\kappa_j(n,t_m)\}_{j\in\mathbb{Q}_0}$ can be determined. In accordance with (5.9) and (5.13), we obtain

$$\Theta(u,n,t_m)_{t_m} = \Theta(u,n,t_m)\Delta\widetilde{\Pi}_m^{(1)}(u,n,t_m).$$

Comparing the coefficients of ς and η, we have

$$\begin{aligned}
\delta_{j,t_m} &= \delta_1\Delta\varrho_{j-1} + \delta_2\Delta\varrho_{j-2} + \cdots + \delta_j\Delta\varrho_0,\\
\kappa_{j,t_m} &= \kappa_0\Delta\chi_j + \kappa_2\Delta\chi_{j-1} + \cdots + \kappa_{j-1}\Delta\chi_1, \qquad j \geqslant 0,
\end{aligned}$$

whence

$$\begin{aligned}
&\Delta\chi_0 = 0, \qquad \Delta\chi_1 = \kappa_{1,t_m} = \Delta\bar{b}_{m+1},\\
&\Delta\varrho_0 = \frac{\Theta_{1,t_m}}{\Theta_1} = \frac{v_{t_m}}{v} = 2\tilde{a}_{m+1},\\
&\Delta\varrho_1 = \frac{1}{\Theta_1}\Theta_{2,t_m} - \frac{\Theta_2}{\Theta_1}\Delta\varrho_0 = \Delta(\tilde{c}_{m+1} + (E-1)^{-1}\tilde{a}_{m+1}).
\end{aligned}$$

It then follows that

$$\begin{aligned}
&\chi_0(n,t_m) = 0, \qquad \chi_1(n,t_m) = \bar{b}_{m+1}(n,t_m),\\
&\varrho_0(n,t_m) = 2(E-1)^{-1}\tilde{a}_{m+1}(n,t_m),\\
&\varrho_1(n,t_m) = -\tilde{c}_{m+1}(n,t_m) + (E-1)^{-1}\tilde{a}_{m+1}(n,t_m).
\end{aligned}$$

Therefore, in view of $\Delta\Delta^{-1} = \Delta^{-1}\Delta = 1$, the following results can be deduced:

$$\begin{aligned}
\widetilde{\Pi}_{m+1}^{(1)}(u,n,t_m) \underset{\varsigma\to 0}{=} &\ \widetilde{\Pi}_m^{(1)}(u,n,t_m)\varsigma^{-2} + \left(\Theta(u,n,t_m) + s_n E\frac{1}{\Theta^-(u,n,t_m)}\right)\tilde{b}_{m+1} +\\
&+ \tilde{a}_{m+1}(n,t_m) + q(n,t_m)s(n,t_m)^2 E\tilde{d}_{m+1}\frac{1}{\Theta^-(u,n,t_m)} =\\
= &\ \varsigma^{-(m+1)} + \varsigma^{-1}\big(\varrho_0 - 2\tilde{a}_{m+1}(n,t_m)\big) + \varrho_1 + \tilde{c}_{m+1}(n,t_m) -\\
&- (E-1)^{-1}\tilde{a}_{m+1}(n,t_m) =\\
= &\ \varsigma^{-(m+1)} + O(\varsigma), \qquad u \to u_{\infty'},\\
\widetilde{\Pi}_{m+1}^{(1)}(u,n,t_m) \underset{\eta\to 0}{=} &\ \widetilde{\Pi}_m^{(1)}(u,n,t_m)\eta^{-2} + \left(\Theta(u,n,t_m) + s(n,t_m)E\frac{1}{\Theta^-(u,n,t_m)}\right)\tilde{b}_{m+1} +\\
&+ \tilde{a}_{m+1}(n,t_m) + q(n,t_m)s(n,t_m)^2 E\tilde{d}_{m+1}\frac{1}{\Theta^-(u,n,t_m)} =\\
= &\ -\eta^{-(m+1)} - \eta^{-1}\chi_0 - \chi_1 + \bar{b}_{m+1}(n,t_m) =\\
= &\ -\eta^{-(m+1)} - O(\eta), \qquad u \to u_{\infty''}.
\end{aligned}$$

We have proved (5.15) for $\widetilde{\Pi}_{m+1}^{(k)}$ for $k=1$; the proof for $k=2$ is similar.

Let $\omega_{u_{\infty k},j}^{(2)}$, $j \geqslant 2$, be the normalized differential of the second kind that is holomorphic on $\mathcal{K}_{l-1}\backslash\{u_{\infty k}\}$ and has a jth-order pole at $u_{\infty k}$ $(k=1,2)$,

$$\begin{aligned}
\omega_{u'_\infty,j}^{(2)} &\underset{\varsigma\to 0}{=} (\varsigma^{-j} + O(1))\,d\varsigma, \qquad u \to u_{\infty'}, \quad \varsigma = \lambda^{-1},\\
\omega_{u''_\infty,j}^{(2)} &\underset{\eta\to 0}{=} (\eta^{-j} + O(1))\,d\eta, \qquad u \to u_{\infty''}, \quad \eta = \lambda^{-1/2},
\end{aligned}$$

and has the $\mathbf{w}$-periods $\int_{\mathbf{w}_\sigma} \omega^{(2)}_{u_{\infty^k},j} = 0$, $\sigma = 1, \ldots, l-1$. Let

$$\widehat{\mho}^{(2)}_m = -\sum_{h=0}^{m} \hat{\alpha}_{m-h}(h+1)\omega^{(2)}_{u_{\infty'},h+2} + \sum_{h=0}^{m} \hat{\beta}_{m-h}(2h+1)\hat{\omega}^{(2)}_{u_{\infty''},2h+2}. \tag{5.16}$$

Integrating (5.16), we obtain

$$\int_{Q_0}^{u} \widehat{\mho}^{(2)}_m \underset{\varsigma\to 0}{=} \sum_{h=0}^{m} \hat{\alpha}_{m-h}\varsigma^{-h-1} + \hat{\ell}^{(2)}_1(Q_0) + O(\varsigma), \qquad u \to u_{\infty'},$$
$$\int_{Q_0}^{u} \widehat{\mho}^{(2)}_m \underset{\eta\to 0}{=} -\sum_{h=0}^{m} \hat{\beta}_{m-h}\eta^{-2h-1} + \hat{\ell}^{(2)}_2(Q_0) + O(\eta), \qquad u \to u_{\infty''}.$$

We next find the explicit Riemann theta function representations for the functions $\Theta(u,n,t_m)$ and $\Xi_1(u,n,n_0,t_m,t_{0m})$.

Theorem 3. *Let $u = (\lambda, f) \in \mathcal{K}_{l-1}\backslash\{u_{\infty'}, u_{\infty''}\}$, $(n, n_0, t_m, t_{0m}) \in \mathbb{Z}^2 \times \mathbb{R}^2$. If $\mathcal{D}_{\underline{\tilde{\mu}}(n,t_m)}$ is nonspecial and $(n,t_m) \in \mathbb{Z}\times\mathbb{R}$, then $\Theta(u,n,t_m)$ and $\Xi_1(u,n,n_0,t_m,t_{0m})$ can be represented as*

$$\Theta(u,n,t_m) = \frac{\theta(\underline{z}(u_{\infty''}, \underline{\tilde{\mu}}(n)))\theta(\underline{z}(u, \underline{\tilde{\mu}}^+(n)))}{\theta(\underline{z}(u_{\infty''}, \underline{\tilde{\mu}}^+(n)))\theta(\underline{z}(u, \underline{\tilde{\mu}}(n)))} \exp\biggl(\int_{Q_0}^{u} \omega^{(3)}_{u_{\infty'},u_{\infty''}} - \ell_2(Q_0)\biggr), \tag{5.17}$$

and

$$\Xi_1(u,n,n_0,t_m,t_{0m}) = \frac{\theta(\underline{z}(u_{\infty''}, \underline{\tilde{\mu}}(n_0,t_{0m})))\theta(\underline{z}(u, \underline{\tilde{\mu}}^+(n,t_m)))}{\theta(\underline{z}(u_{\infty''}, \underline{\tilde{\mu}}^+(n,t_m)))\theta(\underline{z}(u, \underline{\tilde{\mu}}(n_0,t_{0m})))} \exp\biggl((n-n_0)\biggl(\int_{Q_0}^{u} \omega^{(3)}_{u_{\infty'},\infty''} - \ell_2(Q_0)\biggr)\biggr) + \\ + (t_m - t_{0m})(\hat{\ell}^{(2)}_2(Q_0)) - \int_{Q_0}^{u} \widehat{\mho}^{(2)}_m. \tag{5.18}$$

Proof. For $t_{0m} = t_m$, $\Xi_1(u,n,n_0,t_m,t_m)$ has the form

$$\Xi_1(u,n,n_0,t_m,t_m) = \frac{\theta(\underline{z}(u_{\infty''}, \underline{\tilde{\mu}}(n_0,t_{0m})))\theta(\underline{z}(u, \underline{\tilde{\mu}}^+(n,t_m)))}{\theta(\underline{z}(u_{\infty''}, \underline{\tilde{\mu}}^+(n,t_m)))\theta(\underline{z}(u, \underline{\tilde{\mu}}(n_0,t_{0m})))} \exp\biggl((n-n_0)\biggl(\int_{Q_0}^{u} \omega^{(3)}_{u_{\infty'},\infty''} - \ell_2(Q_0)\biggr)\biggr).$$

We also need to verify that

$$\Xi_1(u,n_0,n_0,t_m,t_{0m}) = \exp\biggl(\int_{t_{0m}}^{t_m} \Pi_m(u,n_0,t')dt'\biggr).$$

We let $\mathcal{W}_1(u,n_0,n_0,t_m,t_m)$ denote the right-hand side of (5.18). Then

$$\mathcal{W}_1(u,n_0,n_0,t_m,t_m) = \frac{\theta(\underline{z}(u''_{\infty}, \underline{\tilde{\mu}}(n_0,t_{0m})))\theta(\underline{z}(u, \underline{\tilde{\mu}}^+(n_0,t_m)))}{\theta(\underline{z}(u''_{\infty}, \underline{\tilde{\mu}}^+(n_0,t_m)))\theta(\underline{z}(u, \underline{\tilde{\mu}}(n_0,t_{0m})))} \exp\biggl((t_m - t_{0m})\biggl(\hat{\ell}^{(2)}_2(Q_0) - \int_{Q_0}^{u} \hat{\omega}^{(2)}_m\biggr)\biggr).$$

Next, we prove that

$$\Xi_1(u,n_0,n_0,t_m,t_{0m}) = \mathcal{W}_1(u,n_0,n_0,t_m,t_{0m}).$$

First, we use (3.12), (5.4), and (5.13) to obtain the formula

$$\begin{aligned}\Pi_m(u,n,t_m) &= \widehat{\Gamma}_{11}^m(\lambda,n,t_m)+\widehat{\Gamma}_{12}^m(\lambda,n,t_m)\Theta(u,n,t_m)+\widehat{\Gamma}_{13}^m\frac{1}{\Theta^-(u,n,t_m)}=\\ &=\widehat{\Gamma}_{11}^m+\widehat{\Gamma}_{12}^m\frac{f^2\Gamma_{12}^{(p)}-fB_l+A_l}{F_{l-1}}-\widehat{\Gamma}_{13}^m\frac{f^2\Gamma_{12}^{(p)}-fG_l+H_l}{F_{l-1}}=\\ &=\frac{1}{F_{l-1}}\left(\frac{1}{3}F_{l-1,t_m}+(\widehat{\Gamma}_{12}^m\Gamma_{13}^{(p)}-\widehat{\Gamma}_{13}^m\Gamma_{12}^{(p)})\left(f^2+\frac{2}{3}Y_l\right)-(\widehat{\Gamma}_{12}^mB_l-\widehat{\Gamma}_{12}^mG_l)f\right)=\\ &=-\frac{\mu_{j,t_m}(n,t_m)}{\lambda-\mu_j(n,t_m)}+O(1)=\partial_{t_m}\ln(\lambda-\mu_j(n,t_m))+O(1),\quad \lambda\to\mu_j(n,t_m),\end{aligned}$$

where $O(1)\neq 0$. Consequently,

$$\begin{aligned}\Xi_1(u,n_0,n_0,t_m,t_{0m}) &= \exp\left(\int_{t_{0m}}^{t_m}\partial_{t'}\ln(\lambda-\mu_j(n_0,t')\,dt')\right)=\frac{\lambda-\mu_j(n_0,t_m)}{\lambda-\mu_j(n_0,t_{0m})}O(1)=\\ &=\begin{cases}(\lambda-\mu_j(n_0,t_m))O(1), & u\to\tilde{\mu}_j(n_0,t_m)\neq\tilde{\mu}_j(n_0,t_{0m}),\\ O(1), & u\to\tilde{\mu}_j(n_0,t_m)=\tilde{\mu}_j(n_0,t_{0m}),\\ (\lambda-\mu_j(n_0,t_{0m}))^{-1}O(1), & u\to\tilde{\mu}_j(n_0,t_{0m})\neq\tilde{\mu}_j(n_0,t_m).\end{cases}\end{aligned}$$

Hence, $\Xi_1(u,n_0,n_0,t_m,t_{0m})$ and $\mathcal{W}_1(u,n_0,n_0,t_m,t_{0m})$ have the same poles and zeros on $\mathcal{K}_{l-1}$. In addition, we can find that $\mathcal{K}_{l-1}$, $\Xi_1(u,n_0,n_0,t_m,t_{0m})$ and $\mathcal{W}_1(u,n_0,n_0,t_m,t_{0m})$ have the identical essential singularities. Because of $\mathcal{D}_{\underline{\tilde{\mu}}(n,t_m)}$ is nonspecial, Eqs. (5.17) and (5.18) have been proved.

We let the **o**-periods of $\widehat{\mho}_m^{(2)}$ be denoted as

$$\underline{\widehat{\mathcal{A}}}_m^{(2)}=(\widehat{\mathcal{A}}_{m,1}^{(2)},\ldots,\widehat{\mathcal{A}}_{m,l-1}^{(2)}),\qquad \widehat{\mathcal{A}}_{m,\sigma}^{(2)}=\frac{1}{2\pi i}\int_{\mathbf{o}_j}\widehat{\mho}_m^{(2)},\quad \sigma=1,\ldots,l-1. \tag{5.19}$$

Theorem 4 (straightening out of the flows). *The following equality holds*:

$$\underline{\rho}(n,t_m)=\underline{\rho}(n_0,t_{0m})+\underline{\mathcal{A}}^{(3)}(n-n_0)+\underline{\widehat{\mathcal{A}}}_m^{(2)}(t_m-t_{0m}). \tag{5.20}$$

Proof. Introducing the meromorphic differential

$$\mho(n,n_0,t_m,t_{0m})=\frac{\partial}{\partial\lambda}\ln(\Xi_1(u,n,n_0,t_m,t_{0m}))\,d\lambda,$$

we use (5.18) to obtain

$$\mho(n,n_0,t_m,t_{0m})=(n-n_0)\omega_{u_{\infty'},u_{\infty''}}^{(3)}-(t_m-t_{0m})\widehat{\mho}_m^{(2)}+\sum_{j=1}^{l-1}\omega_{\tilde{\mu}_j(n,t_m),\mu_j(n_0,t_{0m})}^{(3)}+\sum_{j=1}^{l-1}\check{\ell}_j\omega_j,$$

where $\check{\ell}\in\mathbb{C}$, $j=1,\ldots,l-1$. On $\mathcal{K}_{l-1}$, any of the **w**-periods and **o**-periods is an integer multiple of $2\pi i$ because $\Xi_1(u,n,n_0,t_m,t_{0m})$ is single-valued, and hence

$$2\pi i\mathcal{B}_\sigma=\int_{\mathbf{w}_\sigma}\mho(n,n_0,t_m,t_{0m})=\int_{\mathbf{w}_\sigma}\sum_{j=1}^{l-1}\check{\ell}_j\omega_j=\check{\ell}_\sigma,\qquad \sigma=1,\ldots,l-1,$$

where $\mathcal{B}_\sigma \in \mathbb{Z}$. Similarly, for $\mathcal{C}_\sigma \in \mathbb{Z}$ $(\sigma = 1, \dots, l-1)$, we have

$$\begin{aligned}
2\pi i \mathcal{C}_\sigma &= \int_{\mathbf{o}_\sigma} \mho(n, n_0, t_m, t_{0m}) = \\
&= (n - n_0) \int_{\mathbf{o}_\sigma} \omega^{(3)}_{u_{\infty'}, u_{\infty''}} - (t_m - t_{0m}) \int_{\mathbf{o}_\sigma} \widehat{\mho}^{(2)}_m + \\
&\quad + \sum_{j=1}^{l-1} \int_{\mathbf{o}_\sigma} \omega_{\tilde{\mu}_j(n,t_m), \tilde{\mu}_j(n_0,t_{0m})} + \int_{\mathbf{o}_\sigma} \sum_{j=1}^{l-1} \check{\ell}_j \omega_j = \\
&= 2\pi i (n - n_0) \mathcal{A}^{(3)}_\sigma - 2\pi i (t_m - t_{0m}) \int_{\mathbf{o}_\sigma} \widehat{\mho}^{(2)}_m + \\
&\quad + 2\pi i \sum_{j=1}^{l-1} \int_{\tilde{\mu}_j(n_0,t_{0m})}^{\tilde{\mu}_j(n,t_m)} \omega_\sigma + 2\pi i \sum_{j=1}^{l-1} \mathcal{B}_j \int_{\mathbf{o}_\sigma} \omega_j = \\
&= 2\pi i (n - n_0) \mathcal{A}^{(3)}_\sigma - 2\pi i (t_m - t_{0m}) \widehat{\mathcal{A}}^{(2)}_{m,\sigma} + \\
&\quad + 2\pi i \Big(\sum_{j=1}^{l-1} \int_{Q_0}^{\tilde{\mu}_j(n,t_m)} \omega_\sigma - \sum_{j=1}^{l-1} \int_{Q_0}^{\tilde{\mu}_j(n_0,t_{0m})} \omega_\sigma \Big) + 2\pi i \sum_{j=1}^{l-1} \mathcal{B}_j \tau_{j,\sigma},
\end{aligned}$$

whence

$$\underline{\mathcal{C}} = (n - n_0) \underline{\mathcal{A}}^{(3)} - (t_m - t_{0m}) \underline{\widehat{\mathcal{A}}}^{(2)}_m + \sum_{j=1}^{l-1} \int_{Q_0}^{\tilde{\mu}_j(n,t_m)} \underline{\omega} - \sum_{j=1}^{l-1} \int_{Q_0}^{\tilde{\mu}_j(n_0,t_{0m})} \underline{\omega} + \underline{\mathcal{B}} \tau, \tag{5.21}$$

where $\underline{\mathcal{C}} = (\mathcal{C}_1, \dots, \mathcal{C}_{l-1}) \in \mathbb{Z}^{l-1}$ and $\underline{\mathcal{B}} = (\mathcal{B}_1, \dots, \mathcal{B}_{l-1}) \in \mathbb{Z}^{l-1}$. Therefore, we have proved (5.20) because (5.21) is equivalent to (5.20).

From Theorem 4, we have

$$\begin{aligned}
&\theta(\underline{z}(u_{\infty'}, \underline{\tilde{\mu}}(n, t_m))) = \theta(\underline{\widehat{\mathcal{B}}}_0 + \underline{\mathcal{A}}^{(3)} n + \underline{\widehat{\mathcal{A}}}^{(2)}_m t_m), \\
&\theta(\underline{z}(u_{\infty'}, \underline{\tilde{\mu}}^{+}(n, t_m))) = \theta(\underline{\widehat{\mathcal{B}}}_1 + \underline{\mathcal{A}}^{(3)} n + \underline{\widehat{\mathcal{A}}}^{(2)}_m t_m), \\
&\theta(\underline{z}(u_{\infty'}, \underline{\tilde{\mu}}^{++}(n, t_m))) = \theta(\underline{\widehat{\mathcal{B}}}_2 + \underline{\mathcal{A}}^{(3)} n + \underline{\widehat{\mathcal{A}}}^{(2)}_m t_m), \\
&\theta(\underline{z}(u_{\infty''}, \underline{\mu}(n, t_m))) = \theta(\underline{\widehat{\mathcal{B}}}'_0 + \underline{\mathcal{A}}^{(3)} n + \underline{\widehat{\mathcal{A}}}^{(2)}_m t_m), \\
&\theta(\underline{z}(u_{\infty''}, \underline{\mu}^{+}(n, t_m))) = \theta(\underline{\widehat{\mathcal{B}}}'_1 + \underline{\mathcal{A}}^{(3)} n + \underline{\widehat{\mathcal{A}}}^{(2)}_m t_m), \\
&\theta(\underline{z}(u_{\infty''}, \underline{\mu}^{++}(n, t_m))) = \theta(\underline{\widehat{\mathcal{B}}}'_2 + \underline{\mathcal{A}}^{(3)} n + \underline{\widehat{\mathcal{A}}}^{(2)}_m t_m),
\end{aligned} \tag{5.22}$$

where

$$\begin{aligned}
&\underline{\widehat{\mathcal{B}}}_0 = \underline{\widehat{\mathcal{S}}} - \underline{\mathcal{A}}^{(3)}, \qquad \underline{\widehat{\mathcal{B}}}'_0 = \underline{\widehat{\mathcal{S}}}' - \underline{\mathcal{A}}^{(3)}, \qquad \underline{\widehat{\mathcal{B}}}_1 = \underline{\widehat{\mathcal{S}}} + \underline{\mathcal{A}}^{(3)}, \\
&\underline{\widehat{\mathcal{B}}}'_1 = \underline{\widehat{\mathcal{S}}}' + \underline{\mathcal{A}}^{(3)}, \qquad \underline{\widehat{\mathcal{B}}}_2 = \underline{\widehat{\mathcal{S}}} + 2\underline{\mathcal{A}}^{(3)}, \qquad \underline{\widehat{\mathcal{B}}}'_2 = \underline{\widehat{\mathcal{S}}}' + 2\underline{\mathcal{A}}^{(3)}, \\
&\underline{\widehat{\mathcal{S}}} = \underline{\Lambda} - \underline{\mathbb{O}}(u_{\infty'}) + \underline{\rho}(n_0, t_{0m}) - \underline{\mathcal{A}}^{(3)} n_0 - \underline{\mathcal{A}}^{(2)}_m t_{0m}, \\
&\underline{\widehat{\mathcal{S}}}' = \underline{\Lambda} - \underline{\mathbb{O}}(u_{\infty''}) + \underline{\rho}(n_0, t_{0m}) - \underline{\mathcal{A}}^{(3)} n_0 - \underline{\mathcal{A}}^{(2)}_m t_{0m}.
\end{aligned}$$

Theorem 5. *Let $(n,t_m)\in\mathbb{Z}\times\mathbb{R}$ and let the divisor $\mathcal{D}_{\underline{\tilde{\mu}}(n,t_m)}$ be nonspecial. Then*

$$
\begin{aligned}
q(n,t_m) =& -\omega_0^{\infty''} - \sum_{j=1}^{l-1}\left(\frac{1}{3\beta_0}\mathbb{Q}_{j,l-1}+\frac{1}{\alpha_0\beta_0}\mathbb{Q}_{j,2p-1}\right)\frac{\partial}{\partial_{zj}}\ln\frac{\theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}^{++}(n,t_m)))}{\theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}^{+}(n,t_m)))} - \\
&-\frac{\theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}(n,t_m)))\theta(\underline{z}(u_{\infty'},\underline{\mu}^{++}(n,t_m)))}{\theta(\underline{z}(u_{\infty''},\underline{\tilde{\mu}}^{++}(n,t_m)))\theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}^{+}(n,t_m)))}\exp(\ell_1(Q_0)-\ell_2(Q_0)),\\
r(n,t_m) =& -\omega_0^{\infty'} + \sum_{j=1}^{l-1}\left(\frac{2}{3\alpha_0}\mathbb{Q}_{j,l-1}-\frac{1}{\alpha^2}\mathbb{Q}_{j,2p-1}\right)\frac{\partial}{\partial_{z_j}}\ln\frac{\theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}^{+}(n,t_m)))}{\theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}(n,t_m)))},\\
\frac{s(n,t_m)}{s^-(n,t_m)} =& \,\omega_0^{\infty'} - \sum_{j=1}^{l-1}\left(\frac{2}{3\alpha_0}\mathbb{Q}_{j,l-1}-\frac{1}{\alpha^2}\mathbb{Q}_{j,2p-1}\right)\frac{\partial}{\partial_{z_j}}\ln\frac{\theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}^{+}(n,t_m)))}{\theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}(n,t_m)))},\\
v(n,t_m) =& -\frac{\theta(\underline{z}(u_{\infty''},\underline{\mu}(n,t_m)))\theta(\underline{z}(u_{\infty'},\underline{\mu}^{+}(n,t_m)))}{\theta(\underline{z}(u_{\infty'},\underline{\tilde{\mu}}(n,t_m)))^2}\exp(\ell_1(Q_0)-\ell_2(Q_0)).
\end{aligned}
\tag{5.23}
$$

Combining (5.20) and (5.23) shows that the Riemann theta representation for $q(n,t_m)$, $r(n,t_m)$, $s(n,t_m)$ and $v(n,t_m)$ has a remarkable linearity in $(n,t_m)\in\mathbb{Z}\times\mathbb{R}$. Expressions (5.23) can then be rewritten as

$$
\begin{aligned}
q(n,t_m) =& -\omega_0^{\infty''} + \sum_{j=1}^{l-1}\mathbb{K}_{j,0}^{(\infty'')}\frac{\partial}{\partial_{z_j}}\ln\frac{\theta(\underline{\widehat{\mathcal{B}}}_2+\underline{\mathcal{A}}^{(3)}n+\underline{\widehat{\mathcal{A}}}_m^{(2)}t_m)}{\theta(\underline{\widehat{\mathcal{B}}}_1+\underline{\mathcal{A}}^{(3)}n+\underline{\widehat{\mathcal{A}}}_m^{(2)}t_m)} - \\
&-\frac{\theta(\underline{\widehat{\mathcal{B}}}_0+\underline{\mathcal{A}}^{(3)}n+\underline{\widehat{\mathcal{A}}}_m^{(2)}t_m)\theta(\underline{\widehat{\mathcal{B}}}_2+\underline{\mathcal{A}}^{(3)}n+\underline{\widehat{\mathcal{A}}}_m^{(2)}t_m)}{\theta(\underline{\widehat{\mathcal{B}}}_2'+\underline{\mathcal{A}}^{(3)}n+\underline{\widehat{\mathcal{A}}}_m^{(2)}t_m)\theta(\underline{\widehat{\mathcal{B}}}_1+\underline{\mathcal{A}}^{(3)}n+\underline{\widehat{\mathcal{A}}}_m^{(2)}t_m)}\exp(\ell_1(Q_0)-\ell_2(Q_0)),\\
r(n,t_m) =& -\omega_0^{\infty'} + \sum_{j=1}^{l-1}\mathbb{K}_{j,0}^{(\infty')}\frac{\partial}{\partial_{z_j}}\ln\frac{\theta(\underline{\widehat{\mathcal{B}}}_1+\underline{\mathcal{A}}^{(3)}n+\underline{\widehat{\mathcal{A}}}_m^{(2)}t_m)}{\theta(\underline{\widehat{\mathcal{B}}}_0+\underline{\mathcal{A}}^{(3)}n+\underline{\widehat{\mathcal{A}}}_m^{(2)}t_m)},\\
\frac{s(n,t_m)}{s^-(n,t_m)} =& \,\omega_0^{\infty'} - \sum_{j=1}^{l-1}\mathbb{K}_{j,0}^{(\infty')}\frac{\partial}{\partial_{z_j}}\ln\frac{\theta(\underline{\widehat{\mathcal{B}}}_1+\underline{\mathcal{A}}^{(3)}n+\underline{\widehat{\mathcal{A}}}_m^{(2)}t_m)}{\theta(\underline{\widehat{\mathcal{B}}}_0+\underline{\mathcal{A}}^{(3)}n+\underline{\widehat{\mathcal{A}}}_m^{(2)}t_m)},\\
v(n,t_m) =& -\frac{\theta(\underline{\widehat{\mathcal{B}}}_0'+\underline{\mathcal{A}}^{(3)}n+\underline{\widehat{\mathcal{A}}}_m^{(2)}t_m)\theta(\underline{\widehat{\mathcal{B}}}_1+\underline{\mathcal{A}}^{(3)}n+\underline{\widehat{\mathcal{A}}}_m^{(2)}t_m)}{\theta(\underline{\widehat{\mathcal{B}}}_0+\underline{\mathcal{A}}^{(3)}n+\underline{\widehat{\mathcal{A}}}_m^{(2)}t_m)^2}\exp(\ell_1(Q_0)-\ell_2(Q_0)).
\end{aligned}
\tag{5.24}
$$

Hence, formulas (5.23) and (5.24). give algebro-geometric solutions of the discrete hierarchy of the generalized Toda lattice (1.3).

To clarify the algebro-geometric solutions, we consider a simpler example of the Riemann theta representation under the condition $p=1$. The genus of $\mathcal{K}_2$ is therefore equal to 2, and we can obtain the following results by direct calculation:

$$
\begin{aligned}
&\Gamma_{11}^{(1)}=\alpha_0+\beta_0+\alpha_1-\beta_1, && \Gamma_{12}^{(1)}=(3\beta_0-2\alpha_0)q^-, && \Gamma_{13}^{(1)}=-\alpha_0 sq,\\
&\Gamma_{21}^{(1)}=(3\beta_0-2\alpha_0)v, && \Gamma_{22}^{(1)}=\beta_0-\alpha_1+2\beta_1, && \Gamma_{23}^{(1)}=(3\beta_0-\alpha_0)s,\\
&\Gamma_{31}^{(1)}=(2\alpha_0-3\beta_0)\frac{v^-}{s^-q^-}, && \Gamma_{32}^{(1)}=(\alpha_0-3\beta_0)\frac{1}{s^-}, && -\Gamma_{11}^{(1)}-\Gamma_{22}^{(1)}=-\alpha_0-\beta_0-\beta_1.
\end{aligned}
$$

The trigonal curve $\mathcal{F}_3(\lambda,f)=0$, whose degree is $l=3$ ($\alpha_0\beta_0\neq 0$), can then be defined as

$$\mathcal{K}_2\colon\ \mathcal{F}_3(\lambda,f)=f^3-f^2X_3(\lambda)+fY_3(\lambda)-Z_3(\lambda)=0,$$

where

$$
\begin{aligned}
&X_3(\lambda) = 0,\\
&Y_3(\lambda) = (-{\alpha_0}^2 + \alpha_0\beta_0 - 3{\beta_0}^2)\lambda^2 + \imath_1\lambda - ({a_2}^2 + a_2c_2 + {c_2}^2),\\
&Z_3(\lambda) = ({\alpha_0}^2\beta_0 - 3\alpha_0{\beta_0}^2 + 2{\beta_0}^3)\lambda^3 + \imath_2\lambda^2 + \imath_3\lambda - ({a_2}^2c_2 + a_2{c_2}^2),
\end{aligned}
$$

and $\imath_1$, $\imath_2$, and $\imath_3$ are arbitrary constants. Therefore, the polynomials of F_2 and E_2 can be reexpressed as

$$
\begin{aligned}
F_2(\lambda, n, t_m) &= (3\beta_0 - \alpha_0)(\lambda - \mu_1(n))(\lambda - \mu_2(n)),\\
E_2(\lambda, n, t_m) &= (\alpha_0 - 3\beta_0)(\lambda - \mu_1^+(n))(\lambda - \mu_2^+(n)).
\end{aligned}
$$

The Riemann theta representations of the potentials in the case of genus is 2 can therefore be rewritten as

$$
\begin{aligned}
q(n, t_m) =& -\omega_0^{\infty''} + \sum_{j=1}^{2} \mathbb{K}_{j,0}^{(\infty'')} \frac{\partial}{\partial_{z_j}} \ln \frac{\theta(\widehat{\mathcal{B}}_2 + \mathcal{A}_2^{(3)} n + \widehat{\mathcal{A}}_{m,2}^{(2)} t_m)}{\theta(\widehat{\mathcal{B}}_1 + \mathcal{A}_2^{(3)} n + \widehat{\mathcal{A}}_{m,2}^{(2)} t_m)} - \\
& - \frac{\theta(\widehat{\mathcal{B}}_0 + \mathcal{A}_2^{(3)} n + \widehat{\mathcal{A}}_{m,2}^{(2)} t_m)\theta(\widehat{\mathcal{B}}_2 + \mathcal{A}_2^{(3)} n + \widehat{\mathcal{A}}_{m,2}^{(2)} t_m)}{\theta(\widehat{\mathcal{B}}_2' + \mathcal{A}_2^{(3)} n + \widehat{\mathcal{A}}_{m,2}^{(2)} t_m)\theta(\widehat{\mathcal{B}}_1 + \mathcal{A}_2^{(3)} n + \widehat{\mathcal{A}}_{m}^{(2)} t_m)} \exp(\ell_1(Q_0) - \ell_2(Q_0)),\\
r(n, t_m) =& -\omega_0^{\infty'} + \sum_{j=1}^{2} \mathbb{K}_{j,0}^{(\infty')} \frac{\partial}{\partial_{z_j}} \ln \frac{\theta(\widehat{\mathcal{B}}_1 + \mathcal{A}_2^{(3)} n + \widehat{\mathcal{A}}_{m,2}^{(2)} t_m)}{\theta(\widehat{\mathcal{B}}_0 + \mathcal{A}_2^{(3)} n + \widehat{\mathcal{A}}_{m,2}^{(2)} t_m)},\\
\frac{s(n, t_m)}{s^-(n, t_m)} =& \ \omega_0^{\infty'} - \sum_{j=1}^{2} \mathbb{K}_{j,0}^{(\infty')} \frac{\partial}{\partial_{z_j}} \ln \frac{\theta(\widehat{\mathcal{B}}_1 + \mathcal{A}_2^{(3)} n + \widehat{\mathcal{A}}_{m,2}^{(2)} t_m)}{\theta(\widehat{\mathcal{B}}_0 + \mathcal{A}^{(3)} n + \widehat{\mathcal{A}}_{m,2}^{(2)} t_m)},\\
v(n, t_m) =& -\frac{\theta(\widehat{\mathcal{B}}_0' + \mathcal{A}_2^{(3)} n + \widehat{\mathcal{A}}_{m,2}^{(2)} t_m)\theta(\widehat{\mathcal{B}}_1 + \mathcal{A}_2^{(3)} n + \widehat{\mathcal{A}}_{m,2}^{(2)} t_m)}{\theta(\widehat{\mathcal{B}}_0 + \mathcal{A}_2^{(3)} n + \widehat{\mathcal{A}}_{m,2}^{(2)} t_m)^2} \exp(\ell_1(Q_0) - \ell_2(Q_0)),
\end{aligned}
$$

where

$$
\begin{aligned}
&\theta(z) = \sum_{\mathcal{F}_1 \in \mathbb{Z}} \exp\{2\pi i \mathcal{F}_1 z + \pi i \tau_{11} \mathcal{F}_1^2\},\\
&\widehat{\mathcal{B}}_0 = \widehat{\mathcal{S}} - \mathcal{A}_2^{(3)}, \qquad \widehat{\mathcal{B}}_0' = \widehat{\mathcal{S}}' - \mathcal{A}_2^{(3)}, \qquad \widehat{\mathcal{B}}_1 = \widehat{\mathcal{S}} + \mathcal{A}_2^{(3)},\\
&\widehat{\mathcal{B}}_1' = \widehat{\mathcal{S}}' + \mathcal{A}_2^{(3)}, \qquad \widehat{\mathcal{B}}_2 = \widehat{\mathcal{S}} + 2\mathcal{A}_2^{(3)}, \qquad \widehat{\mathcal{B}}_2' = \widehat{\mathcal{S}}' + 2\mathcal{A}_2^{(3)},\\
&\widehat{\mathcal{S}} = \Lambda_2 - \mathbb{O}_2(u_{\infty'}) + \rho_2(n_0, t_{0m}) - \mathcal{A}_2^{(3)} n_0 - \widehat{\mathcal{A}}_{m,2}^{(2)} t_{0m},\\
&\widehat{\mathcal{S}}' = \Lambda_2 - \mathbb{O}_2(u_{\infty''}) + \rho_2(n_0, t_{0m}) - \mathcal{A}_2^{(3)} n_0 - \widehat{\mathcal{A}}_{r,2}^{(2)} t_{0m},\\
&\mathbb{K}_{j,0}^{(\infty')} = \frac{2}{3\alpha_0}\mathbb{Q}_{j,l-1} - \frac{1}{\alpha^2}\mathbb{Q}_{j,2p-1}, \qquad \mathbb{K}_{j,0}^{(\infty'')} = -\frac{1}{3\beta_0}\mathbb{Q}_{j,l-1} - \frac{1}{\alpha_0\beta_0}\mathbb{Q}_{j,2p-1}.
\end{aligned}
$$

These formulas define algebro-geometric solutions of the discrete hierarchy of the generalized Toda lattice (1.3) in the case of genus 2.

6. Conclusions and Remarks

We have found algebro-geometric solutions of the hierarchy of generalized Toda lattices. The hierarchy was generated using the zero-curvature equation, and the functions Ξ and Θ were introduced on the trigonal curve. Based on the Abel differential, the Riemann theta representations of the potentials were constructed in the stationary and time-dependent cases, and solutions of the hierarchy were obtained. Currently, increasingly many researchers focus on trigonal curves and the application of these methods is gaining in popularity. Discussing the algebro-geometric solutions of the 4th-order soliton equations is also interesting, and we plan to address this problem in the future. Equally important is the study of soliton solutions beyond the algebro-geometric solutions, such as the lump–soliton and breather solutions.

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REFERENCES

1. M. Toda, "Vibration of a chain with a nonlinear interaction," *J. Phys. Soc. Japan*, **22**, 431–436 (1967).
2. M. Toda, *Nonlinear Waves and Solitons*, Mathematics and its Applications (Japanese Series), Vol. 5, Kluwer, Dordrecht (1989).
3. M. Toda, *Theory of Nonlinear Lattices*, Springer Series in Solid-State Sciences, Vol. 20, Springer, Berlin, Heidelberg (2012).
4. H. Flaschka, "On the Toda lattice. II. Inverse-scattering solution," *Progr. Theor. Phys.*, **51**, 703–716 (1974).
5. A. Bloch, *Hamiltonian and Gradient Flows, Algorithms, and Control*, AMS, Providence, RI (1994).
6. V. Muto, A. C. Scott, and P. L. Christiansen, "Thermally generated solitons in a Toda lattice model of DNA," *Phys. Lett. A*, **136**, 33–36 (1989).
7. M. J. Ablowitz and J. F. Ladik, "Nonlinear differential-difference equations," *J. Math. Phys.*, **16**, 598–603 (1975).
8. T. Takebe, "Toda lattice hierarchy and conservation laws," *Commun. Math. Phys.*, **129**, 281–318 (1990).
9. D.-J. Zhang, "Conservation laws of the two-dimensional Toda lattice hierarchy," *J. Phys. Soc. Japan*, **71**, 2583–2586 (2002).
10. W.-X. Ma, X.-X. Xu, and Y. Zhang, "Semidirect sums of Lie algebras and discrete integrable couplings," *J. Math. Phys.*, **47**, 053501, 16 pp. (2006).
11. R.-G. Zhou and Q.-Y. Jiang, "A Darboux transformation and an exact solution for the relativistic Toda lattice equation," *J. Phys. A: Math. Gen.*, **38**, 7735–7742 (2005).
12. C.-Z. Li and J.-S. He, "The extended multi-component Toda hierarchy," *Math. Phys. Anal. Geom.*, **17**, 377–407 (2014).
13. A. R. Its and V. B. Matveev, "vHill's operator with finitely many gaps," *Funct. Anal. Appl.*, **9**, 65–66 (1975).
14. I. M. Krichever, "Algebraic-geometric construction of the Zaharov–Shabat equations and their periodic solutions," *Dokl. Akad. Nauk SSSR*, **227**, 394–397 (1976).
15. S. P. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, *Theory of Solitons: The Inverse Scattering Method*, Consultants Bureau [Plenum], New York (1984).
16. W. Bulla, F. Gesztesy, H. Holden, and G. Teschl, "Algebro-geometric quasi-periodic finite-gap solutions of the Toda and Kac–van Moerbeke hierarchies," *Mem. Amer. Math. Soc.*, **135**, 1–79 (1998).
17. C.-W. Cao, X.-G. Geng, and H.-Y. Wang, "Algebro-geometric solution of the 2+1 dimensional Burgers equation with a discrete variable," *J. Math. Phys.*, **43**, 621–643 (2002).
18. X.-G. Geng and Y.-T. Wu, "Finite-band solutions of the classical Boussinesq–Burgers equations," *J. Math. Phys.*, **40**, 2971–2982 (1999).
19. Q.-L. Zhao, C.-X. Li, and X.-Y. Li, "Algebro-geometric constructions of a hierarchy of integrable semi-discrete equations," *J. Nonlinear Math. Phys.* (2022), Publ. online.
20. L. Luo and E.-G. Fan, "Variable separation and algebraic-geometric solutions of modified Toda lattice equation," *Modern Phys. Lett. B.*, **24**, 2041–2055 (2010).
21. X.-G. Geng and D. Gong, "Quasi-periodic solutions of the relativistic Toda hierarchy," *J. Nonlinear Math. Phys.*, **19**, 489–523 (2012).
22. H. Airault, H. P. McKean, and J. Moser, "Rational and elliptic solutions of the Korteweg–de Vries equation and a related many-body problem," *Commun. Pure Appl. Math.*, **30**, 95–148 (1977).
23. J. N. Elgin, V. Z. Enolski, and A. R.Its, "Effective integration of the nonlinear vector Schrödinger equation," *Phys. D*, **225**, 127–152 (2007).
24. R. Dickson, F. Gesztesy, and K. Unterkofler, "A new approach to the Boussinesq hierarchy," *Math. Nachr.*, **198**, 51–108 (1999).
25. X.-G. Geng, L.-H. Wu, and G.-L. He, "Algebro-geometric constructions of the modified Boussinesq flows and quasi-periodic solutions," *Phys. D*, **240**, 1262–1288 (2011).

26. X.-G. Geng, L.-H. Wu, and G.-L. He, "Quasi-periodic solutions of the Kaup–Kupershmidt hierarchy," *J. Nonlinear Sci.*, **23**, 527–555 (2013).
27. J. Wei, X.-G. Geng, and X. Zeng, "Quasi-periodic solutions to the hierarchy of four-component Toda lattices," *J. Geom. Phys.*, **106**, 26–41 (2016).
28. X. Zeng and X.-G. Geng, "Finite-band solutions for the hierarchy of coupled Toda lattices," *Acta Appl. Math.*, **154**, 59–81 (2018).
29. L.-L. Ma and X.-X. Xu, "A family of discrete Hamiltonian equations associated with a discrete three-by-three matrix spectral problem," *Internat. J. Modern Phys. B*, **23**, 3859–3869 (2009).