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LONG-TIME ASYMPTOTICS FOR THE NONLOCAL KUNDU–NONLINEAR-SCHRÖDINGER EQUATION BY THE NONLINEAR STEEPEST DESCENT METHOD

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We study the long-time asymptotics of the nonlocal Kundu–nonlinear-Schrödinger equation with a decaying initial value. The long-time asymptotics of the solution follow from the nonlinear steepest descent method proposed by Deift–Zhou and the Riemann–Hilbert method.

Keywords: long-time asymptotics, nonlocal Kundu–nonlinear-Schrödinger equation, nonlinear steepest descent method, Riemann–Hilbert method

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1. Introduction

As is well known, the parity and time (PT) symmetry is one of the most important symmetries in quantum theory. In 1998, Bender and Boettcher [1] obtained the PT symmetry by replacing the Hermiticity of Hamiltonians in quantum theory and showed that most basic quantum properties are preserved for PT-symmetric Hamiltonians. Subsequently, researchers also applied PT symmetry to optics, electricity, and so on [2]–[8]. Ablowitz proposed the nonlocal nonlinear Schrödinger equation in 2013 [9], and a large number of models of nonlocal integrable systems have been proposed and studied since then [10]–[14].

In this paper, we consider the coupled Kundu–nonlinear-Schrödinger (Kundu–NLS) equations [15]

$$\begin{aligned} &iq_t + q_{xx} + 2\alpha e^{i(\theta-\phi)}q^2 r - (\theta_t + \theta_x^2 - i\theta_{xx})q + 2i\theta_x q_x = 0,\\ -&ir_t + r_{xx} + 2\alpha e^{-i(\phi-\theta)}r^2 q - (\phi_t + \phi_x^2 + i\phi_{xx})r - 2i\phi_x r_x = 0, \end{aligned} \tag{1.1}$$

where $\theta(x,t)$, $\phi(x,t)$ are arbitrary gauge functions. The Lax pair of Eqs. (1.1) can be written as

$$\begin{aligned} v_x &= Mv = (-i\lambda\sigma_3 + \sqrt{\alpha}U_0)v,\\ v_t &= Nv = (-2i\lambda^2\sigma_3 + 2\sqrt{\alpha}\lambda U_0 + U_1)v, \end{aligned} \tag{1.2}$$

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and

$$U_0 = \begin{pmatrix} 0 & qe^{i\theta} \\ -re^{-i\phi} & 0 \end{pmatrix}, \quad U_1 = \begin{pmatrix} i\alpha qre^{i(\theta-\phi)} & i\sqrt{\alpha}(qe^{i\theta})_x \\ i\sqrt{\alpha}(re^{-i\phi})_x & -i\alpha qre^{i(\theta-\phi)} \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Setting $r(x,t) = q^*(-x,t)$ and $\phi(x,t) = \theta(-x,t)$, we reduce Eqs. (1.1), to the nonlocal Kundu–NLS equation [15]

$$iq_t + q_{xx} + 2\alpha e^{i(\theta-\theta(-x,t))}q^2q^*(-x,t) - (\theta_t + \theta_x^2 - i\theta_{xx})q + 2i\theta_x q_x = 0. \tag{1.3}$$

When $\alpha = 1$, nonlocal Kundu–NLS equation (1.3) is focusing, and when $\alpha = -1$, it is defocusing.

The main goal in this paper is to study the long-time asymptotics for the nonlocal Kundu–NLS equation (1.3) with a decaying initial value $q(x,0) = q_0(x) \in \mathbb{S}(\mathbb{R})$, where

$$\mathbb{S}(\mathbb{R}) = \left\{ f(x) \,\middle|\, \int_{-\infty}^{\infty} (1 + |x|^\gamma f(x))\, dx < \infty,\ \gamma > 1 \right\} \tag{1.4}$$

is the Schwartz space. Our interest in the long-time behavior of the initial value problem for the integrable nonlocal Kundu–NLS equation was largely motivated by Rybalko and Shepelsky [16], [17], who studied the long-time behavior of solutions of the nonlocal NLS equation. Generally speaking, the long-time asymptotics of the solutions of integrable systems are a hot topic, with various outstanding approaches having been proposed [18]–[24].

An extremely efficient method to analyze solutions of integrable systems is the nonlinear steepest-descent method [25] proposed by Deift and Zhou based on the preceding studies. The main idea is to reduce the oscillating Riemann–Hilbert (RH) problem to a solvable one through a series of rapidly descending deformation paths. With this effective method, more and more integrable systems have been studied, including the dispersion KdV equation [26], the defocusing NLS equation [27], [28], the Camassa–Holm equation [29], the Kundu–Eckhaus equation [30], the three-component coupled nonlinear Schrödinger system [31], the Fokas–Lenells and derivative NLS equations [32], [33], the MKdV equation in a quarter plane $\{x \geqslant 0, t \geqslant 0\}$ [34], [35], and coupled modified Korteweg–de Vries equations [36].

This paper is organized as follows. In Sec. 2, we construct the RH problem of the nonlocal Kundu–NLS equation via transformation (2.2), Volterra equations (2.3), scattering relation (2.4), and symmetry relations (2.7). Then, using the steepest decent contours, trigonometric decomposition, and a scaling transformation, we obtain the Cauchy problem (1.3) with the decaying value. In the Appendix, we give the proof of Theorem 1 based on the use of the Weber equation and the standard parabolic cylinder function.

2. The RH problem for the nonlocal Kundu–NLS equation

By changing the variable as

$$w = ve^{i\lambda x\sigma_3 + 2i\lambda^2 t\sigma_3}, \qquad |x| \to \infty, \tag{2.1}$$

we reduce Lax pair (1.2) to

$$\begin{aligned} &w_x + i\lambda[\sigma_3, w] = U_0 w, \\ &w_t + 2i\lambda^2[\sigma_3, w] = V_1 w, \end{aligned} \tag{2.2}$$

where $V_1 = 2\sqrt{\alpha}\lambda U_0 + U_1$, $[\sigma_3, w] = \sigma_3 w - w\sigma_3$ is the Lie bracket operation. The tracelessness condition $\operatorname{tr} U_0 = \operatorname{tr} V_1 = 0$ implies that $\det w = 1$.

To construct the RH problem for the nonlocal Kundu–NLS equation, we introduce two Volterra equations

$$\begin{aligned} w_1(x,t;\lambda) &= I + \int_{-\infty}^{x} e^{i\lambda(x-y)\,\mathrm{ad}\,\sigma_3}(U_0 w_1(y,\lambda))\, dy, \\ w_2(x,t;\lambda) &= I - \int_{x}^{+\infty} e^{-i\lambda(x-y)\,\mathrm{ad}\,\sigma_3}(U_0 w_2(y,\lambda))\, dy, \end{aligned} \tag{2.3}$$

where $e^{\mathrm{ad}\sigma_3}(M) = e^{\sigma_3} M e^{-\sigma_3}$ for a matrix M and I is the identity matrix. It follows from Eqs. (2.3) that

$$e^{i\lambda(x-y)\sigma_3}\sqrt{\alpha}U_0 e^{-i\lambda(x-y)\sigma_3} = \begin{pmatrix} 0 & \sqrt{\alpha} q e^{i\theta} e^{2i\lambda(x-y)} \\ -\sqrt{\alpha} q^*(-x,t) e^{-i\theta(-x,t)} e^{-2i\lambda(x-y)} & 0 \end{pmatrix}.$$

Let $w_1(x,t;k) = (w_1^{(1)}, w_1^{(2)})$ and $w_2(x,t;k) = (w_2^{(1)}, w_2^{(2)})$. It follows that $w_1^{(1)}$ and $w_2^{(2)}$ are analytic in the lower half-plane $\mathbb{C}^- = \{\lambda \in \mathbb{C} \mid \operatorname{Im}\lambda < 0\}$, and $w_1^{(2)}$ and $w_2^{(1)}$ are analytic in the upper half-plane $\mathbb{C}^+ = \{\lambda \in \mathbb{C} \mid \operatorname{Im}\lambda > 0\}$.

The matrix solutions of system (1.2) with $\lambda \in \mathbb{R}$ satisfy the relation

$$v_1(x,t;\lambda) = v_2(x,t;\lambda) S(\lambda), \tag{2.4}$$

where $S(\lambda)$ is the scattering matrix. From [37], we have

$$v_1^*(-x,t;-\lambda^*) = \Delta^{-1} v(x,t;\lambda)\Delta, \qquad \Delta = \begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix}, \quad \sigma = \pm 1. \tag{2.5}$$

Based on scattering relation (2.4) and symmetry (2.5), the expression for the scattering matrix $S(\lambda)$ can be written as

$$S(\lambda) = \begin{pmatrix} A_1(\lambda) & -\sigma B^*(-\lambda^*) \\ B(\lambda) & A_2(\lambda) \end{pmatrix}, \qquad \lambda \in \mathbb{R}, \tag{2.6}$$

and the elements $A_1(\lambda)$, $A_2(\lambda)$ of $S(\lambda)$ satisfy the symmetry relations

$$A_1(\lambda) = A_1^*(-\lambda^*), \qquad A_2(\lambda) = A_2^*(-\lambda^*). \tag{2.7}$$

It is obvious that symmetry relation (2.7) for the nonlocal Kundu–NLS equation differ from those in the local case, which highlights the necessity of studying nonlocal integrable systems.

On the other hand, it is worth noting that the scattering matrix $S(\lambda)$ can be uniquely determined as

$$S(\lambda) = v_2^{-1}(x,0;\lambda) v_1(x,0;\lambda) = e^{ix\lambda} w_2^{-1}(x,0;\lambda) w_1(x,0;\lambda) e^{-ix\lambda}, \tag{2.8}$$

where $w_1(x,0;\lambda)$, $w_2(x,0;\lambda)$ are defined by the Volterra equations (2.3). If

$$w_1(x,0;\lambda) = \begin{pmatrix} (w_1)_{11} & (w_1)_{12} \\ (w_1)_{21} & (w_1)_{22} \end{pmatrix}, \tag{2.9}$$

then we have

$$\begin{aligned}
(w_1)_{11}(x;\lambda) &= 1 + \sqrt{\alpha}\int_{-\infty}^{x} q(y) e^{i\theta(y)} (w_1)_{21}(y;\lambda)\, dy, \\
(w_1)_{21}(x;\lambda) &= -\sqrt{\alpha}\int_{-\infty}^{x} q^*(-y) e^{-i\theta(-y)} (w_1)_{11}(y;\lambda) e^{-2i\lambda(x-y)}\, dy, \\
(w_1)_{12}(x;\lambda) &= \sqrt{\alpha}\int_{-\infty}^{x} q(y) e^{i\theta(y)} (w_1)_{22}(y;\lambda) e^{2i\lambda(x-y)}\, dy, \\
(w_1)_{22}(x;\lambda) &= 1 - \sqrt{\alpha}\int_{-\infty}^{x} q^*(-y) e^{-i\theta(-y)} (w_1)_{12}(y;\lambda)\, dy.
\end{aligned} \tag{2.10}$$

Thus, the scattering data $A_1(\lambda)$, $A_2(\lambda)$, $B(\lambda)$ are given by

$$\begin{aligned} A_1(\lambda) &= \lim_{x\to+\infty} (w_1)_{11}(x;\lambda), \\ A_2(\lambda) &= \lim_{x\to+\infty} (w_1)_{22}(x;\lambda), \end{aligned} \qquad B(\lambda) = \lim_{x\to+\infty} e^{-2ix\lambda} (w_1)_{21}(x;\lambda). \tag{2.11}$$

We rewrite relation (2.4) as

$$w_1(x,t;\lambda) = w_2(x,t;\lambda)e^{-i\lambda x-2i\lambda^2 t}S(\lambda)e^{i\lambda x+2i\lambda^2 t}, \qquad \lambda\in\mathbb{R}, \tag{2.12}$$

with

$$\begin{aligned} w_1^{(1)} &= w_2^{(1)}A_1(\lambda) + w_2^{(2)}B(\lambda)e^{2i\lambda x+4i\lambda^2 t},\\ w_1^{(2)} &= -\sigma w_2^{(1)}B^*(-\lambda^*)e^{-2i\lambda x-4i\lambda^2 t} + w_2^{(2)}A_2(\lambda). \end{aligned} \tag{2.13}$$

Equation (2.13) can be written in matrix form

$$\left(\frac{w_1^{(1)}}{A_1(\lambda)}, w_2^{(2)}\right) = \left(w_2^{(1)}, \frac{w_1^{(2)}}{A_2(\lambda)}\right)e^{-i(\lambda x+2\lambda^2 t)\,\mathrm{ad}\,\sigma_3}\begin{pmatrix} 1+\sigma H_1(\lambda)H_2(\lambda) & \sigma H_2(\lambda)\\ H_1(\lambda) & 1\end{pmatrix},$$

where $H_1(\lambda) = B(\lambda)/A_1(\lambda)$ and $H_2(\lambda) = B^*(-\lambda)/A_2(\lambda)$. This follows from

$$H_2^*(-\lambda) = \frac{B(\lambda)}{A_2^*(-\lambda)} = H_1(\lambda)\frac{A_1(\lambda)}{A_2^*(-\lambda)} = H_1(\lambda)\frac{A_1(\lambda)}{A_2(\lambda)}, \tag{2.14}$$

and

$$H_1^*(-\lambda) = H_2(\lambda)\frac{A_2(\lambda)}{A_1(\lambda)}, \qquad 1+\sigma H_1(\lambda)H_2(\lambda) = \frac{1}{A_1(\lambda)A_2(\lambda)}. \tag{2.15}$$

To obtain the original oscillatory RH problem of the nonlocal Kundu–NLS equation (1.3), we define a piecewise analytic function as

$$P(x,t;\lambda) = \begin{cases} \left(\dfrac{w_1^{(1)}}{A_1(\lambda)}, w_2^{(2)}\right), & \lambda\in\mathbb{C}^-,\\ \left(w_2^{(1)}, \dfrac{w_1^{(2)}}{A_1(\lambda)}\right), & \lambda\in\mathbb{C}^+. \end{cases} \tag{2.16}$$

It satisfies the RH problem

$$P_+(x,t;\lambda) = P_-(x,t;\lambda)J(x,t;\lambda), \qquad P(x,t;\lambda)\to I, \quad \lambda\to\infty \tag{2.17}$$

with the jump matrix

$$J(x,t;\lambda) = \begin{pmatrix} 1+\sigma H_1(\lambda)H_2(\lambda) & \sigma H_2(\lambda)e^{-i(2\lambda x+4\lambda^2 t)}\\ H_1(\lambda)e^{i(2\lambda x+4\lambda^2 t)} & 1\end{pmatrix}. \tag{2.18}$$

Therefore, the solution of nonlocal Kundu–NLS equation (1.3) can be written as

$$\begin{aligned} q(x,t)e^{i\theta} &= 2i\lim_{\lambda\to\infty}\lambda\,(P(x,t;\lambda))_{12},\\ -q^*(-x,t)e^{-i\theta(-x,t)} &= 2i\lim_{\lambda\to\infty}\lambda\,(P(x,t;\lambda))_{21}. \end{aligned} \tag{2.19}$$

The approach in this paper extends Deift–Zhou's method to obtain the long-time asymptotic behavior of the solution through the related phase point drop.

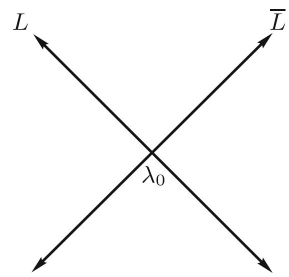

Fig. 1. Contours L and $\overline{L}$.

2.1. The steepest decent contours. let $F(\lambda) = (x/t)\lambda + 2\lambda^2$. From

$$\frac{dF(\lambda)}{d\lambda} = \frac{x}{t} + 4\lambda = 0, \qquad \frac{d^2F(\lambda)}{d\lambda^2} = 4 \neq 0, \tag{2.20}$$

we then obtain a stationary point $\lambda_0 = -x/4t$ and two steepest decent contours (see Fig. 1)

$$\begin{aligned} L &= \{\lambda = \lambda_0 + \mu e^{3i\pi/4},\ \mu \in \mathbb{R}\}, \\ \overline{L} &= \{\lambda = \lambda_0 + \mu e^{-3i\pi/4},\ \mu \in \mathbb{R}\}. \end{aligned} \tag{2.21}$$

Let $\lambda = \lambda_1 + i\lambda_2$. For the above stationary point λ_0, we have

$$F(\lambda) = -4\lambda\lambda_0 + 2\lambda^2 = -4\lambda_1\lambda_0 + 2(\lambda_1^2 - 2\lambda_2^2) + 4i(\lambda_1 - \lambda_0)\lambda_2. \tag{2.22}$$

It thus follows that

- the oscillating factor $e^{itF(\lambda)}$ decays exponentially on $\mathrm{Re}(iF) < 0$,
- the oscillating factor $e^{-itF(\lambda)}$ decays exponentially on $\mathrm{Re}(iF) > 0$.

The constant-sign intervals of $\mathrm{Re}(iF) = -4(\mathrm{Re}\,\lambda - \lambda_0)\,\mathrm{Im}\,\lambda$ are shown in Fig. 2.

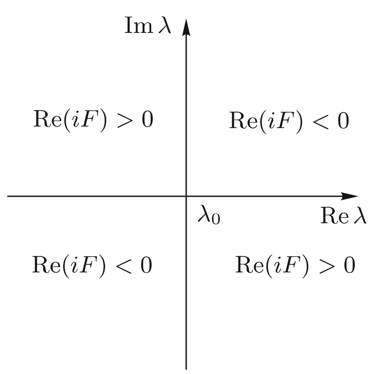

Fig. 2. The constant-sign intervals of $\mathrm{Re}(iF)$ on the complex λ-plane.

2.2. Trigonometric decomposition. In the physically interesting region $|x/t| \leqslant C$, following [25], we can decompose the jump matrix $J(x,t;\lambda)$ in (2.17) as follows:

$$J(x,t;\lambda) = \begin{pmatrix} 1 & \sigma H_2(\lambda) e^{-2itF} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ H_1(\lambda) e^{2itF} & 1 \end{pmatrix}$$

for $\lambda \in (\lambda_0, +\infty)$ and

$$J(x,t;\lambda) = \begin{pmatrix} 1 & 0 \\ \frac{H_1(\lambda)e^{2itF}}{1+\sigma H_1(\lambda)H_2(\lambda)} & 1 \end{pmatrix} \begin{pmatrix} 1+\sigma H_1 H_2 & 0 \\ 0 & \frac{1}{1+\sigma H_1 H_2} \end{pmatrix} \begin{pmatrix} 1 & \frac{\sigma H_2(\lambda)e^{-2itF}}{1+\sigma H_1(\lambda)H_2(\lambda)} \\ 0 & 1 \end{pmatrix}.$$

for $\lambda \in (-\infty, \lambda_0)$.

It is known that the diagonal matrix has to be eliminated for $\lambda < \lambda_0$. We therefore introduce the transformation

$$P^{(1)} = P\delta^{-\sigma_3}(\lambda), \tag{2.23}$$

where $\delta(\lambda)$ satisfies the scalar RH problem and

$$\begin{aligned} &\delta(\lambda) = \begin{cases} \delta_-(\lambda)(1+\sigma H_1(\lambda)H_2(\lambda)), & \lambda \in (-\infty, \lambda_0), \\ \delta_-(\lambda), & \lambda \in (\lambda_0, \infty), \end{cases} \\ &\delta(\lambda) \to 1, \qquad \lambda \to \infty. \end{aligned} \tag{2.24}$$

From the Sokhotski–Plemelj formula, the solution of this scalar RH problem can be expressed as

$$\delta(\lambda) = \exp\left(\frac{1}{2\pi i} \int_{-\infty}^{\lambda_0} \frac{\log(1+\sigma H_1(\xi)H_2(\xi))}{\xi - \lambda}\, d\xi \right). \tag{2.25}$$

It is worth emphasizing that in contrast to the general local integrable system, $1 + \sigma H_1(\xi)H_2(\xi)$ is not real-valued in the nonlocal Kundu–NLS equation. Deformation (2.25) can be expressed as

$$\delta(\lambda) = (\lambda_0 - \lambda)^{i\kappa(\lambda_0)} e^{\tau(\lambda)}, \tag{2.26}$$

where

$$\begin{aligned} \kappa(\lambda_0) &= -\frac{1}{2\pi} \log(1+\sigma H_1(\lambda_0)H_2(\lambda_0)), \\ \tau(\lambda) &= \frac{1}{2i\pi} \int_{-\infty}^{\lambda_0} \log(\xi - \lambda)\, d\left[\log(1+\sigma H_1(\xi)H_2(\xi))\right]. \end{aligned} \tag{2.27}$$

From symmetry relations (2.7) and Eq. (2.15), we have

$$\begin{aligned} &\kappa(-\lambda_0) = \kappa^*(\lambda_0), \\ &\operatorname{Im}\kappa(\lambda_0) = -\frac{1}{2\pi} \int_{-\infty}^{\lambda_0} d\arg(1+\sigma H_1(\xi)H_2(\xi)), \qquad |\operatorname{Im}\kappa(\lambda_0)| < \frac{1}{2}, \\ &\tau^*(-\lambda_0) + \tau(\lambda_0) = \frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{\log[A_1(\lambda_0)A_2(\lambda_0)]}{\xi - \lambda_0}\, d\xi. \end{aligned} \tag{2.28}$$

Therefore, the RH problem (2.17) becomes

$$P_+^{(1)}(x,t;\lambda) = P_-^{(1)}(x,t;\lambda)J^{(1)}(x,t;\lambda), \qquad P^{(1)}(x,t;\lambda) \to I, \quad \lambda \to \infty, \tag{2.29}$$

where

$$J^{(1)}(x,t;\lambda) = \begin{pmatrix} 1 & \sigma H_2(\lambda)\delta^2(\lambda)e^{-2itF} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ H_1(\lambda)\delta^{-2}(\lambda)e^{2itF} & 1 \end{pmatrix}$$

for $\lambda \in (\lambda_0, +\infty)$ and

$$J^{(1)}(x,t;\lambda) = \begin{pmatrix} 1 & 0 \\ \dfrac{H_1(\lambda)e^{2itF}\delta_-^{-2}(\lambda)}{1+\sigma H_1(\lambda)H_2(\lambda)} & 1 \end{pmatrix} \begin{pmatrix} 1 & \dfrac{\sigma H_2(\lambda)e^{-2itF}\delta_+^{2}(\lambda)}{1+\sigma H_1(\lambda)H_2(\lambda)} \\ 0 & 1 \end{pmatrix}$$

for $\lambda \in (-\infty, \lambda_0)$.

Obviously, the jump matrix contains four oscillating factors

$$H_1(\lambda), \quad H_2(\lambda), \quad \frac{H_1(\lambda)}{1+\sigma H_1(\lambda)H_2(\lambda)}, \quad \frac{\sigma H_2(\lambda)}{1+\sigma H_1(\lambda)H_2(\lambda)},$$

where

$$H_1^*(-\lambda) = H_2(\lambda)\frac{A_2(\lambda)}{A_1(\lambda)}, \qquad A_j = 1 + O\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty$$

(with $j = 1, 2$). Following [25], we define a piecewise function

$$\omega(\lambda) = \begin{cases} \dfrac{H_1(\lambda)}{1+\sigma H_1(\lambda)H_2(\lambda)}, & \lambda \in (-\infty, \lambda_0), \\ H_1(\lambda), & \lambda \in (\lambda_0, +\infty). \end{cases} \tag{2.30}$$

Because $\omega^*(-\lambda) = \rho(A_1(\lambda), A_2(\lambda))\omega(\lambda)$, where $\rho(A_1(\lambda), A_2(\lambda))$ is a constant as $\lambda \to \infty$ it follows that $\omega(\lambda)$ defined this way can also be approximated by similar rational functions. For $\lambda \in (-\infty, \lambda_0)$, we write it as

$$\omega(\lambda) = \frac{H_1(\lambda)}{1+\sigma H_1(\lambda)H_2(\lambda)} = H_{\mathrm{e}}(\lambda^2) + \lambda H_{\mathrm{o}}(\lambda^2), \tag{2.31}$$

where $H_{\mathrm{e}}(\,\cdot\,), H_{\mathrm{o}}(\,\cdot\,) \in \mathbb{S}$. For an integer $m \in \mathbb{Z}^+$, it follows from Taylor's formula with remainder that

$$\begin{aligned} H_{\mathrm{e}}(\lambda^2) &= \mu_0^{\mathrm{e}} + \mu_1^{\mathrm{e}}(\lambda^2 - \lambda_0^2) + \cdots + \mu_m^{\mathrm{e}}(\lambda^2 - \lambda_0^2)^m + \\ &\quad + \frac{1}{m!}\int_{\lambda_0^2}^{\lambda^2} H_{\mathrm{e}}^{(m+1)}(\gamma)(\lambda^2 - \gamma)^m\, d\gamma, \\ H_{\mathrm{o}}(\lambda^2) &= \mu_0^{\mathrm{o}} + \mu_1^{\mathrm{o}}(\lambda^2 - \lambda_0^2) + \cdots + \mu_m^{\mathrm{o}}(\lambda^2 - \lambda_0^2)^m + \\ &\quad + \frac{1}{m!}\int_{\lambda_0^2}^{\lambda^2} H_{\mathrm{o}}^{(m+1)}(\gamma)(\lambda^2 - \gamma)^m\, d\gamma. \end{aligned} \tag{2.32}$$

We set

$$R(\lambda) = \sum_{j=0}^{m} \mu_j^{\mathrm{e}}(\lambda^2 - \lambda_0^2)^j + \lambda \sum_{j=0}^{m} \mu_j^{\mathrm{o}}(\lambda^2 - \lambda_0^2)^j. \tag{2.33}$$

Comparing Eqs. (2.31) and (2.33), we then have

$$\left.\frac{d^j\omega(\lambda)}{d\lambda^j}\right|_{\pm\lambda_0} = \left.\frac{d^j R(\lambda)}{d\lambda^j}\right|_{\pm\lambda_0}, \qquad 0 \leqslant j \leqslant m, \tag{2.34}$$

where $\mu_j^{\mathrm{e}} = \mu_j^{\mathrm{e}}(\lambda_0^2)$ and $\mu_j^0 = \mu_j^0(\lambda_0^2)$ decay rapidly as $\lambda_0 \to \infty$, which follows because

$$\mu_j^{\mathrm{e}}(\lambda_0^2) = \frac{1}{j!}\left.\frac{d^j H_{\mathrm{e}}(u)}{du^j}\right|_{\lambda_0^2}, \qquad \mu_j^{\mathrm{o}}(\lambda_0^2) = \frac{1}{j!}\left.\frac{d^j H_{\mathrm{o}}(u)}{du^j}\right|_{\lambda_0^2}. \tag{2.35}$$

Let $\omega(\lambda) = r(\lambda) + R(\lambda)$ for $\lambda \in (-\infty, \lambda_0)$. From (2.34) we then have

$$\left.\frac{d^j r(\lambda)}{d\lambda^j}\right|_{\pm\lambda_0} = 0, \qquad 0 \leqslant j \leqslant m. \tag{2.36}$$

We write $r(\lambda)$ as

$$r(\lambda) = r_1(\lambda) + r_2(\lambda), \tag{2.37}$$

where $r_1(\lambda)$ is small and $r_2(\lambda)$ has an analytic continuation to $\lambda + i0$. Thus,

$$\omega(\lambda) = r_1(\lambda) + r_2(\lambda) + R(\lambda). \tag{2.38}$$

Proposition 1. *Let $m = 4n+1$, $n \in \mathbb{Z}^+$. As $t \to \infty$, the functions $r_1(\lambda)$, $r_2(\lambda)$, $R(\lambda)$ satisfy the estimates*

$$\begin{aligned} &|e^{-2iF(\lambda)} r_1(\lambda)| \leqslant \frac{c}{(1+|\lambda|^2)t^\ell}, && \lambda \in \mathbb{R}, \\ &|e^{-2iF(\lambda)} r_2(\lambda)| \leqslant \frac{c}{(1+|\lambda|^2)t^\ell}, && \lambda \in L, \\ &R(\lambda) \leqslant ce^{-4t\mu^2}, && \lambda \in \mathbb{C}, \qquad \mu = \text{const}, \end{aligned} \tag{2.39}$$

where ℓ is a positive integer. The complex conjugate of $\omega(\lambda)$ yields similar estimates for $r_1^*(\lambda), r_2^*(\lambda), R^*(\lambda)$ on $\mathbb{R} \cup \overline{L}$.

Proof. We define the function

$$\psi(\lambda) = (\lambda^2 - \lambda_0^2)^n, \qquad \lambda < \lambda_0. \tag{2.40}$$

For $\lambda < \lambda_0$, the map $\lambda \mapsto F(\lambda) = -4\lambda\lambda_0 + 2\lambda^2$ is one-to-one, $F(\lambda_0) = -2\lambda_0^2$, and

$$\frac{d\lambda}{dF} = \frac{1}{4(\lambda(F) - \lambda_0)}.$$

We can therefore define a function

$$\left(\frac{r}{\psi}\right)(F) = \begin{cases} \dfrac{r(\lambda(F))}{\psi(\lambda(F))}, & F(\lambda_0) \geqslant -2\lambda_0^2, \\ 0, & F < -2\lambda_0^2. \end{cases} \tag{2.41}$$

Then

$$\left(\frac{r}{\psi}\right)(F) = O[(\lambda^2(F) - \lambda_0^2)^{m+1-n}] \in \mathbb{H}^j, \qquad 0 \leqslant j \leqslant \frac{3n+2}{2},$$

where $\mathbb{H}^j$ is the Hilbert space of rapidly decreasing functions. Using the Fourier transformation, we have

$$\left(\frac{r}{\psi}\right)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isF(\lambda)} \overbrace{\left(\frac{r}{\psi}\right)}(s)\, ds, \qquad \lambda < \lambda_0,$$

where

$$\overbrace{\left(\frac{r}{\psi}\right)}(s) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda_0} e^{-isF(\lambda)} \left(\frac{r}{\psi}\right)(\lambda)\, dF(\lambda), \qquad s \in \mathbb{R}.$$

It follows from Eqs. (2.32) and (2.40) that

$$\left(\frac{r}{\psi}\right)(\lambda) = \frac{(\lambda^2-\lambda_0^2)^{3n+2}}{m!}\bigg(\int_0^1 H_{\rm e}^{(m+1)}[\lambda_0^2+\gamma(\lambda^2-\lambda_0^2)](1-\gamma)^m\,d\gamma + \\ + \int_0^1 H_{\rm o}^{(m+1)}[\lambda_0^2+\gamma(\lambda^2-\lambda_0^2)](1-\gamma)^m\,d\gamma\bigg).$$

For $0 \leqslant j \leqslant (3n+2)/2$, we have the estimate

$$\frac{1}{\sqrt{2\pi}}\int_0^{\lambda_0}\left|\left(\frac{d}{dF}\right)^j\left(\frac{r}{\psi}\right)(\lambda)\right|^2|dF| = \frac{1}{\sqrt{2\pi}}\int_0^{\lambda_0}\left|\left(\frac{1}{4(\lambda-\lambda_0)}\frac{d}{d\lambda}\right)^j\left(\frac{r}{\psi}\right)(\lambda)(\lambda)\right|^2|4(\lambda-\lambda_0)|\,d\lambda \leqslant c_1 < \infty.$$

Using Plancherel's formula, we obtain

$$\int_{-\infty}^{\infty}(1+s^2)^j\left|\overbrace{\left(\frac{r}{\psi}\right)}(s)\right|^2 ds \leqslant c_2 < \infty, \qquad 0 \leqslant j \leqslant \frac{3n+2}{2}. \tag{2.42}$$

In accordance with (2.37) and (2.41), we have

$$r(\lambda) = \frac{\psi(\lambda)}{\sqrt{2\pi}}\int_t^{\infty} e^{isF}\left|\overbrace{\left(\frac{r}{\psi}\right)}(s)\right|^2 ds + \frac{\psi(\lambda)}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{isF}\left|\overbrace{\left(\frac{r}{\psi}\right)}(s)\right|^2 ds \equiv r_1(\lambda)+r_2(\lambda).$$

It hence follows that

$$|e^{-2itF(\lambda)}r_1| = |\psi(\lambda)|\frac{1}{\sqrt{2\pi}}\int_t^{\infty}\left|\overbrace{\left(\frac{r}{\psi}\right)}(s)\right| ds \leqslant \\ \leqslant |\psi(\lambda)|\left[\frac{1}{\sqrt{2\pi}}\int_t^{\infty}(1+s^2)^{-p}\,ds\right]^{1/2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t}(1+s^2)^p\left|\overbrace{\left(\frac{r}{\psi}\right)}(s)\right|^2 ds\right]^{1/2} \leqslant \\ \leqslant \frac{c}{t^{1/2-p}}.$$

It can also be shown that $r_2(\lambda)$ has an analytic continuation to L defined by (2.21). Hence, using formula (2.42) again, we have

$$|e^{-2itF(\lambda)}r_2| = e^{-t\,\mathrm{Re}(iF)}|\psi(\lambda)|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{(s-t)\,\mathrm{Re}(iF)}\left|\overbrace{\left(\frac{r}{\psi}\right)}(s)\right| ds \leqslant ce^{-t\,\mathrm{Re}(iF)}.$$

Because $F(\lambda) = 2(\lambda-\lambda_0)^2 - 2\lambda_0^2$ and hence $\mathrm{Re}(iF) = 2\mu^2$, it follows that

$$|e^{-2itF(\lambda)}r_2| \leqslant C\frac{1}{t^{q/2}}, \qquad C = \text{const}.$$

Finally,

$$|e^{-2itF(\lambda)}R(\lambda)| \leqslant Ce^{-4t\mu^2}.$$

On the other hand, in the case $\lambda > \lambda_0$, we can set $\omega(\lambda) = H_1(\lambda)$. Similarly, from Taylor's formula, we have

$$(\lambda-i)^{m+5}\omega(\lambda) = \sum_{j=0}^{m}\mu_j(\lambda-\lambda_0)^j + \frac{1}{m!}\int_{\lambda_0}^{\lambda}[(\gamma-i)^{m+5}\omega(\gamma)]^{(m+1)}(\lambda-\gamma)^m\,d\gamma. \tag{2.43}$$

We define

$$R(\lambda)=\frac{1}{(\lambda-i)^{m+5}}\sum_{j=0}^{m}\mu_j(\lambda-\lambda_0)^j,\qquad r(\lambda)=\omega(\lambda)-R(\lambda). \tag{2.44}$$

Comparing with Eq. (2.34), we see that

$$\left.\frac{d^j\omega(\lambda)}{d\lambda^j}\right|_{\lambda_0}=\left.\frac{d^jR(\lambda)}{d\lambda^j}\right|_{\lambda_0},\qquad 0\leqslant j\leqslant m.$$

Let $\tilde{\psi}(\lambda)=(\lambda-\lambda_0)^n/(\lambda-i)^{n+2}$. Then

$$\left(\frac{r}{\tilde{\psi}}\right)(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{isF(\lambda)}\,\widehat{\left(\frac{r}{\tilde{\psi}}\right)}(s)\,ds,\qquad \lambda\geqslant\lambda_0,$$

where

$$\widehat{\left(\frac{r}{\tilde{\psi}}\right)}(s)=\int_{\lambda_0}^{\infty}e^{isF(\lambda)}\left(\frac{r}{\tilde{\psi}}\right)(\lambda)\,dF(\lambda).$$

Combining Eqs. (2.43) and (2.44) gives

$$\left(\frac{r}{\tilde{\psi}}\right)(\lambda)=\frac{(\lambda-\lambda_0)^{3n+2}}{(\lambda-i)^{3n+4}}g(\lambda,\lambda_0),$$

where

$$g(\lambda,\lambda_0)=\frac{1}{m!}\int_0^1[(\gamma-i)^{m+5}\omega(\gamma)]^{(\lambda+5)}[\lambda_0+\gamma(\lambda-\lambda_0)](1-\gamma)^m\,d\gamma.$$

We thus see that

$$\left|\frac{d^jg(\lambda,\lambda_0)}{d\lambda^j}\right|\leqslant C,\qquad \lambda\geqslant\lambda_0.$$

This finishes the proof.

Thus, the RH problem (2.29) can be rewritten as

$$P_+^{(2)}(x,t;\lambda)=P_-^{(2)}(x,t;\lambda)J_\delta^{(2)}(x,t;\lambda),\qquad P_+^{(2)}(x,t;\lambda)\to I,\quad \lambda\to\infty, \tag{2.45}$$

where $J_\delta^{(2)}(x,t;\lambda)=\delta_\pm^{\operatorname{ad}\sigma_3}e^{-itF(\lambda)\operatorname{ad}\sigma_3}b_\pm$ with

$$b_+=I+\Phi_+=\begin{pmatrix}1&\omega(\lambda)\\0&1\end{pmatrix},\qquad b_-=I-\Phi_-=\begin{pmatrix}1&0\\\omega^*(\lambda)&1\end{pmatrix}. \tag{2.46}$$

According to decomposition (2.38), $b_\pm$ can be decomposed into two parts:

$$\begin{aligned}b_+&=b_+^{\mathrm{o}}b_+^{\mathrm{a}}=\begin{pmatrix}1&r_1\\0&1\end{pmatrix}\begin{pmatrix}1&r_2+R\\0&1\end{pmatrix},\\ b_-&=b_-^{\mathrm{o}}b_-^{\mathrm{a}}=\begin{pmatrix}1&0\\r_1^*&1\end{pmatrix}\begin{pmatrix}1\\r_2^*+R^*&1\end{pmatrix}.\end{aligned} \tag{2.47}$$

Hence, the jump matrix $J_\delta^{(2)}(x,t;\lambda)$ can be written as

$$J_\delta^{(2)}(x,t;\lambda)=\delta^{\operatorname{ad}\sigma_3}e^{-itF(\lambda)\operatorname{ad}\sigma_3}\underbrace{(b_-^{\mathrm{a}})^{-1}}_{\overline{L}}\underbrace{(b_-^{\mathrm{o}})^{-1}b_+^{\mathrm{o}}}_{\mathbb{R}}\underbrace{b_+^{\mathrm{a}}}_{L}, \tag{2.48}$$

where we indicate that $(b^{\mathrm{a}}_-)^{-1}$ is continued analytically to $\overline{L}$, $(b^{\mathrm{o}}_-)^{-1}b^{\mathrm{o}}_+$ has no analytic continuation but decays rapidly as $t\to\infty$, and b^{a}_+ is continued analytically to L. We introduce the transformation

$$P^{(3)}(x,t;\lambda)=P^{(2)}(x,t;\lambda)T, \tag{2.49}$$

where

$$T=\begin{cases} I, & \lambda\in\Omega_2\cup\Omega_5,\\ (b^{\mathrm{a}}_-)^{-1}, & \lambda\in\Omega_1\cup\Omega_4,\\ (b^{\mathrm{a}}_+)^{-1}, & \lambda\in\Omega_3\cup\Omega_6,\end{cases} \tag{2.50}$$

and Ω_i $(i=1,\dots,6)$ are shown in Fig. 3.

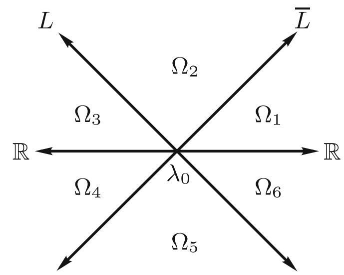

Fig. 3. The domains Ω_i for $i=1,\dots,6$.

Thus, the RH problem on $\mathbb{R}$ can be transformed into a RH problem on $\Omega=\bigcup_i\Omega_i$,

$$P^{(3)}_+(x,t;\lambda)=P^{(3)}_-(x,t;\lambda)J^{(3)}_\delta(x,t;\lambda),\qquad P^{(3)}(x,t;\lambda)\to I,\quad \lambda\to\infty, \tag{2.51}$$

where

$$J^{(3)}_\delta(x,t;\lambda)=\delta^{\mathrm{ad}\,\sigma_3}e^{-itF(\lambda)\,\mathrm{ad}\,\sigma_3}\begin{cases}(b^{\mathrm{o}}_-)^{-1}b^{\mathrm{o}}_+, & \lambda\in\mathbb{R},\\ b^{\mathrm{a}}_+, & \lambda\in L,\\ (b^{\mathrm{a}}_-)^{-1}, & \lambda\in\overline{L}.\end{cases} \tag{2.52}$$

If we take the real axis as an example, we have $P^{(3)}_{R+}=P^{(3)}_{R-}J^{(3)}_\delta$. From transformation (2.49), it follows that

$$P^{(3)}_{R+}=P^{(2)}_{R+}T_{R+},\qquad P^{(3)}_{R-}=P^{(2)}_{R-}T_{R-},\qquad J^{(3)}_\delta=(T_{R-})^{-1}J^{(2)}_\delta T_{R+}. \tag{2.53}$$

If we let $T_{R-}=(b^{\mathrm{a}}_-)^{-1}$ and $T_{R+}=(b^{\mathrm{a}}_+)^{-1}$, then we obtain (2.52) for $\lambda\in\mathbb{R}^+$.

Let

$$b^{(3)}_\pm=\pm\delta^{\mathrm{ad}\,\sigma_3}e^{-itF(\lambda)\,\mathrm{ad}\,\sigma_3}(b_\pm-I),\qquad b^{(3)}=b^{(3)}_+-b^{(3)}_-. \tag{2.54}$$

From the above estimates, we have $b^{(3)}_\pm, b^{(3)}\in L^2(\Omega)\cap L^\infty(\Omega)$. We define a bounded Cauchy operator $C_\pm(f)$ for $f\in L^2(\Omega)$:

$$(C_\pm f)(\lambda)=\frac{1}{2\pi i}\int_\Omega\frac{f(\zeta)}{\zeta-\lambda_\pm}\,d\zeta,\qquad \lambda\in\Omega. \tag{2.55}$$

Thus, the $C_\pm$, as a map from $L^2(\Omega_i)$ to $L^2(\Omega)$, is independent of λ_0 and

$$C_+-C_-=1,\qquad C_{b^{(3)}}f=C_+(f(b^{(3)}_-))+C_-(f(b^{(3)}_+)),$$

where f is a 2×2 matrix-valued function.

If $\chi(x,t;\lambda) \in L^2(\Omega) \cap L^\infty(\Omega)$ is a solution of RH problem (2.51), then, based on [22] and using the Neumann series, we have

$$P^{(3)}(x,t;\lambda) = I + \frac{1}{2\pi i}\int_\Omega \frac{\chi(x,t;\lambda)b^{(3)}(\zeta)}{\zeta-\lambda}\,d\zeta, \qquad \lambda \in \mathbb{C}/\Omega. \tag{2.56}$$

In addition, the solution of nonlocal Kundu–NLS equation (1.3) can be represented as

$$\begin{aligned} q^*(-x,t) &= -2ie^{-i\theta(-x,t)}\lim_{\lambda\to\infty}\left(\lambda\frac{1}{2\pi i}\int_\Omega \frac{\chi(x,t;\lambda)b^{(3)}(\zeta)}{\zeta-\lambda}\,d\zeta\right)_{21} = \\ &= \frac{e^{i\theta(-x,t)}}{\pi}\left(\int_\Omega (1-C_{b^{(3)}})^{-1}I(\zeta)b^{(3)}(\zeta)\,d\zeta\right)_{21}. \end{aligned} \tag{2.57}$$

Let

$$b^{(3)} = b^{\mathrm{e}} + b^R, \tag{2.58}$$

where $b^{\mathrm{e}} = b^{(3)} \restriction \mathbb{R}$ is supported on $\mathbb{R}$ and can be composed of the contributions to $b^{(3)}$ by the terms $r_1(\lambda)$ and $r_1^*(\lambda^*)$, and $b^R = b^{(3)} \restriction L \cup \overline{L}$ is supported on $L \cup \overline{L}$ and can be composed of the contributions to $b^{(3)}$ by the terms $r_2(\lambda)$ and $r_2^*(\lambda^*)$. We give specific expressions below, It is obvious that $b^R = 0$ for $\lambda \in \mathbb{R}$, and we hence have

$$b^{(3)} = \begin{pmatrix} 0 & r_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ r_1^* & 0 \end{pmatrix} = \begin{pmatrix} r_1 + r_1^* & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.59}$$

For $\lambda \in L$, $J_\delta^{(3)}(x,t;\lambda) = b_+^{\mathrm{a}}$, and then

$$b^{(3)} = b_+^{\mathrm{a}} - I = \begin{pmatrix} 0 & r_2 + R \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & r_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}.$$

For $\lambda \in \overline{L}$, $J_\delta^{(3)}(x,t;\lambda) = (b_-^{\mathrm{a}})^{-1}$, and

$$b^{(3)} = (b_-^{\mathrm{a}})^{-1} - I = \begin{pmatrix} 0 & 0 \\ -r_2^* - R^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -r_2^* & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -R^* & 0 \end{pmatrix}.$$

Through careful analysis and verification, we see that the contributions to the solution of the RH problem are the parts of the functions $R(\lambda)$ and $R^*(\lambda^*)$, and the others are infinitesimal at long times. Then

$$\begin{aligned} &\int_\Omega [(1-C_{b^{(3)}})^{-1}I]b^{(3)}\,d\zeta = \int_\Omega [(1-C_{b^{(3)}})^{-1}(1-C_{b^{(3)}}+C_{b^{(3)}})I]b^{(3)}\,d\zeta = \\ &= \int_\Omega b^{(3)}\,d\zeta + \int_\Omega [(1-C_{b^{(3)}})^{-1}C_{b^{(3)}}I]b^{(3)}\,d\zeta = \\ &= \int_\Omega b^{(3)}\,d\zeta + \int_\Omega [(1-C_{b^R})^{-1}(1-C_{b^R})(1-C_{b^{(3)}})^{-1}C_{b^{(3)}}I]b^{(3)}\,d\zeta = \\ &= \int_\Omega b^{(3)}\,d\zeta + \int_\Omega [(1-C_{b^R})^{-1}(1-C_{b^{(3)}}+C_{b^{\mathrm{e}}})(1-C_{b^{(3)}})^{-1}C_{b^{(3)}}I]b^{(3)}\,d\zeta = \\ &= \int_\Omega b^{(3)}\,d\zeta + \int_\Omega [(1-C_{b^R})^{-1}C_{b^{(3)}}I]b^{(3)}\,d\zeta + \\ &\quad + \int_\Omega [(1-C_{b^R})^{-1}C_{b^{\mathrm{e}}}\cdot(1-C_{b^{(3)}})^{-1}C_{b^{(3)}}I]b^{(3)}\,d\zeta. \end{aligned}$$

Using (2.58), we have

$$\int_\Omega [(1-C_{b^{(3)}})^{-1}I]b^{(3)}\,d\zeta = \int_\Omega b^R\,d\zeta + \int_\Omega b^{\mathrm{e}}\,d\zeta + \int_\Omega [(1-C_{b^R})^{-1}C_{b^{(3)}}I]b^{(3)}\,d\zeta + \\ + \int_\Omega [(1-C_{b^R})^{-1}C_{b^{\mathrm{e}}}\cdot(1-C_{b^{(3)}})^{-1}C_{b^{(3)}}I]b^{(3)}\,d\zeta. \tag{2.60}$$

We consider the third integral in (2.60) and write it as

$$\int_\Omega [(1-C_{b^R})^{-1}C_{b^{(3)}}I]b^{(3)}\,d\zeta = \int_\Omega [(1-C_{b^R})^{-1}(C_{b^R}+C_{b^{\mathrm{e}}})I]b^{(3)}\,d\zeta = \\ = \int_\Omega [(1-C_{b^R})^{-1}C_{b^{\mathrm{e}}}I]b^{(3)}\,d\zeta + \int_\Omega [(1-C_{b^R})^{-1}C_{b^R}I]b^{\mathrm{e}} + \\ + \int_\Omega [(1-C_{b^R})^{-1}C_{b^R}I]b^R\,d\zeta = \\ = \int_\Omega [(1-C_{b^R})^{-1}C_{b^{\mathrm{e}}}I]b^{(3)}\,d\zeta + \int_\Omega [(1-C_{b^R})^{-1}C_{b^R}I]b^{\mathrm{e}}\,d\zeta + \\ + \int_\Omega [(1-C_{b^R})^{-1}(1-(1-C_{b^R}))I]b^R\,d\zeta = \\ = \int_\Omega [(1-C_{b^R})^{-1}C_{b^{\mathrm{e}}}I]b^{(3)}\,d\zeta + \int_\Omega [(1-C_{b^R})^{-1}C_{b^R}I]b^{\mathrm{e}}\,d\zeta + \\ + \int_\Omega [(1-C_{b^R})^{-1}I]b^R\,d\zeta - \int_\Omega b^R\,d\zeta. \tag{2.61}$$

Substituting (2.61) in (2.60) yields

$$\int_\Omega [(1-C_{b^{(3)}})^{-1}I]b^{(3)}\,d\zeta = \int_\Omega [(1-C_{b^R})^{-1}I]b^R\,d\zeta + \\ + \int_\Omega b^{\mathrm{e}}\,d\zeta + \int_\Omega [(1-C_{b^R})^{-1}C_{b^{\mathrm{e}}}I]b^{(3)}\,d\zeta + \int_\Omega [(1-C_{b^R})^{-1}C_{b^R}I]b^{\mathrm{e}}\,d\zeta + \\ + \int_\Omega [(1-C_{b^R})^{-1}C_{b^{\mathrm{e}}}(1-C_{b^{(3)}})^{-1}C_{b^{(3)}}I]b^{(3)}\,d\zeta = \\ = \int_\Omega [(1-C_{b^R})^{-1}I]b^R\,d\zeta + \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}. \tag{2.62}$$

Lemma 1. *We have*

$$(1_{\Sigma_1}-C^1_{\mathrm{u}})^{-1} = R_{\Sigma_1}(1_{\Sigma_{12}}-C^{12}_{\mathrm{u}})^{-1}I_{\Sigma_1\to\Sigma_{12}}, \tag{2.63}$$

where Σ_1 and Σ_2 are two oriented lines in $\mathbb{C}$, $\Sigma_{12}=\Sigma_1\cup\Sigma_2$, R_{Σ_1} denotes the restriction map $L^2_{\Sigma_{12}}\to L^2_{\Sigma_1}$, $I_{\Sigma_1\to\Sigma_{12}}$ denotes the embedding $L^2_{\Sigma_1}\to L^2_{\Sigma_{12}}$, C^{12}_{u} denotes the Cauchy operator from $L^2_{\Sigma_{12}}\to L^2_{\Sigma_1}$, C^1_{u} denotes the Cauchy operator from $L^2_{\Sigma_1}\to L^2_{\Sigma_1}$, and 1 denotes the identity operator.

Proof. If $g\in L^2_{\Sigma_{12}}$, then

$$(1_{\Sigma_1}-C^1_{\mathrm{u}})R_{\Sigma_1}g = 1_{\Sigma_1}R_{\Sigma_1}g - C^1_{\mathrm{u}}R_{\Sigma_1}g = g - C^{12}_{\mathrm{u}}g = (1_{\Sigma_{12}}-C^{12}_{\mathrm{u}})g.$$

and the sought relation (2.63) follows.

Hence, for $f\in L^2_{\Sigma_1}$,

$$(1_{\Sigma_1}-C^1_{\mathrm{u}})R_{\Sigma_1}(1_{\Sigma_{12}}-C^{12}_{\mathrm{u}})^{-1}I_{\Sigma_1\to\Sigma_{12}}f = (1_{\Sigma_{12}}-C^{12}_{\mathrm{u}})(1_{\Sigma_{12}}-C^{12}_{\mathrm{u}})^{-1}I_{\Sigma_1\to\Sigma_{12}}f = f.$$

Let $\Sigma_1 = \Omega/\mathbb{R}$, $\Sigma_{12} = \Omega$. By the second resolvent identity, the norm $\|(1 - C_{b^R})^{-1}\|_{L^2(\Omega/\mathbb{R})}$ is equivalent to $\|(1 - C_{b^{(3)}})^{-1}\|_{L^2(\Omega)}$. Then the operator $(1 - C_{b^R})^{-1}$ exists and is uniformly bounded as $t \to \infty$,

$$\|(1 - C_{b^R})^{-1}\|_{L^2(\Omega)} \leqslant c. \tag{2.64}$$

Hence follow the estimates for terms in the right-hand side of (2.62):

$$\begin{aligned}
&|\mathrm{I}| = \left| \int_\Omega b^{\mathrm{e}}\, d\zeta \right| \leqslant \|b^{\mathrm{e}}\|_{L^1} \leqslant ct^{-l},\\
&|\mathrm{II}| = \left| \int_\Omega [(1 - C_{b^R})^{-1} C_{b^{\mathrm{e}}} I] b^{(3)}\, d\zeta \right| \leqslant \|(1 - C_{b^R})^{-1} C_{b^{\mathrm{e}}} I\|_{L^2}\, \|b^{(3)}\|_{L^2} \leqslant\\
&\qquad \leqslant c\|b^{(3)}\|_{L^2}\, \|b^R\|_{L^2} \leqslant ct^{-l},\\
&|\mathrm{III}| = \left| \int_\Omega [(1 - C_{b^R})^{-1} C_{b^R} I] b^{\mathrm{e}}\, d\zeta \right| \leqslant \|(1 - C_{b^R})^{-1} C_{b^R} I\|_{L^2}\, \|b^{\mathrm{e}}\|_{L^2} \leqslant ct^{-l},\\
&|\mathrm{IV}| = \left| \int_\Omega [(1 - C_{b^R})^{-1} C_{b^{\mathrm{e}}} (1 - C_{b^{(3)}})^{-1} C_{b^{(3)}} I] b^{(3)}\, d\zeta \right| \leqslant\\
&\qquad \leqslant \|(1 - C_{b^R})^{-1}\|_{L^2}\, \|b^{\mathrm{e}}\|_{L^2} \|(1 - C_{b^{(3)}})^{-1}\|_{L^2}\, \|b^{(3)}\|_{L^2} \leqslant ct^{-l}
\end{aligned}$$

(where we write $L^2 = L^2(\Omega)$ and $L^1 = L^1(\Omega)$ for brevity). It hence follows that

$$q^*(-x, t) = \frac{e^{i\theta(-x,t)}}{\pi} \left(\int_\Omega (1 - C_{b^R})^{-1} I(\zeta) b^R(\zeta)\, d\zeta \right)_{21} + O(t^{-l}). \tag{2.65}$$

2.3. Scaling transformation. Based on [25], [31], [32], [38], we introduce a scaling transformation

$$\Xi\colon\ \lambda - \lambda_0 = \frac{\tilde\lambda}{\sqrt{8t}}, \quad \Omega \to \Omega^{\Xi}. \tag{2.66}$$

We then have the RH problem

$$P_+^{(4)}(x, t; \tilde\lambda) = P_-^{(4)}(x, t; \tilde\lambda) J^{(4)}(x, t; \tilde\lambda), \qquad P^{(4)}(x, t; \tilde\lambda) \to I, \quad \tilde\lambda \to \infty, \tag{2.67}$$

where $J^{(4)}(x, t; \tilde\lambda) = \Xi(J_\sigma^{(3)}(x, t; \lambda))$ or, explicitly,

$$J^{(4)}(x, t; \tilde\lambda) = (8t)^{-\frac{i\kappa}{2}} e^{2it\lambda_0^2 + i\tau_0} \cdot (-\tilde\lambda)^{i\kappa} e^{-\frac{i\tilde\lambda^2}{4} + \tau(\tilde\lambda) - i\tau_0} = \Xi_1 \cdot \Xi_2, \tag{2.68}$$

where κ and τ ($\tau_0 = \tau(0)$) are obtained from Eq. (2.26). As a result, we have

$$P^{(4)}(\tilde\lambda) = \Xi(P^{(3)}(\lambda)) = I + \Xi(P_1^{(3)}(\lambda)) + \Xi(P_2^{(3)}(\lambda)) + \cdots, \tag{2.69}$$

whence $P_1^{(4)}(\tilde\lambda) = P_1^{(3)}(\lambda)\sqrt{8t}$.

Then, the solution of nonlocal Kundu–NLS equation (1.3) can be expressed as

$$-q^*(x, t) e^{-i\theta(-x,t)} = 2i(P_1^{(4)}(\tilde\lambda))_{12} \frac{1}{\sqrt{8t}} + O(t^{-l}) = \frac{i}{\sqrt{2t}} (P_1^{(4)}(\tilde\lambda))_{12} + O(t^{-l}). \tag{2.70}$$

With the jump matrix $J^{(4)}(x, t; \tilde\lambda)$ in (2.68), we see that Ξ_1 is independent of $\tilde\lambda$, and therefore the transformation $P^{(5)}(x, t; \tilde\lambda) = \Xi_1^{-\operatorname{ad}\sigma_3} P^{(4)}(x, t; \tilde\lambda)$ gives a RH problem on Ω_{Ξ_1} (see Fig. 4),

$$P_+^{(5)}(x, t; \tilde\lambda) = P_-^{(5)}(x, t; \tilde\lambda) J^{(5)}(x, t; \tilde\lambda), \qquad P^{(5)}(x, t; \tilde\lambda) \to I, \quad \tilde\lambda \to \infty, \tag{2.71}$$

where $J^{(5)}(x, t; \tilde\lambda) = \Xi_2^{\operatorname{ad}\sigma_3} (\hat b_-)^{-1} \hat b_+$ and $\hat b_\pm = I \pm b_\pm^R$.

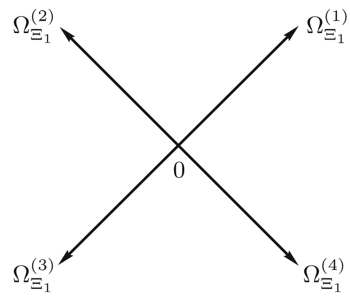

Fig. 4. The contours $\Omega_{\Xi_1(\tilde{\lambda})}$.

On one hand, for $\tilde{\lambda} \in \{\tilde{\lambda} = \mu e^{\pm 3\pi i/4}, \mu \in \mathbb{R}\}$, we have

$$\hat{b}_+(\tilde{\lambda}) = \begin{pmatrix} 0 & \Xi(R(\tilde{\lambda})) \\ 0 & 0 \end{pmatrix}, \qquad \hat{b}_-(\tilde{\lambda}) = \begin{pmatrix} 0 & 0 \\ \Xi(R^*(\tilde{\lambda}^*)) & 0 \end{pmatrix}. \tag{2.72}$$

On the other hand, as $t \to \infty$, we obtain the RH problem with a phase point, which suggests that

$$\Xi_2 = (-\tilde{\lambda})^{i\kappa} e^{\tau\left(\frac{\tilde{\lambda}}{\sqrt{8t}} + \lambda_0\right) - \tau(\lambda_0)} e^{\frac{-i\tilde{\lambda}^2}{4}} \to (-\tilde{\lambda})^{i\kappa} e^{\frac{-i\tilde{\lambda}^2}{4}}, \tag{2.73}$$

and

$$\begin{aligned} &\lim_{t\to\infty}\left(H_1\left(\frac{\tilde{\lambda}}{\sqrt{8t}} + \lambda_0\right)\right) = H_1(\lambda_0), \\ &\lim_{t\to\infty}\left(\frac{H_1}{1+\sigma H_1 H_2}\left(\frac{\tilde{\lambda}}{\sqrt{8t}} + \lambda_0\right)\right) = \frac{H_1(\lambda_0)}{1+\sigma H_1(\lambda_0)H_2(\lambda_0)}, \\ &\lim_{t\to\infty}\left(\sigma H_2\left(\frac{\tilde{\lambda}}{\sqrt{8t}} + \lambda_0\right)\right) = \sigma H_2(\lambda_0), \\ &\lim_{t\to\infty}\left(\frac{\sigma H_2}{1+\sigma H_1 H_2}\left(\frac{\tilde{\lambda}}{\sqrt{8t}} + \lambda_0\right)\right) = \frac{\sigma H_2(\lambda_0)}{1+\sigma H_1(\lambda_0)H_2(\lambda_0)}. \end{aligned} \tag{2.74}$$

We thus arrive at the RH problem on the contour Ω_{Ξ_2} (see Fig. 5),

$$P^{(6)}_+(x,t;\tilde{\lambda}) = P^{(6)}_-(x,t;\tilde{\lambda}) J^{(6)}(x,t;\tilde{\lambda}), \qquad P^{(6)}(x,t;\tilde{\lambda}) \to I, \quad \tilde{\lambda} \to \infty, \tag{2.75}$$

where $J^{(6)}(x,t;\tilde{\lambda}) = (-\tilde{\lambda})^{i\kappa\,\mathrm{ad}\,\sigma_3} e^{\frac{-i\tilde{\lambda}^2}{4}\,\mathrm{ad}\,\sigma_3}$ is given as follows:

$$\begin{aligned} &\tilde{\lambda} \in \Omega^{(1)}_{\Xi_2}: \qquad \check{b}_+ = \begin{pmatrix} 1 & 0 \\ H_1(\lambda_0) & 1 \end{pmatrix}, \\ &\tilde{\lambda} \in \Omega^{(2)}_{\Xi_2}: \qquad \check{b}_+ = \begin{pmatrix} 1 & \dfrac{\sigma H_2(\lambda_0)}{1+\sigma H_1(\lambda_0)H_2(\lambda_0)} \\ 0 & 1 \end{pmatrix}, \\ &\tilde{\lambda} \in \Omega^{(3)}_{\Xi_2}: \qquad (\check{b}_-)^{-1} = \begin{pmatrix} 1 & 0 \\ \dfrac{H_1(\lambda_0)}{1+\sigma H_1(\lambda_0)H_2(\lambda_0)} & 1 \end{pmatrix}, \\ &\tilde{\lambda} \in \Omega^{(4)}_{\Xi_2}: \qquad (\check{b}_-)^{-1} = \begin{pmatrix} 1 & \sigma H_2(\lambda_0) \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

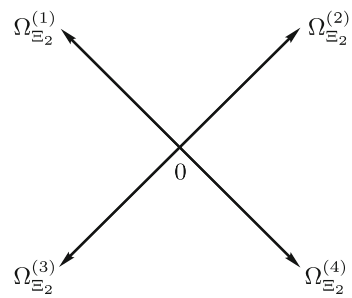

Fig. 5. The contour Ω_{Ξ_2}.

According to [17], [25], [30], [32], [38], the jump matrices $J^{(5)}(x,t;\tilde{\lambda})$ and $J^{(6)}(x,t;\tilde{\lambda})$ satisfy the norm relation

$$\|J^{(5)}(x,t;\tilde{\lambda}) - J^{(6)}(x,t;\tilde{\lambda})\|_{L^1\cap L^\infty(\Omega_{\Xi_2})} \leqslant \begin{cases} ct^{-1+2|\operatorname{Im}\kappa(\lambda_0)|}, & \operatorname{Im}\kappa(\lambda_0) > 0, \\ ct^{-1}\log t, & \operatorname{Im}\kappa(\lambda_0) = 0, \\ t^{-1}, & \operatorname{Im}\kappa(\lambda_0) < 0, \end{cases}$$

whence the solution of nonlocal Kundu–NLS equation (1.3) can be given as

$$-q^*(x,t)e^{-i\theta(-x,t)} = \frac{i}{\sqrt{2t}}(\Xi_1)^2(P_1^{(6)}(x,t;\tilde{\lambda}))_{12} + \begin{cases} O(ct^{-1+2|\operatorname{Im}\kappa(\lambda_0)|}), & \operatorname{Im}\kappa(\lambda_0) > 0, \\ O(ct^{-1}\log t), & \operatorname{Im}\kappa(\lambda_0) = 0, \\ O(t^{-1}), & \operatorname{Im}\kappa(\lambda_0) < 0, \end{cases}$$

where $P_1^{(6)}(x,t;\tilde{\lambda})$ can be obtained by the expansion of $P^{(6)}(x,t;\tilde{\lambda})$.

Next, we introduce a transformation

$$P^{(7)}(x,t;\tilde{\lambda}) = P^{(6)}(x,t;\tilde{\lambda})\digamma^{-1}, \tag{2.76}$$

where $\digamma$ is defined as

$$\digamma = \begin{cases} (-\tilde{\lambda})^{-i\kappa\,\mathrm{ad}\,\sigma_3}, & \tilde{\lambda} \in \Omega_\digamma^{(2)} \cup \Omega_\digamma^{(5)}, \\ (-\tilde{\lambda})^{-i\kappa\,\mathrm{ad}\,\sigma_3}(b_+^\digamma)^{-1}, & \tilde{\lambda} \in \Omega_\digamma^{(1)} \cup \Omega_\digamma^{(3)}, \\ (-\tilde{\lambda})^{-i\kappa\,\mathrm{ad}\,\sigma_3}(b_-^\digamma)^{-1}, & \tilde{\lambda} \in \Omega_\digamma^{(4)} \cup \Omega_\digamma^{(6)}, \end{cases} \tag{2.77}$$

and the domains $\Omega_\digamma^{(i)}$ and contours $\digamma^i$ are shown in Fig. 6. We then have

$$b_+^\digamma = \begin{cases} (-\tilde{\lambda})^{-i\kappa\,\mathrm{ad}\,\sigma_3} e^{-\frac{i\tilde{\lambda}^2}{4}\,\mathrm{ad}\,\sigma_3} \begin{pmatrix} 1 & 0 \\ H_1(\lambda_0) & 1 \end{pmatrix}, & \tilde{\lambda} \in \Omega_\digamma^{(1)}, \\ (-\tilde{\lambda})^{-i\kappa\,\mathrm{ad}\,\sigma_3} e^{-\frac{i\tilde{\lambda}^2}{4}\,\mathrm{ad}\,\sigma_3} \begin{pmatrix} 1 & \dfrac{\sigma H_2(\lambda_0)}{1+\sigma H_1(\lambda_0)H_2(\lambda_0)} \\ 0 & 1 \end{pmatrix}, & \tilde{\lambda} \in \Omega_\digamma^{(3)}, \end{cases}$$

$$(b_-^\digamma)^{-1} = \begin{cases} (-\tilde{\lambda})^{-i\kappa\,\mathrm{ad}\,\sigma_3} e^{-\frac{i\tilde{\lambda}^2}{4}\,\mathrm{ad}\,\sigma_3} \begin{pmatrix} 1 & 0 \\ \dfrac{H_1(\lambda_0)}{1+\sigma H_1(\lambda_0)H_2(\lambda_0)} & 1 \end{pmatrix}, & \tilde{\lambda} \in \Omega_\digamma^{(4)}, \\ (-\tilde{\lambda})^{-i\kappa\,\mathrm{ad}\,\sigma_3} e^{-\frac{i\tilde{\lambda}^2}{4}\,\mathrm{ad}\,\sigma_3} \begin{pmatrix} 1 & \sigma H_2(\lambda_0) \\ 0 & 1 \end{pmatrix}, & \tilde{\lambda} \in \Omega_\digamma^{(6)}. \end{cases}$$

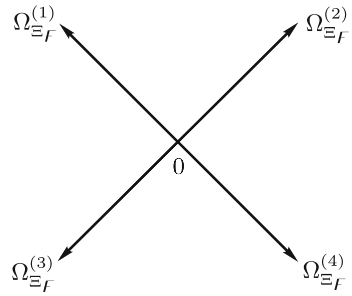

Fig. 6. Domains $\Omega^i_\digamma$ and contours $\digamma^i$.

It hence follows that $P^{(7)}(x,t;\tilde\lambda)$ satisfies the RH problem

$$\begin{gathered} P^{(7)}_+(x,t;\tilde\lambda) = P^{(7)}_-(x,t;\tilde\lambda)J^{(7)}(x,t;\tilde\lambda), \qquad P^{(7)}(x,t;\tilde\lambda) \to I, \quad \tilde\lambda \to \infty, \\ J^{(7)}(x,t;\tilde\lambda) = \begin{cases} e^{-\frac{i\tilde\lambda^2}{4}\operatorname{ad}\sigma_3}\begin{pmatrix} 1+\sigma H_1(\lambda_0)H_2(\lambda_0) & \sigma H_2(\lambda_0) \\ H_1(\lambda_0) & 1 \end{pmatrix}^{-1}, & \tilde\lambda \in \mathbb{R}, \\ I, & \tilde\lambda \in \digamma^1 \cup \digamma^2 \cup \digamma^3 \cup \digamma^4. \end{cases} \end{gathered} \tag{2.78}$$

By transformation (2.76), the formula $\digamma^{-1}(-\tilde\lambda)^{-i\kappa\operatorname{ad}\sigma_3}$ can be expressed as

$$\digamma^{-1}(-\tilde\lambda)^{-i\kappa\operatorname{ad}\sigma_3} = I + O\left(\frac{1}{\tilde\lambda}\right), \qquad \tilde\lambda \to \infty, \tag{2.79}$$

and therefore

$$\begin{aligned} P^{(7)}(x,t;\tilde\lambda) &= P^{(6)}(x,t;\tilde\lambda)\digamma^{-1} = P^{(6)}(\digamma^{-1}(-\tilde\lambda)^{-i\kappa\sigma_3})(-\tilde\lambda)^{-i\kappa\sigma_3} = \\ &= \left(I + \frac{P^{(6)}_1}{\tilde\lambda} + O\left(\frac{1}{\tilde\lambda^2}\right)\right)\left(I + O\left(\frac{1}{\tilde\lambda}\right)\right)(-\tilde\lambda)^{-i\kappa\sigma_3} = \\ &= \left(I + \frac{P^{(6)}_1}{\tilde\lambda} + \frac{P^{(6)}_2}{\tilde\lambda^2} + \cdots\right)(-\tilde\lambda)^{-i\kappa\sigma_3}. \end{aligned}$$

Let $P^{(8)}(x,t;\tilde\lambda) = P^{(7)}(x,t;\tilde\lambda)e^{-\frac{i\tilde\lambda^2}{4}\sigma_3}$. It then follows that $P^{(8)}(x,t;\tilde\lambda)$ satisfies the RH problem

$$\begin{gathered} P^{(8)}_+(x,t;\tilde\lambda) = P^{(8)}_-(x,t;\tilde\lambda)J^{(8)}(x,t;\tilde\lambda), \qquad P^{(8)}e^{\frac{i\tilde\lambda^2}{4}\sigma_3}(-\tilde\lambda)^{-i\kappa\sigma_3} \to I, \quad \tilde\lambda \to \infty, \\ J^{(8)}(x,t;\lambda_0) = \begin{pmatrix} 1+\sigma H_1(\lambda_0)H_2(\lambda_0) & \sigma H_2(\lambda_0) \\ H_1(\lambda_0) & 1 \end{pmatrix}. \end{gathered} \tag{2.80}$$

Theorem 1. *If the spectral functions are defined by Eqs.* (2.10), *the long-time asymptotics of the solution of the nonlocal Kundu–NLS equation* (1.3) *with a decaying initial value* $q_0(x)$ *are given by*

$$q^*(-x,t) = t^{-1/2+\operatorname{Im}\xi(\lambda_0)}\frac{\pi e^{\frac{\pi i - 2\pi\kappa}{4} + i\theta(-x,t)}}{H_1(\lambda_0)\Gamma(-a)} + \begin{cases} O(ct^{-1+2|\operatorname{Im}\kappa(\lambda_0)|}), & \operatorname{Im}\kappa(\lambda_0) > 0, \\ O(ct^{-1}\log t), & \operatorname{Im}\kappa(\lambda_0) = 0, \\ O(t^{-1}), & \operatorname{Im}\kappa(\lambda_0) < 0, \end{cases} \tag{2.81}$$

where $\Gamma(\,\cdot\,)$ *is the Gamma function.*

Appendix: Proof of Theorem 1

To solve the nonlocal Kundu–NLS equation with a decaying initial value, we use the Weber equation and the standard parabolic cylinder function. From the equalities

$$\frac{d}{d\tilde{\lambda}}P_{+}^{(8)} = \frac{d}{d\tilde{\lambda}}P_{-}^{(8)}J^{(8)}(\lambda_0), \qquad \frac{1}{2}i\tilde{\lambda}\sigma_3 P_{+}^{(8)} = \frac{1}{2}i\tilde{\lambda}\sigma_3 P_{-}^{(8)}J^{(8)}(\lambda_0)$$

we have

$$\left(\partial_{\tilde{\lambda}}P^{(8)} + \frac{1}{2}i\tilde{\lambda}\sigma_3 P^{(8)}\right)_{+} = \left(\partial_{\tilde{\lambda}}P^{(8)} + \frac{1}{2}i\tilde{\lambda}\sigma_3 P^{(8)}\right)_{-}J^{(8)}(\lambda_0). \tag{A.1}$$

Obviously, because $J^{(8)}(\lambda_0) = 1$, it follows that $\det P_{\pm}^{(8)} = 1$ for $\tilde{\lambda} \in \mathbb{R}$. Thus, based on the Painlevé's expansion theorem, we see that $\det P^{(8)}$ is analytic and bounded on $\mathbb{C}$. Furthermore,

$$\begin{aligned}\left[\left(\partial_{\tilde{\lambda}}P^{(8)} + \frac{1}{2}i\tilde{\lambda}\sigma_3 P^{(8)}\right)(P^{(8)})^{-1}\right]_{+} &= \left(\partial_{\tilde{\lambda}}P^{(8)} + \frac{1}{2}i\tilde{\lambda}\sigma_3 P^{(8)}\right)_{-}J^{(8)}(\lambda_0)(J^{(8)}(\lambda_0))^{-1}(P_{-}^{(8)})^{-1} = \\ &= \left[\left(\partial_{\tilde{\lambda}}P^{(8)} + \frac{1}{2}i\tilde{\lambda}\sigma_3 P^{(8)}\right)(P^{(8)})^{-1}\right]_{-}\end{aligned} \tag{A.2}$$

is also analytic and bounded on $\mathbb{C}$. From the expression

$$\left(\partial_{\tilde{\lambda}}P^{(8)} + \frac{1}{2}i\tilde{\lambda}\sigma_3 P^{(8)}\right)(P^{(8)})^{-1} = \frac{1}{2}i\tilde{\lambda}[\sigma_3, P^{(8)}] = \frac{1}{2}i[\sigma_3, P_1^{(7)}] + O\left(\frac{1}{\tilde{\lambda}}\right), \tag{A.3}$$

by the Liouville theorem we see that

$$\left(\partial_{\tilde{\lambda}}P^{(8)} + \frac{1}{2}i\tilde{\lambda}\sigma_3 P^{(8)}\right)(P^{(8)})^{-1} = \begin{pmatrix}\Theta_{11} & \Theta_{12}\\ \Theta_{21} & \Theta_{22}\end{pmatrix} \tag{A.4}$$

is a constant matrix. Comparing formulas (A.2) and (A.4), we have

$$i(P_1^{(7)})_{12} = \Theta_{12}, \qquad -i(P_1^{(7)})_{21} = \Theta_{21}, \qquad \Theta_{11} = \Theta_{22} = 0, \tag{A.5}$$

and hence the solution of nonlocal Kundu–NLS equation (1.3) can be written as

$$u(x,y) = -\frac{1}{\sqrt{2t}}(\Xi_1)^2\Theta_{21} + O\left(\frac{\log t}{t}\right). \tag{A.6}$$

From (A.4), we have the system of equations

$$\frac{dP_{11}^{(8)}}{d\tilde{\lambda}} + \frac{1}{2}i\tilde{\lambda}P_{11}^{(8)} = \Theta_{12}P_{21}^{(8)}, \qquad \frac{dP_{21}^{(8)}}{d\tilde{\lambda}} - \frac{1}{2}i\tilde{\lambda}P_{21}^{(8)} = \Theta_{21}P_{11}^{(8)}. \tag{A.7}$$

which reduces to a single equation

$$\frac{dP_{11}^{(8)}}{d\tilde{\lambda}} = \left(-\frac{\tilde{\lambda}^2}{4} - \frac{1}{2}i + \Theta_{12}\Theta_{21}\right)P_{11}^{(8)}. \tag{A.8}$$

For $\operatorname{Im}\tilde{\lambda} > 0$, let

$$(P^{(8)})^{+} = \begin{pmatrix}(P^{(8)})_{11}^{+} & (P^{(8)})_{12}^{+}\\ (P^{(8)})_{21}^{+} & (P^{(8)})_{22}^{+}\end{pmatrix}.$$

Then Eq. (A.8) can be rewritten in the form

$$\frac{d(P^{(8)})^+_{11}}{d\tilde{\lambda}} = \left(-\frac{\tilde{\lambda}^2}{4} - \frac{1}{2}i + \Theta_{12}\Theta_{21}\right)(P^{(8)})^+_{11}, \tag{A.9}$$

which is called the Weber equation. It has two linearly independent solutions $D_a(\zeta)$ and $D_a(-\zeta)$ called the parabolic cylinder functions,

$$(P^{(8)})^+_{11} = c_1 D_a(\zeta) + c_2 D_a(-\zeta), \tag{A.10}$$

where $\zeta = \tilde{\lambda} e^{-3i\pi/4}$, c_1, c_2 are constants.

The parabolic cylinder functions $D_a(\zeta)$ have the following asymptotic property as $\zeta \to \infty$: for $|\arg\zeta| < 3\pi/4$,

$$D_a(\zeta) = \zeta^2 e^{-\zeta^2/4} + \zeta^2 e^{-\zeta^2/4} O(\zeta^{-2});$$

for $\pi/4 < \arg\zeta < 5\pi/4$,

$$D_a(\zeta) = \zeta^2 e^{-\zeta^2/4}[1 + O(\zeta^{-2})] - \sqrt{2\pi}\Gamma^{-1}(-a) e^{ai\pi} \zeta^{-1-a} e^{\zeta^2/4}(1 + O(\zeta^{-2});$$

and for $-5\pi/4 < \arg\zeta < -\pi/4$,

$$D_a(\zeta) = \zeta^2 e^{-\zeta^2/4}[1 + O(\zeta^{-2})] - \sqrt{2\pi}\Gamma^{-1}(-a) e^{-ai\pi} \zeta^{-1-a} e^{\zeta^2/4}(1 + O(\zeta^{-2}).$$

Thus,

$$\begin{aligned} (P^{(8)})^+_{11}(\tilde{\lambda}) &= e^{-3\pi\kappa/4} D_a(\tilde{\lambda} e^{-3i/4}), \\ (P^{(8)})^+_{21}(\tilde{\lambda}) &= \frac{1}{\Theta_{12}}\, e^{-3\pi\kappa/4}\left(\partial_{\tilde{\lambda}} D_a\left(\tilde{\lambda} e^{-3i/4}\right) + \frac{i\tilde{\lambda}}{2} D_a\left(\tilde{\lambda} e^{-3i/4}\right)\right), \end{aligned} \tag{A.11}$$

Similarly, for $\operatorname{Im}\tilde{\lambda} < 0$, let

$$(P^{(8)})^- = \begin{pmatrix} (P^{(8)})^-_{11} & (P^{(8)})^-_{12} \\ (P^{(8)})^-_{21} & (P^{(8)})^-_{22} \end{pmatrix}.$$

Then

$$\begin{aligned} (P^{(8)})^-_{11}(\tilde{\lambda}) &= e^{\pi\kappa/4} D_a(\tilde{\lambda} e^{i/4}), \\ (P^{(8)})^-_{21}(\tilde{\lambda}) &= \frac{1}{\Theta_{12}}\, e^{\pi\kappa/4}\left(\partial_{\tilde{\lambda}} D_a(\tilde{\lambda} e^{i/4}) + \frac{i\tilde{\lambda}}{2} D_a(\tilde{\lambda} e^{i/4})\right). \end{aligned} \tag{A.12}$$

From the RH problem (2.80), we then have

$$\begin{pmatrix} 1 + \sigma H_1(\lambda_0) H_2(\lambda_0) & \sigma H_2(\lambda_0) \\ H_1(\lambda_0) & 1 \end{pmatrix} = \begin{pmatrix} (P^{(8)})^-_{11} & (P^{(8)})^-_{12} \\ (P^{(8)})^-_{21} & (P^{(8)})^-_{22} \end{pmatrix}^{-1} \begin{pmatrix} (P^{(8)})^+_{11} & (P^{(8)})^+_{12} \\ (P^{(8)})_{21} & (P^{(8)})^+_{22} \end{pmatrix}, \tag{A.13}$$

whence

$$H_1(\lambda_0) = -(P^{(8)})^-_{21}(P^{(8)})^+_{11} + (P^{(8)})^-_{11}(P^{(8)})^+_{21} = \frac{\sqrt{2\pi} e^{(i\pi - 2\pi\kappa)/4}}{\Theta_{21}\Gamma(-a)}. \tag{A.14}$$

Conflicts of interest. The authors declare no conflicts of interest.

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