Theoretical and Mathematical Physics, **212**(1): 962–971 (2022)

EXISTENCE AND STABILITY OF A STABLE STATIONARY SOLUTION WITH A BOUNDARY LAYER FOR A SYSTEM OF REACTION–DIFFUSION EQUATIONS WITH NEUMANN BOUNDARY CONDITIONS

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We consider an initial boundary value problem for a singularly perturbed parabolic system of two reaction–diffusion-type equations with Neumann conditions, where the diffusion coefficients are of different degrees of smallness and the right-hand sides need not be quasimonotonic. We obtain an asymptotic approximation of the stationary solution with a boundary layer and prove existence theorems, the asymptotic stability in the sense of Lyapunov, and the local uniqueness of such a solution. The obtained result is applied to a class of problems of chemical kinetics.

Keywords: reaction–diffusion systems, stationary solution, quasimonotonicity conditions, method of differential inequalities, upper and lower solutions, boundary layer, stability in the sense of Lyapunov

DOI: 10.1134/S0040577922070066

1. Introduction

We consider a parabolic system of two reaction–diffusion-type equations with Neumann conditions with the diffusion coefficients of different degrees of smallness. Equations of this type are used to model processes in various applications. In particular, in chemical kinetics, they describe the interaction between substances in an inhomogeneous medium by modeling fast biomolecular reactions at different rates (with different intensities of sources). Their solutions describe the variation in the concentration of reactants in space and time. For example, similar problems were considered in [1]–[4] (also see the references therein) in studying the front-type solutions in problems of urban ecology. Stationary solutions with internal transition layers in such systems were studied in [5], and in problems with discontinuous sources, in [6]. A rigorous asymptotic analysis of such systems is based on the asymptotic method of differential inequalities (see [7]). This method is based on basic theorems of differential inequalities (comparison theorems), which can be found, for example, in book [8]. Such systems were studied under the condition that the right-hand sides of the equations are quasimonotonic. The physical interpretation of such conditions in problems of chemical kinetics, where the quantities have the meaning of concentrations of substances, amounts to specifying how variations in the concentration of one of the components affects the rate of variation in the second component of the system. The relevant studies commonly either resort to one of the four possible versions (see, e.g., [9]) or successively apply all possible combinations, each with specific conditions (see, e.g., [10]). We note that

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This paper is supported by the Russian Science Foundation grant No. 18-11-00042.

Translated from *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 212, No. 1, pp. 83–94, July, 2022. Received January 21, 2022. Revised January 21, 2022. Accepted February 16, 2022.

the study of quasimonotonic singularly perturbed systems started quite a long time ago in applications to systems of Tikhonov-type ordinary differential equations [11], and was then extended to some classes of systems of partial differential equations and boundary value problems (see [12] and the references therein). At the same time, there are several important applications in which the quasimonotonicity condition is not satisfied.

In this paper, we study the problem of the existence and asymptotic stability of a stationary solution of a system with Neumann boundary conditions without requiring the quasimonotonicity of the right-hand sides. We construct an asymptotic approximation of the stationary solution and prove its asymptotic stability in the sense of Lyapunov. The obtained result can be applied to a class of problems of chemical kinetics.

2. Statement of the problem

We consider a system of two reaction–diffusion-type equations with different powers of a small parameter multiplying the differential operator. The problem is considered in a closed simply connected two-dimensional domain D bounded by a sufficiently smooth boundary ∂D:

$$\begin{aligned} &\mathcal{N}_u := \varepsilon^4 \Delta u - \frac{\partial u}{\partial t} - f(u,v,x,\varepsilon) = 0, \qquad x = (x_1, x_2) \in D, \quad t > 0,\\ &\mathcal{N}_v := \varepsilon^2 \Delta v - \frac{\partial v}{\partial t} - g(u,v,x,\varepsilon) = 0,\\ &u(x,0,\varepsilon) = u_{\mathrm{init}}(x,\varepsilon), v(x,0,\varepsilon) = v_{\mathrm{init}}(x,\varepsilon), \qquad x \in \bar{D},\\ &\left.\frac{\partial u}{\partial n}\right|_{\partial D} = h(x), \qquad \left.\frac{\partial v}{\partial n}\right|_{\partial D} = q(x), \qquad x \in \partial D. \end{aligned} \tag{1}$$

Here, $\varepsilon > 0$ is a small parameter, and the functions $f(u,v,x,\varepsilon)$ and $g(u,v,x,\varepsilon)$ are defined for $(u,v,x) \in G \equiv I_u \times I_v \times D$ and $0 < \varepsilon \le \varepsilon_0$, where ε_0 is a positive constant. The derivative in the boundary condition is taken with respect to the inner normal to ∂D. Our goal is to study the existence and Lyapunov stability of the stationary solution of problem (1). This solution is a solution of the elliptic boundary value problem

$$\begin{aligned} &\mathcal{L}_u := \varepsilon^4 \Delta u - f(u,v,x,\varepsilon) = 0, \qquad x \in D,\\ &\mathcal{L}_v := \varepsilon^2 \Delta v - g(u,v,x,\varepsilon) = 0,\\ &\left.\frac{\partial u}{\partial n}\right|_{\partial D} = h(x), \qquad \left.\frac{\partial v}{\partial n}\right|_{\partial D} = q(x), \qquad x \in \partial D. \end{aligned} \tag{2}$$

In what follows, we study the existence and Lyapunov stability of a boundary-layer solution of problem (2), regarded as the stationary solution of problem (1).

We assume that the following conditions are satisfied.

Condition A0. The functions $f(u,v,x,\varepsilon)$, $g(u,v,x,\varepsilon)$, $h(x)$, and $q(x)$ are sufficiently smooth.

Condition A1. The degenerate system

$$\begin{aligned} &f(u,v,x,0) = 0,\\ &g(u,v,x,0) = 0 \end{aligned} \tag{3}$$

has a solution $u = \bar{u}(x)$, $v = \bar{v}(x)$ such that

$$f_u(\bar{u}(x), \bar{v}(x), x, 0) > 0, \qquad g_v(\bar{u}(x), \bar{v}(x), x, 0) > 0 \quad \text{for} \quad x \in \bar{D}.$$

We use the notation $\bar{f}(x) = f(\bar{u}(x), \bar{v}(x), x, 0)$, $\bar{g}(x) = g(\bar{u}(x), \bar{v}(x), x, 0)$ and a similar notation for the derivatives of these functions.

Condition A2. The determinant of the matrix $\begin{pmatrix} \bar{f}_u(x) & \bar{f}_v(x) \\ \bar{g}_u(x) & \bar{g}_v(x) \end{pmatrix}$ is positive for $x \in \bar{D}$.

Condition A3. There exist $\gamma_u(x)$ and $\gamma_v(x)$ such that, for any $-1 \le \Theta \le 1$ in the expression

$$\begin{pmatrix} \bar{f}_u(x) & \Theta\bar{f}_v(x) \\ \Theta\bar{g}_u(x) & \bar{g}_v(x) \end{pmatrix} \begin{pmatrix} \gamma_u(x) \\ \gamma_v(x) \end{pmatrix} = \begin{pmatrix} A(x) \\ B(x) \end{pmatrix},$$

we have $A(x) > 0$ and $B(x) > 0$ for all $x \in \bar{D}$.

Remark 1. In some cases, a solution of system (3) can be found as follows. The equation $f(u, v, x, 0) = 0$ has a root $u = \phi(v, x)$: $f_u(\phi(v, x), v, x, 0) > 0$, $v \in I_v$, $x \in \bar{D}$. The equation $p(v, x) := g(\phi(v, x), v, x, 0) = 0$ has a root $v = v_0(x)$: $p_v(v_0, x) = g_v(\phi(v_0, x), v_0, x, 0) > 0$ for $x \in \bar{D}$, i.e., under more stringent conditions.

In what follows, we prove the existence of a stationary solution, construct and justify its asymptotics, and obtain conditions for its asymptotic stability in the sense of Lyapunov.

3. Construction of the asymptotics

3.1. Local coordinates. We let the boundary ∂D be defined parametrically: $x_1 = \phi(\theta)$, $x_2 = \psi(\theta)$, where $0 \le \theta < \Xi$ is a parameter such that as it increases from 0 to Ξ, the point $(\phi(\theta), \psi(\theta))$ passes through each point of the boundary ∂D. To describe the solution near the boundary ∂D, we introduce a δ-neighborhood $\partial D^\delta := \{P \in D : \operatorname{dist}(P, \partial D) < \delta\}$, $\delta = \text{const} > 0$. In the δ-neighborhood of ∂D, we introduce local coordinates (r, θ), where r is the distance from a given point inside this neighborhood to the point on the boundary ∂D with coordinates $(\phi(\theta), \psi(\theta))$ along the normal to ∂D. It is known that if the boundary is sufficiently smooth (the functions $\phi(\theta)$ and $\psi(\theta)$ have continuous derivatives), then, in a sufficiently small neighborhood of the boundary, there exists a one-to-one correspondence between the initial coordinates (x_1, x_2) and the local coordinates (r, θ), given by the formulas

$$x_1 = \phi(\theta) - r\frac{\psi_\theta}{\sqrt{\phi_\theta^2 + \psi_\theta^2}}, \qquad x_2 = \psi(\theta) + r\frac{\phi_\theta}{\sqrt{\phi_\theta^2 + \psi_\theta^2}}.$$

The unit vector of the tangent $\mathbf{k}$ and the unit vector of the normal $\mathbf{n}$ to ∂D are defined as

$$\mathbf{k} = \begin{pmatrix} \dfrac{1}{\sqrt{1 + \psi_\theta^2/\phi_\theta^2}} \\ \dfrac{\psi_\theta/\phi_\theta}{\sqrt{1 + \psi_\theta^2/\phi_\theta^2}} \end{pmatrix}, \qquad \mathbf{n} = \begin{pmatrix} \dfrac{-\psi_\theta}{\sqrt{\phi_\theta^2 + \psi_\theta^2}} \\ \dfrac{\phi_\theta}{\sqrt{\phi_\theta^2 + \psi_\theta^2}} \end{pmatrix}.$$

Passing to the new variables, we express the differential operator Δ in the variables (r, θ) as

$$\Delta_{r,\theta} = \frac{\partial^2}{\partial r^2} + H_\theta \frac{\partial H_\theta}{\partial r}\frac{\partial}{\partial r} + \frac{1}{H_\theta}\frac{\partial}{\partial \theta}\left(\frac{1}{H_\theta}\right)\frac{\partial}{\partial \theta} + \frac{1}{H_\theta^2}\frac{\partial^2}{\partial \theta^2},$$

where

$$H_\theta = \sqrt{\left(\frac{\partial x_1}{\partial \theta}\right)^2 + \left(\frac{\partial x_2}{\partial \theta}\right)^2}$$

is a Lamé coefficient.

We introduce stretched variables of two scales $\xi = r/\varepsilon$ and $\eta = r/\varepsilon^2$. Expanding the coefficients at the partial derivatives in power series in ε, we then express the differential operators in the stretched variables as

$$\Delta_{\xi,\theta} = \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial\xi^2} + \frac{1}{\varepsilon}\frac{\partial}{\partial\xi}\frac{\phi_{\theta\theta}\psi_\theta - \psi_{\theta\theta}\phi_\theta}{\sqrt{\psi_\theta^2+\phi_\theta^2}} + \sum_{i=0}^{\infty}\varepsilon^{i-1}L_i, \tag{4}$$

$$\Delta_{\eta,\theta} = \frac{1}{\varepsilon^4}\frac{\partial^2}{\partial\xi^2} + \frac{1}{\varepsilon^2}\frac{\partial}{\partial\xi}\frac{\phi_{\theta\theta}\psi_\theta - \psi_{\theta\theta}\phi_\theta}{\sqrt{\psi_\theta^2+\phi_\theta^2}} + \sum_{i=0}^{\infty}\varepsilon^{i-2}L_i, \tag{5}$$

where the L_i are linear differential operators containing the partial derivatives $\partial/\partial\theta$ and $\partial^2/\partial\theta^2$. Because the local coordinate r is introduced as the distance along the inner normal to ∂D, the operator of the boundary condition in the local and stretched coordinates becomes

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial r} = \frac{1}{\varepsilon}\frac{\partial}{\partial\xi} = \frac{1}{\varepsilon^2}\frac{\partial}{\partial\eta}.$$

3.2. General form of the asymptotics. The formal asymptotics of a solution can be constructed standardly by following the steps of the algorithm implementing the method of boundary functions, according to which the sought functions are represented as

$$\begin{aligned} u(x,\varepsilon) &= \bar{u}(x,\varepsilon) + Pu(\xi,\theta,\varepsilon) + Ru(\eta,\theta,\varepsilon),\\ v(x,\varepsilon) &= \bar{v}(x,\varepsilon) + Pv(\xi,\theta,\varepsilon) + Rv(\eta,\theta,\varepsilon). \end{aligned} \tag{6}$$

The nonlinearity is represented similarly,

$$f(u,x,\varepsilon) = \bar{f} + Pf(\xi,\theta,\varepsilon) + Rf(\eta,\theta,\varepsilon),$$

where

$$\begin{aligned} Pf = {}& f(\bar{u}(\varepsilon\xi,\theta,\varepsilon) + Pu(\xi,\theta,\varepsilon), \bar{v}(\varepsilon\xi,\theta,\varepsilon) + Pv(\xi,\theta,\varepsilon), \varepsilon\xi,\theta,\varepsilon) - \\ & - f(\bar{u}(\varepsilon\xi,\theta,\varepsilon), \bar{v}(\varepsilon\xi,\theta,\varepsilon), \varepsilon\xi,\theta,\varepsilon),\\ Rf = {}& f(\bar{u}(\varepsilon^2\eta,\theta,\varepsilon) + Pu(\varepsilon\eta,\theta,\varepsilon) + Ru(\eta,\theta,\varepsilon), \bar{v}(\varepsilon^2\eta,\theta,\varepsilon) + \\ & + Pv(\varepsilon\eta,\theta,\varepsilon) + Rv(\eta,\theta,\varepsilon), \varepsilon^2\eta,\theta,\varepsilon) - \\ & - f(\bar{u}(\varepsilon^2\eta,\theta,\varepsilon) + Pu(\varepsilon\eta,\theta,\varepsilon), \bar{v}(\varepsilon^2\eta,\theta,\varepsilon) + Pv(\varepsilon\eta,\theta,\varepsilon), \varepsilon^2\eta,\theta,\varepsilon). \end{aligned} \tag{7}$$

For brevity, we write the terms in terms of the new coordinates (r,θ) (in what follows, the old or new coordinates are used in the representation of the boundary-layer part of the nonlinearity for reasons of convenience). The function $g(u,v,x,\varepsilon)$ is represented similarly.

The functions $\bar{u}(x,\varepsilon)$ and $\bar{v}(x,\varepsilon)$ are the regular part of the asymptotics and describe the functions u and v far away from the boundary ∂D; the functions $P(\xi,\theta,\varepsilon)$ and $R(\eta,\theta,\varepsilon)$ are the boundary-layer part of the asymptotics and describe the solution near the boundary ∂D on two different scales. All terms of asymptotics (6) can be written as power series in ε:

$$\begin{aligned} \bar{u}(x,\varepsilon) &= \bar{u}_0(x) + \varepsilon\bar{u}_1(x) + \cdots,\\ Pu(\xi,\theta,\varepsilon) &= P_0u(\xi,\theta) + \varepsilon P_1u(\xi,\theta) + \cdots,\\ Ru(\eta,\theta,\varepsilon) &= R_0u(\eta,\theta) + \varepsilon R_1u(\eta,\theta) + \cdots. \end{aligned}$$

The terms of the asymptotics for the function v can be written similarly.

We introduce the notation

$$\begin{aligned} &\bar{f}(x) = f(\bar{u}_0(x), \bar{v}_0(x), x, 0),\\ &\tilde{f}(\xi,\theta) = f(\bar{u}_0(0,\theta) + P_0u(\xi,\theta), \bar{v}_0(0,\theta) + P_0v(\xi,\theta), \theta, 0),\\ &\hat{f}(\eta,\theta) = f(\bar{u}_0(0,\theta) + P_0u(0,\theta) + R_0u(\eta,\theta), \bar{v}_0(0,\theta) + P_0v(0,\theta) + R_0v(\eta,\theta), \theta, 0). \end{aligned}$$

Similar notation is used for the function g and for the derivatives of these functions.

3.3. Regular part of the asymptotics. Substituting (6) in initial problem (2) and standardly dividing this problem into problems for the regular and boundary-layer parts, we obtain a sequence of problems for the coefficients of the regular and boundary-layer parts of the asymptotic approximation. The leading terms of the regular part, $\bar{u}_0(x)$ and $\bar{v}_0(x)$, are determined from degenerate system (2):

$$f(\bar{u}_0(x), \bar{v}_0(x), x, 0) = 0,$$
$$g(\bar{u}_0(x), \bar{v}_0(x), x, 0) = 0.$$

Under condition A1, this system has the solution

$$\bar{u}_0(x) = \bar{u}(x), \qquad \bar{v}_0(x) = \bar{v}(x).$$

The coefficients of the regular part of asymptotic approximation (6) for $k \geq 1$ are determined from the system

$$\begin{aligned} \bar{f}_u(x)\bar{u}_k + \bar{f}_v(x)\bar{v}_k &= F_k(x), \\ \bar{g}_u(x)\bar{u}_k + \bar{g}_v(x)\bar{v}_k &= G_k(x), \end{aligned} \tag{8}$$

where $F_k(x)$ and $G_k(x)$ are functions known at each step and depending on the coefficients of the regular part of the asymptotic approximation of the preceding orders. The determinant of this system satisfies the inequality $\Delta = \bar{f}_u(x)\bar{g}_v(x) - \bar{f}_v(x)\bar{g}_u(x) > 0$ by condition A2. Hence, these systems are uniquely solvable.

3.4. Boundary-layer part of the asymptotics. The system for the functions of the boundary-layer part is obtained by Vasil'eva's method and has the form

$$\begin{aligned} &\frac{\partial^2}{\partial \eta^2} Ru + \varepsilon^2 \frac{\phi_{\theta\theta}\psi_\theta - \psi_{\theta\theta}\phi_\theta}{\sqrt{\phi_\theta^2 + \psi_\theta^2}} \frac{\partial}{\partial \eta} Ru + \sum_{i=0}^{\infty} \varepsilon^{i+2} L_i Ru = Rf, \\ &\frac{1}{\varepsilon^2} \frac{\partial^2}{\partial \eta^2} Rv + \frac{\phi_{\theta\theta}\psi_\theta - \psi_{\theta\theta}\phi_\theta}{\sqrt{\phi_\theta^2 + \psi_\theta^2}} \frac{\partial}{\partial \eta} Rv + \sum_{i=0}^{\infty} \varepsilon^{i} L_i Rv = Rg, \\ &\varepsilon^2 \frac{\partial^2}{\partial \xi^2} Pu + \varepsilon^3 \frac{\phi_{\theta\theta}\psi_\theta - \psi_{\theta\theta}\phi_\theta}{\sqrt{\phi_\theta^2 + \psi_\theta^2}} \frac{\partial}{\partial \xi} Pu + \sum_{i=0}^{\infty} \varepsilon^{i+3} L_i Pu = Pf, \\ &\frac{\partial^2}{\partial \xi^2} Pv + \varepsilon \frac{\phi_{\theta\theta}\psi_\theta - \psi_{\theta\theta}\phi_\theta}{\sqrt{\phi_\theta^2 + \psi_\theta^2}} \frac{\partial}{\partial \xi} Pv + \sum_{i=0}^{\infty} \varepsilon^{i+1} L_i Pv = Pg. \end{aligned} \tag{9}$$

Successively writing the terms at different powers of the small parameter and using the initial boundary condition and the condition that the boundary-layer functions tend to zero at infinity, we obtain problems for determining them.

The problems for the zeroth and first orders of the function Rv become

$$\begin{cases} \dfrac{\partial^2 R_0 v}{\partial \eta^2} = 0, \\ \left.\dfrac{\partial R_0 v}{\partial \eta}\right|_{\eta=0} = 0, \\ R_0 v(\infty) = 0, \end{cases} \qquad \begin{cases} \dfrac{\partial^2 R_1 v}{\partial \eta^2} = 0, \\ \left.\dfrac{\partial R_1 v}{\partial \eta}\right|_{\eta=0} = -\left.\dfrac{\partial P_0 v}{\partial \xi}\right|_{\xi=0}, \\ R_1 v(\infty) = 0. \end{cases}$$

These problems have the solutions $R_0 v(\eta, \theta) = 0$, $R_1 v(\eta, \theta) = 0$.

Using representation (7), we find the functions $P_0u(\xi,\theta)$ and $P_0v(\xi,\theta)$ as

$$f(\bar{u}_0(0,\theta)+P_0u(\xi,\theta),\bar{v}_0(0,\theta)+P_0v(\xi,\theta),\theta,0)=0, \tag{10}$$

$$\begin{cases} \dfrac{\partial^2 P_0v}{\partial\xi^2}=g(\bar{u}_0(0,\theta)+P_0u(\xi,\theta),\bar{v}_0(0,\theta)+P_0v(\xi,\theta),\theta,0),\\ \dfrac{\partial P_0v}{\partial\xi}\bigg|_{\xi=0}=-\dfrac{\partial R_1v}{\partial\eta}\bigg|_{\eta=0}=0,\\ P_0v(\infty,\theta)=0. \end{cases} \tag{11}$$

Problem (10), (11) has the solution $P_0v(\xi,\theta)=0$, $P_0u(\xi,\theta)=0$.

The function $R_0u(\eta,\theta)$ is determined from the problem

$$\begin{cases} \dfrac{\partial^2 R_0u}{\partial\eta^2}=f(\bar{u}_0(0,\theta)+R_0u,\bar{v}_0(0,\theta),0,\theta),\\ \dfrac{\partial R_0u}{\partial\eta}\bigg|_{\eta=0}=0,\\ R_0u(\infty)=0, \end{cases} \tag{12}$$

whose solution is $R_0u(\xi,\theta)=0$. All zeroth-order functions of the boundary-layer part of the asymptotics are thus equal to zero. In the case under study, we therefore have $\bar{f}=\tilde{f}=\hat{f}$ and $\bar{g}=\tilde{g}=\hat{g}$.

For the function $R_2v(\eta)$, we have the equation

$$\frac{\partial^2 R_2v}{\partial\eta^2}=0$$

whence, using the condition that $R_2v(\infty)=0$ at infinity, we obtain $R_2v(\eta)=0$.

The problems for determining $P_1v(\xi,\theta)$ and $P_1u(\xi,\theta)$ have the form

$$P_1u=-\frac{\bar{f}_v}{\bar{f}_u}P_1v(\xi,\theta),$$

$$\begin{cases} \dfrac{\partial^2 P_1v}{\partial\xi^2}+\left(\bar{g}_u\dfrac{\bar{f}_v}{\bar{f}_u}-\bar{g}_v\right)P_1v=0,\\ \dfrac{\partial P_1v}{\partial\xi}\bigg|_{\xi=0}=q(\theta)-\dfrac{\partial\bar{v}_0}{\partial r}\bigg|_{r=0}-\dfrac{\partial R_2v}{\partial\eta}\bigg|_{\eta=0},\\ P_1v(\infty)=0. \end{cases}$$

We hence explicitly obtain

$$P_1v(\xi,\theta)=\left(\frac{\partial\bar{v}_0}{\partial r}\bigg|_{r=0}-q(\theta)\right)\exp\left[-\xi\sqrt{\left(\bar{g}_u\frac{\bar{f}_v}{\bar{f}_u}-\bar{g}_v\right)}\right],$$

and then find $P_1u(\xi,\theta)$.

The system for the function $R_1u(\eta,\theta)$ becomes

$$\begin{cases} \dfrac{\partial^2 R_1u}{\partial\eta^2}-\bar{f}_uR_1u=0,\\ \dfrac{\partial R_1u}{\partial\eta}\bigg|_{\eta=0}=-\dfrac{\partial P_0u}{\partial\xi}\bigg|_{\xi=0}=0,\\ R_1u(\infty)=0, \end{cases}$$

and it has the trivial solution $R_1u(\eta,\theta)=0$.

The boundary function R_2u is determined from the problem

$$\begin{cases} \dfrac{\partial^2 R_2 u}{\partial \eta^2} + \bar{f}_u R_2 u = 0, \\ \left.\dfrac{\partial R_2 u}{\partial \eta}\right|_{\eta=0} = h(\theta) - \left.\dfrac{\partial \bar{u}_0}{\partial r}\right|_{r=0} - \left.\dfrac{\partial P_1 v}{\partial \xi}\right|_{\xi=0}, \\ R_2 u(\infty) = 0, \end{cases}$$

whose solution, just as the solution of the problem for boundary functions of subsequent orders obtained from (9), can be written explicitly and has a standard exponentially decreasing estimate.

4. Justification of the asymptotics

4.1. Existence of the solution. To prove the existence of the stationary solution (the solution of problem (2)) we use the asymptotic method of differential inequalities (see [7] and the references therein). For this, we construct the upper and lower solutions of problem (2) in the domain $\bar{D}$, i.e., (u_α, v_α) and (u_β, v_β), as a modification of the formal asymptotics of order k constructed in the preceding section, i.e., the functions $U_k(x,\varepsilon)$ and $V_k(x,\varepsilon)$ (the kth-order partial sums of representations (6)). By definition, these functions must satisfy the following conditions.

Condition B1. $u_\alpha(x,\varepsilon) \le u_\beta(x,\varepsilon)$ and $v_\alpha(x,\varepsilon) \le v_\beta(x,\varepsilon)$ for $x \in \bar{D}$.

Condition B2. The following inequalities are satisfied:

$$\begin{aligned} &\mathcal{L}_u(u_\beta) \le 0 \le \mathcal{L}_u(u_\alpha), \qquad v_\alpha \le v \le v_\beta, \\ &\mathcal{L}_v(v_\beta) \le 0 \le \mathcal{L}_v(v_\alpha), \qquad u_\alpha \le u \le u_\beta, \quad x \in \bar{D}. \end{aligned}$$

Condition B3. The following inequalities are satisfied:

$$\left.\frac{\partial u_\beta}{\partial n}\right|_{\partial D} \le h(x) \le \left.\frac{\partial u_\alpha}{\partial n}\right|_{\partial D}, \qquad \left.\frac{\partial v_\beta}{\partial n}\right|_{\partial D} \le q(x) \le \left.\frac{\partial v_\alpha}{\partial n}\right|_{\partial D}.$$

We demonstrate the construction of the upper and lower solutions as a modification of the first order of the above-constructed asymptotics (in what follows, these functions are used to estimate the domain of attraction of the stationary solution):

$$\begin{aligned} u_\alpha(x,\varepsilon) &= U_1(x,\varepsilon) - \varepsilon\gamma_u(x) - \varepsilon^2 e^{-\kappa_1 \xi}, \\ v_\alpha(x,\varepsilon) &= V_1(x,\varepsilon) - \varepsilon\gamma_v(x) - \varepsilon^2 e^{-\kappa_2 \xi}, \\ u_\beta(x,\varepsilon) &= U_1(x,\varepsilon) + \varepsilon\gamma_u(x) + \varepsilon^2 e^{-\kappa_1 \xi}, \\ v_\beta(x,\varepsilon) &= V_1(x,\varepsilon) + \varepsilon\gamma_v(x) + \varepsilon^2 e^{-\kappa_2 \xi}. \end{aligned} \tag{13}$$

Ordering condition B1 is satisfied by virtue of representation (13). Because the first-order asymptotics exactly satisfies the boundary conditions of problem (2), it follows that the boundary-layer additions to representation (13) compensate the discrepancies in the boundary conditions for sufficiently large κ_i and thus ensure that the corresponding boundary inequalities in condition B3 are satisfied.

We now verify the first inequality in condition B2. It must be satisfied for all $v_\alpha \le v \le v_\beta$ or, which is the same, for all

$$v \in [V_1 - \varepsilon\gamma_v(x) - \varepsilon^2 e^{-\kappa_2\xi},\ V_1 + \varepsilon\gamma_v(x) + \varepsilon^2 e^{-\kappa_2\xi}].$$

After the substitution in the first inequality, we obtain

$$\mathcal{L}_u(u_\beta) = O(\varepsilon^2) - \varepsilon(\bar{f}_u\gamma_u + \Theta\bar{f}_v\gamma_v), \qquad x \in \bar{D},$$

where $-1 \le \Theta \le 1$. Condition A3 implies the inequality $\mathcal{L}_u(u_\beta) < 0$. For the three remaining inequalities, we similarly obtain

$$L_v v_\beta = O(\varepsilon^2) - \varepsilon(\Theta\bar{g}_u\gamma_u + \bar{g}_v\gamma_v) < 0, \qquad x \in \bar{D},$$
$$L_u u_\alpha = O(\varepsilon^2) + \varepsilon(\bar{f}_u\gamma_u + \Theta\bar{f}_v\gamma_v) > 0, \qquad x \in \bar{D},$$
$$L_v v_\alpha = O(\varepsilon^2) + \varepsilon(\Theta\bar{g}_u\gamma_u + \bar{g}_v\gamma_v) > 0, \qquad x \in \bar{D}.$$

The inequalities in condition B2 are thus satisfied by condition A3. It follows from the comparison theorem (see, e.g., [8]) that there exists a solution of problem (2) for which the inequalities

$$\begin{aligned} u_\alpha(x,\varepsilon) \le u(x,\varepsilon) \le u_\beta(x,\varepsilon), \qquad x \in \bar{D}, \\ v_\alpha(x,\varepsilon) \le v(x,\varepsilon) \le v_\beta(x,\varepsilon), \qquad x \in \bar{D}, \end{aligned} \tag{14}$$

are satisfied. It follows from representations (13) for the lower and upper solutions that $u_\alpha(x,\varepsilon) - u_\beta(x,\varepsilon) = O(\varepsilon)$ and $v_\alpha(x,\varepsilon) - v_\beta(x,\varepsilon) = O(\varepsilon)$. We can similarly use the kth-order asymptotic approximation $U_k(x,\varepsilon)$, $V_k(x,\varepsilon)$ to prove the existence of the solution employing the upper and lower solutions for which the relations $u_\alpha(x,\varepsilon) - u_\beta(x,\varepsilon) = O(\varepsilon^k)$, $v_\alpha(x,\varepsilon) - v_\beta(x,\varepsilon) = O(\varepsilon^k)$ hold. We thus have the following theorem.

Theorem 1. *Let conditions* A0–A3 *be satisfied. Then for sufficiently small* ε*, there exists a solution* $u(x,\varepsilon), v(x,\varepsilon)$ *of problem* (2) *with a boundary layer near* ∂D *for which the functions* $U_n(x,\varepsilon)$*,* $V_n(x,\varepsilon)$ *are a uniform asymptotic approximation up to* ε^{n+1} *for* $x \in \bar{D}$*.*

4.2. Asymptotic stability of the stationary solution. An approach that is typically used to prove the asymptotic stability of the stationary solution of problem (1) is efficient in many classes of problems and involves the upper and lower solutions of a special structure (see [7]). We recall that the vector functions $(U_\beta(x,t,\varepsilon), V_\beta(x,t,\varepsilon))$ and $(U_\alpha(x,t,\varepsilon), V_\alpha(x,t,\varepsilon))$ are respectively called the upper and lower solutions of system (1) if they satisfy the following conditions for sufficiently small ε.

Condition C1. $U_\alpha(x,t,\varepsilon) \le U_\beta(x,t,\varepsilon)$ and $V_\alpha(x,t,\varepsilon) \le V_\beta(x,t,\varepsilon)$ for $x \in \bar{D}$, $t > 0$.

Condition C2. The following inequalities are satisfied:

$$\mathcal{N}_u(U_\beta) \le 0 \le \mathcal{N}_u(U_\alpha), \qquad V_\alpha \le V \le V_\beta, \qquad x \in \bar{D}, \quad t > 0,$$
$$\mathcal{N}_v(V_\beta) \le 0 \le \mathcal{N}_v(V_\alpha), \qquad U_\alpha \le U \le U_\beta, \qquad x \in \bar{D}, \quad t > 0.$$

Condition C3. The following inequalities are satisfied:

$$\left.\frac{\partial U_\beta}{\partial n}\right|_{\partial D} \le h(x) \le \left.\frac{\partial U_\alpha}{\partial n}\right|_{\partial D},$$
$$\left.\frac{\partial V_\beta}{\partial n}\right|_{\partial D} \le q(x) \le \left.\frac{\partial V_\alpha}{\partial n}\right|_{\partial D}.$$

We define the functions $U_\alpha(x,t,\varepsilon)$ and $U_\beta(x,t,\varepsilon)$ as

$$U_\alpha(x,t,\varepsilon) = u(x,\varepsilon) + (u_\alpha(x,\varepsilon) - u(x,\varepsilon))e^{-\kappa t},$$
$$U_\beta(x,t,\varepsilon) = u(x,\varepsilon) + (u_\beta(x,\varepsilon) - u(x,\varepsilon))e^{-\kappa t},$$

where $u(x,\varepsilon)$ is the stationary solution whose existence was proved in Theorem 1, and $u_\alpha(x,\varepsilon)$ and $u_\beta(x,\varepsilon)$ are the lower and upper solutions for the u-component of the stationary problem determined in (13). The functions $V_\alpha(x,t,\varepsilon)$ and $V_\beta(x,t,\varepsilon)$ are determined similarly. Using the obvious estimates

$$|u(x,\varepsilon) - u_\alpha(x,\varepsilon)| \le C\varepsilon, \qquad |v(x,\varepsilon) - v_\alpha(x,\varepsilon)| \le C\varepsilon,$$

we can easily show that conditions C1–C3 are satisfied (similar calculations can be found, for example, in [9]). It follows from [8] that if the conditions

$$u_\alpha(x,\varepsilon) \le u_{\mathrm{init}}(x,\varepsilon) \le u_\beta(x,\varepsilon), \qquad v_\alpha(x,\varepsilon) \le v_{\mathrm{init}}(x,\varepsilon) \le v_\beta(x,\varepsilon)$$

are satisfied, then there exists a unique solution $(U_\varepsilon(x,t,\varepsilon), V_\varepsilon(x,t,\varepsilon))$ of problem (1) such that

$$U_\alpha(x,t,\varepsilon) \le U_\varepsilon(x,t,\varepsilon) \le U_\beta(x,t,\varepsilon), \qquad V_\alpha(x,t,\varepsilon) \le V_\varepsilon(x,t,\varepsilon) \le V_\beta(x,t,\varepsilon)$$

for $x \in \bar{D}$, $t > 0$. From this fact and from the structure of the lower and upper solutions of the parabolic problem, we obtain the following result.

Theorem 2. *Assume that conditions* A0–A3 *are satisfied. Then for sufficiently small* ε, *the stationary solution* $u_\varepsilon(x,\varepsilon)$, $v_\varepsilon(x,\varepsilon)$ *of problem* (1) *is asymptotically stable in the sense of Lyapunov, and the domain of attraction is at least* $(U_\alpha(x,\varepsilon), V_\alpha(x,\varepsilon)) \times (U_\beta(x,\varepsilon), V_\beta(x,\varepsilon))$. *This solution is also locally unique as the solution of problem* (2) *in this domain.*

5. Example of a system of chemical kinetics

We now consider an example of problem (1) for the following system of chemical kinetics describing fast biomolecular and monomolecular reactions (see [12]):

$$\begin{aligned}
&\varepsilon^4 \Delta u - \frac{\partial u}{\partial t} = k_1 uv + k_2 u(v-p)^2 + \varepsilon I_u(x) \equiv f(u,v,x,\varepsilon),\\
&\varepsilon^2 \Delta v - \frac{\partial v}{\partial t} = k_3 uv + k_4 v(v-p) + \varepsilon I_v(x) \equiv g(u,v,x,\varepsilon),\\
&u(x,0,\varepsilon) = u_{\mathrm{init}}(x,\varepsilon), v(x,0,\varepsilon) = v_{\mathrm{init}}(x,\varepsilon), \qquad x \in \bar{D},\\
&\left.\frac{\partial u}{\partial n}\right|_{\partial D} = h(x), \qquad \left.\frac{\partial v}{\partial n}\right|_{\partial D} = q(x), \qquad x \in \partial D, \quad t > 0.
\end{aligned} \tag{15}$$

The functions $u(x,t)$ and $v(x,t)$ describe the concentrations of two substances participating in the reaction, $I_u(x)$ and $I_v(x)$ are nonnegative functions describing the external sources, the function $p = p(x)$ is sufficiently smooth, and $k_i > 0$, $i = \overline{1,4}$, are constants. Two equations enter the system with different powers of a small parameter multiplying the differential operator, which corresponds to different rates of variation in the concentrations of the two reagents or to the diffusion coefficients of different degrees of smallness. We note that none of the four quasimonotonicity conditions that are usually imposed in such problems is satisfied for the system under study, because the derivative $f_v = k_1 u + k_2 u 2(v-p)$ changes sign at p in the domain between the lower and upper solutions.

The degenerate system

$$k_1uv + k_2u(v-p)^2 = 0,$$
$$k_3uv + k_4v(v-p) = 0$$

has a solution

$$\bar{u}(x) = 0, \qquad \bar{v}(x) = p(x).$$

The derivatives are $\bar{f}_u(x) = k_1p(x) > 0$ and $\bar{g}_v(x) = k_4p(x) > 0$, and hence condition A1 is satisfied.

The determinant of the matrix determining the regular part of the asymptotics is positive,

$$\begin{vmatrix} \bar{f}_u & \bar{f}_v \\ \bar{g}_u & \bar{g}_v \end{vmatrix} = k_1k_4p^2 > 0,$$

and if the functions in the expression

$$\begin{pmatrix} \bar{f}_u & \Theta\bar{f}_v \\ \Theta\bar{g}_u & \bar{g}_v \end{pmatrix} \begin{pmatrix} \gamma_u(x) \\ \gamma_v(x) \end{pmatrix} = \begin{pmatrix} k_1p & 0 \\ \Theta k_3p & k_4p \end{pmatrix} \begin{pmatrix} \gamma_u(x) \\ \gamma_v(x) \end{pmatrix} = \begin{pmatrix} A(x) \\ B(x) \end{pmatrix}$$

are chosen such that $\gamma_u(x) > 0$ and $\gamma_v(x) > k_3\gamma_u(x)/k_4 > 0$, then we have $A(x) > 0$ and $B(x) > 0$. Thus, conditions A2 and A3 are satisfied.

The results of Theorems 1 and 2 can therefore be applied to the considered system of chemical kinetics.

Conflicts of interest. The authors declare no conflicts of interest.

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