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QUASIDIFFERENTIAL OPERATOR AND QUANTUM ARGUMENT SHIFT METHOD

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We describe an explicit formula for the first-order quasiderivation of an arbitrary central element of the universal enveloping algebra of a general linear Lie algebra. We apply it to show that derivations of any two central elements of the universal enveloping algebra commute. This contributes to the Vinberg problem of finding commutative subalgebras in universal enveloping algebras with the underlying Poisson algebras determined by the argument shift method.

Keywords: universal enveloping algebra, Lie algebra, quantum argument shift method, deformation quantization

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1. Introduction

The argument shift method introduced by Mischenko and Fomenko [1] is a well-known method devised to construct commutative families of elements in Poisson algebras. Loosely speaking, it consists in shifting the central elements of this algebra (central in the sense of a Poisson bracket) in the direction of a vector field ξ that satisfies an algebraic relation involving the Poisson tensor

$$\mathcal{L}_\xi^2 \pi = 0, \tag{1}$$

where $\mathcal{L}_\xi \pi$ is the Lie derivative of the Poisson tensor π with respect to ξ. This relation is often called the "Nijenhuis equation," or "Nijenhuis property" and can be regarded as a sort of linearity condition. If this relation holds, then the shifted (in the direction of ξ) central elements, although no more central, continue to commute with each other.

The most common nontrivial example of this Nijenhuis property and the one that was originally discovered by Manakov, Mischenko, and Fomenko [1], [2] is the case where the Poisson structure is given by the usual Kirillov–Kostant brackets on the dual space of a Lie algebra and the field is any constant vector field on this space: the Kirillov–Kostant brackets have linear coefficients and hence the Nijenhuis equation holds automatically. This method is quite productive and gives rise to maximal commutative Poisson subalgebras in polynomials on the dual space.

The purpose of this note is to suggest a construction that generalizes the argument shift method to the domain of universal enveloping algebras. Namely, Vinberg asked whether the commutative subalgebras

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generated by the shift can be lifted to the universal enveloping algebra of the Lie algebra. In about 30 years several constructions have appeared that determine such a lift, see, e.g., [3]–[5]. But all these constructions are based on the considerations of generating elements in the center of the Poisson algebras and do not address the question of whether the shift itself can be transfered into the noncommutative situation. In this note, we address this question and suggest a method that hypothetically leads to its solution, at least for the Lie algebra $g = gl_d$.

Our approach is based on the considerations of the so-called "quasiderivations" of the algebra Ugl_d, introduced in [6]. Namely, we prove an explicit formula for the first quasiderivation of such an element (Theorem 1) and thus show that the first quasiderivations in any direction of *any* central element in the universal enveloping algebra commute (Corollary 1). These observations support the conjecture that *quasiderivations induce a shift in the universal enveloping algebras analogous to the shift in Poisson algebras.*

2. Preliminaries and statement of the results

We let $\mathbb{F}$ denote the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$ and let d be a nonnegative integer. The $d \times d$ matrix units are denoted by

$$E_i^j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} 0 \dots 0\, 1\, 0 \dots 0 \end{pmatrix} \tag{2}$$

with 1 in the ith place of the column vector and in the jth place of the row vector. We then have the commutation relations $[E_p^q, E_r^s] = \delta_r^q E_p^s - E_r^q \delta_p^s$. The universal enveloping algebra of the general linear Lie algebra $gl_d = gl_d(\mathbb{F})$ is denoted by $U = Ugl_d$ and its center is denoted by $Z = Z(Ugl_d)$. We let π denote the canonical Lie-algebra homomorphism of the general linear Lie algebra gl_d into its universal enveloping algebra Ugl_d and set $e_i^j = \pi(E_i^j)$ for each i and j. The generators e_i^j satisfy the same commutation relations $[e_p^q, e_r^s] = \delta_r^q e_p^s - e_r^q \delta_p^s$ as the matrix units E_i^j. We define

$$e = \sum_{i,j=1}^{d} e_j^i \otimes E_i^j \in U \otimes M(d, \mathbb{F}) = \begin{pmatrix} e_1^1 & e_2^1 & \dots & e_d^1 \\ e_1^2 & e_2^2 & \dots & e_d^2 \\ \vdots & \vdots & \ddots & \vdots \\ e_1^d & e_2^d & \dots & e_d^d \end{pmatrix} \in M(d, U). \tag{3}$$

We remark that the (i, j) entry of the matrix e is the generator $e_j^i = \pi(E_j^i)$, whereas the (j, i) entry of the corresponding matrix unit E_j^i is equal to 1. The role of the lower and upper indices is opposite here. In other words, we let e denote the *transposed* matrix of generators of the algebra Ugl_d; this choice of notation simplifies our computations below. We have

$$(e^2)_j^i = \sum_{k=1}^{d} e_k^i e_j^k, \qquad [e_p^q, (e^n)_r^s] = [(e^n)_p^q, e_r^s] = \delta_r^q (e^n)_p^s - (e^n)_r^q \delta_p^s. \tag{4}$$

It can be shown that the center of Ugl_d is generated by the set $\left\{\operatorname{tr} e^n\right\}_{n=1}^{d}$.

Definition 1 (see [6]). Quasiderivations of Ugl_d is a unique family of linear operators ∂^i_j on U satisfying the equations

- $\partial^i_j 1 = 0$ for any i and j;
- $\partial^i_j e^q_p = \delta^i_p \delta^q_j$ for any i, j, p, and q;
- the twisted Leibniz rule
$$\partial^i_j(fg) = (\partial^i_j f)g + f(\partial^i_j g) + \sum_{k=1}^{d} (\partial^i_k f)(\partial^k_j g) \tag{5}$$
for any i and j and for any elements f and g of U.

We define a map ∂ of the universal enveloping algebra U into the matrix algebra $M(d, U)$ by $(\partial f)^i_j = \partial^i_j f$ for any $f \in U$. The following theorem is the main result in this paper.

Theorem 1. *We have*

$$\partial \operatorname{tr} e^n = \frac{(e+\delta)^n - (e-\delta)^n}{2} + \sum_{k=0}^{n-1} (\operatorname{tr} e^k) \frac{(e+\delta)^{n-k-1} - (e-\delta)^{n-k-1}}{2} \tag{6}$$

for any n, where δ is the $d \times d$ identity matrix.

Corollary 1. *We suppose that ξ is an arbitrary element of the Lie algebra gl_d and let*

$$\partial_\xi = \sum_{i,j=1}^{d} \xi^j_i \partial^i_j. \tag{7}$$

Then we have $[\partial_\xi f, \partial_\xi g] = 0$ for any central elements f and g.

Proof. It is sufficient to show that $[\operatorname{tr}(\xi e^m), \operatorname{tr}(\xi e^n)] = 0$ because we have

$$\partial \prod_{k=1}^{m} \operatorname{tr} e^{n_k} = (\operatorname{tr} e^{n_1} + \partial \operatorname{tr} e^{n_1}) \ldots (\operatorname{tr} e^{n_m} + \partial \operatorname{tr} e^{n_m}) - \prod_{k=1}^{m} \operatorname{tr} e^{n_k}. \tag{8}$$

From

$$[\operatorname{tr}(\xi e^m), e^i_j] = \sum_{p,q=1}^{d} \xi^q_p [(e^m)^p_q, e^i_j] = \sum_{p,q=1}^{d} \xi^q_p \big(\delta^p_j (e^m)^i_q - (e^m)^p_j \delta^i_q\big) = [e^m, \xi]^i_j,$$

we deduce

$$[\operatorname{tr}(\xi e^m), \operatorname{tr}(\xi e^n)] = \sum_{k=1}^{n} \operatorname{tr}(\xi e^{k-1} [e^m, \xi] e^{n-k}) = \sum_{k=1}^{n} \operatorname{tr}[\xi e^{m+n-k}, \xi e^{k-1}]. \tag{9}$$

We can assume that $m < n$. Then

$$[\operatorname{tr}(\xi e^m), \operatorname{tr}(\xi e^n)] = \frac{[\operatorname{tr}(\xi e^m), \operatorname{tr}(\xi e^n)] - [\operatorname{tr}(\xi e^n), \operatorname{tr}(\xi e^m)]}{2} =$$
$$= \frac{1}{2} \sum_{k=m+1}^{n} \operatorname{tr}[\xi e^{m+n-k}, \xi e^{k-1}] = 0. \tag{10}$$

3. The proof of the main theorem

We begin with the following observation: due to the twisted Leibniz rule in Definition 1, we have

$$\begin{aligned}\partial^i_j(e^n)^p_q &= \sum_{r=1}^d \partial^i_j(e^p_r(e^{n-1})^r_q) = \\ &= \sum_{r=1}^d(\partial^i_j e^p_r)(e^{n-1})^r_q + \sum_{r=1}^d e^p_r\partial^i_j(e^{n-1})^r_q + \sum_{k,r=1}^d(\partial^i_k e^p_r)(\partial^k_j(e^{n-1})^r_q) = \\ &= (e^{n-1})^i_q\delta^p_j + \sum_{r=1}^d e^p_r\partial^i_j(e^{n-1})^r_q + \partial^p_j(e^{n-1})^i_q. \qquad (11)\end{aligned}$$

for $n > 0$.

Lemma 1. *The quasiderivation $\partial^i_j(e^n)^p_q$ has the form*

$$\partial^i_j(e^n)^p_q = \sum_{k+\ell<n}\big(a^{(n)}_{k,\ell}(e^k)^i_q(e^\ell)^p_j + b^{(n)}_{k,\ell}(e^k)^p_q(e^\ell)^i_j\big), \qquad (12)$$

where $a^{(n)}_{k,\ell}$ and $b^{(n)}_{k,\ell}$ are nonnegative integers.

Proof. The proof is by induction on n. We have

$$\begin{aligned}\partial^i_j(e^n)^p_q &= (e^{n-1})^i_q\delta^p_j + \sum_{r=1}^d e^p_r\partial^i_j(e^{n-1})^r_q + \partial^p_j(e^{n-1})^i_q = \\ &= (e^{n-1})^i_q\delta^p_j + \sum_{k,\ell}a^{(n-1)}_{k,\ell}\sum_{r=1}^d e^p_r(e^k)^i_q(e^\ell)^r_j + \sum_{k,\ell}b^{(n-1)}_{k,\ell}(e^{k+1})^p_q(e^\ell)^i_j + \\ &\quad + \sum_{k,\ell}\big(a^{(n-1)}_{k,\ell}(e^k)^p_q(e^\ell)^i_j + b^{(n-1)}_{k,\ell}(e^k)^i_q(e^\ell)^p_j\big). \qquad (13)\end{aligned}$$

by the induction hypothesis. Because

$$\sum_{r=1}^d e^p_r(e^k)^i_q(e^\ell)^r_j = (e^k)^i_q(e^{\ell+1})^p_j + \sum_{r=1}^d[e^p_r,(e^k)^i_q](e^\ell)^r_j = (e^k)^i_q(e^{\ell+1})^p_j + \delta^p_q(e^{k+\ell})^i_j - (e^k)^p_q(e^\ell)^i_j,$$

we have

$$\partial^i_j(e^n)^p_q = (e^{n-1})^i_q\delta^p_j + \sum_{k,\ell}a^{(n-1)}_{k,\ell}\big((e^k)^i_q(e^{\ell+1})^p_j + \delta^p_q(e^{k+\ell})^i_j\big) + \sum_{k,\ell}b^{(n-1)}_{k,\ell}\big((e^{k+1})^p_q(e^\ell)^i_j + (e^k)^i_q(e^\ell)^p_j\big). \quad (14)$$

This equation gives rise to a recursive process

$$\cdots \to (a^{(n-1)}_{k,\ell}, b^{(n-1)}_{k,\ell})_{k,\ell} \to (a^{(n)}_{k,\ell}, b^{(n)}_{k,\ell})_{k,\ell} \to \cdots. \qquad (15)$$

We have the polynomial identities

$$\sum_{k+\ell<n}a^{(n)}_{k,\ell}x^ky^\ell = x^{n-1} + \sum_{k+\ell<n-1}(a^{(n-1)}_{k,\ell}x^ky^{\ell+1} + b^{(n-1)}_{k,\ell}x^ky^\ell), \qquad (16)$$

$$\sum_{k+\ell<n}b^{(n)}_{k,\ell}x^ky^\ell = \sum_{k+\ell<n-1}\big(a^{(n-1)}_{k,\ell}y^{k+\ell} + b^{(n-1)}_{k,\ell}x^{k+1}y^\ell\big). \qquad (17)$$

Using

$$\partial_j^i \operatorname{tr} e^n = \sum_{k+\ell<n} \big(a_{k,\ell}^{(n)} (e^{k+\ell})_j^i + (\operatorname{tr} e^k) b_{k,\ell}^{(n)} (e^\ell)_j^i\big),$$

we can define polynomials $f^{(n)}(x)$ and $g_k^{(n)}(x)$ with integer coefficients by

$$f^{(n)}(x) = \sum_{k,\ell} a_{k,\ell}^{(n)} x^{k+\ell}, \qquad g_k^{(n)}(x) = \sum_\ell b_{k,\ell}^{(n)} x^\ell. \tag{18}$$

Then we have

$$\begin{aligned} f^{(n)}(x) &= x^{n-1} + \sum_{k,\ell} a_{k,\ell}^{(n-1)} x^{k+\ell+1} + \sum_{k,\ell} b_{k,\ell}^{(n-1)} x^{k+\ell} = \\ &= x^{n-1} + x f^{(n-1)}(x) + \sum_{k<n-1} x^k g_k^{(n-1)}(x), \end{aligned} \tag{19}$$

by identity (16), and also

$$\begin{aligned} g_0^{(n)}(x) &= \sum_{k,\ell} a_{k,\ell}^{(n-1)} x^{k+\ell} = f^{(n-1)}(x), \\ g_{k+1}^{(n)}(x) &= \sum_\ell b_{k,\ell}^{(n-1)} x^\ell = g_k^{(n-1)}(x) = \cdots = g_0^{(n-k-1)}(x) = f^{(n-k-2)}(x), \end{aligned}$$

by identity (17). In this way, we obtain the formulas

$$\partial_j^i \operatorname{tr} e^n = \Big(f^{(n)}(e) + \sum_{k<n} (\operatorname{tr} e^k) g_k^{(n)}(e)\Big)_j^i = \Big(f^{(n)}(e) + \sum_{k<n} (\operatorname{tr} e^k) f^{(n-k-1)}(e)\Big)_j^i \tag{20}$$

and

$$f^{(n)}(x) = x^{n-1} + x f^{(n-1)}(x) + \sum_{k<n-1} x^k f^{(n-k-2)}(x). \tag{21}$$

The solution of this recursive equation is

$$f^{(n)}(x) = \frac{(x+1)^n - (x-1)^n}{2} \tag{22}$$

and we have

$$\partial_j^i \operatorname{tr} e^n = \left(\frac{(e+\delta)^n - (e-\delta)^n}{2} + \sum_{k=0}^{n-1} (\operatorname{tr} e^k) \frac{(e+\delta)^{n-k-1} - (e-\delta)^{n-k-1}}{2}\right)_j^i.$$

Appendix: Exact formula for $\partial_j^i (e^n)_q^p$

The methods that we used above can be generalized to give formulas for quasiderivations of more general elements.

We proceed to obtain $a_{k,\ell}^{(n)}$ and $b_{k,\ell}^{(n)}$ and derive an exact formula for $\partial_j^i (e^n)_q^p$. We set

$$f_\pm^{(n)}(x) = \frac{(x+1)^n \pm (x-1)^n}{2} = \sum_{k=0}^n \frac{1 \pm (-1)^{n-k}}{2} \binom{n}{k} x^k \tag{A.1}$$

and

$$h_k^{(n)}(x) = \sum_{\ell=0}^{n-k-1} a_{k,\ell}^{(n)} x^\ell. \tag{A.2}$$

Then we have

$$\begin{aligned} \partial_j^i (e^n)_q^p &= \sum_{k=0}^{n-1} \big((e^k)_q^i h_k^{(n)} (e)_j^p + (e^k)_q^p g_k^{(n)} (e)_j^i \big) = \\ &= \sum_{k=0}^{n-1} \big((e^k)_q^i h_k^{(n)} (e)_j^p + (e^k)_q^p f_-^{(n-k-1)} (e)_j^i \big), \end{aligned} \tag{A.3}$$

whence

$$\sum_{k=0}^{n-1} y^k h_k^{(n)}(x) = y^{n-1} + \sum_{k=0}^{n-2} y^k \big(x h_k^{(n-1)}(x) + f_-^{(n-k-2)}(x) \big) \tag{A.4}$$

by identity (16). We have $h_{n-1}^{(n)}(x) = 1$ and

$$\begin{aligned} h_k^{(n)}(x) &= x h_k^{(n-1)}(x) + f_-^{(n-k-2)}(x) = \cdots = \\ &= x^{n-k-1} h_k^{(k+1)}(x) + \sum_{\ell=0}^{n-k-2} x^\ell f_-^{(n-k-\ell-2)}(x) = \\ &= x^{n-k-1} + \sum_{\ell=0}^{n-k-2} x^\ell f_-^{(n-k-\ell-2)}(x) = \\ &= f_-^{(n-k)}(x) - x f_-^{(n-k-1)}(x) = f_+^{(n-k-1)}(x). \end{aligned} \tag{A.5}$$

for $k < n$ by identity (21). Furthermore,

$$a_{k,\ell}^{(n)} = \frac{1 - (-1)^{n-k-\ell}}{2} \binom{n-k-1}{\ell}, \qquad b_{k,\ell}^{(n)} = \frac{1 + (-1)^{n-k-\ell}}{2} \binom{n-k-1}{\ell} \tag{A.6}$$

and

$$\begin{aligned} \partial_j^i (e^n)_q^p &= \sum_{k=0}^{n-1} \big((e^k)_q^i f_+^{(n-k-1)} (e)_j^p + (e^k)_q^p f_-^{(n-k-1)} (e)_j^i \big) = \\ &= \sum_{k=0}^{n-1} \Bigg((e^k)_q^i \bigg(\frac{(e+\delta)^{n-k-1} + (e-\delta)^{n-k-1}}{2} \bigg)_j^p + \\ &\qquad + (e^k)_q^p \bigg(\frac{(e+\delta)^{n-k-1} - (e-\delta)^{n-k-1}}{2} \bigg)_j^i \Bigg) = \\ &= \sum_{k+\ell=0}^{n-1} \binom{n-k-1}{\ell} \bigg(\frac{1 - (-1)^{n-k-\ell}}{2} (e^k)_q^i (e^\ell)_j^p \, \frac{1 + (-1)^{n-k-\ell}}{2} (e^k)_q^p (e^\ell)_j^i \bigg). \end{aligned}$$

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