Theoretical and Mathematical Physics, **211**(1): 558–566 (2022)

A NEW CLASS OF SPHERICALLY SYMMETRIC GRAVITATIONAL COLLAPSE

R. Kumar* **and A. Jaiswal***

We discuss the gravitational collapse of a spherically symmetric perfect fluid distribution of uniformly contracting stars. In a uniformly contracting star, the relative volume element (relative distance in the case of spherical symmetry) between any two neighboring fluid particles is preserved irrespective of the radial coordinate. The physical meaning is that during collapse each small volume element of the fluid distribution preserves its spatial position. This new class of gravitational collapse is analogous to the reverse phenomenon of motion of galaxies during the expansion of the Universe. We discuss the shearing solution of a perfect fluid distribution executing the uniform expansion, which is a scalar and obeys the equation of state $p = p(\rho)$. The field equation is solved in complete generality, such that that the Oppenheimer–Snyder solution with homogeneous density and the Thompson–Whitrow shear-free solution arise as particular cases.

Keywords: spherical symmetry, gravitational collapse, uniformly contracting star, equation of state, perfect fluid distribution

DOI: 10.1134/S0040577922040092

1. Introduction

In general relativity and cosmology, the gravitational collapse of massive stars and the modeling of the structure of compact objects such as neutron stars, quasars, supernovae, black holes , etc., have been investigated under various conditions [1]–[4]. The work of Oppenheimer and Snyder [5] on the problem of general relativistic gravitational collapse of a massive star has attracted considerable attention of researchers in this field. Spherically symmetric perfect fluid distributions in general relativity have been the subject of extensive studies because of their possible applications in gravitational collapse of stars and the dynamical evolution of the Universe (see [6]–[9] and the references therein). The problems of a spherically symmetric fluid distribution, that can either collapse onto the center (gravitational collapse) or explode from that point (spherical blast), has been studied by various authors. Most of the studies provided the models of a gravitating sphere but different kinds of obstacles appeared depending on the approach adopted for the modeling [1]–[9].

The nonlinearity and complexity of Einstein's equations make it very difficult to find their solutions in closed analytic form unless additional assumptions are used. One of the main difficulties in the study of

*Department of Mathematics and Statistics, Deen Dayal Upadhyaya Gorakhpur University, Gorakhpur, India, e-mails: rkmath09@gmail.com (corresponding author), annujais3012@gmail.com.

R. Kumar is thankful to an UGC-BSR Startup grant, India, for financial assistance. A. Jaiswal is thankful to the Council of Science and Technology, UP, India (vide letter no. CST/D-2289).

Prepared from an English manuscript submitted by the authors; for the Russian version, see *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 211, No. 1, pp. 136–146, April, 2022. Received December 20, 2021. Revised January 5, 2022. Accepted January 7, 2022.

 0040-5779/22/2111-0558 © 2022 Pleiades Publishing, Ltd.

gravitational collapse is that only a few exact solutions of Einstein's equations are known that are relevant to collapse with reasonable matter [10]–[13]. As is well known, the spherically symmetric gravitational collapse is characterized by the four-acceleration vector, the shear tensor, and the expansion scalar. The relevance of the shear tensor in the evolution of collapsing systems and the consequences emerging from its vanishing have been emphasized by many authors [14]–[17].

In this paper, we study the shearing solution for a spherically symmetric perfect fluid distribution with a uniform expansion scalar Θ. Because the expansion scalar describes the rate of change of a small volume of the fluid, it is intuitively clear that the evolution of a uniformly collapsing star delimits the spatial volume elements of the fluid distribution. Thus, the evolution of a uniformly contracting system implies interesting astrophysical consequences and this scenario allows studying a wide range of models for spherically symmetric fluid distributions. It is well known that to discuss the class of adiabatically contracting/expanding systems in sufficient generality requires invoking additional assumptions based on mathematical simplicity or physical grounds. As noted above, this necessarily involves some assumption on the uniform expansion scalar, which may be general in nature.

After the introduction, some basic equations are presented in Sec. 2. In Sec. 3, we introduce one of the basic assumptions, that the expansion of the four-velocity vector is uniform, $\Theta = \Theta(t)$. We also introduce two other assumptions, which are identically satisfied for a homogeneous density ($\rho = \rho(t)$) and shear-free motion $\sigma = 0$, and show that this latter assumption is equivalent to the former one. However, they allow exploring general situations with $\rho \neq \rho(t)$ and $\sigma \neq 0$. The analysis of dynamical equations is carried out under the assumption that $\Theta = \Theta(t)$ obeys the equation of state $p = p(\rho)$; the solutions are investigated in complete generality. The discussion and concluding remarks on the results are presented in Sec. 4, where prospective applications and scope of these results are also briefly mentioned.

2. The basic equations

For a spherically symmetric fluid distribution, the interior metric in the comoving coordinates is

$$ds^2 = A^2\,dt^2 - B^2\,dr^2 - R^2\,d\Omega^2, \tag{1}$$

where A, B and R are functions of (t, r), $d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2$ is the 2-sphere, and the comoving coordinate system is taken as $x^i = (t, r, \theta, \phi)$, $i = 0, 1, 2, 3$. We regard the source of matter distribution as a perfect fluid represented by the energy–momentum tensor

$$T^i_j = (p + \rho)V^i V_j - p\delta^i_j, \tag{2}$$

where ρ and p are the energy density and the pressure of the perfect fluid distribution. The four-velocity vector V^i is

$$V^i = (A^{-1}, 0, 0, 0), \qquad V^i V_i = 1. \tag{3}$$

The four-acceleration vector a^i, the expansion scalar Θ, and the shear σ of the fluid distribution can be expressed as

$$a^i = \left(0, \frac{A'}{AB}, 0, 0\right), \qquad \Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{R}}{R}\right), \qquad \sigma = \frac{1}{A}\left(\frac{\dot{B}}{B} - \frac{\dot{R}}{R}\right), \tag{4}$$

where σ is the shear scalar defined by

$$\sigma_{ij} = V_{(i;j)} + a_{(i}V_{j)} - \frac{1}{3}\Theta(g_{ij} + V_i V_j), \qquad \sigma^2 = \frac{3}{2}\sigma^{ij}\sigma_{ij}. \tag{5}$$

The Einstein field equations

$$G_{ij} = -\frac{8\pi G}{c^4}T_{ij} \tag{6}$$

yield the independent equations

$$8\pi p B^2 = \frac{B^2}{R^2} + \frac{B^2}{A^2}\left(2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} - 2\frac{\dot{A}\dot{R}}{AR}\right) - \frac{R'}{R}\left(\frac{R'}{R} + 2\frac{A'}{A}\right), \tag{7}$$

$$\begin{aligned} 8\pi p R^2 = {} & \frac{R^2}{A^2}\left[\frac{\ddot{B}}{B} + \frac{\ddot{R}}{R} - \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{R}}{R}\right) + \frac{\dot{B}}{B}\frac{\dot{R}}{R}\right] - \\ & - \frac{R^2}{B^2}\left[\frac{A''}{A} + \frac{R''}{R} - \frac{A'B'}{AB} + \left(\frac{A'}{B} - \frac{B'}{B}\right)\frac{R'}{R}\right], \end{aligned} \tag{8}$$

$$8\pi\rho A^2 = \frac{A^2}{B^2}\left(2\frac{R''}{R} + \frac{R'^2}{R^2} - 2\frac{B'R'}{BR} - \frac{B^2}{R^2}\right) - \frac{\dot{R}}{R}\left(\frac{2\dot{B}}{B} + \frac{\dot{R}}{R}\right), \tag{9}$$

$$\frac{\dot{R}'}{R} - \frac{A'\dot{R}}{AR} - \frac{R'\dot{B}}{RB} = 0, \tag{10}$$

where the gravitational constant G and c are taken equal to unity. Here and hereafter, the dot and the prime denote the respective partial derivatives with respect to t and r.

We use the set of equations obtained from the Bianchi identities, which are equivalent to set of equations (7)–(10) (if $\dot{R} \neq 0$),

$$p' = -\frac{A'}{A}(p+\rho), \tag{11}$$

$$\dot{\rho} = -\left(\frac{\dot{B}}{B} + 2\frac{\dot{R}}{R}\right)(p+\rho), \tag{12}$$

$$\dot{m} = -4\pi p \dot{R} R^2, \tag{13}$$

$$m' = 4\pi\rho R' R^2,, \tag{14}$$

where $m(t,r)$ is the mass function of the fluid distribution [18]:

$$8m(t,r) = \frac{1}{2}R\left(1 + \frac{\dot{R}^2}{A^2} - \frac{R'^2}{B^2}\right). \tag{15}$$

Thompson and Whitrow [19] have obtained the following useful relation using Eqs. (7), (8) and (10):

$$\frac{\partial}{\partial t}\left[\left(\frac{1}{B}\frac{R'}{R}\right)' R^2\right] + R^2 R' \frac{1}{\dot{R}}\left(\frac{\dot{R}}{R} - \frac{\dot{B}}{B}\right)\frac{\partial}{\partial t}\frac{R'}{BR} + \frac{\dot{R}}{AR}\frac{\partial}{\partial t}\left[\frac{BR^2}{A}\left(\frac{\dot{R}}{R} - \frac{\dot{B}}{B}\right)\right] + B\frac{\dot{R}}{R} = 0. \tag{16}$$

This equation is obtained by equating Eqs. (7) and (8) and eliminating A' with the help of Eq. (10) (for the details, we refer the reader to [19], where Eq. (16) was first derived). We also note that Eqs. (11) and (12) are differential equations for ρ and p and hence their solutions have to be fed back into the corresponding parent equations.

We note that the condition $\dot{B}/B = \dot{R}/R$ restricts the motion of the fluid to be shear-free. In the shear-free case, Eq. (16) can easily be integrated with respect to t, yielding

$$2\left(\frac{R''}{R} - \frac{R'^2}{R^2} - 2\frac{B'R'}{BR}\right) = B + K(r)B^2, \tag{17}$$

where $K(r)$ is an arbitrary constant of integration.

There are two physical variables p and ρ and three metric coefficient variables A, B, and R to be determined, but the number of equations available is only four. Besides, Eqs. (7)–(10) are second-order coupled differential equations, and it is not possible to integrate them in the general case. Hence, in order to find any solution, some simplifying assumptions are needed based either on physical requirements or on mathematical simplicity. The most frequently used assumption is the validity of the mathematical relation $\dot{B}/B = \dot{R}/R$. This condition has also been obtained as from the requirement of isotropic coordinates or equivalently by assuming the vanishing of the shear of the 4-velocity of the fluid [14]–[17], [19]–[21]. Our objective here is to obtain shearing solutions, and we therefore assume that $\sigma \neq 0$. In the literature, symmetries of these equations have also been explored and used to achieve the desired objective.

3. Dynamics of a uniformly contracting star

In a collapsing configuration, there are two different velocity contributions: one is due to the circumferential velocity $\dot{R}$, which is related to the change of the geometric radius R of the star, and the other is related to the $(\dot{\delta l})$ relative velocity between neighboring layers of fluid particles, where δl is the infinitesimal distance between two neighboring fluid particles. The expansion scalar Θ describes the rate of change of small volumes of the fluid distribution. Here, we analyze the spherically symmetric gravitational collapse under the assumption of a uniform expansion scalar. The contraction of a spherical distribution of fluid is said to be uniform if the relative distance between any two fluid particles δl changes in the same way as the circumferential radius R.

For the uniform expansion scalar, the rate of change of the relative distance should be independent of direction and of the radial coordinate r. Under this kinematical condition, the distance between any two neighboring fluid particles in a collapsing spherically symmetric star remains unchanged.

The assumption can expressed as

$$\Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{R}}{R}\right) = F(t), \tag{18}$$

where $F(t)$ is an arbitrary function of t. The idea is to answer the question of whether the assumption leads to a general scenario where the perfect fluid obeys an equation of state and the homogeneous-density solution occurs as a particular case. However, behind this assumption there is another motivation based on mathematical simplicity, which we present in what follows. We also note that we temporarily present the analysis that is independent of the above assumption.

Under the assumption in (18), Eq. (10) can be integrated with respect to r, with the result

$$\frac{\dot{B}}{B} = \frac{1}{3}A\left[F(t) - \frac{3G(t)}{R^3}\right], \tag{19}$$

where $G(t)$ is an arbitrary function of integration. Equations (18) and (19) can now be used to obtain

$$\frac{\dot{R}}{R} = \frac{1}{6}\left[2F(t) + \frac{3G(t)}{R^3}\right]. \tag{20}$$

The shear of the four-velocity vector then reduces to

$$\sigma = \frac{3}{2}\frac{G(t)}{R^3}. \tag{21}$$

We note for future reference that

$$G(t) = 0 \quad \Longrightarrow \quad \frac{\dot{B}}{B} = \frac{\dot{R}}{R} \quad \Longleftrightarrow \quad \sigma = 0. \tag{22}$$

In this context, it is important to note that in the analysis of perfect fluid distributions obeying an equation of state, a key role is played by the following equation that arises under shear-free motion [14]–[17]. We consider Eq. (10) and eliminate A' using Eq. (11), which yields

$$p' = -(p+\rho)\left[\frac{\dot{R}'}{\dot{R}} - \frac{R'}{R} + \frac{R'}{R}\left(\frac{\dot{R}}{R} - \frac{\dot{B}}{B}\right)\right]. \tag{23}$$

We next differentiate Eq. (12) with respect to r and eliminate p' using Eq. (23) to obtain

$$\dot{\rho}' = \frac{\dot{\rho}\rho'}{p+\rho} + (p+\rho)\left[\frac{\dot{R}}{R}\left(1 - \frac{\dot{B}/B}{\dot{R}/R}\right)' + \frac{R'}{R}\left(1 - \frac{\dot{B}/B}{\dot{R}/R}\right)\left(2\frac{\dot{R}}{R} + \frac{\dot{B}}{B}\right)\right]. \tag{24}$$

By rearranging the last term in the right-hand side, we obtain

$$\dot{\rho}'(p+\rho) = \rho'\dot{\rho} + (p+\rho)^2\dot{R}\left[\left(1 - \frac{\dot{B}/B}{\dot{R}/R}\right)' - R'\left(1 - \frac{\dot{B}/B}{\dot{R}/R}\right)^2 + 3R'\left(1 - \frac{\dot{B}/B}{\dot{R}/R}\right)\right]. \tag{25}$$

We assume the validity of the equation

$$\left(1 - \frac{\dot{B}/B}{\dot{R}/R}\right)' - R'\left(1 - \frac{\dot{B}/B}{\dot{R}/R}\right)^2 + 3R'\left(1 - \frac{\dot{B}/B}{\dot{R}/R}\right) = 0, \tag{26}$$

which leads to

$$\dot{\rho}'(p+\rho) = \dot{\rho}\rho'. \tag{27}$$

The assumption stated in Eq. (26) is motivated by the fact that Eq. (27) is well known to have played an important role in obtaining shear-free solutions describing perfect fluid distributions obeying an equation of state [20], [21]. We show in what follows that assumption (26) is equivalent to the assumption stated in Eq. (18).

In particular, we note that $B = c(r)R$ (where $c(r)$ is arbitrary) is one of the solution of Eq. (26), which describes shear-free motion. For the shearing solution ($\sigma \neq 0$), this equation can be integrated with respect to r as

$$\frac{\dot{R}}{R} - \frac{\dot{B}}{B} = \frac{3\mathcal{G}(t)}{\mathcal{G}(t) - R^3}\frac{\dot{R}}{R}, \tag{28}$$

where $\mathcal{G}(t)$ is a constant of integration. Using this equation, we can integrate Eq. (10) with respect to r:

$$\frac{\dot{R}}{R} = \mathcal{F}(t)A\left(1 - \frac{\mathcal{G}(t)}{R^3}\right) \tag{29}$$

where $\mathcal{F}(t)$ is a constant of integration. Equations (28) and (29) can now be compared respectively with (19) and (20). In view of (18), it is evident that $\Theta = \mathcal{F}(t)$ and the identifications are

$$\mathcal{F}(t) = F(t), \qquad \mathcal{G}(t) = -\frac{3}{2}\frac{G(t)}{F(t)}, \tag{30}$$

which establishes the claim that assumption (26) *is equivalent to the assumption stated in Eq.* (18).

We note that Eq. (27) is satisfied identically for a homogeneous density $\rho = \rho(t)$, whereas Eq. (26) is satisfied identically in the shear-free case. As anticipated earlier, the analysis being pursued here includes these cases as particular cases. These cases are important as a class and have been discussed extensively and exclusively in the literature [5], [19]–[21]. We therefore assume that $\rho \neq \rho(t)$ throughout this study.

Equation (27) can be integrated with respect to t by using Eq. (12), with the result

$$\rho' B R^2 = U(r), \tag{31}$$

where $U(r)$ is an arbitrary function of integration. We note that for the requirement $\rho \neq \rho(t)$ to be satisfied, $U(r)$ must be a nonzero function. Further, in view of (11), Eq. (27) is amenable to integration with respect to r, yielding

$$A(p+\rho) = K(t)\dot{\rho}, \tag{32}$$

where $K(t)$ is an arbitrary function of integration. This equation in view of Eqs. (12) and (18) yields the identification

$$K(t) = -\frac{1}{F(t)} \quad \Longrightarrow \quad A(p+\rho) = -\frac{\dot{\rho}}{F(t)}. \tag{33}$$

Equation (16), together with (9), (19), (27), and (31), reduces to the form

$$\frac{\partial}{\partial t}\left[\left(\frac{\rho' R' R}{U(r)}\right)' R^2\right] - \frac{3}{2}\frac{\mathcal{G}(t)}{R^3 - \mathcal{G}(t)}\frac{U(r)}{\rho'}\frac{\partial}{\partial t}\left(\frac{\rho' R' R}{U}\right)^2 + \frac{U(r)}{\rho'}\frac{\dot{R}}{R^3} - \\ - \frac{1}{3}F(t)\left(1 - \frac{\mathcal{G}(t)}{R^3}\right)\frac{\partial}{\partial t}\left[\frac{F(t)\mathcal{G}(t)}{R^3}\frac{U(r)}{\rho'}\right] = 0, \tag{34}$$

$$8\pi\rho = \frac{1}{3}F^2(t)\left(1 - \frac{\mathcal{G}^2(t)}{R^6}\right) - 3\left(\frac{\rho' R' R}{U(r)}\right)^2 - \frac{R}{R'}\left[\left(\frac{\rho' R' R}{U(r)}\right)^2\right]' + \frac{1}{R^2}. \tag{35}$$

Thus, once these are explicitly known as functions of t and r, the pressure of the fluid p is given by Eqs. (13) and (14). However, it is necessary to specify R and B as explicit functions of t and r. We note that Eq. (16) with A eliminated via Eq. (20) is a differential equation for B and R. Thus, Eqs. (16) and (28) are coupled differential equations for B and R, but it seems quite unlikely that their integration is possible in general. Therefore, for the further analysis of solutions, we adapt the general equation of state [20], [21]

$$p = p(\rho), \qquad \frac{1}{p+\rho} = \frac{dq}{d\rho}, \qquad q = q(\rho). \tag{36}$$

We consider

$$\rho = \rho(v), \qquad v = f(r) + g(t) \tag{37}$$

where $f(t)$ and $g(r)$ are arbitrary functions. In view of (36), (37), Eq. (27) yields

$$e^q = \rho_v = \frac{1}{X(v)}; \tag{38}$$

the integration constant is here absorbed into the equation. In view of (37), Eq. (31) now yields ($U(r) = 1$) $\rho_v f' B R^2 = 1$. Similarly, in view of (27) and (37), Eq. (33) results in

$$A = \frac{X_v}{X}\frac{1}{F(t)}, \tag{39}$$

where we use the freedom of choosing the t-coordinate such that $g(t) = t$. Also, Eq. (29) leads to

$$\frac{X_v}{X} = 3\frac{\dot{R}R^2}{R^3 - \mathcal{G}(t)}. \tag{40}$$

Because Eq. (16) is very difficult to integrate in its complete generality, we have to make some simplifying assumption. We note that Eqs. (28) and (40) point to the feasibility of taking $\mathcal{G}(t) = \mathcal{G}_0 = \text{const}$.

It is straightforward to integrate Eqs. (28) and (40) with respect to t, with the result

$$B = J(r)R\left(1 - \frac{\mathcal{G}_0}{R^3}\right), \qquad R = [\mathcal{G}_0 + P(r)X]^{1/3}, \tag{41}$$

where $J(r)$ and $P(r)$ are constants of integration. It also follows from Eqs. (28) and (41) that

$$\left(\frac{\dot{R}}{R} - \frac{\dot{B}}{B}\right)BR^3 = -3\mathcal{G}_0 J(r)\dot{R}. \tag{42}$$

Equation (16) can now be integrated with respect to t, yielding

$$\mathcal{G}_0\left(\frac{R'^2}{B^2R^2} + \frac{\dot{R}^2}{A^2R^2} + \frac{1}{R^2}\right) + \frac{2}{3}\frac{R^2}{BJ(r)}\left[\frac{R''}{R} - \frac{R'}{R}\left(\frac{R'}{R} + \frac{B'}{B}\right) + \frac{B^2}{R^2}\right] = Q(r), \tag{43}$$

where $Q(r)$ is an arbitrary function of integration. By using Eq. (15), we can write the above equation as

$$\mathcal{G}_0 J(r)\left(\frac{m}{R^3} + \frac{1}{A^2}\frac{\dot{R}^2}{R^2}\right) + \frac{1}{3}\frac{R^2}{B}\left[\frac{R''}{R} - \frac{R'}{R}\left(\frac{R'}{R} + \frac{B'}{B}\right) + \frac{B^2}{R^2}\right] = Q(r)J(r). \tag{44}$$

In view of (42), Eq. (9) can now be written as

$$8\pi\rho = \frac{3}{R^2}\left(\frac{\dot{R}^2}{A^2} - \frac{R'^2}{B^2}\right) + 6\frac{\mathcal{G}_0 J(r)}{B^2A^2R^2}\frac{\dot{R}^2}{R^2} - \frac{2}{B^2}\left[\frac{R''}{R} - \frac{R'}{R}\left(\frac{R'}{R} + \frac{B'}{B}\right)\right] + \frac{1}{R^2}. \tag{45}$$

Now, by Eqs. (42) and (45), we obtain

$$m = \frac{4\pi}{3}\rho P(r)X + \frac{Q(r)}{24}. \tag{46}$$

Differentiating this equation with respect to r and using (14), it is straightforward to obtain

$$\rho' BR^2 = -\frac{1}{32\pi}J(r)Q'(r). \tag{47}$$

In view of Eq. (38), (39), and (41), this equation yields

$$P(r) = -\frac{1}{32\pi}\frac{Q'(r)}{f'}. \tag{48}$$

We note that Eq. (13) is satisfied identically. Thus, collecting the results, we have

$$\begin{aligned} &A = \frac{1}{F(t)}\frac{X_v}{X}, \qquad B = P(r)J(r)X[\mathcal{G}_0 + P(r)X]^{-2/3}, \qquad R = [\mathcal{G}_0 + P(r)X]^{1/3}, \\ &\rho = \int \frac{1}{X}\,dv, \qquad p + \rho = -\frac{1}{X_v}, \end{aligned} \tag{49}$$

where X and v are given by (37) and (38).

The metric in Eq. (1) now becomes

$$ds^2 = \left(\frac{X_v}{X}\right)^2 \frac{dt^2}{F^2(t)} - P^2(r)J^2(r)(\mathcal{G}_0 + P(r)X)^{-4/3}\,dr^2 - (\mathcal{G}_0 + P(r)X)^{2/3}\,d\Omega^2. \tag{50}$$

We note that $F(t)$, $P(r)$, and $J(r)$ are arbitrary parameters here; after the t-transformation $dt/F(t) \to d\tilde{t}$ and the r-transformation $P(r)J(r)\,dr \to d\tilde{r}$, the metric takes the form

$$ds^2 = \left(\frac{X_v}{X}\right)^2 d\tilde{t}^{\,2} - (\mathcal{G}_0 + P(r)X)^{-4/3}[d\tilde{r}^2 + (\mathcal{G}_0 + P(r)X)^2\, d\Omega^2]. \tag{51}$$

This metric represents the geometry of a spherically symmetric gravitating system undergoing uniform contraction/expansion.

In view of (49), Eq. (34) reduces to

$$\frac{\partial}{\partial t}\left[\left(\frac{RR'f'}{X}\right)' f'R^2\right] - \frac{3}{4}\frac{\mathcal{G}_0}{R^3 - \mathcal{G}_0}\frac{\partial}{\partial t}\left(\frac{RR'f'}{X}\right)^2 + \frac{1}{3}\frac{R^3 - \mathcal{G}_0}{R^3}\left[\frac{X_v}{R^2} - \mathcal{G}_0 F(t)\frac{\partial}{\partial t}\frac{F(t)X}{R^3}\right] = 0.$$

This equation allows us to investigate the constraint, if any, on the functional dependence of the unknown functions $F(t)$, $\mathcal{G}_0$, $P(r)$, $f(r)$, and $X(v)$.

4. Discussion and the concluding remarks

The uniformly expanding (contracting) system in general relativity has received considerable attention in cosmology and has been applied to describe the physical features of the accelerating (oscillating) universe [22]. In the collapsing configuration, Herrera et al. [23] have systematically studied the spherically symmetric fluid distribution under the expansion-free condition ($\Theta = 0$) and shown that such kind of dynamics exhibit the existence of a cavity, or void, surrounding the fluid distribution center [24]. In this paper, we study the uniformly contracting (exploding) system, which allows the future discussion of a wide range of models for the evolution of spherically symmetric collapsing stars (or cosmological models) describing uniform motion.

We started with the assumption of a uniform expansion scalar following the nonzero shear and inhomogeneous density of the fluid distribution. Actually, the investigation in this paper was possible because of the starting assumption leading to the integrability of the Einstein equation $G_{01} = 0$. We can therefore generalize our results to all situations involving adiabatically contracting and expanding systems. The additional assumption in (26) was considered for the shearing solution and was then found to be equivalent to assumption (18), $\Theta = \Theta(t)$. These two equivalent equations made the integration of Thompson–Whitrow equation (16) possible in its complete generality. We adopted the validity of the equation of state in Eqs. (36)–(38) for the analysis of the complete solution of field equations. This allowed obtaining explicit expressions for all physical and geometric parameters as functions of $X(v)$, which is an implicit function of t and r.

We studied in detail the consequences of assumption (18) in search for solutions of field equations for a spherically symmetric perfect fluid distribution in their complete generality. The results in this paper are very desirable in studying spherically symmetric gravitational collapse and cosmological models describing uniform motion. As expected, in a general study presented here, a great deal of further analysis remains to be done. The future objective include the analysis of particular choices for arbitrary constants $P(r)$, $f(r)$, and $F(t)$ to obtain more explicit solution. Some particular solutions will also provide singularities (black holes or naked singularities) in such kind of dynamical stars. This new collapse solutions are therefore very useful, even if they are simplified ones.

Acknowledgments. The authors are thankful to the referee for the valuable suggestions and comments to improve the manuscript.

Conflicts of interest. The authors declare no conflicts of interest.

REFERENCES

1. G. C. McVittie, "Gravitational motions of collapse or of expansion in general relativity," *Ann. Inst. H. Poincaré Sect. A (N. S.)*, **6**, 1–15 (1967).
2. A. H. Taub, "Restricted motions of gravitating spheres," *Ann. Inst. H. Poincaré Sect. A (N. S.)*, **9**, 153–178 (1968).
3. M. C. Faulkes, "Non-static fluid spheres in general relativity," *Prog. Theor. Phys.*, **42**, 1139–1142 (1969).
4. E. N. Glass and B. Mashhoon, "On a spherical star system with a collapsed core," *Astrophys. J.*, **205**, 570–577 (1976).
5. J. R. Oppenheimer and H. Snyder, "On continued gravitational contraction," *Phys. Rev.*, **56**, 455–459 (1939).
6. G. C. McVittie, "The mass-particle in an expanding universe," *Mon. Not. R. Astron. Soc.*, **93**, 325–339 (1933).
7. B. Nolan, "Sources for McVittie's mass particle in an expanding universe," *J. Math. Phys.*, **34**, 178–185 (1933).
8. A. Ori and T. Piran, "Naked singularities and other features of self-similar general-relativistic gravitational collapse," *Phys. Rev. D*, **42**, 1068–1090 (1990).
9. P. S. Joshi, N. Dadhich, and R. Maartens, "Why do naked singularities form in gravitational collapse?," *Phys. Rev. D*, **65**, 101501, 4 pp. (2002), arXiv: gr-qc/0109051.
10. W. B. Bonnor and M. C. Faulkes, "Exact solutions for oscillating spheres in general relativity," *Mon. Not. R. Astron. Soc.*, **137**, 239–251 (1967).
11. H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations*, Cambridge Univ. Press, Cambridge (2003).
12. H. Nariai, "A simple model for gravitational collapse with pressure gradient," *Prog. Theor. Phys.*, **38**, 92–106 (1967).
13. V. Husain, E. A. Martinez, and D. Núñez, "Exact solution for scalar field collapse," *Phys. Rev. D*, **50**, 3783–3786 (1994), arXiv: gr-qc/9402021.
14. C. B. Collins and J. Wainwright, "Role of shear in general-relativistic cosmological and stellar models," *Phys. Rev. D*, **27**, 1209–1218 (1983).
15. E. N. Glass, "Shear-free gravitational collapse," *J. Math. Phys.*, **20**, 1508–1513 (1979).
16. R. Chan, "Collapse of a radiating star with shear," *Mon. Not. R. Astron. Soc.*, **288**, 589–595 (1997); "Erratum: Collapse of a radiating star with shear," **299**, 811–811 (1998).
17. L. Herrera and N. O. Santos, "Shear-free and homology conditions for self-gravitating dissipative fluids," *Mon. Not. R. Astron. Soc.*, **343**, 1207–1212 (2003).
18. C. W. Misner and D. H. Sharp, "Relativistic equations for adiabatic, spherically symmetric gravitational collapse," *Phys. Rev. D*, **136**, B571–B576 (1964).
19. I. H. Thompson and G. J. Whitrow, "Time-dependent internal solutions for spherically symmetrical bodies in general relativity: I. Adiabatic collapse," *Mon. Not. R. Astron. Soc.*, **136**, 207–217 (1967).
20. M. Wyman, "Equations of state for radially symmetric distributions of matter," *Phys. Rev. D*, **70**, 396–400 (1946).
21. B. Mashhoon and M. Hossein Partovi, "On the gravitational motion of a fluid obeying an equation of state," *Ann. Phys.*, **130**, 99–138 (1980).
22. J. V. Narlikar, *An Introduction to Cosmology*, Cambridge Univ. Press, Cambridge (2002).
23. L. Herrera, N. O. Santos, and A. Wang, "Shearing expansion-free spherical anisotropic fluid evolution," *Phys. Rev. D*, **78**, 084026, 10 pp. (2008), arXiv: 0810.1083.
24. V. A. Skripkin, "A point discontinuity in a perfect incompressible fluid in the general theory of relativity," *Sov. Phys. Dokl.*, **5**, 1183–1186 (1961).