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ON PERIODIC GIBBS MEASURES OF THE ISING MODEL CORRESPONDING TO NEW SUBGROUPS OF THE GROUP REPRESENTATION OF A CAYLEY TREE

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*We give a full description of all index-*4 *subgroups of the group representation of a Cayley tree. Also, we give new weakly periodic Gibbs measures of the Ising model corresponding to index-*4 *subgroups of the group representation of the Cayley tree.*

Keywords: Cayley tree, group representation of Cayley tree, subgroups, Ising model, Gibbs measure

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1. Introduction

There are many open problems of group theory that arise in studying problems in natural sciences, such as physics, mechanics, coding theory, biology, and so on. For instance, a configuration of a physical system on trees can be regarded as a function defined on the set of vertices of a Cayley tree (see, e.g., [1]–[3]).

There are several main directions in the theory of Gibbs measure, such as splitting Gibbs measures, Euclidean Gibbs measures, gradient Gibbs measures, and so on. For instance, by Kolmogorov's extension theorem, we define a special family of Gibbs measures for Hamiltonians—the splitting Gibbs measures (see, e.g., [4]). Also, a Gibbs measure that satisfies the DLR equilibrium equations is called a Euclidean Gibbs measure (see, e.g., [5], [6]). It is known that the set of periodic and weakly periodic Gibbs measures is a subset of the set of splitting Gibbs measures, and this paper is devoted to such measures.

On the set of configurations of a model, one defines a Gibbs measure. The theory of periodic and weakly periodic Gibbs measures is one of the main directions in the theory of splitting Gibbs measures. To give a definition of periodic and weakly periodic Gibbs measures on Cayley trees, one needs subgroups of the group representation of the trees (see [7], [3]). As usual, by using the invariance property of subgroups of the group representation of Cayley trees, the description of the set of periodic or weakly periodic Gibbs measures for Hamiltonians with finite spin values on Cayley trees can be reduced to solving a system of algebraic equations (see [3]). Also, the description of the set of periodic or weakly periodic Gibbs measures for Hamiltonians with infinite spin values on Cayley trees reduces to solving a system of algebraic equations (see, e.g., [8]–[10]). If the invariance property holds, then it allows finding periodic and weakly periodic Gibbs measures corresponding to an arbitrary finite-index subgroup of the group representation of the Cayley tree. Also, for any normal finite-index subgroup of the group representation of a Cayley tree,

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the invariance property holds. For this reason, the theory of periodic and weakly periodic Gibbs measures corresponding to normal subgroups is well developed (see, e.g., [11], [3]). On the other hand, for any (not normal) subgroup of the group representation of the Cayley tree, the invariance property does not hold in general (see [12]).

In this paper, we give a full description of all (normal and not normal) index-4 subgroups of the group representation of the Cayley tree (see Theorem 2). In Theorem 4, we give new weakly periodic Gibbs measures of the Ising model with nearest-neighbor interaction corresponding to the (not normal) index-4 subgroups of the group representation of the Cayley tree.

2. Index-4 subgroups for the group representation of a Cayley tree

Cayley tree. A Cayley tree (Bethe lattice) Γ^k of order $k \geqslant 1$ is an infinite homogeneous tree, i.e., a graph without cycles, such that exactly $k+1$ edges originate from each vertex. Let $\Gamma^k = (V, L)$, where V is the set of vertices and L is the set of edges. Two vertices x and y are called *nearest neighbors* if there exists an edge $l \in L$ connecting them. We use the notation $l = \langle x, y \rangle$. A collection of nearest-neighbor pairs $\langle x, x_1 \rangle, \langle x_1, x_2 \rangle, \ldots, \langle x_{d-1}, y \rangle$ is called a *path* from x to y. The distance $d(x; y)$ on the Cayley tree is the number of edges of the shortest path from x to y.

Group representation of the Cayley tree. Let G_k be the free product of $k+1$ cyclic groups of the order 2 with respective generators $a_1, a_2, \ldots, a_{k+1}$. It is known that there exists a one-to-one correspondence between the set of vertices V of the Cayley tree of order $k \geqslant 1$ and elements of the group G_k. To give this correspondence, we fix an arbitrary element $x_0 \in V$ and let it correspond to the unit element e of G_k. Using $a_1, a_2, \ldots, a_{k+1}$, we label the nearest-neighbors of e, moving in the positive direction. We next label the nearest neighbors of each a_i, $i = 1, 2, \ldots, k+1$, by $a_i a_j$, $j = 1, 2, \ldots, k+1$. Because all a_i have $e$as a the common neighbor, we assign $a_i a_i = a_i^2 = e$ to it. Other neighbors are labeled starting from $a_i a_i$ in the positive direction. We label the set of all nearest neighbors of each $a_i a_j$ by words $a_i a_j a_q$, $q = 1, 2, \ldots, k+1$, starting from $a_i a_j a_j = a_i$ in the positive direction. Iterating this argument yields a one-to-one correspondence between the set of vertices V of the Cayley tree Γ^k and the group G_k (see, e.g., [13]).

In [11], [13], a full description of normal index-4 subgroups for the group representation of the Cayley tree is given. Also, all index-3 subgroups were constructed in [12]. In this section, we continue these investigations and give all forms of (not normal) index-4 subgroups of the group representation of the Cayley tree.

Normal finite-index subgroups of the group G_k. Any (minimally represented) element $x \in G_k$ has the form $x = a_{i_1} a_{i_2} \ldots a_{i_n}$, where $1 \leqslant i_m \leqslant k+1$, $m = 1, 2, \ldots, n$. The number n is called the length of the word x and is denoted by $l(x)$. The number of letters a_i, $i = 1, 2, \ldots, k+1$, that enter a noncontractible representation of a word x is denoted by $w_x(a_i)$.

Definition 1. Let $M_1, M_2, \ldots, M_m$ be some sets and $M_i \neq M_j$ for $i \neq j$. We say that the intersection $\bigcap_{i=1}^{m} M_i$ is *contractible* if there exists i_0 ($1 \leqslant i_0 \leqslant m$) such that

$$\bigcap_{i=1}^{m} M_i = \Bigl(\bigcap_{i=1}^{i_0-1} M_i\Bigr) \bigcap \Bigl(\bigcap_{i=i_0+1}^{m} M_i\Bigr).$$

Let $N_k = \{1, 2, \ldots, k+1\}$. We set

$$H_A = \Bigl\{ x \in G_k \Bigm| \sum_{i \in A} \omega_x(a_i) \text{ is even} \Bigr\}. \tag{2.1}$$

The following theorem [3], [13] describes several normal subgroups of G_k.

Theorem 1. *For any A, $\varnothing \neq A \subseteq N_k$, the set $H_A \subset G_k$ satisfies the following properties:*

- *H_A is a normal subgroup and $|G_k : H_A| = 2$;*
- *$H_A \neq H_B$ for all $A \neq B \subseteq N_k$;*
- *let $A_1, A_2, \ldots, A_m \subseteq N_k$; if $\bigcap_{i=1}^{m} H_{A_i}$ is noncontractible, then it is a normal subgroup of index 2^m;*
- *any normal index-4 subgroup has the form $H_A \cap H_B$, i.e.,*

$$\{H \mid |G_k : H| = 4\} = \{H_A \cap H_B \mid A, B \subseteq N_k, A \neq B\};$$

- *the group G_k does not have normal subgroups of an odd index ($\neq 1$);*
- *the group G_k has normal subgroups of an arbitrary even index.*

By Theorem 1, we can conclude that any subgroup of an odd index ($\neq 1$) of the group G_k is not a normal subgroup. In addition, all normal index-4 subgroups of G_k are described. We now prove that there exists (not normal) index-4 subgroups of G_k and give a full description of such subgroups.

Let $A_0 \subset N_k$, $0 \leqslant |A_0| \leqslant k-2$. Let also (A_1, A_2, A_3) be a partition of the set $N_k \backslash A_0$ and m_j be the minimal element of A_j, $j = 1, 2, 3$. Then we consider the homomorphism $u_{A_1 A_2 A_3} \colon G_k \to \{e, a_{m_1}, a_{m_2}, a_{m_3}\}$ given by

$$u_{A_1 A_2 A_3}(x) = \begin{cases} e, & \text{if } x = a_i, \quad i \in N_k \backslash (A_1 \cup A_2 \cup A_3), \\ a_{m_j}, & \text{if } x = a_i, \quad i \in A_j, \ j = 1, 2, 3, \end{cases} \tag{2.2}$$

For $i \in \{1, 2, \ldots, 8\}$, we define the maps $\gamma_i \colon \langle a_{m_1}, a_{m_2}, a_{m_3} \rangle \to \{e, a_{m_1}, a_{m_2}, a_{m_3}\}$ by the following formulas:

$$\gamma_1(x) = \begin{cases} e, & x = e, \\ a_{m_1}, & x \in \{a_{m_3} a_{m_1}, a_{m_2} a_{m_3}\}, \\ a_{m_2}, & x \in \{a_{m_1} a_{m_3}, a_{m_3} a_{m_2}\}, \\ a_{m_3}, & x \in \{a_{m_1} a_{m_2}, a_{m_2} a_{m_1}\}, \\ \gamma_1(a_{i_1} a_{i_2} \ldots a_{i_{n-2}} \gamma_1(a_{i_{n-1}} a_{i_n})), & x = a_{i_1} a_{i_2} \ldots a_{i_n}, \quad l(x) > 2, \end{cases} \tag{2.3}$$

$$\gamma_2(x) = \begin{cases} e, & x = e, \\ a_{m_1}, & x \in \{a_{m_3} a_{m_2}, a_{m_2} a_{m_1}\}, \\ a_{m_2}, & x \in \{a_{m_1} a_{m_3}, a_{m_3} a_{m_1}\}, \\ a_{m_3}, & x \in \{a_{m_1} a_{m_2}, a_{m_2} a_{m_3}\}, \\ \gamma_2(a_{i_1} a_{i_2} \ldots a_{i_{n-2}} \gamma_2(a_{i_{n-1}} a_{i_n}), & x = a_{i_1} a_{i_2} \ldots a_{i_n}, \quad l(x) > 2, \end{cases} \tag{2.4}$$

$$\gamma_3(x) = \begin{cases} e, & x = e, \\ a_{m_1}, & x \in \{a_{m_3} a_{m_1}, a_{m_2} a_{m_1}\}, \\ a_{m_2}, & x \in \{a_{m_1} a_{m_3}, a_{m_3} a_{m_2}\}, \\ a_{m_3}, & x \in \{a_{m_1} a_{m_2}, a_{m_2} a_{m_3}\}, \\ \gamma_3(a_{i_1} a_{i_2} \ldots a_{i_{n-2}} \gamma_3(a_{i_{n-1}} a_{i_n})), & x = a_{i_1} a_{i_2} \ldots a_{i_n}, \quad l(x) > 2, \end{cases} \tag{2.5}$$

$$\gamma_4(x)=\begin{cases} e, & x=e,\\ a_{m_1}, & x\in\{a_{m_2}a_{m_3},a_{m_3}a_{m_2}\},\\ a_{m_2}, & x\in\{a_{m_1}a_{m_2},a_{m_3}a_{m_1}\},\\ a_{m_3}, & x\in\{a_{m_1}a_{m_3},a_{m_2}a_{m_1}\},\\ \gamma_4(a_{i_1}a_{i_2}\dots a_{i_{n-2}}\gamma_4(a_{i_{n-1}}a_{i_n})), & x=a_{i_1}a_{i_2}\dots a_{i_n},\quad l(x)>2,\end{cases}\tag{2.6}$$

$$\gamma_5(x)=\begin{cases} e, & x=e,\\ a_{m_1}, & x\in\{a_{m_2}a_{m_1},a_{m_3}a_{m_1}\},\\ a_{m_2}, & x\in\{a_{m_1}a_{m_2},a_{m_3}a_{m_2}\},\\ a_{m_3}, & x\in\{a_{m_1}a_{m_3},a_{m_2}a_{m_3}\},\\ \gamma_5(a_{i_1}a_{i_2}\dots a_{i_{n-2}}\gamma_5(a_{i_{n-1}}a_{i_n})), & x=a_{i_1}a_{i_2}\dots a_{i_n},\quad l(x)>2,\end{cases}\tag{2.7}$$

$$\gamma_6(x)=\begin{cases} e, & x=e,\\ a_{m_1}, & x\in\{a_{m_2}a_{m_3},a_{m_3}a_{m_1}\},\\ a_{m_2}, & x\in\{a_{m_1}a_{m_2},a_{m_3}a_{m_2}\},\\ a_{m_3}, & x\in\{a_{m_1}a_{m_3},a_{m_2}a_{m_1}\},\\ \gamma_6(a_{i_1}a_{i_2}\dots a_{i_{n-2}}\gamma_6(a_{i_{n-1}}a_{i_n})), & x=a_{i_1}a_{i_2}\dots a_{i_n},\quad l(x)>2,\end{cases}\tag{2.8}$$

$$\gamma_7(x)=\begin{cases} e, & x=e,\\ a_{m_1}, & x\in\{a_{m_2}a_{m_1},a_{m_3}a_{m_2}\},\\ a_{m_2}, & x\in\{a_{m_1}a_{m_2},a_{m_3}a_{m_1}\},\\ a_{m_3}, & x\in\{a_{m_1}a_{m_3},a_{m_2}a_{m_3}\},\\ \gamma_7(a_{i_1}a_{i_2}\dots a_{i_{n-2}}\gamma_7(a_{i_{n-1}}a_{i_n})), & x=a_{i_1}a_{i_2}\dots a_{i_n},\quad l(x)>2,\end{cases}\tag{2.9}$$

$$\gamma_8(x)=\begin{cases} e, & x=e,\\ a_{m_1}, & x\in\{a_{m_2}a_{m_3},a_{m_3}a_{m_2}\},\\ a_{m_2}, & x\in\{a_{m_1}a_{m_3},a_{m_3}a_{m_1}\},\\ a_{m_3}, & x\in\{a_{m_1}a_{m_2},a_{m_2}a_{m_1}\},\\ \gamma_8(a_{i_1}a_{i_2}\dots a_{i_{n-2}}\gamma_8(a_{i_{n-1}}a_{i_n})), & x=a_{i_1}a_{i_2}\dots a_{i_n},\quad l(x)>2,\end{cases}\tag{2.10}$$

where $l(x)$ is the length of x and $i_s\in\{m_1,m_2,m_3\}$ for $s\in\{1,2,\dots,n\}$.

We set

$$\Im^j_{A_1A_2A_3}(G_k)=\{x\in G_k\mid \gamma_j(u_{A_1A_2A_3}(x))=e\},\qquad j=1,2,\dots,8.\tag{2.11}$$

Proposition 1. *Let $A_0\subset N_k$, $0\leqslant|A_0|\leqslant k-2$, and (A_1,A_2,A_3) be a partition of the set $N_k\backslash A_0$. Then $\Im^j_{A_1A_2A_3}(G_k)$ is a subgroup of the group G_k.*

Proof. It is known that $\Im^j_{A_1A_2A_3}(G_k)$ is a subgroup if and only if $xy\in\Im^j_{A_1A_2A_3}(G_k)$ and $y^{-1}\in\Im^j_{A_1A_2A_3}(G_k)$ for all $x,y\in\Im^j_{A_1A_2A_3}(G_k)$. For any $j=1,2,\dots,8$, we consider two elements $x=a_{i_1}a_{i_2}\dots a_{i_n}\in\Im^j_{A_1A_2A_3}(G_k)$ and $y=a_{s_1}a_{s_2}\dots a_{s_m}\in\Im^j_{A_1A_2A_3}(G_k)$. If

$$u(x)=a_{j_1}a_{j_2}\dots a_{j_s},\qquad s\leqslant n,\quad j_i\in\{m_1,m_2,m_3\},\quad i\in\{1,2\dots,s\},$$
$$u(y)=a_{t_1}a_{t_2}\dots a_{t_r},\qquad r\leqslant m,\quad t_i\in\{m_1,m_2,m_3\},\quad i\in\{1,2\dots,r\},$$

then wee have $\gamma_j(a_{j_1}a_{j_2}\dots a_{j_s})=e$ and $\gamma_j(a_{t_1}a_{t_2}\dots a_{t_r})=e$.

We first prove that $y^{-1} \in \Im^j_{A_1A_2A_3}(G_k)$. Indeed,

$$\gamma_j(u(y)) = e \quad \Longleftrightarrow \quad a_{t_1}a_{t_2}\dots a_{t_r}\Im^j_{A_1A_2A_3}(G_k) = \Im^j_{A_1A_2A_3}(G_k).$$

We multiply both sides of the last equality by a_{t_1} from the left to obtain

$$a_{t_2}\dots a_{t_r}\Im^j_{A_1A_2A_3}(G_k) = a_{t_1}\Im^j_{A_1A_2A_3}(G_k).$$

Multiplying both sides of the last equality by a_{t_2} gives

$$a_{t_3}\dots a_{t_r}\Im^j_{A_1A_2A_3}(G_k) = a_{t_2}a_{t_1}\Im^j_{A_1A_2A_3}(G_k).$$

After continuing this process $r-2$ times, we have

$$a_{t_r}a_{t_{r-1}}\dots a_{t_1}\Im^j_{A_1A_2A_3}(G_k) = \Im^j_{A_1A_2A_3}(G_k) \quad \Longrightarrow \quad y^{-1} \in \Im^j_{A_1A_2A_3}(G_k).$$

We now show that $\gamma_j(xy) = e$. The proof of this is equivalent to showing the equality

$$xy\Im^j_{A_1A_2A_3}(G_k) = a_{j_1}a_{j_2}\dots a_{j_k}a_{t_1}a_{t_2}\dots a_{t_r}\Im^j_{A_1A_2A_3}(G_k) = \Im^j_{A_1A_2A_3}(G_k). \tag{2.12}$$

From the foregoing, we have

$$a_{t_1}a_{t_2}\dots a_{t_r}\Im^j_{A_1A_2A_3}(G_k) = a_{j_k}a_{j_{k-1}}\dots a_{j_1}\Im^j_{A_1A_2A_3}(G_k).$$

We multiply both sides of the last equality by a_{j_k} from the left and obtain

$$a_{j_k}a_{t_1}a_{t_2}\dots a_{t_r}\Im^j_{A_1A_2A_3}(G_k) = a_{j_{k-1}}\dots a_{j_1}\Im^j_{A_1A_2A_3}(G_k).$$

After continuing this process $k-1$ times (multiplying by $a_{j_{k-1}}, \dots, a_{j_1}$), we obtain equality (2.12).

Theorem 2. *For the group G_k, the following equality holds:*

$$\{K \mid K \text{ is a subgroup of } G_k \text{ of index 4}\} = \\ = \bigcup_{j=1}^{8}\{\Im^j_{A_1A_2A_3}(G_k) \mid (A_1,A_2,A_3) \text{ is a partition of } N_k\backslash A_0\}.$$

Proof. Let K be a subgroup of the group G_k with $|G_k : K| = 4$. Then it is easy to verify that there exist $a_p, a_q, a_r \in G_k$ such that the cosets K, a_pK, a_qK, and a_rK are disjoint. We set

$$A_0 = \{i \in N_k \mid a_i \in K\}, \qquad A_1 = \{i \in N_k \mid a_i \in a_pK\},$$
$$A_2 = \{i \in N_k \mid a_i \in a_qK\}, \qquad A_3 = \{i \in N_k \mid a_i \in a_rK\}.$$

Then we can conclude that (A_1, A_2, A_3) is a partition of $N_k\backslash A_0$. Let m_i be the minimal element of A_i, $i = 1, 2, 3$. By Proposition 1, we obtain that $\Im^j_{A_1A_2A_3}(G_k)$, $j = 1, 2, \dots, 8$ is a subgroup of G_k.

Let $a_{m_j}a_{m_s}K \in \{K, a_{m_j}K,\}$ for any $j \in \{1, 2, 3\}$ and $s \neq j$. Then it is easy to verify that $|G_k : K| < 4$. If $a_{m_1}a_{m_2}K = a_{m_3}K$ and $a_{m_1}a_{m_3}K = a_{m_3}K$, then

$$a_{m_1}a_{m_2}K = a_{m_1}a_{m_3}K \quad \Longrightarrow \quad a_{m_2}K = a_{m_3}K \quad \Longrightarrow \quad |G_k : K| < 4.$$

Now, we consider the case $a_{m_1}a_{m_2}K = a_{m_2}K$ and $a_{m_1}a_{m_3}K = a_{m_2}K$. In this case, we obtain

$$a_{m_1}a_{m_2}K = a_{m_1}a_{m_3}K \quad \Longrightarrow \quad a_{m_2}K = a_{m_3}K \quad \Longrightarrow \quad |G_k : K| < 4.$$

Hence, only two cases remain:

$$a_{m_1}a_{m_2}K = a_{m_2}K, \qquad a_{m_1}a_{m_3}K = a_{m_3}K$$

and

$$a_{m_1}a_{m_2}K = a_{m_2}K, \qquad a_{m_1}a_{m_3}K = a_{m_3}K.$$

Case 1. Let $a_{m_1}a_{m_2}K = a_{m_2}K$ and $a_{m_1}a_{m_3}K = a_{m_3}K$. If we consider the coset $a_{m_2}a_{m_3}K$, it can be equal to either $a_{m_1}K$ or $a_{m_3}K$. If $a_{m_2}a_{m_3}K = a_{m_1}K$, then $a_{m_2}a_{m_1}K = a_{m_3}K$, i.e., we have the kernels of the respective functions γ_1 and γ_8. In the case $a_{m_2}a_{m_3}K = a_{m_3}K$, the coset $a_{m_3}a_{m_2}K$ can be equal to $a_{m_1}K$ or $a_{m_2}K$. Namely, we obtain the kernel of the respective functions γ_2 and γ_3.

Case 2. If $a_{m_1}a_{m_2}K = a_{m_2}K$ and $a_{m_1}a_{m_3}K = a_{m_3}K$, then $a_{m_2}a_{m_3}K \in \{a_{m_1}K, a_{m_3}K\}$. If $a_{m_2}a_{m_3}K = a_{m_1}K$, then $a_{m_2}a_{m_1}K = a_{m_3}K$, i.e., we have the kernel of the respective function γ_4 and γ_5. Also, for the case $a_{m_2}a_{m_3}K = a_{m_3}K$, we have only one subcase $a_{m_2}a_{m_1}K = a_{m_1}K$ and $a_{m_2}a_{m_1}K = a_{m_3}K$. It gives the kernels of the respective functions γ_6 and γ_7.

Remark. Let $A = \{m_1, m_2\}$ and $B = \{m_1, m_3\}$. Then

$$\Im^8_{A_1A_2A_3}(G_k) = \{x \in G_k \mid \gamma_8(u_{A_1A_2A_3}(x)) = e\} = H_A \cap H_B.$$

Namely, $\Im^8_{A_1A_2A_3}(G_k)$ is a normal subgroup of G_k. For any $j = 1, 2, \ldots, 7$, the subgroup $\Im^j_{A_1A_2A_3}(G_k)$ is not a normal subgroup of G_k.

3. Weakly periodic Gibbs measures for the Ising model on Cayley trees

In this section, we give new weakly periodic Gibbs measures for the Ising model corresponding to index-4 subgroups of the group G_k.

We consider models where spin takes values in the set $\Phi := \{-1, 1\}$, and is assigned to the vertices of the Cayley tree. For $A \subset V$, a configuration σ_A on A is an arbitrary function $\sigma_A \colon A \to \Phi$. The set of all configurations on A is denoted by $\Omega_A = \Phi^A$.

The (formal) Hamiltonian of the Ising model is

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \sigma(x)\sigma(y), \tag{3.1}$$

where $J \in \mathbb{R}\backslash\{0\}$ is a coupling constant and $\langle x, y\rangle$ stands for nearest-neighbor vertices. For a fixed $x^0 \in V$, called the root, we set

$$W_n = \{x \in V \mid d(x^0, x) = n\}, \qquad V_n = \bigcup_{m=0}^{n} W_m. \tag{3.2}$$

Let $S(x)$ be direct successors of x, i.e.,

$$S(x) = \{y \in G_k \mid d(y, x^0) = d(x, x^0) + 1\}.$$

For any $x \in G_k$, the set $\{y \in G_k \mid \langle x, y\rangle\}\backslash S(x)$ has a single element, which is denoted by $x_\downarrow$.

We define a finite-dimensional distribution of a probability measure μ in the volume V_n as

$$\mu_n(\sigma_n) = Z_n^{-1} \exp\Big\{-\beta H_n(\sigma_n) + \sum_{x\in W_n} h_x\sigma_n(x)\Big\}, \tag{3.3}$$

where $\beta = 1/T$, $T > 0$ is the temperature and Z_n^{-1} is the normalization factor.

Let $\{h_x \in \mathbb{R}, x \in V\}$ be a collection of real numbers and

$$H_n(\sigma_n) = -J \sum_{\langle x,y\rangle\in L} \sigma_n(x)\sigma_n(y).$$

We say that probability distributions (3.3) are compatible if for all $n \geqslant 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$,

$$\sum_{\omega_n\in\Phi^{W_n}} \mu_n(\sigma_{n-1} \vee \omega_n) = \mu_{n-1}(\sigma_{n-1}). \tag{3.4}$$

Here, $\sigma_{n-1} \vee \omega_n$ is the concatenation of configurations. In this case, according to the Kolmogorov extension theorem (see [14]), there exists a unique measure μ on Φ^{V_n},

$$\mu(\{\sigma|_{V_n} = \sigma_n\}) = \mu_n(\sigma_n).$$

Such a measure is called the splitting Gibbs measure corresponding to Hamiltonian (3.1) and the function h_x, $x \in V$. The following statement [15], [16] describes conditions on h_x guaranteeing the compatibility of $\mu_n(\sigma_n)$.

Theorem 3. *The probability distributions $\mu_n(\sigma_n)$, $n = 1, 2, \dots$, in (3.3) are compatible if and only if the following equation holds for any $x \in V$:*

$$h_x = \sum_{y\in S(x)} f(h_y, \theta), \quad \textit{where} \quad \theta = \tanh(J\beta), \quad f(h, \theta) = \operatorname{artanh}(\theta \tanh h). \tag{3.5}$$

Definition 2. Let K be a subgroup of G_k, $k \geqslant 1$. We say that a function $h = \{h_x \in \mathbb{R} \colon x \in G_k\}$ is *K-periodic* if $h_{yx} = h_x$ for all $x \in G_k$ and $y \in K$. A G_k-periodic function is called *translation invariant*. A Gibbs measure μ is called *K-periodic* if it corresponds to a K-periodic function h.

Let $G_k : K = \{K_1, K_2, \dots, K_r\}$ be a family of cosets and K a subgroup of index $r \in \mathbb{N}$.

Definition 3. A set of quantities $h = \{h_x,\ x \in G_k\}$ is said to be *K-weakly periodic* if $h_x = h_{ij}$, for any $x \in K_i$, $x_\downarrow \in K_j$. A Gibbs measure μ is said to be K-weakly periodic if it corresponds to a K-weakly periodic set of h.

We note that the weakly periodic set of h coincides with an ordinary periodic one (see Definition 2) if h_x is independent of $x_\downarrow$.

The K-weakly periodic Gibbs measure of some models were mainly studied only in the case where K is a normal subgroup of G_k. We consider K-weakly periodic Gibbs measures of the Ising model on the Cayley tree in the case where K is not a normal index-4 subgroup of G_k.

Let $A_0 = \{4, 5, \dots, k+1\}$, $A_s = \{s\}$, where $s \in \{1, 2, 3\}$, i.e., $m_i = i$ for $i \in \{1, 2, 3\}$. We consider the functions $u_{\{1\}\{2\}\{3\}} \colon \{a_1, a_2, \dots, a_{k+1}\} \to \{e, a_1, a_2, a_3\}$ (defined in (2.2)) and $\gamma_1 \colon \langle a_1, a_2, a_3\rangle \to \{e, a_1, a_2, a_3\}$ (defined in (2.3)):

$$u_{\{1\},\{2\},\{3\}}(x) = \begin{cases} e, & x = a_i \ \ i = N_k \backslash \{1, 2, 3\}, \\ a_i, & x = a_i \ \ i \in \{1, 2, 3\}, \end{cases} \tag{3.6}$$

and

$$\gamma_1(x) = \begin{cases} e, & x = e, \\ a_1, & x \in \{a_3a_1, a_2a_3\}, \\ a_2, & x \in \{a_1a_3, a_3a_2\}, \\ a_3, & x \in \{a_1a_2, a_2a_1\}, \\ \gamma_1(a_{i_1}a_{i_2}\dots a_{i_{n-2}}\gamma_1(a_{i_{n-1}}a_{i_n})), & x = a_{i_1}a_{i_2}\dots a_{i_n}, \quad l(x) > 2. \end{cases} \tag{3.7}$$

Let $K_0^* := \Im^1_{\{1\}\{2\}\{3\}}(G_k)$ (defined in (2.11)), i.e.,

$$K_0^* = \{x \in G_k \mid \gamma_1(u_{\{1\}\{2\}\{3\}}(x)) = e\}.$$

By Theorem 2, we have that K_0^* is an index-4 subgroup of the group G_k. We set

$$G_2/K_0^* = \{K_0^*, K_1^*, K_2^*, K_3^*\},$$

where

$$\begin{aligned} K_1^* &= \{x \in G_2 \mid \gamma(u_{\{1\}\{2\}\{3\}}(x)) = a_1\}, \\ K_2^* &= \{x \in G_2 \mid \gamma(u_{\{1\}\{2\}\{3\}}(x)) = a_2\}, \\ K_3^* &= \{x \in G_2 \mid \gamma(u_{\{1\}\{2\}\{3\}}(x)) = a_3\}. \end{aligned}$$

Then we have

$$\begin{aligned} q_0(K_0^*) &:= |\{a_1, a_2, \dots, a_{k+1}\} \cap K_0^*| = |\{a_4, a_5, \dots, a_{k+1}\}| = k-2, \\ q_1(K_0^*) &:= |\{a_1, a_2, \dots, a_{k+1}\} \cap K_1^*| = |\{a_1\}| = 1, \\ q_2(K_0^*) &:= |\{a_1, a_2, \dots, a_{k+1}\} \cap K_2^*| = |\{a_2\}| = 1, \\ q_3(K_0^*) &:= |\{a_1, a_2, \dots, a_{k+1}\} \cap K_3^*| = |\{a_3\}| = 1 \end{aligned}$$

and

$$Q(K_0^*) := (q_0(K_0^*), q_1(K_0^*), q_2(K_0^*), q_3(K_0^*)) = (k-2, 1, 1, 1).$$

We assume that $x \in K_1^*$ (the cases $x \in K_2^*$ and $x \in K_3^*$ are similar), i.e., $\gamma(u_{\{1\}\{2\}\{3\}}(x)) = a_1$. Then it is easy to verify that

$$\{\gamma(u_{\{1\}\{2\}}(xa_1)), \gamma(u_{\{1\}\{2\}\{3\}}(xa_2)), \gamma(u_{\{1\}\{2\}\{3\}}(xa_3))\} = \{e, a_2, a_3\}.$$

In addition, we have $\gamma(u_{\{1\}\{2\}\{3\}}(xa_i)) = a_1, i \in \{4, 5, \dots, k+1\}$. Consequently,

$$\begin{aligned} q_0(x) &:= |\{xa_1, xa_2, \dots, xa_{k+1}\} \cap K_0^*| = 1, \\ q_1(x) &:= |\{xa_1, xa_2, \dots, xa_{k+1}\} \cap K_1^*| = k-2, \\ q_2(x) &:= |\{xa_1, xa_2, \dots, xa_{k+1}\} \cap K_2^*| = 1, \\ q_3(x) &:= |\{xa_1, xa_2, \dots, xa_{k+1}\} \cap K_3^*| = 1. \end{aligned}$$

Hence, $Q(x) = (1, k-2, 1, 1)$. Clearly (see the details in [3], [12]), for any $x \in G_k$, there exists a permutation π_x of the coordinates of the vector $Q(K_0)$ such that $\pi_x Q(K_0) = Q(x)$.

Let $h = (h_1, h_2, \ldots, h_{12}) \in \mathbb{R}^{12}$ and $h' = (h'_1, h'_2, \ldots, h'_{12}) \in \mathbb{R}^{12}$. Then we define the operator $W_2\colon \mathbb{R}^{12} \to \mathbb{R}^{12}$ as

$$W(h) = h' \quad \Longleftrightarrow \quad \begin{cases} h'_1 = f(h_7,\theta) + f(h_{10},\theta), & h'_2 = f(h_4,\theta) + f(h_{10},\theta), \\ h'_3 = f(h_7,\theta) + f(h_4,\theta), & h'_4 = f(h_8,\theta) + f(h_{11},\theta), \\ h'_5 = f(h_1,\theta) + f(h_{11},\theta), & h'_6 = f(h_1,\theta) + f(h_8,\theta), \\ h'_7 = f(h_5,\theta) + f(h_{12},\theta), & h'_8 = f(h_2,\theta) + f(h_{12},\theta), \\ h'_9 = f(h_2,\theta) + f(h_5,\theta), & h'_{10} = f(h_6,\theta) + f(h_9,\theta), \\ h'_{11} = f(h_3,\theta) + f(h_9,\theta), & h'_{12} = f(h_3,\theta) + f(h_6,\theta). \end{cases} \tag{3.8}$$

We note that K_0^* is an index-4 subgroup of the group G_k and the set $h = \{h_x, x \in G_k\}$ is called K_0^*-weakly periodic if $h_x = h_{ij}$ for any $x \in K_i^*$, $x_\downarrow \in K_j^*$, $i,j \in \{0,1,2,3\}$. We use the operator W to study K_0^*-weakly periodic Gibbs measures for Ising models on the Cayley tree of order two. Because $Q(x) = (1,0,1,1)$, the cases $h_x = h_{i,i}$, $i \in \{0,1,2,3\}$, do not occur. Then a K_0^*-weakly periodic set of h has the form

$$h_x = \begin{cases} h_{0,1} := h_1, & x \in K_0^*, x_\downarrow \in K_1^*, \\ h_{0,2} := h_2, & x \in K_0^*, x_\downarrow \in K_2^*, \\ h_{0,3} := h_3, & x \in K_0^*, x_\downarrow \in K_3^*, \\ h_{1,0} := h_4, & x \in K_1^*, x_\downarrow \in K_0^*, \\ h_{1,2} := h_5, & x \in K_1^*, x_\downarrow \in K_2^*, \\ h_{1,3} := h_6, & x \in K_1^*, x_\downarrow \in K_3^*, \\ h_{2,0} := h_7, & x \in K_2^*, x_\downarrow \in K_0^*, \\ h_{2,1} := h_8, & x \in K_2^*, x_\downarrow \in K_1^*, \\ h_{2,3} := h_9, & x \in K_2^*, x_\downarrow \in K_3^*, \\ h_{3,0} := h_{10}, & x \in K_3^*, x_\downarrow \in K_0^*, \\ h_{3,1} := h_{11}, & x \in K_3^*, x_\downarrow \in K_1^*, \\ h_{3,2} := h_{12}, & x \in K_3^*, x_\downarrow \in K_2^*, \end{cases}$$

where $h = (h_1, h_2, \ldots, h_{12})$, in view of (3.5), satisfies the equations

$$Wh = h. \tag{3.9}$$

Finding all solutions of Eq. (3.9) is not easy. That is why we solve the equation on invariant sets on the Cayley tree of order two. It is easy to verify that the following sets are invariant with respect to the operator W:

$$\begin{aligned} I_1 &= \{h \in \mathbb{R}^{12} \mid h_1 = h_2 = h_3 = h_4 = h_5 = h_6 = h_7 = h_8 = h_9 = h_{10} = h_{11} = h_{12}\}, \\ I_2 &= \{h \in \mathbb{R}^{12} \mid h_1 = h_2 = h_{11} = h_{12},\ h_4 = h_6 = h_7 = h_9,\ h_3 = h_{10},\ h_5 = h_8\}, \\ I_3 &= \{h \in \mathbb{R}^{12} \mid h_1 = h_3 = h_8 = h_9,\ h_4 = h_5 = h_{10} = h_{12},\ h_2 = h_7,\ h_6 = h_{11}\}, \\ I_4 &= \{h \in \mathbb{R}^{12} \mid h_2 = h_3 = h_5 = h_6,\ h_7 = h_8 = h_{10} = h_{11},\ h_1 = h_4,\ h_9 = h_{12}\}, \\ I_5 &= \{h \in \mathbb{R}^{12} \mid h_1 = h_5 = h_7,\ h_2 = h_4 = h_8,\ h_3 = h_6 = h_9,\ h_{10} = h_{11} = h_{12}\}, \\ I_6 &= \{h \in \mathbb{R}^{12} \mid h_1 = h_8 = h_{11},\ h_2 = h_9 = h_{10},\ h_3 = h_7 = h_{12},\ h_4 = h_5 = h_6\}, \\ I_7 &= \{h \in \mathbb{R}^{12} \mid h_1 = h_6 = h_{10},\ h_2 = h_5 = h_{12},\ h_3 = h_4 = h_{11},\ h_7 = h_8 = h_9\}, \end{aligned}$$

$$I_8 = \{h \in \mathbb{R}^{12} \mid h_1 = h_2 = h_3,\ h_4 = h_7 = h_{10},\ h_6 = h_8 = h_{12},\ h_5 = h_9 = h_{11}\},$$
$$I_9 = \{h \in \mathbb{R}^{12} \mid h_1 = h_2 = h_4 = h_5 = h_7 = h_8,\ h_3 = h_6 = h_9,\ h_{10} = h_{11} = h_{12}\},$$
$$I_{10} = \{h \in \mathbb{R}^{12} \mid h_1 = h_{12},\ h_2 = h_{11},\ h_3 = h_{10},\ h_4 = h_9,\ h_5 = h_8,\ h_6 = h_7\}.$$

We consider Eq. (3.9) on the invariant sets I_2, I_3, and I_4. On I_2, this equation can be written as

$$\begin{aligned} h_1 &= f(h_3,\theta) + f(h_4,\theta), \qquad & h_3 &= 2f(h_4,\theta), \\ h_4 &= f(h_1,\theta) + f(h_5,\theta), \qquad & h_5 &= 2f(h_1,\theta). \end{aligned} \tag{3.10}$$

Also, on the set I_3 Eq. (3.9) is equivalent to

$$\begin{aligned} h_1 &= f(h_2,\theta) + f(h_4,\theta), \qquad & h_2 &= 2f(h_4,\theta), \\ h_4 &= f(h_1,\theta) + f(h_6,\theta), \qquad & h_6 &= 2f(h_1,\theta). \end{aligned} \tag{3.11}$$

If we change variables in (3.10), then we can obtain (3.11), i.e., $h_1 \to h_1$, $h_4 \to h_4$, $h_5 \to h_6$, and $h_3 \to h_2$. on the set I_4, Eq. (3.9) is equivalent to

$$\begin{aligned} h_1 &= 2f(h_7,\theta), \qquad & h_2 &= f(h_1,\theta) + f(h_7,\theta), \\ h_7 &= f(h_2,\theta) + f(h_9,\theta), \qquad & h_9 &= 2f(h_2,\theta). \end{aligned} \tag{3.12}$$

If we change variables in (3.10), then we can obtain (3.12), i.e., $h_1 \to h_7$, $h_3 \to h_9$, $h_4 \to h_2$, and $h_5 \to h_1$. Hence, it is sufficient to solve Eq. (3.9) on I_2 (the cases I_3 and I_4 are similar).

We consider Eq. (3.9) on the invariant sets I_j, $j = 5, 6, 7, 8$. On I_5, this equation can be written as

$$\begin{aligned} h_1 &= f(h_1,\theta) + f(h_{10},\theta), \qquad & h_2 &= f(h_2,\theta) + f(h_{10},\theta), \\ h_3 &= f(h_1,\theta) + f(h_2,\theta), \qquad & h_{10} &= 2f(h_3,\theta). \end{aligned} \tag{3.13}$$

Also, on the set I_6, Eq. (3.9) is equivalent to

$$\begin{aligned} h_1 &= f(h_2,\theta) + f(h_3,\theta), \qquad & h_2 &= f(h_2,\theta) + f(h_4,\theta), \\ h_3 &= f(h_3,\theta) + f(h_4,\theta), \qquad & h_4 &= 2f(h_1,\theta). \end{aligned} \tag{3.14}$$

If we change variables in (3.13), then we can obtain (3.14), i.e., $h_1 \to h_2$, $h_2 \to h_3$, $h_3 \to h_1$, $h_{10} \to h_4$. On the set I_7, Eq. (3.9) is equivalent to

$$\begin{aligned} h_1 &= f(h_1,\theta) + f(h_7,\theta, \qquad & h_2 &= f(h_1,\theta) + f(h_3,\theta), \\ h_3 &= f(h_3,\theta) + f(h_7,\theta), \qquad & h_7 &= 2f(h_2,\theta). \end{aligned} \tag{3.15}$$

In this case, we change the variables as $h_1 \to h_1$, $h_2 \to h_3$, $h_3 \to h_2$, and $h_{10} \to h_7$. Similarly, on the set I_8 Eq. (3.9) is equivalent to

$$\begin{aligned} h_1 &= 2f(h_4,\theta), \qquad & h_4 &= f(h_5,\theta) + f(h_6,\theta), \\ h_5 &= f(h_1,\theta) + f(h_5,\theta), \qquad & h_6 &= f(h_1,\theta) + f(h_6,\theta). \end{aligned} \tag{3.16}$$

The correspondence between (3.13) and (3.16) is given by $h_1 \to h_6$, $h_2 \to h_5$, $h_3 \to h_4$, and $h_{10} \to h_1$. Thus, we can conclude that it is sufficient to solve Eq. (3.9) on I_5 (other cases are similar).

Theorem 4. *For the Ising model on Γ^2, the following assertions hold:*

1. *All K_0^*-weakly periodic Gibbs measures on the invariant sets I_1, I_9, and I_{10} are translation invariant.*
2. *Let $a_{\rm cr} \approx 1.69562077$, then for $\theta \in \left[-\infty, -\frac{1}{3}\right) \cup \left(\frac{1-a_{\rm cr}}{1+a_{\rm cr}}, \infty\right)$, there exists a K_0^*-weakly periodic (not translation-invariant) Gibbs measure on the invariant sets I_2, I_3, and I_4.*

Proof. 1. It is known that solutions of the system of equations (3.9) on the invariant set I_1 are translation invariant. Translation-invariant Gibbs measures for the Ising model are well studied (see [3]). We consider K_0^*-weakly periodic Gibbs measures on the invariant set I_9. Then the system of equations (3.9) can be written as

$$h_1 = f(h_1,\theta) + f(h_{10},\theta), \qquad h_3 = 2f(h_1,\theta), \qquad h_{10} = 2f(h_3,\theta). \tag{3.17}$$

Using the fact that $f(h,\theta)$ is monotonically increasing over h, we solve system (3.17). Let $\max\{h_1, h_3, h_{10}\} = h_1$, then

$$h_3 = 2f(h_1,\theta) \geqslant f(h_1,\theta) + f(h_{10},\theta) = h_1 \quad \Longrightarrow \quad h_1 = h_3.$$

Consequently, $h_3 = 2f(h_1,\theta) = 2f(h_3,\theta) = h_{10}$, i.e., $h_1 = h_3 = h_{10}$. If $\max\{h_1, h_3, h_{10}\} = h_3$, and we have

$$h_{10} = 2f(h_3,\theta) \geqslant 2f(h_1,\theta) = h_3 \quad \Longrightarrow \quad h_{10} = h_3.$$

Thus, we have

$$2f(h_1,\theta) = h_3 = h_{10} = 2f(h_3,\theta) \quad \Longrightarrow \quad h_1 = h_3 \quad \Longrightarrow \quad h_1 = h_3 = h_{10}.$$

Finally, let $\max\{h_1, h_3, h_{10}\} = h_{10}$, then we obtain

$$2f(h_3,\theta) = h_{10} \geqslant h_3 = 2f(h_1,\theta) \quad \Longrightarrow \quad h_3 \geqslant h_1,$$

whence

$$f(h_1,\theta) \geqslant f(h_{10},\theta) \quad \Longrightarrow \quad h_1 \geqslant h_{10}.$$

Namely, we obtain $h_1 = h_3 = h_{10}$.

We now consider K_0^*-weakly periodic Gibbs measures on the invariant set I_{10}. Then the system of equations (3.9) can be written as

$$\begin{aligned} h_1 &= f(h_3,\theta) + f(h_6,\theta), \quad h_2 = f(h_3,\theta) + f(h_4,\theta), \quad h_3 = f(h_4,\theta) + f(h_6,\theta), \\ h_4 &= f(h_5,\theta) + f(h_2,\theta), \quad h_5 = f(h_1,\theta) + f(h_2,\theta), \quad h_6 = f(h_1,\theta) + f(h_5,\theta). \end{aligned} \tag{3.18}$$

It suffices to consider the case $h_1 = \max\{h_1, h_2, h_3, h_4, h_5, h_6\}$, other cases are proved similarly:

$$f(h_3,\theta) + f(h_6,\theta) = h_1 \geqslant h_2 = f(h_3,\theta) + f(h_4,\theta) \quad \Longrightarrow \quad h_6 \geqslant h_4.$$

Similarly, we obtain that

$$h_1 \geqslant h_3 \quad \Longrightarrow \quad h_3 \geqslant h_, \qquad f(h_1,\theta) \geqslant f(h_5,\theta) \quad \Longrightarrow \quad h_5 \geqslant h_4.$$

From $f(h_3,\theta)+f(h_6,\theta)=h_1 \geqslant h_5 = f(h_1,\theta)+f(h_2,\theta)$, we deduce that

$$f(h_2,\theta) \leqslant \min\{f(h_3,\theta), f(h_6,\theta)\} \quad \Longrightarrow \quad h_2 \leqslant h_3, h_2 \leqslant h_6.$$

Because $f(h_3,\theta)+f(h_6,\theta) = h_1 \geqslant h_6 = f(h_1,\theta)+f(h_5,\theta)$, we have $h_5 \leqslant h_3$, $h_5 \leqslant h_6$, and $h_2 \leqslant h_3$, and hence $h_3 \leqslant h_6$. Finally, from $h_3 \geqslant h_5$ and consequently $h_2 \leqslant h_4$, we can conclude that

$$h_1 \geqslant h_6 \geqslant h_3 \geqslant h_5 \geqslant h_4 \geqslant h_2.$$

From the last inequality,

$$h_2 = f(h_3,\theta)+f(h_4,\theta) \geqslant f(h_5,\theta)+f(h_2,\theta) = h_4 \quad \Longrightarrow \quad h_2 = h_4$$

and $h_5 \geqslant h_3$, whence $h_5 = h_3$. We thus obtain $h_2 = h_4$ and $h_5 = h_3$ whence $h_1 = h_6$. As a result, it is easy to verify that

$$h_1 = h_2 = h_3 = h_4 = h_5 = h_6.$$

2. We consider Eq. (3.9) on the invariant set I_2 (it follows from the foregoing that the cases I_3 and I_4 are similar). On the invariant set I_2, Eq. (3.9) can be written as

$$\begin{aligned} h_1 &= f(h_3,\theta)+f(h_4,\theta), \qquad & h_3 &= 2f(h_4,\theta), \\ h_4 &= f(h_1,\theta)+f(h_5,\theta), \qquad & h_5 &= 2f(h_1,\theta). \end{aligned} \tag{3.19}$$

Using the fact that

$$f(h,\theta) = \frac{1}{2}\ln\frac{(1+\theta)e^{2h}+1-\theta}{(1-\theta)e^{2h}+1+\theta}$$

and setting $z_i = e^{2h_i}$, $i = \{1,2,\dots,9\}$, and

$$a = \frac{1-\theta}{1+\theta}, \qquad g(z) = \frac{z+a}{az+1}$$

we obtain the following system of equations instead: of (3.10):

$$z_1 = g(z_3)g(z_4), \qquad z_3 = g^2(z_4), \qquad z_4 = g(z_1)g(z_5), \qquad z_5 = g^2(z_1).$$

This system can be rewritten in the form

$$z_4 = g[g(g^2(z_4))\cdot g(z_4)]\cdot g[g^2(g(g^2(z_4))\cdot g(z_4))]. \tag{3.20}$$

Let $z_4 := x$, then after simple calculations we can rewrite (3.20) as

$$\begin{aligned} &(a+1)(x-1)(x+1)\big((a^2-a+1)x^2+(a^3-2a^2+3a)x+a^2-a+1\big)\times \\ &\times\big(a^2x^2+(a^2-2a+1)x+a^2\big)Q(x)=0, \end{aligned}$$

where

$$\begin{aligned} Q(x) := {}&(a^5+3a^4+4a^2-a+1)x^4+(6a^5+20a^3+6a)x^3+ \\ &+(2a^6-2a^5+24a^4+22a^2+2a)x^2+ \\ &+(6a^5+20a^3+6a)x+a^5+3a^4+4a^2-a+1. \end{aligned}$$

If $x = 1$, then we have $z_1 = z_3 = z_4 = z_5 = 1$, i.e., a translation-invariant solution. If $x = -1$, then $z_1 = z_4 = -1$ and $z_3 = z_5 = 1$, i.e., a periodic solution.

It is easy to verify that if $a \in (-\infty; -1] \cup [2; \infty)$, then the equation

$$(a^2 - a + 1)x^2 + (a^3 - 2a^2 + 3a)x + a^2 - a + 1 = 0$$

has at least one solution, i.e., a weakly periodic solution.

Let $a \in [-1 - \sqrt{2}; -1 + \sqrt{2}]$, then the equation

$$a^2x^2 + (a^2 - 2a + 1)x + a^2 = 0$$

has at least one solution, i.e., a weakly periodic solution. We set $x + 1/x = t$, then

$$Q(x) = (a^5 + 3a^4 + 4a^2 - a + 1)t^2 + (6a^5 + 20a^3 + 6a)t + 2a^6 - 4a^5 + 18a^4 + 14a^2 + 4a - 2 = 0.$$

For $a \in (-1 + \sqrt{2}, a_{\mathrm{cr}})$, after simple calculations, we obtain that the polynomial $Q(x)$ has at least one solution, i.e., weakly a periodic solution. Hence, for

$$a \in (-\infty; a_{\mathrm{cr}}) \cup [2; \infty) \quad \Longleftrightarrow \quad \theta \in \left[-\infty, -\frac{1}{3}\right) \cup \left(\frac{1 - a_{\mathrm{cr}}}{1 + a_{\mathrm{cr}}}, \infty\right),$$

there exists a K_0^*-weakly periodic (not translation-invariant) Gibbs measure on I_2.

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