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DIFFERENTIAL EQUATIONS FOR THE MAJORANA PARTICLE IN 3 + 1 AND 1 + 1 DIMENSIONS

S. De Vincenzo*

The relativistic wave equation considered to mathematically describe the Majorana particle is the Dirac equation with a real Lorentz scalar potential plus the Majorana condition. Certainly, depending on the representation that one uses, the resulting differential equation changes. It could be a real or a complex system of coupled equations, or it could even be a single complex equation for a single component of the entire wave function. Any of these equations or systems of equations could be referred to as a Majorana equation or Majorana system of equations because it can be used to describe the Majorana particle. For example, in the Weyl representation in $3 + 1$ *dimensions, we can have two nonequivalent explicitly covariant complex first-order equations; in contrast, in* $1 + 1$ *dimensions, we have a complex system of coupled equations. In any case, whichever equation or system of equations is used, the wave function that describes the Majorana particle in* $3 + 1$ *or* $1 + 1$ *dimensions is determined by four or two real quantities. The aim of this paper is to study and discuss all these issues from an algebraic standpoint, highlighting the similarities and differences that arise between these equations in the cases of* $3+1$ *and* $1+1$ *dimensions in the Dirac, Weyl, and Majorana representations. In addition, we rederive and use results that follow from a procedure already introduced by Case to obtain a two-component Majorana equation in* $3 + 1$ *dimensions. We for the first time introduce a similar procedure in* $1 + 1$ *dimensions and then use the obtained results.*

Keywords: relativistic quantum mechanics of a single particle, Dirac equation, equations for a Majorana particle, Dirac representation, Weyl representation, Majorana representation

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I would like to dedicate this paper to the memory of my beloved father Carmine De Vincenzo Di Fresca, who passed away unexpectedly on March 16, 2018. *That day something inside of me also died.*

1. Introduction

In general, the relativistic wave equation considered to mathematically describe a first-quantized Majorana particle (an electrically neutral fermion in $3+1$ dimensions that is its own antiparticle, i.e., a 3D Majorana particle) is the Dirac equation with a real Lorentz scalar potential together with the so-called Majorana condition [1], [2]. This condition requires that the Dirac wave function be equal to its charge-conjugate wave function, i.e., $\Psi = \Psi_{\mathrm{C}}$. This way of characterizing the Majorana particle can be implemented in $3+1$

*Escuela de Física, Facultad de Ciencias, Universidad Central de Venezuela, Caracas, Venezuela, e-mail: salvatore.devincenzo@ucv.ve.

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and $1+1$ dimensions, although in the latter case one would be describing the one-dimensional Majorana particle (i.e., the 1D Majorana particle).

As might be expected, the resulting differential equation depends on the representation that one uses when writing the Dirac equation and the Majorana condition, even without distinguishing between $3+1$ and $1+1$ dimensions. It could be a real or a complex system of coupled equations or even a single complex equation for a single component of the entire wave function, whose solution, together with the relation that emerges from the Majorana condition, would allow one to build the entire wave function [3]–[6].

Unexpectedly, the equation generally known in the literature as the Majorana equation is a relativistic wave equation similar to the free Dirac equation,

$$i\hat{\gamma}^{\mu}\,\partial_{\mu}\Psi - \frac{mc}{\hbar}\hat{1}\Psi = 0,$$

($\hat{1}$ is the identity matrix, which is a 4×4 matrix in $3+1$ dimensions and a 2×2 matrix in $1+1$ dimensions), but in addition to the Dirac wave function Ψ, the Majorana equation also includes the charge-conjugate wave function Ψ_{C}. The equation in question is usually written as [7]

$$i\hat{\gamma}^{\mu}\,\partial_{\mu}\Psi - \frac{mc}{\hbar}\hat{1}\Psi_{\mathrm{C}} = 0.$$

In writing the Majorana equation, it is important to remember that Ψ_{C} has the same transformation properties as Ψ under proper Lorentz transformations; hence, this equation is Lorentz covariant. Likewise, the Majorana condition is Lorentz covariant [3]. The Majorana equation could describe hypothetical particles that have been called Majoranons [7]. Clearly, the Majorana equation together with the Majorana condition can also lead to equations for the Majorana particle [1]. We note in passing that the Majorana equation can also admit a Lorentz scalar potential.

In general, when characterizing a Majorana particle with the help of complex four-component wave functions (in $3+1$ dimensions) or two-component wave functions (in $1+1$ dimensions), these components are not all independent because the Majorana condition must be satisfied. In the Majorana representation, the Majorana condition becomes the reality condition for the wave function, i.e., $\Psi = \Psi^*$; therefore, we can conclude that in $3+1$ or $1+1$ dimensions, the wave function that describes the Majorana particle has four or two independent real components, and these real components can be accommodated just in two or one independent complex components or component [3]. Then, to describe the Majorana particle in $3+1$ or $1+1$ dimensions, a four-component or two-component wave function is not absolutely necessary, i.e., a four-component or two-component scheme or formalism is not absolutely necessary; the Majorana particle can also be described by two-component or one-component wave functions, i.e., a two-component or one-component scheme or formalism in $3+1$ or $1+1$ dimensions is sufficient.

Returning to the issue of the equations for the Majorana particle that emerge from the Dirac equation and the Majorana condition when a representation is chosen, it is important to realize that in those cases where a complex first-order equation for the upper or lower single component of the entire wave function can be written (for example, in the Dirac representation), the respective lower or upper single component is automatically determined by the Majorana condition (depending on the space–time dimension, this single component can be a two-component or a one-component wave function). The entire wave function that describes the Majorana particle can be immediately constructed from these two components (the upper and the lower components). However, as explained above, the entire wave function is not absolutely needed to describe the Majorana particle; in fact, although the upper or lower component and its respective lower or upper component are not independent of each other, each of them satisfies its own equation, and either of these two can be considered as modeling the Majorana particle.

In $3+1$ dimensions, there exists an equation for the upper component and another for the lower component that stand out above the rest (in this case, these components are two-component wave functions); these are the ones that arise when the Weyl representation is used. In fact, each of these equations can also be written in an explicitly Lorentz-covariant form and can describe a specific type of a Majorana particle. These equations have been named the two-component Majorana equations and tend to the usual Weyl equations when the mass of the particle and the scalar potential go to zero [8], [9]. In $1+1$ dimensions and also in the Weyl representation, we have instead a complex system of coupled equations, i.e., in this case, we cannot write a first-order equation for any of the components of the wave function.

On the other hand, in $3+1$ and $1+1$ dimensions and in the Majorana representation, we also have a real system of coupled equations, and again, no first-order equation for any of the components of the wave function exists. In this paper, besides clarifying how the Majorana particle is described (in first quantization), we also attempt to show the different forms of the equations that can arise when describing it, both in $3+1$ and in $1+1$ dimensions. We believe that a detailed discussion on these issues could be useful and quite pertinent.

The article is organized as follows. In Sec. 2, we present the most basic results related to the relativistic wave equation commonly used to describe a Majorana particle, namely, the Dirac equation with a real Lorentz scalar potential. These results are presented for the cases of $3+1$ and $1+1$ dimensions.

In Sec. 3, we introduce the charge conjugation matrix in each of the representations that we consider. We use only three representations: Dirac (or the standard representation), Weyl (or the chiral or spinor representation), and Majorana. These are the most used ones in practice. In Sec. 3, the charge-conjugation matrices are obtained from a good formula that relates the matrices of charge conjugation in any two representations with the respective similarity matrix that changes the gamma matrices between these two representations. However, we specifically use the fact that in the Majorana representation, the charge-conjugation matrix is the identity matrix; thus, the charge-conjugation matrix in any representation is a function of the similarity matrix that takes us from that representation to the Majorana representation. Again, all these results are presented for the cases of $3+1$ and $1+1$ dimensions.

In Sec. 4, we first present the condition that defines the Majorana particle, i.e., the Majorana condition. We then present the equations and systems of equations that follow from the Dirac equation with a real Lorentz scalar potential and the restriction imposed by the Majorana condition. Again, we consider the Dirac, Weyl, and Majorana representations, in both $3+1$ and $1+1$ dimensions. We also highlight the similarities and differences that arise between these equations in a specific representation but in different space–time dimensions. In this regard, we note that in the Weyl representation, there is a deeper and unexpected difference between these equations. Likewise, we highlight in this section the procedure that in certain cases leads us to writing the entire wave function from the solution of a single equation and the Majorana condition. As an example, throughout Sec. 4, we also introduce various results related to the boundary conditions that can be imposed on the respective wave function that describes the 1D Majorana particle in a box. The results corresponding to the Weyl representation are introduced for the first time.

To complete our study, in Sec. 5 we first rederive in detail an algebraic procedure introduced some time ago by Case to obtain one of the two two-component Majorana equations from the Dirac equation in $3+1$ dimensions and the Majorana condition [8]. In fact, we also obtain the latter two equations after using the Weyl representation in our results, as expected. Then we also use the Dirac and Majorana representations in our results. In addition, we for the first time introduce an algebraic procedure somewhat analogous to that of Case, but in $1+1$ dimensions. Then we repeat the previous program by using the Weyl, Dirac, and Majorana representations. Throughout Sec. 5, we rederive the most important results presented in Sec. 4. Finally, in Sec. 6, we write our conclusions.

An extended version of this paper is available in preprint form [10].

2. Basic results

The equation for a single Dirac particle in $3+1$ dimensions, in a real-valued Lorentz scalar potential $V_{\mathrm{S}} = V_{\mathrm{S}}(x, y, z, t) = V_{\mathrm{S}}(\mathbf{r}, t)$,

$$\left[i\hat{\gamma}^{\mu}\,\partial_{\mu} - \frac{1}{\hbar c}(V_{\mathrm{S}} + mc^2)\hat{1}_4 \right]\Psi = 0, \tag{1}$$

is satisfied by the (generally) complex Dirac wave function of four components Ψ. The matrix $\hat{1}_4$ is the 4-dimensional unit matrix. The matrices $\hat{\gamma}^{\mu} = (\hat{\gamma}^0, \hat{\gamma}^j) \equiv (\hat{\beta}, \hat{\beta}\hat{\alpha}_j)$, with $\mu = 0, j$ and $j = 1, 2, 3$ are the gamma matrices, and the matrices $\hat{\alpha}_j$ and $\hat{\beta}$ are the Dirac matrices. The latter are Hermitian and satisfy the relations $\{\hat{\alpha}_j, \hat{\beta}\} \equiv \hat{\alpha}_j\hat{\beta} + \hat{\beta}\hat{\alpha}_j = \hat{0}_4$ ($\hat{0}_4$ is the 4-dimensional zero matrix), $\{\hat{\alpha}_j, \hat{\alpha}_k\} = 2\delta_{jk}\hat{1}_4$ and $\hat{\beta}^2 = \hat{1}_4$ (δ_{jk} is the Kronecker delta). Therefore, $\{\hat{\gamma}^{\mu}, \hat{\gamma}^{\nu}\} = 2g^{\mu\nu}\hat{1}_4$, where $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the metric tensor, and $(\hat{\gamma}^{\mu})^{\dagger} = \hat{\gamma}^0\hat{\gamma}^{\mu}\hat{\gamma}^0$ ($\dagger$ denotes the Hermitian conjugate, or the adjoint, of a matrix and an operator, as usual). The last two relations imply that the gamma matrices are unitary, but only $\hat{\gamma}^0$ is Hermitian, while $\hat{\gamma}^j$, $j = 1, 2, 3$ is anti-Hermitian.

Multiplying Eq. (1) from the left by the operator $i\hat{\gamma}^{\mu}\,\partial_{\mu} + (1/\hbar c)(V_{\mathrm{S}} + mc^2)\hat{1}_4$ leads to the second-order equation

$$\left[\hat{1}_4\,\partial^{\mu}\,\partial_{\mu} + \frac{1}{\hbar c}(\partial_{\mu}V_{\mathrm{S}})i\hat{\gamma}^{\mu} + \frac{(V_{\mathrm{S}} + mc^2)^2}{\hbar^2 c^2}\hat{1}_4 \right]\Psi = 0. \tag{2}$$

We note that the term containing $\hat{\gamma}^{\mu}$ is not generally a diagonal matrix. In the free case ($V_{\mathrm{S}} = \mathrm{const}$), all the components satisfy the same equation, namely, the Klein–Fock–Gordon equation with the mass $mc^2 + \mathrm{const}$.

The Dirac equation, written in its canonical form, is

$$\left(i\hbar\hat{1}_4\,\frac{\partial}{\partial t} - \widehat{H} \right)\Psi = 0, \tag{3}$$

where the Hamiltonian operator $\widehat{H}$ is

$$\widehat{H} = -i\hbar c\left(\hat{\alpha}_1\,\frac{\partial}{\partial x} + \hat{\alpha}_2\,\frac{\partial}{\partial y} + \hat{\alpha}_3\,\frac{\partial}{\partial z} \right) + (V_{\mathrm{S}} + mc^2)\hat{\beta}. \tag{4}$$

Equation (3) is obtained from Eq. (1) by multiplying it by the matrix $\hbar c\hat{\gamma}^0 = \hbar c\hat{\beta}$ from the left, and using the relations $(\hat{\gamma}^0)^2 = \hat{1}_4$ and $\hat{\gamma}^0\hat{\gamma}^j = \hat{\alpha}_j$.

Likewise, Eq. (1) is also satisfied by the charge-conjugate wave function Ψ_{C}, but this yields

$$\widehat{S}_{\mathrm{C}}(-\hat{\gamma}^{\mu})^*(\widehat{S}_{\mathrm{C}})^{-1} = \hat{\gamma}^{\mu}, \qquad \Psi_{\mathrm{C}} \equiv \widehat{S}_{\mathrm{C}}\Psi^*, \tag{5}$$

and $\widehat{S}_{\mathrm{C}}$ is the charge-conjugation matrix (the asterisk * represents the complex conjugate) [11], [12]. This matrix is obviously determined up to a phase factor. As noted above, the matrices $\hat{\gamma}^{\mu}$ are unitary. More specifically, this is because $\{\hat{\gamma}^0, \hat{\gamma}^j\} = \hat{0}_4$ and $(\hat{\gamma}^0)^2 = -(\hat{\gamma}^j)^2 = \hat{1}_4$ and because $(\hat{\gamma}^0)^{\dagger} = \hat{\gamma}^0\hat{\gamma}^0\hat{\gamma}^0$ and $(\hat{\gamma}^j)^{\dagger} = \hat{\gamma}^0\hat{\gamma}^j\hat{\gamma}^0$. Likewise, the matrices $(-\hat{\gamma}^{\mu})^*$ are also unitary. In effect, $g^{\mu\nu}$ is real; thus, we can write $(-\hat{\gamma}^{\mu})^*(-\hat{\gamma}^{\nu})^* + (-\hat{\gamma}^{\nu})^*(-\hat{\gamma}^{\mu})^* = 2g^{\mu\nu}\hat{1}_4$, and $((-\hat{\gamma}^{\mu})^*)^{\dagger} = (-\hat{\gamma}^0)^*(-\hat{\gamma}^{\mu})^*(-\hat{\gamma}^0)^*$; therefore, $((-\hat{\gamma}^0)^*)^{\dagger} = ((-\hat{\gamma}^0)^*)^{-1}$ and $((-\hat{\gamma}^j)^*)^{\dagger} = ((-\hat{\gamma}^j)^*)^{-1}$. Thus, because the matrices $\hat{\gamma}^{\mu}$ and $(-\hat{\gamma}^{\mu})^*$ are linked via the relation in the left-hand side of Eq. (5), the matrix $\widehat{S}_{\mathrm{C}}$ can be chosen to be unitary (for more details on this result, see, e.g., Ref. [13], p. 899). For example, in the Majorana representation, we have that $\Psi_{\mathrm{C}} = \Psi^*$, i.e., $\widehat{S}_{\mathrm{C}} = \hat{1}_4$, and that $\hat{\gamma}^{\mu} = (-\hat{\gamma}^{\mu})^* = i\,\mathrm{Im}(\hat{\gamma}^{\mu})$ (by virtue of Eq. (5)), i.e., all entries of the gamma matrices are purely imaginary. Also, we have $i\hat{\gamma}^{\mu} = (i\hat{\gamma}^{\mu})^* = \mathrm{Re}(i\hat{\gamma}^{\mu})$, and hence the operator acting on Ψ in Eq. (1) is real. The last condition implies only that Eq. (1) could have real-valued solutions. In the same way, Eq. (2) could also have real solutions.

All the equations and relations that we have written so far in $3+1$ dimensions and which are dependent on Greek and Latin indices maintain their form in $1+1$ dimensions. Certainly, these indices are now restricted to $\mu, \nu, \ldots = 0, 1$, and $j, k, \ldots = 1$. The Dirac wave function Ψ then has only two components and satisfies Eqs. (1)–(3), with $\hat{1}_4 \rightarrow \hat{1}_2$ ($\hat{1}_2$ is the 2×2 identity matrix) and $V_\mathrm{S} = V_\mathrm{S}(x,t)$. The gamma matrices are just $\hat{\gamma}^0 \equiv \hat{\beta}$ and $\hat{\gamma}^1 \equiv \hat{\beta}\hat{\alpha}$, where the (Hermitian) Dirac matrices $\hat{\alpha}$ and $\hat{\beta}$ satisfy the relations $\{\hat{\alpha}, \hat{\beta}\} = \hat{0}_2$ ($\hat{0}_2$ is the 2-dimensional zero matrix), $\hat{\alpha}^2 = \hat{1}_2$ and $\hat{\beta}^2 = \hat{1}_2$. Thus, $\{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} = 2g^{\mu\nu}\hat{1}_2$, where $g^{\mu\nu} = \mathrm{diag}(1,-1)$, and $(\hat{\gamma}^\mu)^\dagger = \hat{\gamma}^0\hat{\gamma}^\mu\hat{\gamma}^0$. As before, the two gamma matrices are unitary, but $\hat{\gamma}^0$ is Hermitian, and $\hat{\gamma}^1$ is anti-Hermitian. Likewise, the Hamiltonian operator for Dirac equation (3) is simply given by

$$\hat{H} = -i\hbar c\hat{\alpha}\,\frac{\partial}{\partial x} + (V_\mathrm{S} + mc^2)\hat{\beta}. \tag{6}$$

3. Charge conjugation in the Dirac, Weyl, and Majorana representations

As is well known, in choosing a representation one is choosing a set of Dirac and gamma matrices that satisfy the Clifford relations. As was demonstrated, for instance, in Ref. [6], if one has written the charge-conjugation matrix in a representation, e.g., $\widehat{S}_\mathrm{C}$, then one can write it in any other representation, say, $\widehat{S}'_\mathrm{C}$, using the relation

$$\widehat{S}'_\mathrm{C} = \widehat{S}\widehat{S}_\mathrm{C}(\widehat{S}^*)^{-1}, \tag{7}$$

where $\widehat{S}$ is precisely the unitary similarity matrix that allows mapping the unitary gamma matrices between these two representations: $\hat{\gamma}^{\mu\prime} = \widehat{S}\hat{\gamma}^\mu\widehat{S}^{-1}$. The result in Eq. (7) follows simply because the wave functions Ψ and Ψ_C transform under $\widehat{S}$ as $\Psi' = \widehat{S}\Psi$ and $\Psi'_\mathrm{C} = \widehat{S}\Psi_\mathrm{C}$, but in each representation we also have $\Psi_\mathrm{C} \equiv \widehat{S}_\mathrm{C}\Psi^*$ and $\Psi'_\mathrm{C} \equiv \widehat{S}'_\mathrm{C}(\Psi')^*$. Obviously, if we change the phase factor of the matrix $\widehat{S}_\mathrm{C}$, the matrix $\widehat{S}'_\mathrm{C}$ obtained from Eq. (7) changes by a factor that is also a phase. However, all the matrices involved in Eq. (7) are always determined up to an arbitrary phase factor. If we specify the formula in Eq. (7) to the case where $\widehat{S}_\mathrm{C}$ is written in an arbitrary representation and $\widehat{S}'_\mathrm{C}$ is written in the Majorana representation, i.e., $\widehat{S}'_\mathrm{C} = \hat{1}_4$ in $3+1$ dimensions, or $\widehat{S}'_\mathrm{C} = \hat{1}_2$ in $1+1$ dimensions, then we obtain the result

$$\widehat{S}_\mathrm{C} = \widehat{S}^\dagger\widehat{S}^*, \tag{8}$$

where $\widehat{S}$ is the unitary matrix that takes us from that arbitrary representation to the Majorana representation. From Eq. (8), and because $\widehat{S}_\mathrm{C}$ is a unitary matrix, we deduce that $(\widehat{S}_\mathrm{C})^{-1} = (\widehat{S}_\mathrm{C})^*$. This can also be obtained just by requiring that $(\Psi_\mathrm{C})_\mathrm{C} = \Psi$.

Table 1

Representation	$\hat{\boldsymbol{\alpha}}$	$\hat{\beta} \equiv \hat{\gamma}^0$	$\hat{\beta}\hat{\boldsymbol{\alpha}} \equiv \hat{\boldsymbol{\gamma}}$	$\hat{\gamma}^5 \equiv i\hat{\gamma}^0\hat{\gamma}^1\hat{\gamma}^2\hat{\gamma}^3$	$\widehat{S}_\mathrm{C} = \widehat{S}^\dagger\widehat{S}^*$
Dirac	$\hat{\sigma}_x \otimes \hat{\boldsymbol{\sigma}}$	$\hat{\sigma}_z \otimes \hat{1}_2$	$i\hat{\sigma}_y \otimes \hat{\boldsymbol{\sigma}}$	$\hat{\sigma}_x \otimes \hat{1}_2$	$-i\hat{\sigma}_y \otimes \hat{\sigma}_y$
Weyl	$\hat{\sigma}_z \otimes \hat{\boldsymbol{\sigma}}$	$-\hat{\sigma}_x \otimes \hat{1}_2$	$i\hat{\sigma}_y \otimes \hat{\boldsymbol{\sigma}}$	$\hat{\sigma}_z \otimes \hat{1}_2$	$-i\hat{\sigma}_y \otimes \hat{\sigma}_y$
Majorana	$\hat{\alpha}_1 = -\hat{\sigma}_x \otimes \hat{\sigma}_x$ $\hat{\alpha}_2 = \hat{\sigma}_z \otimes \hat{1}_2$ $\hat{\alpha}_3 = -\hat{\sigma}_x \otimes \hat{\sigma}_z$	$\hat{\sigma}_x \otimes \hat{\sigma}_y$	$\hat{\gamma}^1 = i\hat{1}_2 \otimes \hat{\sigma}_z$ $\hat{\gamma}^2 = -i\hat{\sigma}_y \otimes \hat{\sigma}_y$ $\hat{\gamma}^3 = -i\hat{1}_2 \otimes \hat{\sigma}_x$	$\hat{\sigma}_z \otimes \hat{\sigma}_y$	$\hat{1}_2 \otimes \hat{1}_2$

The results relating to the representations usually identified as Dirac, Weyl, and Majorana in $3+1$ dimensions are given in Table 1. The table also shows the charge-conjugation matrix $\widehat{S}_\mathrm{C}$ in each representation derived from Eq. (8), the respective matrices $\widehat{S}$ being

$$\widehat{S} = \frac{1}{\sqrt{2}}(\hat{\sigma}_x \otimes \hat{\sigma}_y + \hat{\sigma}_z \otimes \hat{1}_2), \tag{9}$$

which permits us to pass from the Dirac representation to the Majorana representation, and

$$\widehat{S} = \frac{1}{2}(\hat{\sigma}_x \otimes \hat{\sigma}_y + \hat{\sigma}_z \otimes \hat{\sigma}_y + \hat{\sigma}_z \otimes \hat{1}_2 - \hat{\sigma}_x \otimes \hat{1}_2), \tag{10}$$

which permits us to pass from the Weyl representation to the Majorana representation. Obviously, the matrix $\widehat{S} = \hat{1}_4 = \hat{1}_2 \otimes \hat{1}_2$ permits us to pass from the Majorana representation to the Majorana representation itself. We note that in $3+1$ dimensions, the charge-conjugation matrix in the Dirac representation is equal to the charge-conjugation matrix in the Weyl representation (up to a phase factor). For completeness, the matrix $\widehat{S}$ that allows us to pass precisely from the Dirac representation to that of Weyl is also given here:

$$\widehat{S} = \frac{1}{\sqrt{2}}(\hat{1}_2 \otimes \hat{1}_2 + i\hat{\sigma}_y \otimes \hat{1}_2). \tag{11}$$

This matrix links the matrices $\widehat{S}_{\mathrm{C}}$ (in the Dirac representation) and $\widehat{S}'_{\mathrm{C}}$ (in the Weyl representation) also via Eq. (7).

In reading Table 1, the following definitions should be considered: $\hat{\boldsymbol{\alpha}} \equiv (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)$, $\hat{\boldsymbol{\gamma}} \equiv (\hat{\gamma}^1, \hat{\gamma}^2, \hat{\gamma}^3)$, and the usual Pauli matrices are $\hat{\boldsymbol{\sigma}} \equiv (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$. Also, $\otimes$ denotes the Kronecker product of matrices

$$\widehat{A} \otimes \widehat{B} \equiv \begin{bmatrix} a_{11}\widehat{B} & \dots & a_{1n}\widehat{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\widehat{B} & \dots & a_{mn}\widehat{B} \end{bmatrix} \tag{12}$$

which satisfies the following properties: 1) $(\widehat{A} \otimes \widehat{B})(\widehat{C} \otimes \widehat{D}) = (\widehat{A}\widehat{C} \otimes \widehat{B}\widehat{D})$; 2) $(\widehat{A} \otimes \widehat{B})^* = \widehat{A}^* \otimes \widehat{B}^*$; and 3) $(\widehat{A} \otimes \widehat{B})^\dagger = \widehat{A}^\dagger \otimes \widehat{B}^\dagger$ (see, e.g., Ref. [14]). We note that we here have $\widehat{S}_{\mathrm{C}} = -\hat{\gamma}^2$ in both the Dirac and Weyl representations. However, when considering these two representations, it is also common to write $\widehat{S}_{\mathrm{C}} = +\hat{\gamma}^2$, and in particle physics, it is more common to set $\widehat{S}_{\mathrm{C}} = +i\hat{\gamma}^2$ and $\widehat{S}_{\mathrm{C}} = -i\hat{\gamma}^2$.

Table 2

Representation	$\hat{\alpha}$	$\hat{\beta} \equiv \hat{\gamma}^0$	$\hat{\beta}\hat{\alpha} \equiv \hat{\gamma}^1$	$\widehat{\Gamma}^5 \equiv -i\hat{\gamma}^5 = \hat{\gamma}^0\hat{\gamma}^1$	$\widehat{S}_{\mathrm{C}} = \widehat{S}^\dagger \widehat{S}^*$
Dirac	$\hat{\sigma}_x$	$\hat{\sigma}_z$	$i\hat{\sigma}_y$	$\hat{\sigma}_x$	$-i\hat{\sigma}_x$
Weyl	$\hat{\sigma}_z$	$\hat{\sigma}_x$	$-i\hat{\sigma}_y$	$\hat{\sigma}_z$	$-i\hat{\sigma}_z$
Majorana	$\hat{\sigma}_x$	$\hat{\sigma}_y$	$-i\hat{\sigma}_z$	$\hat{\sigma}_x$	$\hat{1}_2$

In the same way, the results relating to the representations commonly considered as the Dirac, Weyl, and Majorana representations in $1+1$ dimensions are given in Table 2. The table also shows the charge-conjugation matrix $\widehat{S}_{\mathrm{C}}$ in each representation calculated from Eq. (8). The respective matrices $\widehat{S}$ are

$$\widehat{S} = \frac{1}{\sqrt{2}}(\hat{1}_2 + i\hat{\sigma}_x), \tag{13}$$

which permits us to pass from the Dirac representation to the Majorana representation, and

$$\widehat{S} = \frac{1}{2}(i\hat{1}_2 + \hat{\sigma}_x + \hat{\sigma}_y + \hat{\sigma}_z), \tag{14}$$

which permits us to pass from the Weyl representation to the Majorana representation. We note that in $1+1$ dimensions, the charge-conjugation matrix in the Dirac representation is not equal to the charge-conjugation matrix in the Weyl representation. For completeness, the matrix $\widehat{S}$, which allows us to pass precisely from the Dirac representation to that of Weyl, is also given here:

$$\widehat{S} = \frac{1}{\sqrt{2}}(\hat{\sigma}_x + \hat{\sigma}_z). \tag{15}$$

This matrix links the matrices $\widehat{S}_{\mathrm{C}}$ (in the Dirac representation) and $\widehat{S}'_{\mathrm{C}}$ (in the Weyl representation) also via Eq. (7).

4. Equations for the Majorana single particle. I

The condition that defines a Majorana particle, called the Majorana condition, is given by

$$\Psi = \Psi_{\mathrm{C}} = \widehat{S}_{\mathrm{C}}\,\Psi^{*}. \tag{16}$$

It is important to note that the wave functions $\Psi \equiv [\mathrm{top}, \mathrm{bottom}]^{\mathrm{T}}$ and $\Psi_{\mathrm{C}} \equiv [\mathrm{top}_{\mathrm{C}}, \mathrm{bottom}_{\mathrm{C}}]^{\mathrm{T}}$ (where the words "top" and "bottom" indicate the upper and lower components of the wave function) are similarly transformed under proper Lorentz transformations (T represents the transpose of a matrix). Thus, the upper components of these two wave functions, as well as the lower components, transform similarly. Obviously, this is true in any representation and is unrelated to the Majorana condition. If in addition the Majorana condition holds in Eq. (16), then the upper components of Ψ and Ψ_{C}, as well as their lower components, are equal. In passing, the Majorana condition is sometimes written as $\Psi = \omega\Psi_{\mathrm{C}}$, where ω is an arbitrary unobservable phase factor, and it is still a Lorentz covariant condition [3], as expected. Below, we present the equations or systems of equations for the Majorana particle in the Dirac, Weyl, and Majorana representations in both $3+1$ and $1+1$ dimensions.

4.1. Dirac representation.

3 + 1 dimensions. We write the four-component Dirac wave function (or Dirac spinor) Ψ in the form

$$\Psi \equiv \begin{bmatrix} \varphi \\ \chi \end{bmatrix}, \tag{17}$$

where the upper two-component wave function can be written as $\varphi \equiv [\varphi_1\ \ \varphi_2]^{\mathrm{T}}$ and the lower one as $\chi \equiv [\chi_1\ \ \chi_2]^{\mathrm{T}}$. In $3+1$ dimensions, a two-component wave function such as Ψ is also called a bispinor. The Dirac equation takes the form

$$i\hbar\,\frac{\partial}{\partial t}\begin{bmatrix}\varphi \\ \chi\end{bmatrix} = \widehat{H}\begin{bmatrix}\varphi \\ \chi\end{bmatrix} = \begin{bmatrix}(V_{\mathrm{S}}+mc^2)\hat{1}_2 & -i\hbar c\hat{\boldsymbol{\sigma}}\cdot\nabla \\ -i\hbar c\hat{\boldsymbol{\sigma}}\cdot\nabla & -(V_{\mathrm{S}}+mc^2)\hat{1}_2\end{bmatrix}\begin{bmatrix}\varphi \\ \chi\end{bmatrix}. \tag{18}$$

Majorana condition (16) for the Dirac wave function imposes the following relation among the components of Ψ:

$$\chi = \hat{\sigma}_y\varphi^{*} \equiv \chi_{\mathrm{C}} \qquad (\Leftrightarrow \varphi = -\hat{\sigma}_y\chi^{*} \equiv \varphi_{\mathrm{C}}). \tag{19}$$

Substituting this χ in Eq. (18), we are left with an equation for the two-component wave function φ:

$$i\hbar\hat{1}_2\,\frac{\partial}{\partial t}\varphi = -i\hbar c\hat{\boldsymbol{\sigma}}\cdot\nabla(\hat{\sigma}_y\varphi^{*}) + (V_{\mathrm{S}}+mc^2)\hat{1}_2\varphi. \tag{20}$$

Certainly, two equations arise from the last replacement: one is Eq. (20) and the other is an equation that can also be obtained from Eq. (20) by making the substitutions $\varphi \to \hat{\sigma}_y\varphi^{*}$, $\hat{\sigma}_y\varphi^{*} \to \varphi$, and $V_{\mathrm{S}}+mc^2 \to -(V_{\mathrm{S}}+mc^2)$. It can be algebraically shown that the latter equation and Eq. (20) are equivalent.

Alternatively, if we substitute φ (from Eq. (19)) in Eq. (18), we obtain the following equation for the two-component wave function χ:

$$i\hbar\hat{1}_2\,\frac{\partial}{\partial t}\chi = -i\hbar c\hat{\boldsymbol{\sigma}}\cdot\nabla(-\hat{\sigma}_y\chi^{*}) - (V_{\mathrm{S}}+mc^2)\hat{1}_2\chi. \tag{21}$$

Again, by making the last replacement, two equations arise: one is Eq. (21), and the other is an equation that can also be obtained from Eq. (21) by the replacements $\chi \to -\hat{\sigma}_y\chi^{*}$, $-\hat{\sigma}_y\chi^{*} \to \chi$, and $-(V_{\mathrm{S}}+mc^2) \to V_{\mathrm{S}}+mc^2$. Again, it can be algebraically shown that the latter equation and Eq. (21) are absolutely equivalent. Clearly, if we assume that the wave function that describes the Majorana particle has four components, it is sufficient to solve at least one of the two last (decoupled) two-component equations, namely, Eqs. (20) and (21). This is because φ and χ are algebraically related by Eq. (19). Thus, Eq. (20) (or Eq. (21)) alone can be regarded as a two-component equation that models the 3D Majorana particle.

1 + 1 dimensions. We write the two-component Dirac wave function Ψ in the form given in Eq. (17), but in this case φ and χ are functions of a single component. The Dirac equation (Eq. (3)) with Hamiltonian (6) take the form

$$i\hbar\frac{\partial}{\partial t}\begin{bmatrix}\varphi\\ \chi\end{bmatrix} = \widehat{H}\begin{bmatrix}\varphi\\ \chi\end{bmatrix} = \begin{bmatrix}V_{\mathrm{S}}+mc^2 & -i\hbar c\,\dfrac{\partial}{\partial x}\\ -i\hbar c\,\dfrac{\partial}{\partial x} & -(V_{\mathrm{S}}+mc^2)\end{bmatrix}\begin{bmatrix}\varphi\\ \chi\end{bmatrix}. \tag{22}$$

The Majorana condition (Eq. (16)) for the Dirac wave function imposes the following relation between the two components of Ψ:

$$\chi = -i\varphi^* \equiv \chi_{\mathrm{C}} \qquad (\Leftrightarrow \varphi = -i\chi^* \equiv \varphi_{\mathrm{C}}). \tag{23}$$

Substituting χ from (23) in Eq. (22), we are left with an equation for the one-component wave function φ:

$$i\hbar\frac{\partial}{\partial t}\varphi = -i\hbar c\frac{\partial}{\partial x}(-i\varphi^*) + (V_{\mathrm{S}}+mc^2)\varphi \tag{24}$$

(the other equation that results from the previous substitution in Eq. (22) is essentially the complex conjugate equation of Eq. (24)). In contrast to $3+1$ dimensions, the equation for the lower component χ is equal to the equation for the upper component (Eq. (24)) but with the replacement $V_{\mathrm{S}}+mc^2 \to -(V_{\mathrm{S}}+mc^2)$. In any case, it is sufficient to solve at least one of these one-component equations because φ and χ are algebraically linked via Eq. (23). Thus, for example, it can be said that Eq. (24) (or the equation for χ) alone models the Majorana particle in $1+1$ dimensions [6].

On the other hand, the only four boundary conditions that φ can support when the 1D Majorana particle is inside an impenetrable box (we call them confining boundary conditions) were encountered in Ref. [15]. Likewise, these conditions were found in Ref. [6], but it was shown there that these are just the conditions that can arise mathematically from the general linear boundary condition used in the MIT bag model for a hadronic structure in $1+1$ dimensions (the latter four boundary conditions are also subject to the Majorana condition). In relation to the MIT bag model, see, e.g., Ref. [6] and the references therein, and also Ref. [16]. Specifically, for a box of size L with ends, for example, at $x=0$ and $x=L$, the four confining boundary conditions can be written in the form $f(0,t)=g(L,t)=0$, where f and g are the functions $\mathrm{Im}\,\varphi$ and $\mathrm{Re}\,\varphi$. This is a nice result because the entire two-component Dirac wave function does not support this type of boundary condition at the walls of the box [17]. In addition, two one-parameter families of nonconfining boundary conditions were also found in Ref. [6]. It is even possible (by taking some convenient limits) that these two families also include the four confining boundary conditions. Consequently, these two families actually make up the most general set of boundary conditions for the 1D Majorana particle in a box; see Eq. (93) in Ref. [6]. In detail, we write below, for the first time, these two families of boundary conditions but in the Weyl representation.

Clearly, in the Dirac representation, the procedure for finding single equations for the Majorana particle is similar in $3+1$ and $1+1$ dimensions. However, this representation is not so commonly used to write the equation for the Majorana particle, be it in $3+1$ or in $1+1$ dimensions; rather, the Weyl representation is used (at least in $3+1$ dimensions).

4.2. Weyl representation.

3 + 1 dimensions. We write the four-component Dirac wave function (or spinor) Ψ as

$$\Psi \equiv \begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}, \tag{25}$$

where the upper (lower) two-component wave function can be written as $\varphi_1 \equiv [\xi_1 \ \xi_2]^{\mathrm{T}}$ ($\varphi_2 \equiv [\xi_3 \ \xi_4]^{\mathrm{T}}$). Dirac equation (3) takes the form

$$i\hbar \frac{\partial}{\partial t} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \widehat{H} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} -i\hbar c \hat{\boldsymbol{\sigma}} \cdot \nabla & -(V_{\mathrm{S}} + mc^2)\hat{1}_2 \\ -(V_{\mathrm{S}} + mc^2)\hat{1}_2 & +i\hbar c \hat{\boldsymbol{\sigma}} \cdot \nabla \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}. \tag{26}$$

Majorana condition (16) imposed on Ψ leads to the following relation among its components:

$$\varphi_2 = \hat{\sigma}_y \varphi_1^* \equiv (\varphi_2)_{\mathrm{C}} \qquad (\Leftrightarrow \varphi_1 = -\hat{\sigma}_y \varphi_2^* \equiv (\varphi_1)_{\mathrm{C}}). \tag{27}$$

Substituting this φ_2 in Eq. (26), we are left with an equation for the two-component wave function φ_1:

$$i\hbar \hat{1}_2 \frac{\partial}{\partial t} \varphi_1 = -i\hbar c \hat{\boldsymbol{\sigma}} \cdot \nabla \varphi_1 - (V_{\mathrm{S}} + mc^2)\hat{\sigma}_y \varphi_1^*. \tag{28}$$

Instead, if we substitute φ_1 (from Eq. (27)) in Eq. (26), we obtain the following equation for the two-component wave function φ_2:

$$i\hbar \hat{1}_2 \frac{\partial}{\partial t} \varphi_2 = +i\hbar c \hat{\boldsymbol{\sigma}} \cdot \nabla \varphi_2 + (V_{\mathrm{S}} + mc^2)\hat{\sigma}_y \varphi_2^*. \tag{29}$$

Again, to obtain the wave function Ψ (25), it is sufficient to first solve Eq. (28) (Eq. (29)) to obtain φ_1 (φ_2) and then obtain φ_2 (φ_1) by using Majorana condition (27). We note that the substitution that gave us Eq. (28) for φ_1 also generates another equation, namely, Eq. (29) for $\hat{\sigma}_y \varphi_1^*$ (these two equations are algebraically equivalent). Likewise, the substitution that gave us Eq. (29) for φ_2 also generates another equation, namely, Eq. (28) for $-\hat{\sigma}_y \varphi_2^*$ (again, both equations are equivalent). Thus, the wave function $\varphi_1 = \varphi_1(\mathbf{r}, t)$ satisfies Eq. (28), but unexpectedly, $i\hat{\sigma}_y \varphi_1^*(-\mathbf{r}, t)$ also satisfies Eq. (28) (if the relation $V_{\mathrm{S}}(\mathbf{r}, t) = V_{\mathrm{S}}(-\mathbf{r}, t)$ holds). Similarly, the wave function $\varphi_2 = \varphi_2(\mathbf{r}, t)$ satisfies Eq. (29), but $-i\hat{\sigma}_y \varphi_2^*(-\mathbf{r}, t)$ also satisfies Eq. (29) (and again, the scalar potential must be an even function of $\mathbf{r}$).

Setting $mc^2 = V_{\mathrm{S}} = 0$ in Eq. (26), we obtain two (decoupled) equations

$$i\hbar \hat{1}_2 \frac{\partial}{\partial t} \varphi_1 = -i\hbar c \hat{\boldsymbol{\sigma}} \cdot \nabla \varphi_1, \qquad i\hbar \hat{1}_2 \frac{\partial}{\partial t} \varphi_2 = +i\hbar c \hat{\boldsymbol{\sigma}} \cdot \nabla \varphi_2, \tag{30}$$

These are the well-known Weyl equations. For instance, the first of these two-component equations can be assigned to the (right-handed, or right-helical) massless antineutrino, while the second can be assigned to the (left-handed, or left-helical) massless neutrino (even though it is possible that only one of these two equations is sufficient for the description of a massless fermion, in which case one is led to the so-called Weyl theory [12], [18]). On the other hand, setting $mc^2 = V_{\mathrm{S}} = 0$ in Eqs. (28) and (29), we obtain two equations (in fact, the same equations as given in Eq. (30)), but this time, φ_1 and φ_2 are related by the Majorana condition in Eq. (27). In fact, the four-component Majorana wave functions corresponding to the two-component wave functions φ_1 and φ_2 are given by

$$\Psi = \begin{bmatrix} \varphi_1 \\ \hat{\sigma}_y \varphi_1^* \end{bmatrix} (= \Psi_{\mathrm{C}}), \qquad \Psi = \begin{bmatrix} -\hat{\sigma}_y \varphi_2^* \\ \varphi_2 \end{bmatrix} (= \Psi_{\mathrm{C}}). \tag{31}$$

The four-component Weyl wave functions corresponding to the two-component wave functions φ_1 and φ_2 are given by

$$\Psi = \begin{bmatrix} \varphi_1 \\ 0 \end{bmatrix}, \qquad \Psi = \begin{bmatrix} 0 \\ \varphi_2 \end{bmatrix}. \tag{32}$$

In $3+1$ dimensions, the Weyl representation is definitely the most used. As we show in Sec. 5, two-component equations (28) and (29) can also be written explicitly in covariant form, and each of them can describe a 3D Majorana particle.

1 + 1 dimensions. We write the two-component Dirac wave function Ψ in the form given in Eq. (25), but in this case, φ_1 and φ_2 are wave functions of a single component. Dirac equation (3) takes the form

$$i\hbar\,\frac{\partial}{\partial t}\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix} = \widehat{H}\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix} = \begin{bmatrix}-i\hbar c\,\dfrac{\partial}{\partial x} & V_{\rm S}+mc^2\\ V_{\rm S}+mc^2 & +i\hbar c\,\dfrac{\partial}{\partial x}\end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}. \tag{33}$$

Majorana condition (16) imposed upon Ψ gives the relations

$$\varphi_1 = -i\varphi_1^* \equiv (\varphi_1)_{\rm C}, \qquad \varphi_2 = +i\varphi_2^* \equiv (\varphi_2)_{\rm C}. \tag{34}$$

Obviously, these relations do not allow us to write a one-component first-order equation for the Majorana particle (and from Eq. (2), neither can a standard one-component second-order equation be written). That is, unlike the case in $3+1$ dimensions, the equation that describes the Majorana particle in $1+1$ dimensions is a complex system of coupled equations, i.e., Eq. (33) with the restriction given in Eq. (34).

In this representation, we can also write the most general set of boundary conditions for the 1D Majorana particle inside a box with ends at $x = 0$ and $x = L$. This set consists of two one-parameter families of boundary conditions. In fact, using the results given in Eqs. (67) and (68) of Ref. [6] (written in the Majorana representation) and the fact that the two-component wave functions in the Weyl and Majorana representations satisfy the relation $[\phi_1\ \phi_2]^{\rm T} = \widehat{S}[\varphi_1\ \varphi_2]^{\rm T}$, where the matrix $\widehat{S}$ is given in Eq. (14), we obtain (with the variable t eliminated from in the boundary conditions here and hereafter)

$$\begin{bmatrix}\varphi_1(L)\\ \varphi_2(L)\end{bmatrix} = \begin{bmatrix}-\dfrac{1}{m_2} & -i\dfrac{m_0}{m_2}\\ -i\dfrac{m_0}{m_2} & \dfrac{1}{m_2}\end{bmatrix}\begin{bmatrix}\varphi_1(0)\\ \varphi_2(0)\end{bmatrix}, \tag{35}$$

where $(m_0)^2 + (m_2)^2 = 1$, and

$$\begin{bmatrix}\varphi_1(L)\\ \varphi_2(L)\end{bmatrix} = \begin{bmatrix}\dfrac{1}{m_1} & -i\dfrac{m_3}{m_1}\\ i\dfrac{m_3}{m_1} & \dfrac{1}{m_1}\end{bmatrix}\begin{bmatrix}\varphi_1(0)\\ \varphi_2(0)\end{bmatrix}, \tag{36}$$

where $(m_1)^2 + (m_3)^2 = 1$. We note that the 2×2 matrix in (35) is equal to its own inverse and that the inverse matrix of the 2×2 matrix in (36) is obtained from the latter by making the substitution $m_3 \to -m_3$. We obtain two boundary conditions for an impenetrable box (i.e., two confining boundary conditions) from Eq. (35) and its inverse by letting $m_2 \to 0$:

$$\varphi_1(L) = -i\varphi_2(L), \qquad \varphi_1(0) = -i\varphi_2(0) \tag{37}$$

with $m_0 = 1$, and

$$\varphi_1(L) = +i\varphi_2(L), \qquad \varphi_1(0) = +i\varphi_2(0) \tag{38}$$

with $m_0 = -1$. Likewise, we obtain two other confining boundary conditions from Eq. (36) and its inverse by letting $m_1 \to 0$:

$$\varphi_1(L) = -i\varphi_2(L), \qquad \varphi_1(0) = +i\varphi_2(0), \tag{39}$$

with $m_3 = 1$, and

$$\varphi_1(L) = +i\varphi_2(L), \qquad \varphi_1(0) = -i\varphi_2(0) \tag{40}$$

with $m_3 = -1$. We note that the wave function $[\varphi_1\ \varphi_2]^{\rm T}$ can satisfy any of the boundary conditions included in Eqs. (35) and (36), but then the wave function $[-i\varphi_1^*\ +i\varphi_2^*]^{\rm T}$ also automatically satisfies this boundary condition. This is due to the Majorana condition. Because in this case the Majorana condition is a pair of independent relations, the boundary conditions are presented in terms of the two components of the wave function.

4.3. Majorana representation.

3 + 1 dimensions. The four-component Dirac wave function (or spinor) Ψ can be written as

$$\Psi \equiv \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{41}$$

where the upper (lower) two-component wave function can be written as $\phi_1 \equiv [\zeta_1 \ \zeta_2]^{\mathrm{T}}$ ($\phi_2 \equiv [\zeta_3 \ \zeta_4]^{\mathrm{T}}$). The Dirac equation (Eq. (3)) takes the form

$$i\hbar \frac{\partial}{\partial t} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \widehat{H} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} =$$
$$= \begin{bmatrix} -i\hbar c \hat{1}_2 \frac{\partial}{\partial y} & i\hbar c\big(\hat{\sigma}_x \frac{\partial}{\partial x} + \hat{\sigma}_z \frac{\partial}{\partial z}\big) + (V_{\mathrm{S}} + mc^2)\hat{\sigma}_y \\ i\hbar c\big(\hat{\sigma}_x \frac{\partial}{\partial x} + \hat{\sigma}_z \frac{\partial}{\partial z}\big) + (V_{\mathrm{S}} + mc^2)\hat{\sigma}_y & i\hbar c \hat{1}_2 \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}. \tag{42}$$

Clearly, Eq. (42) is a real system of two coupled equations for the two-component wave functions ϕ_1 and ϕ_2. Thus, we can obtain real-valued solutions of this equation, but complex-valued solutions can also be obtained (although these do not describe a Majorana particle) [19]. Majorana condition (16) imposed on Ψ leads to the relation

$$\Psi = \Psi^* \qquad (\Leftrightarrow \phi_1 = \phi_1^* \equiv (\phi_1)_{\mathrm{C}},\ \phi_2 = \phi_2^* \equiv (\phi_2)_{\mathrm{C}}). \tag{43}$$

That is, the Majorana condition imposed on the Dirac wave function in the Majorana representation implies the realness of this wave function.

1 + 1 dimensions. We write the two-component Dirac wave function Ψ in the form given in Eq. (41), but in this case, ϕ_1 and ϕ_2 are functions of a single component. The Dirac equation (Eq. (3)) has the form

$$i\hbar \frac{\partial}{\partial t} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \widehat{H} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 0 & -i\hbar c \frac{\partial}{\partial x} - i(V_{\mathrm{S}} + mc^2) \\ -i\hbar c \frac{\partial}{\partial x} + i(V_{\mathrm{S}} + mc^2) & 0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}. \tag{44}$$

Again, the Dirac equation in this representation is a real system of two coupled equations for the wave functions ϕ_1 and ϕ_2. However, it is precisely the Majorana condition (16) imposed on Ψ that leads to the realness condition for the wave function:

$$\Psi = \Psi^* \qquad (\Leftrightarrow \phi_1 = \phi_1^* \equiv (\phi_1)_{\mathrm{C}},\ \phi_2 = \phi_2^* \equiv (\phi_2)_{\mathrm{C}}). \tag{45}$$

Recently, distinct real-valued general solutions of the time-dependent Dirac equation in Eq. (44) (i.e., subject to the constraint (45), for distinct scalar potentials and boundaries) were constructed [19]. Certainly, all these solutions describe a 1D Majorana particle in its respective physical situation.

The most general set of boundary conditions for the 1D Majorana particle inside a box in the Majorana representation was written in detail in Ref. [6]. This set consists of two real one-parameter families of boundary conditions (see Eqs. (67) and (68) in [6]). The Majorana condition in the Majorana representation leads very easily to the Majorana condition in any other representation. In fact, we know that wave functions in the Dirac and Majorana representations are linked through the relation $[\phi_1 \ \phi_2]^{\mathrm{T}} = \widehat{S}[\varphi \ \chi]^{\mathrm{T}}$, where the matrix $\widehat{S}$ is given in Eq. (13); in addition, wave functions in the Weyl and Majorana representations are linked through the relation $[\phi_1 \ \phi_2]^{\mathrm{T}} = \widehat{S}[\varphi_1 \ \varphi_2]^{\mathrm{T}}$, where the matrix $\widehat{S}$ is given in Eq. (14). Thus, by imposing the Majorana condition (Eq. (45)) on the last two relations, we obtain the Majorana condition in the Dirac and Weyl representations, i.e., Eqs. (23) and (34) respectively. Certainly, this general discussion is also valid in $3 + 1$ dimensions.

5. Equations for the Majorana single-particle. II

3 + 1 dimensions. We define, as Case did [8], the wave functions and matrices

$$\Psi_{\pm} \equiv \frac{1}{2}(\hat{1}_4 \pm \hat{\gamma}^5)\Psi \tag{46}$$

and

$$\hat{\gamma}^{\mu}_{\pm} \equiv \frac{1}{2}(\hat{1}_4 \pm \hat{\gamma}^5)\hat{\gamma}^{\mu}, \tag{47}$$

where the matrix $\hat{\gamma}^5 \equiv i\hat{\gamma}^0\hat{\gamma}^1\hat{\gamma}^2\hat{\gamma}^3 = -i\hat{\alpha}_1\hat{\alpha}_2\hat{\alpha}_3$ is Hermitian and satisfies the relations $(\hat{\gamma}^5)^2 = \hat{1}_4$, and $\{\hat{\gamma}^5, \hat{\gamma}^{\mu}\} = \hat{0}_4$. In addition, $\hat{\gamma}^5$ satisfies the relation $\widehat{S}_{\mathrm{C}}(-\hat{\gamma}^5)^*(\widehat{S}_{\mathrm{C}})^{-1} = \hat{\gamma}^5$ (i.e., $\hat{\gamma}^5$, just as $\hat{\gamma}^{\mu}$, satisfies Eq. (5)), and

$$\left[\frac{1}{2}(\hat{1}_4 \pm \hat{\gamma}^5)\right]^2 = \frac{1}{2}(\hat{1}_4 \pm \hat{\gamma}^5), \qquad \frac{1}{2}(\hat{1}_4 \pm \hat{\gamma}^5)\frac{1}{2}(\hat{1}_4 \mp \hat{\gamma}^5) = \hat{0}_4. \tag{48}$$

We note that the charge conjugates of the wave functions in (46) satisfy $(\Psi_{\pm})_{\mathrm{C}} = (\Psi_{\mathrm{C}})_{\mp}$. The matrix $\hat{\gamma}^5$ is called the chirality matrix and its eigenstates are precisely Ψ_+ (the right-chiral state), with eigenvalue $+1$, and Ψ_- (the left-chiral state), with eigenvalue -1 [3] (the last two results can easily be demonstrated by multiplying Eq. (46) by $\hat{\gamma}^5$ from the left). However, we also note that $(\Psi_+)_{\mathrm{C}}$ is the eigenstate of $\hat{\gamma}^5$ with eigenvalue -1 (i.e., it is a left-chiral state), and $(\Psi_-)_{\mathrm{C}}$ is the eigenstate of $\hat{\gamma}^5$ with eigenvalue $+1$ (i.e., it is a right-chiral state). The matrices $\hat{\gamma}^5$ and the wave functions $\Psi_{\pm}$ in the three representations that we use in this paper are shown in Tables 1 and 3.

Table 3

Representation	Ψ_+	Ψ_-
Dirac	$\frac{1}{2}\begin{bmatrix}\varphi+\chi \\ \varphi+\chi\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}\varphi-\chi \\ -\varphi+\chi\end{bmatrix}$
Weyl	$\begin{bmatrix}\varphi_1 \\ 0\end{bmatrix}$	$\begin{bmatrix}0 \\ \varphi_2\end{bmatrix}$
Majorana	$\frac{1}{2}\begin{bmatrix}(\hat{1}_2+\hat{\sigma}_y)\phi_1 \\ (\hat{1}_2-\hat{\sigma}_y)\phi_2\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}(\hat{1}_2-\hat{\sigma}_y)\phi_1 \\ (\hat{1}_2+\hat{\sigma}_y)\phi_2\end{bmatrix}$

First, multiplying Eq. (1) by $(\hat{1}_4 + \hat{\gamma}^5)/2$ from the left, we obtain the equation

$$i\hat{\gamma}^{\mu}_{+}\,\partial_{\mu}\Psi_- - \frac{1}{\hbar c}(V_{\mathrm{S}} + mc^2)\hat{1}_4\Psi_+ = 0. \tag{49}$$

and similarly, multiplying Eq. (1) by $(\hat{1}_4 - \hat{\gamma}^5)/2$, we obtain the equation

$$i\hat{\gamma}^{\mu}_{-}\,\partial_{\mu}\Psi_+ - \frac{1}{\hbar c}(V_{\mathrm{S}} + mc^2)\hat{1}_4\Psi_- = 0. \tag{50}$$

Because $\Psi = \Psi_+ + \Psi_-$ and $\hat{\gamma}^{\mu} = \hat{\gamma}^{\mu}_{+} + \hat{\gamma}^{\mu}_{-}$, it follows that Eqs. (49) and (50) are completely equivalent to Dirac equation (1). Likewise, because Eq. (1) is also satisfied by the charge-conjugate wave function, we also have two equations that are equivalent to the Dirac equation for Ψ_{C}. In effect, multiplying the latter by $(\hat{1}_4 + \hat{\gamma}^5)/2$ and $(\hat{1}_4 - \hat{\gamma}^5)/2$, we respectively obtain

$$\begin{aligned} &i\hat{\gamma}^{\mu}_{+}\,\partial_{\mu}(\Psi_+)_{\mathrm{C}} - \frac{1}{\hbar c}(V_{\mathrm{S}} + mc^2)\hat{1}_4(\Psi_-)_{\mathrm{C}} = 0, \\ &i\hat{\gamma}^{\mu}_{-}\partial_{\mu}(\Psi_-)_{\mathrm{C}} - \frac{1}{\hbar c}(V_{\mathrm{S}} + mc^2)\hat{1}_4(\Psi_+)_{\mathrm{C}} = 0 \end{aligned} \tag{51}$$

(we recall that $(\Psi_{\pm})_{\mathrm{C}} = \widehat{S}_{\mathrm{C}}\Psi_{\pm}^{*}$). We note that because $(\Psi_{+})_{\mathrm{C}} = (\Psi_{\mathrm{C}})_{-}$ and $(\Psi_{-})_{\mathrm{C}} = (\Psi_{\mathrm{C}})_{+}$, the wave functions Ψ_{+} and Ψ_{-} as well as $(\Psi_{\mathrm{C}})_{+}$ and $(\Psi_{\mathrm{C}})_{-}$ satisfy the same system of coupled equations, namely, Eqs. (49) and (50) (or the system in Eq. (51), as expected.

In the case where $mc^2 = V_{\mathrm{S}} = 0$, Eqs. (49) and (50) are decoupled, and we have

$$i\hat{\gamma}_{+}^{\mu}\,\partial_{\mu}\Psi_{-} = 0 \qquad (\Rightarrow i\hat{\gamma}^{\mu}\,\partial_{\mu}\Psi_{-} = 0),$$
$$i\hat{\gamma}_{-}^{\mu}\,\partial_{\mu}\Psi_{+} = 0 \qquad (\Rightarrow i\hat{\gamma}^{\mu}\,\partial_{\mu}\Psi_{+} = 0).$$

In the Weyl representation, the last two four-component equations give the usual Weyl equations (30). In the same way, if we set $mc^2 = V_{\mathrm{S}} = 0$ in the system in Eq. (51), we obtain

$$i\hat{\gamma}_{+}^{\mu}\,\partial_{\mu}(\Psi_{+})_{\mathrm{C}} = 0 \qquad (\Rightarrow i\hat{\gamma}^{\mu}\,\partial_{\mu}(\Psi_{+})_{\mathrm{C}} = 0),$$
$$i\hat{\gamma}_{-}^{\mu}\,\partial_{\mu}(\Psi_{-})_{\mathrm{C}} = 0 \qquad (\Rightarrow i\hat{\gamma}^{\mu}\,\partial_{\mu}(\Psi_{-})_{\mathrm{C}} = 0).$$

Certainly, in the Weyl representation, the last two equations also give the usual Weyl equations (30).

The Majorana condition in Eq. (16) takes the form

$$\Psi_{-} = (\Psi_{+})_{\mathrm{C}} \qquad (\Leftrightarrow \Psi_{+} = (\Psi_{-})_{\mathrm{C}}) \tag{52}$$

(we recall that $(\widehat{S}_{\mathrm{C}})^{-1} = (\widehat{S}_{\mathrm{C}})^{*}$), i.e., $\Psi_{-} = (\Psi_{\mathrm{C}})_{-}$ $(\Leftrightarrow \Psi_{+} = (\Psi_{\mathrm{C}})_{+})$. Substituting the wave function Ψ_{-} in Eq. (50), we obtain an equation for the four-component wave function Ψ_{+}:

$$i\hat{\Gamma}^{\mu}\,\partial_{\mu}\Psi_{+} - \frac{1}{\hbar c}(V_{\mathrm{S}} + mc^2)\hat{1}_4\Psi_{+}^{*} = 0, \tag{53}$$

where

$$\widehat{\Gamma}^{\mu} \equiv (\widehat{S}_{\mathrm{C}})^{*}\hat{\gamma}_{-}^{\mu}, \qquad (\widehat{\Gamma}^{\mu})^{*}\widehat{\Gamma}^{\nu} + (\widehat{\Gamma}^{\nu})^{*}\widehat{\Gamma}^{\mu} = -2g^{\mu\nu}\frac{1}{2}(\hat{1}_4 + \hat{\gamma}^5) \tag{54}$$

(the equation for Ψ_{+} that results after making the last substitution but in Eq. (49) is absolutely equivalent to Eq. (53)).

Alternatively, substituting the wave function Ψ_{+} (52) in Eq. (49), we obtain an equation for the four-component wave function Ψ_{-}:

$$i\widehat{\Lambda}^{\mu}\,\partial_{\mu}\Psi_{-} - \frac{1}{\hbar c}(V_{\mathrm{S}} + mc^2)\hat{1}_4\Psi_{-}^{*} = 0, \tag{55}$$

where

$$\widehat{\Lambda}^{\mu} \equiv (\widehat{S}_{\mathrm{C}})^{*}\hat{\gamma}_{+}^{\mu}, \qquad (\widehat{\Lambda}^{\mu})^{*}\,\widehat{\Lambda}^{\nu} + (\widehat{\Lambda}^{\nu})^{*}\widehat{\Lambda}^{\mu} = -2g^{\mu\nu}\frac{1}{2}(\hat{1}_4 - \hat{\gamma}^5) \tag{56}$$

(again, the equation for Ψ_{-} that results after making the last substitution but into Eq. (50) is absolutely equivalent to Eq. (55)). Naturally, by imposing Majorana condition (52) on the equations in (51), we again obtain Eqs. (53) and (55).

On the other hand, setting $mc^2 = V_{\mathrm{S}} = 0$ in Eq. (53) leads us to the relation $i\hat{\gamma}_{-}^{\mu}\,\partial_{\mu}\Psi_{+} = 0$ $(\Rightarrow i\hat{\gamma}^{\mu}\,\partial_{\mu}\Psi_{+} = 0)$, and as can be seen in Eq. (51), we also have $i\hat{\gamma}_{-}^{\mu}\,\partial_{\mu}(\Psi_{-})_{\mathrm{C}} = 0$ $(\Rightarrow i\hat{\gamma}^{\mu}\,\partial_{\mu}(\Psi_{-})_{\mathrm{C}} = 0)$, but in this case we also have $\Psi_{+} = (\Psi_{-})_{\mathrm{C}}$ (due to the Majorana condition). Similarly, setting $mc^2 = V_{\mathrm{S}} = 0$ in Eq. (55) leads us to the relation $i\hat{\gamma}_{+}^{\mu}\,\partial_{\mu}\Psi_{-} = 0$ $(\Rightarrow i\hat{\gamma}^{\mu}\,\partial_{\mu}\Psi_{-} = 0)$, but from Eq. (51) we also have $i\hat{\gamma}_{+}^{\mu}\,\partial_{\mu}(\Psi_{+})_{\mathrm{C}} = 0$ $(\Rightarrow i\hat{\gamma}^{\mu}\,\partial_{\mu}(\Psi_{+})_{\mathrm{C}} = 0)$, where $\Psi_{-} = (\Psi_{+})_{\mathrm{C}}$ (this is also due to the Majorana condition).

To obtain the four-component wave function that describes the Majorana particle, namely, $\Psi = \Psi_{+} + \Psi_{-}$, it is sufficient to solve the equation for Ψ_{+} (Eq. (53)), and then obtain Ψ_{-} from this solution, and using the Majorana condition in (52). Alternatively, we could also solve the equation for Ψ_{-} (55), and then, from this solution, and using the Majorana condition (52), obtain Ψ_+. We note that in the former case, $\Psi = \Psi_+ + (\Psi_+)_{\rm C}$, and therefore $\Psi = \Psi_{\rm C}$ (we recall that $((\Psi_+)_{\rm C})_{\rm C} = \Psi_+$); similarly, in the latter case, $\Psi = (\Psi_-)_{\rm C} + \Psi_-$, and therefore $\Psi = \Psi_{\rm C}$ (with $((\Psi_-)_{\rm C})_{\rm C} = \Psi_-$), as expected. Clearly, the four-component wave function Ψ depends only on the solution of Eq. (53) (or of Eq. (55)); thus, we can assume that Eq. (53) (or Eq. (55)) alone models the 3D Majorana particle and in a form independent of the choice of the representation.

Certainly, the above procedure to obtain Ψ is general, but in each representation, it has its own specific features. In relation to this, we can now obtain different results. In the rest of this subsection, we fully use Tables 3–5.

Table 4

Representation	$\widehat{\Gamma}^0$	$\widehat{\Gamma}^1$	$\widehat{\Gamma}^2$	$\widehat{\Gamma}^3$
Dirac	$\frac{1}{2}\begin{bmatrix}-\hat\sigma_y & -\hat\sigma_y\\ -\hat\sigma_y & -\hat\sigma_y\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}i\hat\sigma_z & i\hat\sigma_z\\ i\hat\sigma_z & i\hat\sigma_z\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}-\hat 1_2 & -\hat 1_2\\ -\hat 1_2 & -\hat 1_2\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}-i\hat\sigma_x & -i\hat\sigma_x\\ -i\hat\sigma_x & -i\hat\sigma_x\end{bmatrix}$
Weyl	$\begin{bmatrix}-\hat\sigma_y & \hat 0_2\\ \hat 0_2 & \hat 0_2\end{bmatrix}$	$\begin{bmatrix}i\hat\sigma_z & \hat 0_2\\ \hat 0_2 & \hat 0_2\end{bmatrix}$	$\begin{bmatrix}-\hat 1_2 & \hat 0_2\\ \hat 0_2 & \hat 0_2\end{bmatrix}$	$\begin{bmatrix}-i\hat\sigma_x & \hat 0_2\\ \hat 0_2 & \hat 0_2\end{bmatrix}$
Majorana	$\frac{1}{2}\begin{bmatrix}\hat 0_2 & \hat\sigma_y - \hat 1_2\\ \hat\sigma_y + \hat 1_2 & \hat 0_2\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}i\hat\sigma_z + \hat\sigma_x & \hat 0_2\\ \hat 0_2 & i\hat\sigma_z - \hat\sigma_x\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}\hat 0_2 & -\hat\sigma_y + \hat 1_2\\ \hat\sigma_y + \hat 1_2 & \hat 0_2\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}-i\hat\sigma_x + \hat\sigma_z & \hat 0_2\\ \hat 0_2 & -i\hat\sigma_x - \hat\sigma_z\end{bmatrix}$

Table 5

Representation	$\hat\Lambda^0$	$\hat\Lambda^1$	$\hat\Lambda^2$	$\hat\Lambda^3$
Dirac	$\frac{1}{2}\begin{bmatrix}\hat\sigma_y & -\hat\sigma_y\\ -\hat\sigma_y & \hat\sigma_y\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}i\hat\sigma_z & -i\hat\sigma_z\\ -i\hat\sigma_z & i\hat\sigma_z\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}-\hat 1_2 & \hat 1_2\\ \hat 1_2 & -\hat 1_2\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}-i\hat\sigma_x & i\hat\sigma_x\\ i\hat\sigma_x & -i\hat\sigma_x\end{bmatrix}$
Weyl	$\begin{bmatrix}\hat 0_2 & \hat 0_2\\ \hat 0_2 & \hat\sigma_y\end{bmatrix}$	$\begin{bmatrix}\hat 0_2 & \hat 0_2\\ \hat 0_2 & i\hat\sigma_z\end{bmatrix}$	$\begin{bmatrix}\hat 0_2 & \hat 0_2\\ \hat 0_2 & -\hat 1_2\end{bmatrix}$	$\begin{bmatrix}\hat 0_2 & \hat 0_2\\ \hat 0_2 & -i\hat\sigma_x\end{bmatrix}$
Majorana	$\frac{1}{2}\begin{bmatrix}\hat 0_2 & \hat\sigma_y + \hat 1_2\\ \hat\sigma_y - \hat 1_2 & \hat 0_2\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}i\hat\sigma_z - \hat\sigma_x & \hat 0_2\\ \hat 0_2 & i\hat\sigma_z + \hat\sigma_x\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}\hat 0_2 & -\hat\sigma_y - \hat 1_2\\ \hat\sigma_y - \hat 1_2 & \hat 0_2\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}-i\hat\sigma_x - \hat\sigma_z & \hat 0_2\\ \hat 0_2 & -i\hat\sigma_x + \hat\sigma_z\end{bmatrix}$

1. In the Weyl representation, the covariant four-component equation for $\Psi_+ = [\varphi_1 \; 0]^{\rm T}$ (53) leads to the explicitly covariant two-component equation for the two-component wave function φ_1

$$\hat\eta^\mu \, \partial_\mu \varphi_1 - \frac{1}{\hbar c}(V_{\rm S} + mc^2)\hat 1_2 \varphi_1^* = 0, \tag{57}$$

where the matrices $\hat\eta^0 = -i\hat\sigma_y$, $\hat\eta^1 = -\hat\sigma_z$, $\hat\eta^2 = -i\hat 1_2$, and $\hat\eta^3 = \hat\sigma_x$ satisfy the relation

$$(\hat\eta^\mu)^* \hat\eta^\nu + (\hat\eta^\nu)^* \hat\eta^\mu = -2g^{\mu\nu}\hat 1_2 \tag{58}$$

(this last relation arises from Eq. (54)). After multiplying Eq. (57) by $-\hat\sigma_y$, this equation takes an alternative form,

$$i\hat\sigma^\mu \, \partial_\mu \varphi_1 + \frac{1}{\hbar c}(V_{\rm S} + mc^2)\hat\sigma_y \varphi_1^* = 0, \tag{59}$$

where $\hat{\sigma}^0 = \hat{1}_2$, $\hat{\sigma}^1 = \hat{\sigma}_x$, $\hat{\sigma}^2 = \hat{\sigma}_y$, and $\hat{\sigma}^3 = \hat{\sigma}_z$ (or, as it is commonly written, $\hat{\sigma}^\mu = (\hat{1}_2, +\hat{\boldsymbol{\sigma}})$) [8]. Equation (59) is precisely Eq. (28), as expected. Now, if we use Majorana condition (52), we can obtain $\Psi_- = [0\ \varphi_2]^{\mathrm{T}}$ from $\Psi_+ = [\varphi_1\ 0]^{\mathrm{T}}$, and the result is $\Psi_- = [0\ \hat{\sigma}_y \varphi_1^*]^{\mathrm{T}}$ (which is in agreement with the result in Eq. (27)). Finally, we can write the four-component wave function for the Majorana particle, namely, $\Psi = \Psi_+ + \Psi_- = [\varphi_1\ \hat{\sigma}_y \varphi_1^*]^{\mathrm{T}}$. It is clear that this four-component solution is dependent only on the two-component complex wave function φ_1, which is a solution of Eq. (59), i.e., here, we have only four independent real quantities. Because we have $\hat{\gamma}^5 \Psi_+ = (+1)\Psi_+$, Eq. (59) is referred to as the right-chiral two-component Majorana equation.

Similarly, the covariant four-component equation for $\Psi_- = [0\ \varphi_2]^{\mathrm{T}}$ (55) leads to the explicitly covariant two-component equation for the two-component wave function φ_2

$$\hat{\xi}^\mu \,\partial_\mu \varphi_2 - \frac{1}{\hbar c}(V_{\mathrm{S}} + mc^2)\hat{1}_2 \varphi_2^* = 0, \tag{60}$$

where the matrices $\hat{\xi}^0 = -\hat{\eta}^0$ and $\hat{\xi}^j = \hat{\eta}^j$, $j = 1, 2, 3$, also satisfy Eq. (58) (in this case, the last relation arises from Eq. (56)). Multiplying Eq. (60) by $\hat{\sigma}_y$, this equation takes the alternative form

$$i\hat{\bar{\sigma}}^\mu \,\partial_\mu \varphi_2 - \frac{1}{\hbar c}(V_{\mathrm{S}} + mc^2)\hat{\sigma}_y \varphi_2^* = 0, \tag{61}$$

where $\hat{\bar{\sigma}}^0 = \hat{\sigma}^0$, $\hat{\bar{\sigma}}^1 = -\hat{\sigma}^1$, $\hat{\bar{\sigma}}^2 = -\hat{\sigma}^2$, and $\hat{\bar{\sigma}}^3 = -\hat{\sigma}^3$ (i.e., $\hat{\bar{\sigma}}^\mu = (\hat{1}_2, -\hat{\boldsymbol{\sigma}})$). Equation (61) is precisely Eq. (29), as expected. Again, if we use Majorana condition (52), we can obtain $\Psi_+ = [\varphi_1\ 0]^{\mathrm{T}}$ from $\Psi_- = [0\ \varphi_2]^{\mathrm{T}}$, and the result is $\Psi_+ = [-\hat{\sigma}_y \varphi_2^*\ 0]^{\mathrm{T}}$ (which is in agreement with the result in Eq. (27)). Thus, we can write the four-component wave function for the Majorana particle, namely, $\Psi = \Psi_+ + \Psi_- = [-\hat{\sigma}_y \varphi_2^*\ \varphi_2]^{\mathrm{T}}$. The last four-component solution depends only on the two-component complex wave function φ_2, which is a solution of Eq. (61), i.e., here, we have only four independent real quantities, as expected for a Majorana particle. Because we have $\hat{\gamma}^5 \Psi_- = (-1)\Psi_-$, Eq. (61) is referred to as the left-chiral two-component Majorana equation.

In summary, Eq. (59) alone can be regarded as a Majorana equation for the Majorana particle, even for a particular type of the Majorana particle. Likewise, Eq. (61) alone can also be regarded as a Majorana equation for the Majorana particle, even for a Majorana particle different from the previous one (for example, with a different mass). Thus, Eqs. (59) and (61), although similar, are nonequivalent two-component equations. Specifically, this is because φ_1 and φ_2 transform in two precise and different ways under Lorentz boosts, i.e., they transform according to two inequivalent representations of the Lorentz group [9]. Certainly, Eqs. (59) and (61) tend to the pair of Weyl equations when $mc^2 = V_{\mathrm{S}} = 0$ (Eq. (30)). Equations (59) and (61) comprise the so-called two-component theory of Majorana particles [8].

Again, in the Weyl representation that we consider in this paper ($\hat{\gamma}^0 = \hat{\beta} = -\hat{\sigma}_x \otimes \hat{1}_2$, $\hat{\boldsymbol{\gamma}} = \hat{\beta}\hat{\boldsymbol{\alpha}} = +i\hat{\sigma}_y \otimes \hat{\boldsymbol{\sigma}}$ and $\hat{\gamma}^5 = +\hat{\sigma}_z \otimes \hat{1}_2$), we used $\widehat{S}_{\mathrm{C}} = -\hat{\gamma}^2 = -i\hat{\sigma}_y \otimes \hat{\sigma}_y$, but this is only because we decided to derive this result from Eq. (8) (with $\widehat{S}$ given by Eq. (10)). We could, for example, write $\widehat{S}_{\mathrm{C}} = -i\hat{\gamma}^2 = +\hat{\sigma}_y \otimes \hat{\sigma}_y$. In the latter case, the equations for φ_1 and φ_2 are simply Eqs. (59) and (61) with the replacement $\hat{\sigma}_y \to +i\hat{\sigma}_y$:

$$i\hat{\sigma}^\mu \,\partial_\mu \varphi_1 + \frac{1}{\hbar c}(V_{\mathrm{S}} + mc^2) i\hat{\sigma}_y \varphi_1^* = 0 \tag{62}$$

and

$$i\hat{\bar{\sigma}}^\mu \,\partial_\mu \varphi_2 - \frac{1}{\hbar c}(V_{\mathrm{S}} + mc^2) i\hat{\sigma}_y \varphi_2^* = 0. \tag{63}$$

Equation (62) is appropriately named the right-chiral two-component Majorana equation, and Eq. (63) is named the left-chiral two-component Majorana equation (see [9], [10]). Incidentally, by linearizing the standard relativistic energy–momentum relation, and without recourse to the Dirac equation, a good derivation of Eq. (62), with $V_{\mathrm{S}} = 0$, was obtained in Ref. [20]. In Ref. [10], we compare the pair of equations (62), (63) with other pairs usually presented in the literature, in particular, with those presented in Refs. [3], [21], [22].

Unsurprisingly, Eqs. (62), (63) for φ_1, φ_2 can be written jointly in the form

$$i\hat{\gamma}^{\mu}\,\partial_{\mu}\Psi - \frac{1}{\hbar c}(V_{\mathrm{S}} + mc^2)\hat{1}_4\Psi_{\mathrm{C}} = 0, \tag{64}$$

where $\Psi = [\varphi_1\ \varphi_2]^{\mathrm{T}}$ and $\Psi_{\mathrm{C}} \equiv \widehat{S}_{\mathrm{C}}\Psi^*$ with $\widehat{S}_{\mathrm{C}} = -i\hat{\gamma}^2 = +\hat{\sigma}_y \otimes \hat{\sigma}_y$. Specifically, Eq. (64) is the (four-component) Majorana equation with a scalar potential (see the discussion on this equation in the introduction). However, if this equation is considered to describe a Majorana particle with a four-component wave function, $\Psi = [\varphi_1\ \varphi_2]^{\mathrm{T}}$, it should be remembered that due to the Majorana condition, $\Psi = \Psi_{\mathrm{C}}$, φ_1 and φ_2 are not independent two-component wave functions. Therefore, in this case, it would be sufficient to solve just one of the two two-component Majorana equations, and then, with the relation between φ_1 and φ_2, we could reconstruct the entire wave function Ψ. However, if Eq. (64) is considered to describe a Majoranon [7], [23], then the two two-component Majorana equations must be solved, the solutions of which are simply the top and bottom components of the wave function Ψ in Eq. (64). This result is somewhat unexpected.

2. In the Dirac representation, the covariant four-component equation for Ψ_+ (Eq. (53)) leads to the covariant two-component Eq. (57) with the replacement $\varphi_1 \to \varphi + \chi$. Likewise, the Majorana condition in Eq. (52) leads to Eq. (27) with the latter replacement plus $\varphi_2 \to -\varphi + \chi$, namely, $-\varphi + \chi = \hat{\sigma}_y(\varphi + \chi)^*$ (Eq. (19)). We recall that the four-component wave functions in the Dirac and Weyl representations are related as $[\varphi_1\ \varphi_2]^{\mathrm{T}} = \widehat{S}[\varphi\ \chi]^{\mathrm{T}}$, where the matrix $\widehat{S}$ is given in Eq. (11). Thus, from Eq. (53), we obtain the two-component wave function $\varphi + \chi$, from which we can construct Ψ_+, and using the Majorana condition, we obtain $-\varphi+\chi$, from which we can construct Ψ_- (see Table 3). Finally, the four-component wave function for the Majorana particle $\Psi = \Psi_+ + \Psi_- = [\varphi\ \chi]^{\mathrm{T}}$ can be written immediately. Similarly, the covariant four-component equation for Ψ_- (Eq. (55)) leads to Eq. (60) with the replacement $\varphi_2 \to -\varphi + \chi$. Thus, from Eq. (55), we obtain the two-component wave function $-\varphi + \chi$, from which we can construct Ψ_-, and using the Majorana condition we obtain $\varphi + \chi$, from which we can construct Ψ_+ (see Table 3). Finally, the four-component wave function for the Majorana particle can be written immediately.

Alternatively, adding and subtracting the former equations that result from Eqs. (53) and (55) and once again using (conveniently) the Majorana condition given in Eq. (19), we obtain an equation for the two-component wave function φ,

$$\hat{\eta}^0\,\partial_0\varphi + \sum_{k=1}^{3}\hat{\eta}^k\,\partial_k(\hat{\sigma}_y\varphi^*) + \frac{1}{\hbar c}(V_{\mathrm{S}} + mc^2)\hat{\sigma}_y\,\varphi = 0 \tag{65}$$

and another equation for the two-component wave function χ,

$$\hat{\xi}^0\,\partial_0\chi + \sum_{k=1}^{3}\hat{\xi}^k\,\partial_k(\hat{\sigma}_y\chi^*) + \frac{1}{\hbar c}(V_{\mathrm{S}} + mc^2)\hat{\sigma}_y\,\chi = 0. \tag{66}$$

Certainly, Eq. (65) leads to Eq. (20), and Eq. (66) leads to Eq. (21). Likewise, from the solution of Eq. (65) or Eq. (66), and properly using Majorana condition (19) in each case, we can obtain the respective four-component wave function $\Psi = [\varphi\ \chi]^{\mathrm{T}}$.

3. In the Majorana representation, the covariant four-component equation for Ψ_+ (53) is precisely Eq. (50), and the covariant four-component equation for Ψ_- (55) is precisely Eq. (49); additionally, the latter equation is the complex conjugate of the former equation. This is shown by the following results.

We recall that in this representation, $\widehat{S}_{\rm C}=\hat{1}_4$; therefore, $\widehat{\Gamma}^\mu=\hat{\gamma}^\mu_-$, $\widehat{\Lambda}^\mu=\hat{\gamma}^\mu_+$, and, from the Majorana condition in Eq. (52), we have $\Psi_-=\Psi_+^*$ (and therefore $\Psi=\Psi_++\Psi_-=\Psi^*$, as expected). In this representation, we also have $\hat{\gamma}^\mu=-(\hat{\gamma}^\mu)^*$ and $\hat{\gamma}^5=-(\hat{\gamma}^5)^*$, and therefore $\hat{\gamma}^\mu_-=-(\hat{\gamma}^\mu_+)^*$. Thus, in the Majorana representation, the equation for the Majorana particle is essentially Eq. (50), where $\hat{\gamma}^\mu_-=-(\hat{\gamma}^\mu_+)^*$ and $\Psi_-=\Psi_+^*$ (in fact, substituting the last relations in the complex conjugate equation of Eq. (50), we obtain Eq. (49)). Specifically, Eq. (50) leads to Eq. (57) with the replacement $\varphi_1\rightarrow(\hat{1}_2+\hat{\sigma}_y)\phi_1-(\hat{1}_2-\hat{\sigma}_y)\phi_2$. We recall that the four-component wave functions in the Majorana and Weyl representations are related by $[\varphi_1\ \varphi_2]^{\rm T}=\widehat{S}^{-1}[\phi_1\ \phi_2]^{\rm T}$, where the matrix $\widehat{S}$ is given in Eq. (10) (additionally, φ_1 and φ_2 are related by Eq. (27), i.e., the Majorana condition, which implies that ϕ_1 and ϕ_2 are real-valued wave functions, as expected). Finally, the equation obtained here and its complex conjugate can be written in the form given in Eq. (42).

1 + 1 dimensions. We introduce the wave functions and matrices

$$\Psi_\pm\equiv\frac{1}{2}(\hat{1}_2\pm\widehat{\Gamma}^5)\Psi,\qquad \hat{\gamma}^\mu_\pm\equiv\frac{1}{2}(\hat{1}_2\pm\widehat{\Gamma}^5)\hat{\gamma}^\mu, \tag{67}$$

where the matrix $\widehat{\Gamma}^5\equiv-i\hat{\gamma}^5$ is Hermitian because $\hat{\gamma}^5\equiv i\hat{\gamma}^0\hat{\gamma}^1=i\hat{\alpha}$ is anti-Hermitian, and satisfies the relations $(\widehat{\Gamma}^5)^2=\hat{1}_2$ and $\{\widehat{\Gamma}^5,\hat{\gamma}^\mu\}=\hat{0}_2$. In addition, $\widehat{\Gamma}^5$ satisfies the relation $\widehat{S}_{\rm C}(\widehat{\Gamma}^5)^*(\widehat{S}_{\rm C})^{-1}=\widehat{\Gamma}^5$ (which is different from the analogous relation satisfied by $\hat{\gamma}^5$ in $3+1$ dimensions), and

$$\left[\frac{1}{2}(\hat{1}_2\pm\widehat{\Gamma}^5)\right]^2=\frac{1}{2}(\hat{1}_2\pm\widehat{\Gamma}^5),\qquad \frac{1}{2}(\hat{1}_2\pm\widehat{\Gamma}^5)\frac{1}{2}(\hat{1}_2\mp\widehat{\Gamma}^5)=\hat{0}_2. \tag{68}$$

We note that in $1+1$ dimensions, $\widehat{\Gamma}^5=\hat{\alpha}$ acts similarly to the standard fifth gamma matrix in $3+1$ dimensions, i.e., as the chirality matrix [24], [25]. However, in this case, the charge conjugates of the wave functions in (67) satisfy $(\Psi_\pm)_{\rm C}=(\Psi_{\rm C})_\pm$. Thus, although it is true that $\widehat{\Gamma}^5\Psi_\pm=(\pm1)\Psi_\pm$, we now have $\widehat{\Gamma}^5(\Psi_\pm)_{\rm C}=(\pm1)(\Psi_\pm)_{\rm C}$, i.e., $\Psi_\pm$ and $(\Psi_\pm)_{\rm C}$ are eigenstates of $\widehat{\Gamma}^5$ with eigenvalues ±1. The matrices $\widehat{\Gamma}^5$ and the wave functions $\Psi_\pm$ in each of the three representations that we use in this paper are shown in Table 2 and Table 6.

Table 6

Representation	Ψ_+	Ψ_-	$\hat{\gamma}^0_+(=-\hat{\gamma}^1_+)$	$\hat{\gamma}^0_-(=\hat{\gamma}^1_-)$
Dirac	$\frac{1}{2}\begin{bmatrix}\varphi+\chi\\ \varphi+\chi\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}\varphi-\chi\\ -\varphi+\chi\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1&-1\\1&-1\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1&1\\-1&-1\end{bmatrix}$
Weyl	$\begin{bmatrix}\varphi_1\\0\end{bmatrix}$	$\begin{bmatrix}0\\ \varphi_2\end{bmatrix}$	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\1&0\end{bmatrix}$
Majorana	$\frac{1}{2}\begin{bmatrix}\phi_1+\phi_2\\ \phi_1+\phi_2\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}\phi_1-\phi_2\\ -\phi_1+\phi_2\end{bmatrix}$	$\frac{i}{2}\begin{bmatrix}1&-1\\1&-1\end{bmatrix}$	$\frac{i}{2}\begin{bmatrix}-1&-1\\1&1\end{bmatrix}$

1. We note that multiplying the Dirac equation in Eq. (1) (but particularized to the case of $1+1$ dimensions) by $(\hat{1}_2+\widehat{\Gamma}^5)/2$ from the left, we obtain the equation

$$i\hat{\gamma}^\mu_+\,\partial_\mu\Psi_- -\frac{1}{\hbar c}(V_{\rm S}+mc^2)\hat{1}_2\Psi_+=0. \tag{69}$$

and similarly, multiplying Eq. (1) by $(\hat{1}_2 - \widehat{\Gamma}^5)/2$, we obtain the equation

$$i\hat{\gamma}_-^{\mu}\,\partial_\mu\Psi_+ - \frac{1}{\hbar c}(V_\mathrm{S} + mc^2)\hat{1}_2\Psi_- = 0. \tag{70}$$

The latter pair of equations is completely equivalent to the Dirac equation and similar to the pair of Eqs. (49) and (50) in 3+1 dimensions. However, only in the present case do the gamma matrices in Eqs. (67) and (68) satisfy the relations

$$\hat{\gamma}_\pm^{\mu}\hat{\gamma}_\mp^{\nu} + \hat{\gamma}_\pm^{\nu}\hat{\gamma}_\mp^{\mu} = 2g^{\mu\nu}\frac{1}{2}(\hat{1}_2 \pm \widehat{\Gamma}^5), \qquad \{\hat{\gamma}_+^{\mu}, \hat{\gamma}_+^{\nu}\} = \{\hat{\gamma}_-^{\mu}, \hat{\gamma}_-^{\nu}\} = \hat{0}_2. \tag{71}$$

The charge-conjugate wave function also satisfies the Dirac equation; thus, we also have two equations equivalent to the last equation. Specifically, by multiplying the Dirac equation for Ψ_C by $(\hat{1}_2 + \widehat{\Gamma}^5)/2$ and $(\hat{1}_2 - \widehat{\Gamma}^5)/2$, we respectively obtain

$$\begin{aligned} &i\hat{\gamma}_+^{\mu}\,\partial_\mu(\Psi_-)_\mathrm{C} - \frac{1}{\hbar c}(V_\mathrm{S} + mc^2)\hat{1}_2(\Psi_+)_\mathrm{C} = 0, \\ &i\hat{\gamma}_-^{\mu}\,\partial_\mu(\Psi_+)_\mathrm{C} - \frac{1}{\hbar c}(V_\mathrm{S} + mc^2)\hat{1}_2(\Psi_-)_\mathrm{C} = 0. \end{aligned} \tag{72}$$

(we recall that $(\Psi_\pm)_\mathrm{C} = \widehat{S}_\mathrm{C}\Psi_\pm^*$). We note that just as Ψ_- and Ψ_+ satisfy Eqs. (69) and (70), so $(\Psi_\mathrm{C})_-$ and $(\Psi_\mathrm{C})_+$ also satisfy them (this is because $(\Psi_\pm)_\mathrm{C} = (\Psi_\mathrm{C})_\pm$). In the case $mc^2 = V_\mathrm{S} = 0$, we obtain

$$\begin{aligned} &i\hat{\gamma}_+^{\mu}\,\partial_\mu\Psi_- = i\hat{\gamma}_+^{\mu}\,\partial_\mu(\Psi_-)_\mathrm{C} = 0 \qquad (\Rightarrow i\hat{\gamma}^{\mu}\,\partial_\mu\Psi_- = i\hat{\gamma}^{\mu}\,\partial_\mu(\Psi_-)_\mathrm{C} = 0), \\ &i\hat{\gamma}_-^{\mu}\,\partial_\mu\Psi_+ = i\hat{\gamma}_-^{\mu}\,\partial_\mu(\Psi_+)_\mathrm{C} = 0 \qquad (\Rightarrow i\hat{\gamma}^{\mu}\,\partial_\mu\Psi_+ = i\hat{\gamma}^{\mu}\,\partial_\mu(\Psi_+)_\mathrm{C} = 0). \end{aligned}$$

The Majorana condition imposed on the two-component wave function Ψ gives the relations

$$\Psi_+ = (\Psi_+)_\mathrm{C}, \qquad \Psi_- = (\Psi_-)_\mathrm{C}, \tag{73}$$

i.e., $\Psi_+ = (\Psi_\mathrm{C})_+$ and $\Psi_- = (\Psi_\mathrm{C})_-$. Thus, in contrast to 3+1 dimensions, Ψ_+ and Ψ_- satisfy the Majorana condition. Clearly, the equation that describes a Majorana particle in 1 + 1 dimensions is the pair of Eqs. (69), (70) (with matrix relations (71)) and the pair of relations, or restrictions, in (73)) (the Majorana condition). Naturally, by imposing the latter condition on the equations in (72), we again obtain Eqs. (69) and (70).

On the other hand, setting $mc^2 = V_\mathrm{S} = 0$ in Eq. (69) leads to the relation $i\hat{\gamma}_+^{\mu}\,\partial_\mu\Psi_- = 0$ $(\Rightarrow i\hat{\gamma}^{\mu}\,\partial_\mu\Psi_- = 0)$, and as can be seen in Eq. (72), we also have $i\hat{\gamma}_+^{\mu}\,\partial_\mu(\Psi_-)_\mathrm{C} = 0$ $(\Rightarrow i\hat{\gamma}^{\mu}\,\partial_\mu(\Psi_-)_\mathrm{C} = 0)$; in this case also, we have $\Psi_- = (\Psi_-)_\mathrm{C}$ (due to the Majorana condition). Similarly, setting $mc^2 = V_\mathrm{S} = 0$ in Eq. (70) leads to the relation $i\hat{\gamma}_-^{\mu}\,\partial_\mu\Psi_+ = 0$ $(\Rightarrow i\hat{\gamma}^{\mu}\,\partial_\mu\Psi_+ = 0)$, but from Eq. (72) we also have $i\hat{\gamma}_-^{\mu}\,\partial_\mu(\Psi_+)_\mathrm{C} = 0$ $(\Rightarrow i\hat{\gamma}^{\mu}\,\partial_\mu(\Psi_+)_\mathrm{C} = 0)$, where $\Psi_+ = (\Psi_+)_\mathrm{C}$ (because of the Majorana condition).

Thus, to obtain the two-component wave function that describes the 1D Majorana particle, $\Psi = \Psi_+ + \Psi_-$, we must solve the system of equations formed by Eqs. (69) and (70), but Ψ_+ and Ψ_- must satisfy the relations in Eq. (73), i.e., the Majorana condition. We note that $\Psi = \Psi_+ + \Psi_- = (\Psi_+)_\mathrm{C} + (\Psi_-)_\mathrm{C}$, and therefore $\Psi = \Psi_\mathrm{C}$, as expected.

We can prove the following results. We use Table 6 here.

1. In the Weyl representation, covariant equation (69) for the two-component wave functions $\Psi_+ = [\varphi_1\ 0]^\mathrm{T}$ and $\Psi_- = [0\ \varphi_2]^\mathrm{T}$ leads only to an equation for the one-component wave functions φ_1 and φ_2:

$$i\hbar\frac{\partial}{\partial t}\varphi_2 = +i\hbar c\frac{\partial}{\partial x}\varphi_2 + (V_\mathrm{S} + mc^2)\varphi_1. \tag{74}$$

Similarly, covariant equation (70) leads to

$$i\hbar\frac{\partial}{\partial t}\varphi_1 = -i\hbar c\frac{\partial}{\partial x}\varphi_1 + (V_{\mathrm{S}} + mc^2)\varphi_2. \tag{75}$$

This pair of equations comprises a complex system of coupled equations; it is just Eq. (33), as expected. Likewise, the Majorana condition in Eq. (73) leads to the pair of relations in Eq. (34), also as expected. Thus, we do not have a first-order equation for a single component of the wave function in the Weyl representation. Clearly, the four real degrees of freedom present in the solutions of Eqs. (74) and (75) reduce to only two due to the two relations that emerge from the Majorana condition.

Incidentally, in $1+1$ dimensions, φ_1 and φ_2 also transform in two different ways under a Lorentz boost. In effect, we write the Lorentz boost along the x-axis in the form $[ct'\ x']^{\mathrm{T}} = e^{-\omega\hat{\sigma}_x}[ct\ x]^{\mathrm{T}}$ (i.e., $x^{\mu\prime} = \Lambda^{\mu}_{\nu}x^{\nu}$), where, as usual, $\tanh\omega = v/c \equiv \beta$ and $\cosh\omega = (1-\beta^2)^{-1/2} \equiv \gamma$, with the speed v' of the (inertial) reference frame with respect to the unprimed (inertial) reference frame being v. Then, under this Lorentz boost, the wave function transforms as $\Psi'(x',t') = \widehat{S}(\Lambda)\Psi(x,t)$, where $\widehat{S}(\Lambda) = e^{-\omega\widehat{\Gamma}^5/2}$ with $\Lambda^{\mu}_{\nu}\hat{\gamma}^{\nu} = \widehat{S}^{-1}(\Lambda)\hat{\gamma}^{\mu}\widehat{S}(\Lambda)$. Then, just in the Weyl representation, the matrix $\widehat{S}(\Lambda)$ is diagonal, and we obtain the results

$$\varphi_1'(x',t') = \left[\cosh\frac{\omega}{2} - \sinh\frac{\omega}{2}\right]\varphi_1(x,t), \qquad \varphi_2'(x',t') = \left[\cosh\frac{\omega}{2} + \sinh\frac{\omega}{2}\right]\varphi_2(x,t). \tag{76}$$

Thus, we have two different kinds of one-component wave functions in $1+1$ dimensions. Certainly, not only φ_1 and φ_2 satisfy the relations in (76) but so also do $(\varphi_1)_{\mathrm{C}}$ and $(\varphi_2)_{\mathrm{C}}$. This is because Ψ and Ψ_{C} transform similarly under Lorentz boosts (i.e., $\Psi_{\mathrm{C}}'(x',t') = \widehat{S}(\Lambda)\Psi_{\mathrm{C}}(x,t)$). Interestingly, in the case where $mc^2 = V_{\mathrm{S}} = 0$, the wave functions with definite chirality, Ψ_+ and Ψ_-, each satisfy the one-dimensional Dirac equation and their own Majorana conditions. Also, in the Weyl representation, the nonzero component of each of these two chiral wave functions satisfies the Weyl equation (see Eqs. (74) and (75)). Thus, we could call the particles described by Ψ_+ and Ψ_- Weyl–Majorana particles [26].

2. In the Dirac representation, Eq. (69) leads to Eq. (74), and Eq. (70) leads to Eq. (75) with the replacements $\varphi_1 \to \varphi + \chi$ and $\varphi_2 \to \varphi - \chi$. Likewise, Majorana condition (73) leads precisely to the pair of relations in Eq. (34) with the last replacements, namely, $\varphi + \chi = -i(\varphi + \chi)^*$ and $\varphi - \chi = +i(\varphi - \chi)^*$ (these two relations imply the result given in Eq. (23)). We recall that the two-component wave functions in the Dirac and Weyl representations are related as $[\varphi_1\ \varphi_2]^{\mathrm{T}} = \widehat{S}[\varphi\ \chi]^{\mathrm{T}}$, where the matrix $\widehat{S}$ is given in Eq. (15). Certainly, adding and subtracting the two equations obtained here and conveniently using the Majorana condition again, we obtain an equation for the component φ of the wave function, namely, Eq. (24), and an equation for the component χ of the wave function, namely, the same Eq. (24) but with the replacements $\varphi \to \chi$ and $V_{\mathrm{S}} + mc^2 \to -(V_{\mathrm{S}} + mc^2)$. As explained before, it is sufficient to solve only one of these two equations because the remaining component can be obtained from the Majorana condition. Thus, only two real quantities, or real degrees of freedom, are sufficient to fully describe the Majorana particle.

3. In the Majorana representation, Eq. (69) leads to Eq. (74), and Eq. (70) leads to Eq. (75) with the replacements $\varphi_1 \to (1-i)(\phi_1 + \phi_2)$ and $\varphi_2 \to (1+i)(\phi_1 - \phi_2)$. We recall that the two-component wave functions in the Majorana and Weyl representations are related by $[\varphi_1\ \varphi_2]^{\mathrm{T}} = \widehat{S}^{-1}[\phi_1\ \phi_2]^{\mathrm{T}}$, where the matrix $\widehat{S}$ is given in Eq. (14). In this representation, $\widehat{S}_{\mathrm{C}} = \hat{1}_2$; therefore, Majorana representation (73) is simply $\Psi_+ = \Psi_+^*$ and $\Psi_- = \Psi_-^*$, i.e., the last condition immediately yields the pair of relations $\phi_1 + \phi_2 = \phi_1^* + \phi_2^*$ and $\phi_1 - \phi_2 = \phi_1^* - \phi_2^*$ (which implies the result in Eq. (43), i.e., the entire two-component wave function must be real). Finally, adding and subtracting the two equations obtained here (but before multiplying the equation that arises from Eq. (74) by $i(1-i)$ and multiplying the one that emerges from Eq. (75) by $i(1+i)$), we obtain a real system of coupled equations, namely, the system in Eq. (44). Because the solutions of this system are real-valued, the wave function has two real degrees of freedom, as expected.

6. Conclusions

Distinct differential equations can be used to describe a Majorana particle in $3+1$ and $1+1$ dimensions. We can have a complex single equation for a single component of the Dirac wave function, as in the Dirac and Weyl representations in $3+1$ dimensions (in these cases, the single component itself is a two-component wave function), and in the Dirac representation in $1+1$ dimensions (in this case, the single component itself is a one-component wave function). In the Weyl representation in $3+1$ dimensions, we can have two complex single equations, each being invariant under its own type of Lorentz transformation (or Lorentz boost); these two two-component covariant equations are therefore nonequivalent equations, and each of them can describe a specific type of 3D Majorana particle. Certainly, because of the Majorana condition, the solutions of these two equations are not independent of each other, that is, in the concrete description of the Majorana particle, two plus two (complex) components are not absolutely necessary (the solution of only one of the two two-component Majorana equations is what is needed to fully describe each type of the Majorana particle). Unexpectedly, in the Weyl representation in $1+1$ dimensions, we have a complex system of coupled equations, i.e., no first-order equation for any of the components of the wave function can be written. On the other hand, we can also have a real system of coupled equations, as it is in the Majorana representation in $3+1$ and $1+1$ dimensions.

All these equations and systems of equations emerge from the Dirac equation and the Majorana condition when a representation is chosen. Certainly, both the Dirac equation and the Majorana condition look different written in their component forms when different representations are used. In any case, whichever equation or system of equations is used to describe the Majorana particle, the wave function that describes it in $3+1$ or $1+1$ dimensions is determined by four or two real quantities (real components, real and imaginary parts of complex components, or just real or just imaginary parts of complex components), i.e., only four or two real quantities are sufficient.

Likewise, in $3+1$ dimensions, the algebraic procedure introduced by Case (and reexamined by us) allows writing two covariant equations of four components for the Majorana particle, in a form independent of the choice of a particular representation for the matrices $\widehat{\Gamma}^{\mu}$ and $\widehat{\Lambda}^{\mu}$ (see Eqs. (53) and (55)). Each of these equations provides one of the two covariant two-component Majorana equations that arise when choosing the Weyl representation. In contrast, in $1+1$ dimensions, the algebraic procedure introduced by us leads to only a covariant system of coupled first-order equations of two components, and these components have their complex degrees of freedom restricted by two conditions that arise from the Majorana condition. This system of equations immediately yields a complex system of coupled first-order equations for one component that emerges when using the Weyl representation, with the restriction given by the Majorana condition. However, in the Dirac representation, the same system of equations, together with the Majorana condition, can lead to two one-component equations (each for a single component of the two-component wave function).

It is hoped that our results can be useful in studying the distinct differential equations that can arise when describing the Majorana particle in $1+1$ and $3+1$ dimensions. As we have seen, the results obtained in these two space–time dimensions are not completely analogous. It is to be expected that these results also exhibit important differences from results in $2+1$ dimensions. However, in the latter case other difficulties can also arise. Definitely, these issues should be treated in another publication.

Conflicts of interest. The author declares no conflicts of interest.

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