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# ON THE FIRST NEWTON LAW, THE EXISTENCE OF THE REFERENCE SYSTEM CORRESPONDING TO THE REST, AND THE GALILEI GROUP

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*In the Russian scientific literature, the authors carefully avoided mentioning the fact that Ernst Mach's assertion of 1904 about the meaninglessness of the concepts of uniform motion and absolute time in and of themselves has never been challenged in classical* (*not quantum*) *mechanics. Using hydrodynamics as an example, V. I. Arnold showed that a system of coordinates in which some finite volume of the medium is at rest exists almost always. We interpret this result as the existence of a system of coordinates in which all particles are at rest* (*the rest frame*).

**Keywords:** continuum, Lagrangian and Eulerian descriptions, reference frame and system of coordinates, Galilei group, Noether theorem

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## 1. Introduction

Newton formulated his first law in Definition III of his fundamental book [1]: "Inherent force of matter is the power of resisting by which every body, so far as it is able, *perseveres in its state either of resting or of moving uniformly straight forward*" (the italics are mine, used to select the text that is called Newton's first law in modern physics).

A question arises immediately: whence is it known that there exists a reference frame in which this state of rest is realized? Newton did not have to answer this question: he adhered to the concept of absolute time and the absolute space. He formulated it below in Scholium (Newton's italics here):

I. *Absolute, true, and mathematical time*, of itself, and from its own nature, flows equably without relation to anything external, and by another name is called duration.

II. *Absolute space*, in its own nature, without relation to anything external, remains always similar and immovable.

Modern classical mechanics has abandoned the concept of absolute time and absolute space, and universal *symmetry* properties are postulated as a fundamental property of the geometrical object of *space–time*: covariance with respect to the Galilei group.

Nevertheless, the question of the existence of the rest frame remains. Until relatively recently, the answers to it in the physical literature were confined to the ideological legacy of the epoch of the historical materialism. For example, in the "Physical Encyclopedia" published in 1988, in the article "Inertial reference frame," we read: "The inertial reference frame is a reference frame where the law of inertia is valid: in the case where the material point is not subjected to any forces (or the mutually balanced forces), it is in the state of rest or uniform rectilinear motion. . . . The concept of the inertial reference frame is a scientific abstraction. A real reference frame is always associated with some body."

*Deceased.

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A more intelligible approach, from our standpoint, was proposed by Fock in his monograph [2]: "In different reference frames, the mathematical forms of the laws of nature is generally different. ... There are reference frames in which the laws of motion have an especially simple form and which (in a certain sense) bear the closest correspondence to the nature. We mean inertial reference frames, in which the body moves rectilinearly and uniformly in the absence of forces acting on it. (Here, the question arises as to how to ensure that there are no forces acting on the body; we assume that there are no forces if all bodies that can have any effect are sufficiently far from the chosen body.)"

And only in 2000, an Izhevsk publishing house ventured to publish (without any comments) the Mach book [3], which is fundamental in this field (the text given below was published in 1904). Mach stated (Mach's italics): "We may not forget that all things are inseparably linked with one another and that we with all our thoughts are only a part of the nature. We absolutely cannot *measure, by means of the time*, the change in things. On the contrary, the time is an abstraction, at which we arrive using the change in things, because we have no *definite* measure, since all of them are related to one another. We call the motion uniform if equal increments of the path correspond to those of another motion that was chosen for comparison (the Earth rotation). The motion can be uniform relative to another one. The problem whether the motion *in itself* is uniform, has no meaning. We also cannot speak about the 'absolute time' (irrespective of any change). This absolute time cannot be measured by means of any motion and, therefore, is of no practical, scientific importance, nobody can say that he knows something about such a time, this concept is useless and 'metaphysical'."

In this paper, we demonstrate that Mach and his implicit followers were wrong: in 1971, Arnold published monograph [4], where, using the hydrodynamics, he proved a theorem on rectification, which guarantees the existence of the state of rest almost always.

## 2. Lagrangian and Eulerian descriptions

I suspect that the authors of textbooks and monographs did not fully realize how much the situation related to the term "reference frame" is confused in the Russian literature. I take a certain risk to begin with its heuristic definition proposed by Truesdell in his fundamental monograph [5]: "For purposes of classical mechanics, the reference frame is represented in the form of the absolutely solid body supplied with a clock." Then Truesdell gave rigorous mathematical definitions of the measurement unit and the reference point of time, of the absolutely solid body (in terms of "rigid motion"), of the freedom in choosing the *coordinate system* on that body, of the naturalness of Cartesian coordinates on it, etc. It seems important to me that in this case, there are fundamental restrictions on the algebra of observables "living" in the described reference frames: addition is defined in this algebra only for quantities that have the same dimension and only if they refer to the same reference frame.

In this scheme, the concepts of the *reference frame* and the *coordinate system* are different. It allows easily and consistently formulating the achievements in theoretical physics such as the least action principle, Noether's theorem, mechanics with Dirac's nonholonomic constraints, as well as giving meaning to the concept of the energy of the Universe (see, e.g., Faddeev's review [6] ).

The opposite standpoint goes back to Sedov [7]. He asserted the identity of the concepts of the reference frame and the coordinate system at least twice, on pages 14 and 18 of the first volume of his monograph "A Course in Continuum Mechanics." There, Sedov introduces the concept of comoving coordinate system (it is different at different instants of time): the coordinates of a point of space are defined as the Cartesian coordinates at the initial instant of time $x^i(t_0) = \xi^i$ of the particle that turned out to be at this point at the instant $t$. In this case, the motion of medium particles is perceived as pure deformation without violating the continuity; for example, the boundary of a moving region consists of the same particles, between which only distances can change.

Following Sedov, some of his students and followers take the concept of absolute time and easily declare that the comoving coordinate system is a reference frame, and state that the medium particles are at rest in this reference frame. It is worth noting that their textbooks and monographs have do not mention the least action principle and Noether's theorem.

A depressing example is given by the beginning of § 5 of the overall unique treatise [8], where arguments in favor of the "comoving reference frame" are adduced in the framework of the Newtonian axiomatics; it is apparent that the author read Newton's writings poorly (if at all).

We pass to the kinematics of the continuum. It is necessary to define more precisely what underlies the purely kinematic problem of the comparison between the Lagrangian and Eulerian descriptions of hydrodynamics means and to refine the meaning of these contents.

We choose a hydrodynamic model (it does not matter which) and its evolution equations. In the Lagrangian description, these equations represent the localized Newton's second law in Cartesian coordinates $x^i_{\xi^i}(t)$, $i = 1, 2, 3$, of particles labeled with the initial data $x^i(t_0) = \xi^i$ of the Cauchy problem for that law (these initial data are called Lagrangian coordinates, which is a misnomer in my opinion). Let $U \in \mathbb{R}^3$ and $T = [t_0, t_1]$ be the region and the time interval where the solution of the Cauchy problem exists and is unique. The Lagrangian description amounts to specifying the coordinates $x^i_{\xi^i}(t)$ of particles in the region $U$ at any instant of time $t \in T$. The particle velocity in $U \times T$ is defined by the formula

$$v^i(x^i_{\xi^i}, t) = \frac{\partial x^i_{\xi^i}(t)}{\partial t} \equiv \dot{x}^i(x^i_{\xi^i}, t). \tag{1}$$

In the Eulerian description of this chosen hydrodynamic model, the evolution equation is the Euler equation for the flow velocity $v^i(t)$ with the Cauchy data $v^i(t_0) = \dot{x}^i(x^i, t_0)$. As a rule, this Cauchy problem has a unique solution in the same region as the Cauchy problem of the Lagrangian description. Thus, the Eulerian description amounts to specifying velocity field $v^i(t)$ in the region $U$ at any instant of time $t \in T$.

The passage from the Lagrangian description to the Eulerian one is trivial and reduces to the definition of particle velocity (1). The passage from the Eulerian description to the Lagrangian one is less trivial and requires proofs of the existence and uniqueness of the solution of the system of ordinary differential equations

$$\dot{x}^i(x^i_{\xi^i}, t) = v^i(x^i_{\xi^i}, t) \quad \text{with the initial condition} \quad x^i(t_0) = \xi^i. \tag{2}$$

The solution of this problem is expounded in detail in classical monographs [9], [10]. However, only in Arnold's book [4], *the theorem on rectification* was also formulated and proved. It seems natural for the autonomous analogue of system (2)

$$\dot{x}_i(x_i, t) = v^i(x_i). \tag{3}$$

Let $x^i$ be the solution of system of equations (3). Then for any sufficiently smooth vector field $v^i(x^i, t)$, there exists a nonsingular change of variables, a map $Z\colon \{x^i\} \to \{y^i\}$, such that system of equations (3) is equivalent to the system

$$\dot{y}^i(y^i, t) = e^i, \tag{4}$$

where $e^i$ is a constant vector with the single nonzero component, for example, $e^i = \{1, 0, 0\}$. In the physical language, this means that the change of variables allows representing the stationary liquid flow as the uniform motion of all particles in the same direction.

For nonautonomous system (2), Arnold proposed the trick to consider it in the space $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ by adding the equation $\dot{t} = 1$ to system (3); then $U$ still lies in $\mathbb{R}^3$, and $T$ is in $\mathbb{R}$. The obtained system of four equations turns out to be autonomous, the analogue of the map $Z$ leaves time unchanged, $\tilde{Z}\colon \{x^i, t\} \to \{y^i, t\}$, and the only nonzero component of the constant vector $\tilde{e}$ is the time component:

$\tilde{e} = \{0, 0, 0, 1\}$. In the physical language, this means that there exists a transformation of coordinates in $T \times U$ such that in the new reference frame, all particles are at rest, and hence this is the *rest frame*.

It is hardly worthwhile to present the rigorous mathematical realization of this idea in this paper. I refer the reader directly to the original source, Arnold's book [4].

## 3. Discussion

We recall that in classical mechanics, the symmetry properties of space–time are a fundamental postulate: the uniformity of time, the homogeneity and isotropy of space, and the equivalence of mutually inertial reference frames. More precisely, the Galilean relativity principle is postulated in the form of the covariance of dynamical variables and dynamical equations under Galilei group transformations

$$t \Rightarrow t' = t + \tau, \qquad x^i \Rightarrow x'^i = x^i + \alpha^i + \varepsilon^{ijk}\omega^j x^k + \nu^i t, \tag{5}$$

where $\tau$, $\alpha^i$, $\omega^j$, and $\nu^i$ are constants, with $\omega^j$ being small. The Galilean relativity principle does not postulate the existence of a rest frame; as a matter of fact, it only introduces the concept of mutually inertial reference frames.

So far so good, but the corollary of the Arnold rectification theorem formulated in the preceding section shatters the basic principles, and it is worth discussing how "to live in the face of newly discovered evidence."

We have some experience in this case. In particular, the remarkable relation between Galilei group transformations and conservations laws appears for the class of mechanical models whose dynamics is determined by the least action principle. Because Noether's theorem is constructive, we can define energy as a dynamical variable that is conserved for a closed system because of the uniformity of time, as a consequence of the invariance under the first transformation in (5) with the parameter $\tau$, and we can define the momentum as a dynamic variable that is conserved for a closed system because of the homogeneity of space, the covariance under the second transformation in (5) with the vector parameter $\alpha^i$, and so on. Because the conservation laws for energy, momentum, and so on are confirmed experimentally with the available accuracy, this fact can be regarded as experimental proof of the Galilean relativity principle (of the Poincaré relativity principle replacing it in the relativistic domain).

Another property of the Galilei group transformations is their demonstrativeness: we can trace them visually or they can be represented as a diagram.

Nothing of the kind appears to be the case for the rectification theorem. All experiments are conducted in the laboratory reference frame accessible to us. In the Eulerian description, we assume that the velocity field $v^i(x^i, t)$ is known in the region $U \times T$. Then the rectified velocity field $u^i(y^i, t) = \{1, 1, 1\}$ (we add the unit shift from the Galilei group in the new coordinates) is related to the initial derived map $\tilde{Z}_*^{-1}$:

$$\dot{x}^i = C^i_j u^j, \qquad C^i_j = \frac{\partial x^i}{\partial y^j}.$$

The nondegenerate matrix $C$ can be represented in the form $C = UKW$, where $U$ and $W$ are unitary matrices dependent on the coordinates, and $K$ is a diagonal matrix of point-dependent scaling transformations. The sole obvious fact is that the transformation $\tilde{Z}^{-1}$ applies to the Cartesian coordinates of the region $U \in \mathbb{R}^3$ and is not contained in the Galilei group. It is unclear how to extract the information about the matrix $C = UKW$ from experimental data. The existence of the transformation $\tilde{Z}^{-1}$ is apparently not at a property of space–time (such as the symmetry under the Galilei group), but is a mathematical consequence of the fact that (in Arnold's example) the relation between the Lagrangian and Eulerian descriptions is determined by the system of ordinary differential equations with a sufficiently smooth right-hand side.

To conclude, I am puzzled by how several generations of physicists failed to notice the assertion, contained in § 139 of "Fluid Mechanics," Volume 6 of the theoretical physics course [11], that the Galilean relativity principle allows constructing the momentum flux density tensor of the liquid using its value in a reference frame arbitrarily moving with respect to the selected one.

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