

RELATIVISTIC LINEAR OSCILLATOR UNDER THE ACTION OF A CONSTANT EXTERNAL FORCE. WAVE FUNCTIONS AND DYNAMICAL SYMMETRY GROUP

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An exactly solvable relativistic model of a linear oscillator is considered in detail in the presence of a constant external force in both the momentum representation and the relativistic configuration representation. It is found that in contrast to the nonrelativistic case, depending on the magnitude of the force, both discrete and continuous energy spectra are possible. It is shown that in the case of a discrete spectrum, the wave functions in the momentum representation are expressed in terms of the Laguerre polynomials, and in the relativistic configuration representation, in terms of the Meixner–Pollaczek polynomials. Integral and differential–difference formulas are found connecting the Laguerre and Meixner–Pollaczek polynomials. A dynamical symmetry group is constructed.

Keywords: relativistic linear oscillator model, uniform field, finite-difference equation, dynamical symmetry group, relation between orthogonal polynomials

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1. Introduction

The harmonic oscillator model played a fundamental role in the emergence and development of nonrelativistic quantum mechanics. This is one of the few exactly solvable quantum mechanical problems. The harmonic oscillator potential is important for applications in nonrelativistic quantum mechanics. It finds wide application in atomic and molecular physics, statistical mechanics, quantum chemistry, nuclear theory, quantum electrodynamics, hadronic physics, etc. (see, e.g., [1]–[3]). The emergence of quark models for describing the properties and structures of hadrons has increased the interest of physicists in the harmonic potential. The development of quark models led to the need to construct relativistic wave functions of composite particles and, in particular, relativistic models of a harmonic oscillator. The wave function contains all the information about a quantum system. Knowledge of the relativistic wave functions, for example, allows calculating the form factors of elastic scattering, the probability and width of meson decays, and the structure functions of hadrons.

The problem of generalizing the quantum nonrelativistic harmonic oscillator to the relativistic case has been discussed for a long time in the literature, but there is still no unambiguous definition of a relativistic harmonic oscillator. In other words, in contrast to the theory of the nonrelativistic harmonic oscillator, the theory of the relativistic harmonic oscillator is still far from completion.

This problem is usually formulated within the framework of various approaches using equations such as Klein–Gordon, Dirac, Salpeter, finite-difference equations in relativistic configuration space, etc. The

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first models of a relativistic harmonic oscillator used in elementary particle physics appeared in the 1950s in [4], [5]. These were followed by the work by Feynman et al. [6]. The impetus for the study of relativistic oscillatory models was the desire to solve the problem of the mass spectrum of elementary particles. These four-dimensional oscillatory models were described by relativistic equations of the Klein–Gordon type, in which the direct relativistic generalization $x^\mu x_\mu = x_0^2 - \mathbf{x}^2$ of the nonrelativistic harmonic oscillator potential was used. Such equations have spurious solutions associated with oscillations along the time axis (see also reviews [7], [8]). In this regard, we especially note paper [6], where the authors used a four-dimensional relativistic oscillator model to describe the spectrum and decays of hadrons and faced the problem of nonphysical degrees of freedom. These nonphysical degrees of freedom primarily occur because the four-dimensional oscillator includes timelike states with a negative norm. Excluding these states from consideration leads, in particular, to violating the unitarity condition of the theory, because the complete set of states is then not used. Subsequently, a four-dimensional oscillatory model was developed in [9]–[11] (see also [12]). In [13], a special relativistic harmonic oscillator was proposed, modeled by the Klein-Gordon theory in the anti-de Sitter space (also see [14]).

In [15], by replacing the momentum in the Dirac equation \mathbf{p} with $\mathbf{p} - i\beta m\omega \mathbf{r}$, where β is the standard Dirac matrix, m is the particle mass, and ω is the oscillator frequency, a new exactly solvable potential of the relativistic harmonic oscillator was obtained. The relativistic oscillator obtained in this way is called the Dirac oscillator and describes a particle with spin 1/2. In the nonrelativistic limit, it transforms into a three-dimensional isotropic harmonic oscillator with spin–orbit coupling. The Dirac oscillator has found wide application in the physics of the nucleus and elementary particles (see, e.g., works [16]–[20]).

Relativistic harmonic oscillators described by the spinless Salpeter equation [21], [22] are considered in [22], [23]. We recall that in the case of one quantum particle, the Hamiltonian of the spinless Salpeter equation with a static interaction potential $V(\hat{\mathbf{r}})$ has the form $H = \sqrt{\hat{\mathbf{p}}^2 + m^2} + V(\hat{\mathbf{r}})$, i.e., it is a simple and straightforward generalization of the corresponding Schrödinger Hamiltonian. For $l = 0$, wave functions in the momentum representation are found as infinite series in [22], and exactly, in [23].

On the other hand, the concept of a relativistic configuration \mathbf{r} -representation introduced in [24], [25] provided the basis for constructing a finite-difference version of relativistic quantum mechanics, which has many important features of nonrelativistic quantum mechanics. The main difference between this version of the theory and quantum mechanics is that the wave function in it satisfies a finite-difference equation with a step equal to the Compton wavelength of a particle $\lambda = \hbar/mc$. We emphasize that the momentum space canonically conjugate to the \mathbf{r} -space is the Lobachevsky space, realized on the upper sheet of the particle mass shell $p_0^2 - \mathbf{p}^2 = m^2c^2$, $p_0 > 0$.

In [26]–[37], various finite-difference equations were investigated, giving a generalization of the harmonic oscillator problem to the relativistic case. In [26], [29], [33] relativistic models of a three-dimensional harmonic oscillator were studied in the relativistic configuration \mathbf{r} -representation. We note that these three-dimensional relativistic oscillator models are free from the disadvantages inherent in four-dimensional relativistic oscillators [4]–[8]. For example, their wave functions are normalized and allow a probabilistic interpretation.

In [27], [28], an exactly solvable linear oscillator model in the relativistic configuration x -representation was considered in detail. Preprint [30] is devoted to the study of this model in the presence of a uniform external field $V_g(x) = gx$, which corresponds to a constant external force $F(x) = -\nabla_x V_g(x) = -g$, where $\nabla_x = \partial_x$. It is shown that in contrast to the corresponding nonrelativistic case, depending on the magnitude of the force $|g|$, both a discrete spectrum and a continuous energy spectrum are possible.

The purpose of this paper is to present the results of the unpublished preprint [30]. This preprint has been referenced in a number of papers (see, e.g., [34]–[37]), i.e., its results are of physical and mathematical interest and have not lost their relevance to this day. For the model under consideration, generalized coherent

states were constructed in [34] and Wigner functions for stationary states, as well as for thermodynamic equilibrium states, were constructed in [36].

It was shown in preprint [30] that in the presence of a constant external force, the wave functions of the relativistic oscillator belonging to the discrete spectrum in the relativistic configuration x -space are expressed in terms of the Meixner–Pollaczek polynomials. The dynamical symmetry group and coherent states were found, as was a new finite-difference analogue of the Rodrigues formula for the Meixner–Pollaczek polynomials. A bilinear generating function for the Meixner–Pollaczek polynomials was obtained, with the help of which the Green’s function was calculated. In this work, we present only part of these results.

The paper is structured as follows. Section 2 is devoted to a brief description of the nonrelativistic linear oscillator in an external uniform field. Section 3 details the relativistic model of a linear oscillator in an external uniform field. In Sec. 4, a dynamical symmetry group is constructed. The results are discussed in Sec. 5. The appendix contains the proof of a limit formula for the Meixner–Pollaczek polynomials.

2. Nonrelativistic linear oscillator in an external uniform field

Of the many different properties of a harmonic oscillator, we note the following two. A harmonic oscillator has 1) an infinite set of bound states, whose corresponding energy levels are equidistant; 2) a class of solutions in the form of coherent states (CS). CSs minimize the product of the coordinate and momentum uncertainty by Gaussian wave packets, whose shape is preserved over time. In nonrelativistic quantum mechanics, the Hamiltonian of a linear oscillator in a uniform external field

$$H_N = -\frac{\hbar^2}{2m}\nabla_x^2 + \frac{1}{2}m\omega^2 x^2 + gx \quad (1)$$

has the eigenfunctions [1]

$$\psi_{Nn}(x) = C_{Nn} H_n \left((x + x_0) \sqrt{\frac{m\omega}{\hbar}} \right) e^{-(m\omega/2\hbar)(x+x_0)^2}, \quad (2)$$

where $x_0 = g/m\omega^2$, $n = 0, 1, 2, \dots$ and $H_n(x)$ are Hermite polynomials. Wave functions (2) correspond to the energy levels

$$E_{Nn} = E_{Nn}^{(0)} - \frac{m\omega^2}{2}x_0^2, \quad E_{Nn}^{(0)} = \hbar\omega \left(n + \frac{1}{2} \right) \quad (3)$$

and satisfy the orthonormality condition

$$\int_{-\infty}^{\infty} \psi_{Nn}^*(x) \psi_{Nm}(x) dx = \delta_{nm}. \quad (4)$$

From this condition for the normalization constant, we obtain

$$C_{Nn} = \frac{C_{N0}}{\sqrt{2^n n!}}, \quad C_{N0} = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4}.$$

The wave functions $\psi_{Nn}(x)$ in (2) in the x -representation can be obtained from the wave functions $\psi_{Nn}^{(0)}(x)$ of a nonrelativistic linear oscillator without a field ($g = 0$) by a simple shift,

$$\psi_{Nn}(x) = U_N \psi_{Nn}^{(0)}(x) = \psi_{Nn}^{(0)}(x + x_0), \quad (5)$$

using the shift operator

$$U_N = e^{ix_0\hat{p}/\hbar} = e^{x_0\nabla_x}. \quad (6)$$

In momentum representation, this shift reduces to multiplication by a simple phase factor:

$$\Phi_{Nn}(p) = e^{ix_0 p/\hbar} \Phi_{Nn}^{(0)}(p). \quad (7)$$

As is well known, if we introduce the bosonic creation and annihilation operators

$$b^\pm = U_N a^\pm U_N^{-1} = a^\pm + \frac{\xi_0}{\sqrt{2}}, \quad [b^-, b^+] = 1, \quad (8)$$

where $a^\pm = (\xi \mp \nabla_\xi)/\sqrt{2}$ and $\xi = x\sqrt{m\omega/\hbar}$, $\xi_0 = x_0\sqrt{m\omega/\hbar}$, then we can find eigenfunctions (2) and eigenvalues (3) of Hamiltonian (1) in a purely algebraic way. However, here we give only two formulas: a formula for the Hamiltonian in Eq. (1) expressed in terms of b^+ and b^- (or a^+ and a^-),

$$H_N = \hbar\omega \left(b^+ b^- + \frac{1}{2} \right) - \frac{1}{2} m\omega^2 x_0^2 = \hbar\omega \left[a^+ a^- + \frac{\xi_0}{\sqrt{2}} (a^+ + a^-) + \frac{1}{2} \right] \quad (9)$$

and a formula for normalized eigenfunctions (2),

$$\psi_{Nn}(x) = \frac{1}{\sqrt{n!}} (b^+)^n \psi_{N0}(x). \quad (10)$$

The action of the operators b^- and b^+ on eigenfunctions (10) is given by the relations

$$b^- \psi_{Nn} = \sqrt{n} \psi_{N,n-1}, \quad b^+ \psi_{Nn} = \sqrt{n+1} \psi_{N,n+1}. \quad (11)$$

3. Relativistic linear oscillator in a uniform external field

We consider a finite-difference equation describing a relativistic linear oscillator in the presence of a constant external force $F(x) = -g$ [30]:

$$H(x)\psi(x) \equiv mc^2 \left[\cosh(i\lambda\nabla_x) + \frac{m\omega^2}{2} x(x + i\lambda) e^{i\lambda\nabla_x} + gx \right] \psi(x) = E\psi(x). \quad (12)$$

It is known that solutions of this equation, in principle, can contain arbitrary functions of the variable x with the period $i\lambda$ (so-called $i\lambda$ -periodic constants, which appear in solutions due to the finite-difference nature of Hamiltonian (12)), i.e., if $\psi(x)$ is a solution of Eq. (12), then $C(x)\psi(x)$ is a solution of this equation, with $C(x \pm i\lambda) = C(x)$ (see also [38]–[41]). These $i\lambda$ -periodic constants can affect the asymptotic behavior of the wave function. Based on this property of the finite-difference equation, an interesting explanation for quark confinement was given in [42]. We fix the form of $i\lambda$ -of periodic constants based on the condition that the relativistic wave function have the correct nonrelativistic limit. Thus, we consider the wave function $\psi(x)$ in the domain $C_0^\infty(-\infty, +\infty)$ —in the space of infinitely differentiable functions with a compact support contained in the interval $(-\infty, +\infty)$. This means that as $|x| \rightarrow \infty$, $\psi(x)$ and all its derivatives of any order tend to zero faster than any power of $|x|^{-1}$.

The Hamiltonian $H(x)$ of Eq. (12) is a Hermitian operator with respect to the inner product

$$(\psi_1, \psi_2) = \int_{-\infty}^{\infty} \psi_1^*(x) \psi_2(x) dx.$$

From the Hermiticity condition for the Hamiltonian, $H^+(x) = H(x)$, it follows that the parameter g (force) is real. To solve Eq. (12), we pass to the p -representation. In our case, the momentum space p is the

one-dimensional Lobachevsky space, realized on the mass-shell hyperbola $p_0^2 - p^2 = m^2c^2$, $p_0 > 0$. The transition to the x -representation

$$\psi(x) = \frac{mc}{\sqrt{2\pi\hbar}} \int \frac{dp}{p_0} \xi(p, x)\psi(p), \quad p_0 = \sqrt{m^2c^2 + p^2}, \quad (13)$$

and the inverse transition to the p -representation

$$\psi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int \xi^*(p, x)\psi(x) dx \quad (14)$$

are carried out using the expansion of the wave functions in terms of a complete orthogonal system of functions [24], [25], [27]

$$\xi(p, x) = \left(\frac{p_0 - p}{mc}\right)^{-ix/\lambda} = e^{ix\chi/\lambda}, \quad \chi = \ln \frac{p_0 + p}{mc}. \quad (15)$$

They make up a basis of an irreducible unitary representation of the group of motions of the one-dimensional Lobachevsky space and satisfy the completeness and orthogonality conditions

$$\begin{aligned} \frac{mc}{2\pi\hbar} \int \frac{dp}{p_0} \xi^*(p, x)\xi(p, x') &= \delta(x - x'), \\ \frac{1}{2\pi\hbar} \int \xi^*(p, x)\xi(p', x) dx &= \frac{1}{mc}\delta(\chi - \chi'). \end{aligned} \quad (16)$$

We note that the Hamiltonian of Eq. (12) and relativistic plane waves (15) have the correct nonrelativistic limit:

$$\lim_{c \rightarrow \infty} [H(x) - mc^2] = H_N(x), \quad \lim_{c \rightarrow \infty} \xi(p, x) = e^{ipx/\hbar}.$$

In the momentum representation, Eq. (12) takes the form of a second-order differential equation

$$\begin{aligned} H(\zeta)\psi(\zeta) &\equiv \hbar\omega \left[-\zeta\nabla_\zeta^2 + i\rho\zeta\nabla_\zeta + \frac{\zeta}{4} + \frac{1}{\omega_0^2\zeta} \right] \psi(\zeta) = E\psi(\zeta), \\ \zeta = \frac{2c(p_0 + p)}{\hbar\omega} = \frac{2}{\omega_0} e^\chi, \quad 0 \leq \zeta < \infty, \quad \rho = \frac{g}{mc\omega}, \quad \omega_0 = \frac{\hbar\omega}{mc^2}, \end{aligned} \quad (17)$$

with the boundary conditions $\zeta^{-1/2}\psi(\zeta)|_{\zeta=0} = 0$ and $\psi(\infty) = 0$. Hence, the functions $\psi(\zeta)$ are defined in the space of square-integrable functions with weight ζ^{-1} on the interval $(0, +\infty)$. Setting (17)

$$\psi(\zeta) = \zeta^\nu e^{(i\rho - \delta)\zeta/2} \varphi(\zeta), \quad \delta = \sqrt{1 - \rho^2}, \quad \nu = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\omega_0^2}}, \quad (18)$$

we arrive at an equation for $\varphi(\zeta)$,

$$\zeta\varphi''(\zeta) + (2\nu - \delta\zeta)\varphi'(\zeta) + \left(\frac{E}{\hbar\omega} - \nu\delta\right)\varphi(\zeta) = 0 \quad (19)$$

with the boundary condition $\varphi(\zeta) < \infty$. This is the confluent hypergeometric equation [43]. Because the parameter $\nu > 1$ takes noninteger values, the general solution of Eq. (19) has the form [1], [43]

$$\varphi(\zeta) = C_1 \Phi\left(\nu - \frac{E}{\hbar\omega\delta}, 2\nu; \delta\zeta\right) + C_2 \zeta^{1-2\nu} \Phi\left(1 - \nu - \frac{E}{\hbar\omega\delta}, 2 - 2\nu; \delta\zeta\right),$$

where $\Phi(a, c; x)$ is the confluent hypergeometric function. Now taking the boundary condition $\varphi(\zeta) < \infty$ into account, we find that $C_2 = 0$. From this, we obtain an explicit form of the wave function in the momentum p -representation for the considered relativistic linear oscillator in an external field:

$$\psi(\zeta) = C' \zeta^\nu e^{(i\rho-\delta)\zeta/2} \Phi\left(\nu - \frac{E}{\hbar\omega\delta}, 2\nu; \delta\zeta\right). \quad (20)$$

In this case, in contrast to the nonrelativistic case, the requirement $\psi(\infty) = 0$ (boundary condition) for wave function (20) imposes the constraint $|g| < mc\omega$ (i.e., $|\rho| < 1$, $0 < \delta \leq 1$) on the value of the parameter g (force) and leads to the energy quantization condition $\nu - E/\hbar\omega\delta = -n$, $n = 0, 1, 2, \dots$. Therefore, the energy levels of the system are equidistant and equal to

$$E_n = \hbar\omega\delta(n + \nu), \quad n = 0, 1, 2, \dots \quad (21)$$

It is clear that this expression for $c \rightarrow \infty$ also has the correct nonrelativistic limit, i.e., matches expression (3). Wave functions (20) corresponding to energy levels (21) in the p -representation

$$\psi_n(\zeta) = C'_n \zeta^\nu e^{(i\rho-\delta)\zeta/2} L_n^{2\nu-1}(\delta\zeta), \quad C'_n = \delta^\nu \sqrt{\frac{n!}{mc\Gamma(n+2\nu)}}, \quad (22)$$

can be expressed in terms of the Laguerre polynomials [44]

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!} \Phi(-n, \alpha+1; x), \quad (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}.$$

By virtue of the orthogonality condition for the Laguerre polynomials [45]

$$\int_0^\infty x^\alpha e^{-x} L_n^\alpha(x) L_m^\alpha(x) dx = \frac{\Gamma(n+\alpha+1)}{n!} \delta_{nm}, \quad \alpha > -1, \quad (23)$$

functions (22) satisfy the normalization condition

$$mc \int_0^\infty \psi_n^*(\zeta) \psi_m(\zeta) \frac{d\zeta}{\zeta} = \delta_{nm}. \quad (24)$$

Because $\delta^2 + \rho^2 = 1$ and $0 < \delta \leq 1$, we can set $\rho = \cos \varphi$ and $\delta = \sin \varphi$, with $\varphi \in (0, \pi)$.

Using integral formula [43]

$$\int_0^\infty t^{\alpha-1} e^{-ct(\cos\beta+i\sin\beta)} dt = \Gamma(\alpha) c^{-\alpha} e^{-i\alpha\beta}, \quad c > 0, \quad |\beta| < \frac{\pi}{2}, \quad \text{Re } \alpha > 0,$$

we now easily find wave functions (22) in the relativistic configuration x -representation:

$$\begin{aligned} \psi_n(x) &= C_n \omega_0^{ix/\lambda} \Gamma(\nu + ix/\lambda) e^{(x/\lambda)(\varphi-\pi/2)} P_n^\nu\left(\frac{x}{\lambda}; \varphi\right), \\ C_n &= (2i\delta)^\nu e^{-i(n+\nu)\varphi} \sqrt{\frac{n!}{2\pi\lambda\Gamma(n+2\nu)}}. \end{aligned} \quad (25)$$

Here, the functions

$$P_n^\nu(x; \varphi) = \frac{(2\nu)_n}{n!} e^{in\varphi} F(-n, \nu + ix; 2\nu; 1 - e^{-2i\varphi}) \quad (26)$$

are the Meixner–Pollaczek polynomials [44], [45]. The orthonormality of wave functions (25),

$$\int_{-\infty}^{\infty} \psi_n^*(x) \psi_m(x) dx = \delta_{nm},$$

follows from the formula [45]

$$\int_{-\infty}^{\infty} P_n^\nu(x; \varphi) P_m^\nu(x; \varphi) \rho^{(\nu)}(x) dx = \frac{(2\nu)_n}{n!} \delta_{nm}, \quad (27)$$

where the weight function has the form

$$\rho^{(\nu)}(x) = \frac{(2\delta)^{2\nu}}{2\pi \Gamma(2\nu)} |\Gamma(\nu + ix)|^2 e^{x(2\varphi - \pi)}.$$

Formula (27) also follows from (23).

It can be proved that as $c \rightarrow \infty$, wave functions (22) and (25) in respective p - and x -representations transform into wave functions of the nonrelativistic linear oscillator in an external uniform field. For the proof, we start from the following limit formulas for the Laguerre and Meixner–Pollaczek polynomials:

$$\lim_{c \rightarrow \infty} \nu^{-n/2} L_n^{2\nu-1}(2\nu + 2x\sqrt{\nu}) = \frac{(-1)^n}{n!} H_n(x), \quad (28)$$

$$\lim_{c \rightarrow \infty} \nu^{-n/2} P_n^\nu\left(x\sqrt{\nu}; \arccos \frac{x_0}{\sqrt{\nu}}\right) = \frac{1}{n!} H_n(x + x_0). \quad (29)$$

The first formula is given in [45], and the proof of the second formula is given in the appendix.

Using relativistic Fourier transformations (13) and (14) and formulas (22) and (25), we obtain the integral relations between the Laguerre and Meixner–Pollaczek polynomials

$$\int_0^\infty e^{-(1-i\cot\varphi)t/2} t^{\nu+ix-1} L_n^{2\nu-1}(t) dt = (2\sin\varphi)^{\nu+ix} e^{(\pi/2-\varphi)(i\nu-x)} \Gamma(\nu+ix) e^{-in\varphi} P_n^\nu(x; \varphi), \quad (30)$$

$$\int_{-\infty}^\infty t^{-ix} \Gamma(\nu+ix) e^{x(\varphi-\pi/2)} P_n^\nu(x; \varphi) dx = 2\pi e^{in\varphi} e^{i(\varphi-\pi/2)\nu} e^{ite^{i\varphi}} t^\nu L_n^{2\nu-1}(2t\sin\varphi), \quad (31)$$

where $t > 0$, $\nu > 0$. Using the equalities $(it\nabla_t)^n t^{-ix} = x^n t^{-ix}$ and $e^{-in\nabla_x} t^{ix} = t^{n+ix}$, we can rewrite formulas (30) and (31) in a “local” form

$$L_n^{2\nu-1}(e^{-i\nabla_x})(2\sin\varphi)^{ix} e^{x(\varphi-\pi/2)} \Gamma(\nu+ix) = (2\sin\varphi)^{ix} e^{x(\varphi-\pi/2)} \Gamma(\nu+ix) e^{-in\varphi} P_n^\nu(x; \varphi), \quad (32)$$

$$P_n^\nu(it\nabla_t; \varphi) t^\nu e^{ite^{i\varphi}} = t^\nu e^{ite^{i\varphi}} e^{in\varphi} L_n^{2\nu-1}(2t\sin\varphi). \quad (33)$$

To conclude this section, we note that the Hamiltonian in the p -representation

$$H(\zeta) = \hbar\omega[a_\zeta(\sigma, \sigma') a_\zeta^\dagger(\sigma, \sigma') + c(\sigma, \sigma')] \quad (34)$$

can be factored in four ways using the operators

$$\begin{aligned} a_\zeta(\sigma, \sigma') &= \frac{1}{2} \left[2\sqrt{\zeta} \nabla_\zeta + Z_\sigma \sqrt{\zeta} + \frac{(2\nu-1)\sigma'}{\sqrt{\zeta}} \right], \\ a_\zeta^\dagger(\sigma, \sigma') &= \frac{1}{2} \left[-2\sqrt{\zeta} \nabla_\zeta + Z_\sigma^* \sqrt{\zeta} + \frac{1+(2\nu-1)\sigma'}{\sqrt{\zeta}} \right]. \end{aligned} \quad (35)$$

Here, $\sigma = \pm 1$, $\sigma' = \pm 1$, $Z_\sigma = \sigma\delta - i\rho$, and $c(\sigma, \sigma') = -\sigma\delta((1 - \sigma')/2 + \nu\sigma')$.

In the x -representation, operators (35) take the form

$$\begin{aligned} a_x(\sigma, \sigma') &= \frac{1}{\sqrt{2\omega_0}} \left\{ Z_\sigma e^{-i\lambda\nabla_x/2} + \omega_0 e^{i\lambda\nabla_x/2} \left[\sigma' \left(\nu - \frac{1}{2} \right) - \frac{ix}{\lambda} \right] \right\}, \\ a_x^+(\sigma, \sigma') &= \frac{1}{\sqrt{2\omega_0}} \left\{ Z_\sigma^* e^{-i\lambda\nabla_x/2} + \omega_0 \left[\sigma' \left(\nu - \frac{1}{2} \right) + \frac{ix}{\lambda} \right] e^{i\lambda\nabla_x/2} \right\}. \end{aligned} \quad (36)$$

We also write the Hamiltonian in factored form in the x -representation:

$$H(x) = \hbar\omega [a_x(\sigma, \sigma') a_x^+(\sigma, \sigma') + c(\sigma, \sigma')]. \quad (37)$$

We note that as $c \rightarrow \infty$, operators (35) and (36) have the asymptotic form

$$\begin{aligned} a_\zeta(\sigma, \sigma') &\cong \frac{1}{\sqrt{2}} \left(\nabla_\eta + \frac{\sigma - \sigma'}{2} \eta - i\xi_0 \right) + \sqrt{\frac{mc^2}{\hbar\omega}} (\sigma + \sigma'), \\ a_\zeta^+(\sigma, \sigma') &\cong \frac{1}{\sqrt{2}} \left(-\nabla_\eta + \frac{\sigma - \sigma'}{2} \eta + i\xi_0 \right) + \sqrt{\frac{mc^2}{\hbar\omega}} (\sigma + \sigma') \end{aligned} \quad (38)$$

and

$$\begin{aligned} a_x(\sigma, \sigma') &\cong \frac{i}{\sqrt{2}} \left(-\frac{\sigma - \sigma'}{2} \nabla_\xi - \xi - \xi_0 \right) + \sqrt{\frac{mc^2}{\hbar\omega}} (\sigma + \sigma'), \\ a_x^+(\sigma, \sigma') &\cong \frac{i}{\sqrt{2}} \left(-\frac{\sigma - \sigma'}{2} \nabla_\xi + \xi + \xi_0 \right) + \sqrt{\frac{mc^2}{\hbar\omega}} (\sigma + \sigma'), \end{aligned} \quad (39)$$

whence it follows that they have the correct nonrelativistic limit only for $\sigma + \sigma' = 0$, i.e., for $\sigma - \sigma' = \pm 2$, where $\eta = p/\sqrt{m\hbar\omega}$ and $\xi = x\sqrt{m\omega/\hbar}$. We also present the asymptotic form of the number $c(\sigma, \sigma')$, which is independent of σ and σ' :

$$c(\sigma, \sigma') \cong \frac{1}{2} \hbar\omega - \frac{1}{2} m\omega^2 x_0^2 + mc^2.$$

4. Dynamic symmetry group

To construct the dynamical symmetry group of the system described by Eq. (12), we consider the cases $|g| < mc\omega$ and $|g| \geq mc\omega$ separately.

A. Let $|\rho| = |g|/mc\omega < 1$, i.e., $0 < \delta \leq 1$ (discrete spectrum). We introduce the following Hermitian operators in the p -representation:

$$K_0 \equiv \Gamma_0(\zeta) = \frac{H(\zeta)}{\hbar\omega\delta}, \quad K_1 \equiv \Gamma_4(\zeta) = \frac{1}{2} \delta\zeta - \Gamma_0(\zeta), \quad K_2 \equiv T(\zeta) = \frac{1}{2} \rho\zeta + i\zeta\nabla_\zeta. \quad (40)$$

They satisfy the commutation relations for the Lie algebra of the group $SU(1, 1)$:

$$[\Gamma_0, \Gamma_4] = iT, \quad [T, \Gamma_0] = i\Gamma_4, \quad [\Gamma_4, T] = -i\Gamma_0. \quad (41)$$

The Casimir operator $C_2 = \Gamma_0^2 - \Gamma_4^2 - T^2 = s(s+1)I$, where I is the unit operator, has the value $C_2 = \nu(\nu-1)$, i.e., $s = \nu - 1$ or $s = -\nu$. The value $s = -\nu < 0$ corresponds to the unitary irreducible representation $D^+(-\nu)$ of the $SU(1, 1)$ group [46]–[48], in which the eigenvalues of the compact generator Γ_0 are equal to $-s + n = n + \nu$, $n = 0, 1, 2, \dots$. Thus, we obtain the correct spectrum (21) for the operator $H = \omega\delta\Gamma_0$. Its eigenfunctions (22) and (25) form a basis of the irreducible representation $D^+(-\nu)$.

We also present the form of operators (40) in the x -representation:

$$\Gamma_0(x) = \frac{H(x)}{\hbar\omega\delta}, \quad \Gamma_4(x) = \frac{\delta}{\omega_0}e^{-i\lambda\nabla_x} - \Gamma_0(x), \quad T(x) = \frac{x}{\lambda} + \frac{\rho}{\omega_0}e^{-i\lambda\nabla_x}. \quad (42)$$

B. We now let $|\rho| \geq 1$ (continuous spectrum). In the case $|\rho| > 1$, the generators of the Lie algebra of the dynamical group $SU(1, 1)$ are related to generators (40) (or (42)) as

$$\Gamma'_0 = -i\Gamma_4, \quad \Gamma'_4 = i\Gamma_0, \quad T' = -T, \quad \delta = i\delta', \quad \delta' = \sqrt{\rho^2 - 1}. \quad (43)$$

Because the spectrum of the noncompact generator Γ'_4 is continuous and equal to $\lambda \in \mathbb{R}$ [46], we conclude that for $|\rho| > 1$, the spectrum of $H = \hbar\omega\delta'\Gamma'_4$ is also continuous.

In the case where $|\rho| = 1$, we can introduce the operators

$$\Gamma''_0(\zeta) = \frac{H(\zeta)}{\hbar\omega} - i\rho\zeta\nabla_\zeta, \quad \Gamma''_4(\zeta) = i\rho\zeta\nabla_\zeta, \quad T''(\zeta) = -\rho\left(\frac{1}{2}\zeta - \Gamma''_0(\zeta)\right), \quad (44)$$

which form the same closed algebra (41), where we still have $C''_2 = \nu(\nu - 1)$. In the x -representation, they can be written as

$$\Gamma''_0(x) = \frac{H(x)}{\hbar\omega} - \rho\frac{x}{\lambda}, \quad \Gamma''_4(x) = \rho\frac{x}{\lambda}, \quad T''(x) = -\rho\left(\frac{1}{\omega_0}e^{-i\lambda\nabla_x} - \Gamma''_0(x)\right). \quad (45)$$

Therefore, $H = \hbar\omega(\Gamma''_0 + \Gamma''_4)$. As is known [46], this operator has a continuous and positive spectrum. Thus, the dynamical symmetry group of the system under consideration is the group $SU(1, 1)$.

5. Conclusion

In this paper, we considered the model of a relativistic linear oscillator in the presence of a constant external force in detail both in the Lobachevsky momentum space and in the relativistic configuration space. Some physical and mathematical results have been obtained. It is interesting to note that in this case, in contrast to the corresponding nonrelativistic problem, bound states are possible only in a finite region of the magnitude of the force, namely, for $|g| < mc\omega$, while only the continuous energy spectrum exists for $|g| \geq mc\omega$. The discrete-spectrum energy levels are equidistant. We showed that the wave functions in the x -representation are expressed in terms of the Meixner–Pollaczek polynomials and constructed a dynamical algebra, using which, as in the nonrelativistic case, allows finding the energy spectrum in a purely algebraic way and constructing the wave functions. Knowing the raising and lowering operators, it is possible to construct the coherent states of the system. We established limit relation (29) connecting the Meixner–Pollaczek and Hermite polynomials.

The connection we noticed between the Laguerre and Meixner–Pollaczek polynomials (formulas (30)–(33)) can be used to find a bilinear generating function for the Meixner–Pollaczek polynomials. We also note that Hamiltonian (12) is an example of a difference operator, and Hamiltonian (17) is an example of a differential operator, whose spectra cannot be found within the perturbation theory in the vicinity of the respective points $|g| = mc\omega$ and $|\rho| = 1$.

Appendix

Here, we prove formula (29). We proceed from the recursion relations for the Meixner–Pollaczek and Hermite polynomials [39]

$$P''_{n+1}(x; \varphi) = A_n P''_n(x; \varphi) + B_n P''_{n-1}(x; \varphi), \quad (A.1)$$

$$H_{n+1}(z) = 2zH_n(z) - 2nH_{n-1}(z), \quad (A.2)$$

where

$$A_n = \frac{2[(n + \nu) \cos \varphi + x \sin \varphi]}{n + 1}, \quad B_n = -\frac{n - 1 + 2\nu}{n + 1}, \quad n = 0, 1, 2, 3, \dots$$

Let

$$Q_n = n! \nu^{-n/2} P_n^\nu \left(x\sqrt{\nu}; \arccos \frac{x_0}{\sqrt{\nu}} \right), \quad Q'_n = \lim_{\nu \rightarrow \infty} Q_n. \quad (\text{A.3})$$

From (A.1), we then obtain the recursion relation for the polynomials Q_n

$$Q_{n+1} = A'_n Q_n + B'_n Q_{n-1}, \quad (\text{A.4})$$

where $A'_n = (n+1)A_n/\sqrt{\nu}$ and $B'_n = n(n+1)B_n/\nu$. Because $\lim_{\nu \rightarrow \infty} A'_n = 2(x+x_0)$ and $\lim_{\nu \rightarrow \infty} B'_n = -2n$, passing to the limit $\nu \rightarrow \infty$ in (A.4) yields the equality

$$Q'_{n+1} = 2(x+x_0)Q'_n - 2nQ'_{n-1}, \quad (\text{A.5})$$

which coincides with recursion relation (A.2) for the Hermite polynomials for $z = x + x_0$. Hence, $Q'_n = H_n(x + x_0)$. This completes the proof.

Conflict of interests. The authors declare no conflicts of interest.

REFERENCES

1. L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics*, Vol. 3: *Quantum Mechanics: Nonrelativistic Theory* (1973).
2. A. I. Baz, Ya. B. Zeldovich, and A. M. Perelomov, *Scattering, Reactions and Decay in Non-relativistic Quantum Mechanics*, Nauka, Moscow (1971).
3. M. Moshinsky and Yu. F. Smirnov, *The Harmonic Oscillator in Modern Physics*, Contemporary Concepts in Physics, Vol. 9, Harwood Academic Publ., Amsterdam (1996).
4. H. Yukawa, "Structure and mass spectrum of elementary particles. II. Oscillator model," *Phys. Rev.*, **91**, 416–417 (1953).
5. M. Markov, "On dynamically deformable form factors in the theory of elementary particles," *Nuovo Cimento*, **3**, 760–772 (1956).
6. R. P. Feynman, M. Kislinger, and F. Ravndal, "Current matrix elements from a relativistic quark model," *Phys. Rev. D*, **3**, 2706–2732 (1971).
7. V. L. Ginzburg and V. I. Man'ko, "Relativistic wave equations with internal degrees of freedom, and partons," *Sov. J. Part. Nucl.*, **7**, 3–20 (1976).
8. P. N. Bogoljubov, "Equations for bound states (quarks)," *Sov. J. Part. Nucl.*, **3** (1972), 144–174.
9. T. De, Y. S. Kim, and M. E. Noz, "Radial effects in the symmetric quark model," *Nuovo Cimento A*, **13**, 1089–1101 (1973).
10. Y. S. Kim and M. E. Noz, "Group theory of covariant harmonic oscillators," *Am. J. Phys.*, **46**, 480–483 (1978).
11. Y. S. Kim and M. E. Noz, "Relativistic harmonic oscillators and hadronic structure in the quantum-mechanics curriculum," *Am. J. Phys.*, **46**, 484–488 (1978).
12. I. Bars, "Relativistic harmonic oscillator revisited," *Phys. Rev. D*, **79**, 045009, 22 pp. (2009), arXiv:0810.2075.
13. D. J. Navarro and J. Navarro-Salas, "Special-relativistic harmonic oscillator modeled by Klein–Gordon theory in anti-de Sitter space," *J. Math. Phys.*, **37**, 6060–6073 (1996).
14. I. I. Cotaescu, "Geometric models of the relativistic harmonic oscillator," *Internat. J. Modern Phys. A*, **12**, 3545–3550 (1997), arXiv:physics/9704009.
15. M. Moshinsky and A. Szczepaniak, "The Dirac oscillator," *J. Phys. A: Math. Gen.*, **22**, L817–L819 (1989).
16. O. L. de Lange, "Shift operators for a Dirac oscillator," *J. Math. Phys.*, **32**, 1296–1300 (1991).

17. M. Moreno and A. Zentella, “Covariance, CPT and the Foldy–Wouthuysen transformation for the Dirac oscillator,” *J. Phys. A: Math. Gen.*, **22**, L8221–L825 (1989).
18. R. Lisboa, M. Malheiro, A. S. de Castro, P. Alberto, and M. Fiolhais, “Pseudospin symmetry and the relativistic harmonic oscillator,” *Phys. Rev. C*, **69**, 024319, 15 pp. (2004), arXiv:nucl-th/0310071.
19. M. Moshinsky, G. Loyola, A. Szczepaniak, C. Villegas, and N. Aquino, “The Dirac oscillator and its contribution to the baryon mass formula,” in: *Proceedings of the Rio De Janeiro International Workshop on Relativistic Aspects of Nuclear Physics* (Centro Brasileiro de Pesquisas Fisicas, Rio de Janeiro, Brasil, August 28–30, 1989) (T. Kodama, K. C. Chung S. J. B. Duarte, and M. C. Nemes, eds.), World Sci., Singapore (1990), pp. 271–308.
20. R. P. Martinez-y-Romero, H. N. Núñez Yépez, and A. L. Salas-Brito, “Relativistic quantum mechanics of a Dirac oscillator,” *Eur. J. Phys.*, **16**, 135–141 (1995), arXiv:quant-ph/9908069.
21. E. E. Salpeter, “Mass corrections to the fine structure of hydrogen-like atoms,” *Phys. Rev.*, **87**, 328–343 (1952).
22. Z.-F. Li, J.-J. Liu, W. Lucha, W.-G. Ma, and F. F. Schöberl, “Relativistic harmonic oscillator,” *J. Math. Phys.*, **46**, 103514, 11 pp. (2005), arXiv:hep-ph/0501268.
23. K. Kowalski and J. Rembieliński, “Relativistic massless harmonic oscillator,” *Phys. Rev. A*, **81**, 012118, 6 pp. (2010), arXiv:1002.0474.
24. V. G. Kadyshevsky, R. M. Mir-Kasimov, and N. B. Skachkov, “Quasi-potential approach and the expansion in relativistic spherical functions,” *Nuovo Cimento A*, **55**, 233–257 (1968).
25. V. G. Kadyshevskii, R. M. Mir-Kasimov, and N. B. Skachkov, “Three-dimensional formulation of the relativistic two-body problem,” *Part. Nucl.*, **2**, 635–690 (1972).
26. A. D. Donkov, V. G. Kadyshevskii, M. D. Matveev, and R. M. Mir-Kassimov, “Quasipotential equation for a relativistic harmonic oscillator,” *Theoret. and Math. Phys.*, **8**, 673–681 (1971).
27. N. M. Atakishiyev, R. M. Mir-Kassimov, and Sh. M. Nagiyev, “Quasipotential models of a relativistic oscillator,” *Theoret. and Math. Phys.*, **44**, 592–603 (1980).
28. N. M. Atakishiyev, “Quasipotential wave functions of a relativistic harmonic oscillator and Pollaczek polynomials,” *Theoret. and Math. Phys.*, **58**, 166–171 (1984).
29. N. M. Atakishiyev, R. M. Mir-Kasimov, and Sh. M. Nagiyev, “A relativistic model of the isotropic oscillator,” *Ann. Phys.*, **497**, 25–30 (1985).
30. R. M. Mir-Kasimov, Sh. M. Nagiev, and E. Dzh. Kagramanov, Relyativistskiy lineynny ostsillyator pod deystviem postoyannoy vneshney sily i bilineynaya proizvodnyashchaya funktsiya dlya polinomov Pollacheka, Preprint No. 214 [in Russian], SKB IFAN AzSSR, Baku (1987).
31. E. D. Kagramanov, R. M. Mir-Kasimov, and Sh. M. Nagiyev, “The covariant linear oscillator and generalized realization of the dynamical $SU(1, 1)$ symmetry algebra,” *J. Math. Phys.*, **31**, 1733–1738 (1990).
32. R. M. Mir-Kasimov, “ $SU_q(1, 1)$ and the relativistic oscillator,” *J. Phys. A: Math. Gen.*, **24**, 4283–4302 (1991).
33. Yu. A. Grishchkin and V. N. Kapshai, “Solution of the Logunov–Tavkhelidze equation for the three-dimensional oscillator potential in the relativistic configuration representation,” *Russ. Phys. J.*, **61**, 1645–1652 (2019).
34. N. M. Atakishiyev and K. B. Wolf, “Generalized coherent states for a relativistic model of the linear oscillator in a homogeneous external field,” *Rep. Math. Phys.*, **27**, 305–311 (1989).
35. N. M. Atakishiyev, Sh. M. Nagiyev, and K. B. Wolf, “Wigner distribution functions for a relativistic linear oscillator,” *Theoret. and Math. Phys.*, **114**, 322–334 (1998).
36. S. M. Nagiyev, G. H. Guliyeva, and E. I. Jafarov, “The Wigner function of the relativistic finite-difference oscillator in an external field,” *J. Phys. A: Math. Theor.*, **42**, 454015, 10 pp. (2009).
37. N. M. Atakishiyev, Sh. M. Nagiyev, and K. B. Wolf, “Realization of $Sp(2, r)$ by finite-difference operators: the relativistic oscillator in an external field,” *J. Group Theory Phys.*, **3**, 61–70 (1995).
38. E. A. Dei, V. N. Kapshai, and N. B. Skachkov, “Exact solutions of a class of quasipotential equations for a superposition of one-boson exchange quasipotentials,” *Theoret. and Math. Phys.*, **82**, 130–138 (1990).
39. A. A. Atanasov and E. S. Pisanova, “Perturbation theory for relativistic three-dimensional two-particle quasipotential equations,” *Theoret. and Math. Phys.*, **89**, 1169–1173 (1991).
40. A. A. Atanasov and A. T. Marinov, “ \hbar -expansion for bound states described by the relativistic three-dimensional two-particle quasi-potential equation,” *Theoret. and Math. Phys.*, **129**, 1400–1407 (2001).

41. O. P. Solovtsova and Yu. D. Chernichenko, “The resummation L -factor in the relativistic quasipotential approach,” *Theoret. and Math. Phys.*, **166**, 194–209 (2011).
42. E. D. Kagramanov, R. M. Mir-Kasimov, and Sh. M. Nagiyev, “Can we treat confinement as a pure relativistic effect?,” *Phys. Lett. A*, **140**, 1–4 (1989).
43. G. Bateman and A. Erdélyi, *Higher Transcendental Functions*, Vol. 1: *The Hypergeometric Function. Legendre Functions*, McGraw-Hill, New York (1953).
44. H. Bateman and A. Erdélyi, *Higher Transcendental Functions*, Vol. 2, McGraw-Hill, New York–Toronto–London (1953).
45. R. Koekoek, P. A. Lesky, and R. F. Swarttouw, *Hypergeometric Orthogonal Polynomials and Their q -Analogues*, Springer, Berlin (2010).
46. A. O. Barut, “Some unusual applications of Lie algebra representations in quantum theory,” *SIAM J. Appl. Math.*, **25**, 247–259 (1973).
47. I. A. Malkin and V. I. Man’ko, *Dynamic Symmetries and Coherent States of Quantum Systems*, Nauka, Moscow (1979).
48. A. O. Barut and R. Raczka, *Theory of Group Representations and Applications*, Polish Sci. Publ. PWN, Warszawa (1986).