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SPIN NONCLASSICALITY VIA VARIANCE

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Although variance, as one of the most fundamental and ubiquitous quantities in quantifying uncertainty, has been widely used in both classical and quantum physics, there are still new applications awaiting exploration. In this work, by interchanging the roles of the state variable and the observable variable, i.e., by formally regarding any state as an observable (which is rational because any state is a priori a Hermitian operator) and considering the average variance of this state (now in the position of an observable) in all spin coherent states, we introduce a quantifier of spin nonclassicality with respect to a resolution of identity induced by spin coherent states. This quantifier is easy to compute and it admits various operational interpretations, such as the purity deficit, the Tsallis 2-entropy deficit, and the squared norm deficit between the Wigner function and the Husimi function. We reveal several intuitive properties of this quantifier, connect it to the phase-space distribution uncertainty, and illustrate it with some prototypical examples. Various extensions are further indicated.

Keywords: spin nonclassicality, spin coherent states, variance, convexity, resolution of identity

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1. Introduction

Quantum systems can be roughly classified into continuous and discrete. Typical continuous systems include bosonic fields (quantum harmonic oscillators), while typical discrete systems include spin or atomic systems. Ever since Mandel introduced the Q parameter [1], [2], quantification of nonclassicality of bosonic field states has been extensively studied [3]–[22]. However, the spin nonclassicality (nonclassicality of discrete systems) is relatively less explored, although several important quantifiers of spin nonclassicality have been introduced [23]–[28]. Due to the complex, subtle, but at the same time relevant nature of nonclassicality, it is desirable to characterize and quantify spin nonclassicality from various perspectives.

In this paper, we introduce a quantifier of spin nonclassicality in terms of the variance of spin states (regarded as observables) in spin coherent states, reveal its basic properties, and further illustrate the results with various typical spin states. The key idea here is to change the perspective by inputting the spin states as observables in the expression of variance.

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More precisely, we recall that for a quantum system in a state τ, the variance of any observable (Hermitian operator) X is defined as

$$V_\tau(X) = \mathrm{tr}\,\tau X^2 - (\mathrm{tr}\,\tau X)^2. \tag{1}$$

It is well known that the variance $V_\tau(X)$ is a *concave* function of τ in the sense that

$$V_{c_1\tau_1+c_2\tau_2}(X) \geqslant c_1 V_{\tau_1}(X) + c_2 V_{\tau_2}(X)$$

for any quantum states τ_i and constants $c_i \geqslant 0$, $c_1 + c_2 = 1$. However, it seems relatively less known that the variance $V_\tau(X)$ is a *convex* function of X in the sense that

$$V_\tau(c_1X_1 + c_2X_2) \leqslant c_1 V_\tau(X_1) + c_2 V_\tau(X_2)$$

for any observables X_i and constants $c_i \geqslant 0, c_1 + c_2 = 1$. By noting that the variance is nonnegative, the above convexity follows readily from the identity

$$V_\tau(c_1X_1 + c_2X_2) + c_1c_2V_\tau(X_1 - X_2) = c_1V_\tau(X_1) + c_2V_\tau(X_2),$$

which can be verified directly.

By considering $V_{|\zeta\rangle\langle\zeta|}(\rho)$, we exploit the above convexity and the resolution of identity induced by the spin coherent states $|\zeta\rangle$ to quantify spin nonclassicality of any spin state ρ. It turns out that the resulting quantifier of spin nonclassicality has intrinsic relations with spin phase-space functions and enjoys a variety of remarkable properties. In particular, a convenient and useful criterion for spin nonclassicality follows.

This paper is structured as follows. In Sec. 2, we review basic features of spin systems. In Sec. 3, we quantify spin nonclassicality via averaged variance, and explore its basic properties. We illustrate the results with various examples in Sec. 4. Finally, we summarize the results and discuss some extensions and perspectives in Sec. 5.

2. Spin systems

For $j = 0, 1/2, 1, 3/2, \ldots$, a spin-$j$ system can be described by a finite-dimensional Hilbert space $\mathbb{C}^{2j+1}$ together with the spin operators (angular momentum operators) $\mathbf{J} = (J_x, J_y, J_z)$ satisfying the commutation rules

$$[J_x, J_y] = iJ_z, \qquad [J_y, J_z] = iJ_x, \qquad [J_z, J_x] = iJ_y,$$

which characterize the $su(2)$ Lie algebra. We note that J_z and the total spin operator $\mathbf{J}^2 = J_x^2 + J_y^2 + J_z^2$ commute, and their joint eigenvectors (the so-called Dicke states) $|j, m\rangle$, $m = -j, -j+1, \ldots, j-1, j$, satisfy

$$J_z|j,m\rangle = m|j,m\rangle, \qquad \mathbf{J}^2|j,m\rangle = j(j+1)|j,m\rangle,$$

and furnish an orthogonal resolution of identity:

$$\sum_{m=-j}^{j} |j,m\rangle\langle j,m| = \mathbf{1}.$$

Let $J_- = J_x - iJ_y$, $J_+ = J_x + iJ_y$ be the ladder operators, then

$$\begin{aligned} J_+|j,m\rangle &= \sqrt{(j-m)(j+m+1)}\,|j,m+1\rangle, \quad m \leqslant j-1, \qquad J_+|j,j\rangle = 0, \\ J_-|j,m\rangle &= \sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle, \quad m \geqslant -j+1, \qquad J_-|j,-j\rangle = 0. \end{aligned}$$

In a spin-j system, the spin coherent states (also called $SU(2)$ coherent states, atomic coherent states, or Bloch coherent states) can be defined in various equivalent forms as [29]–[34]

$$|\zeta\rangle = e^{\xi J_+ - \xi^* J_-}|j,j\rangle = \sum_{m=-j}^{j} \sqrt{\binom{2j}{j+m}} \left(\cos\frac{\theta}{2}\right)^{j+m} \left(e^{i\phi}\sin\frac{\theta}{2}\right)^{j-m} |j,m\rangle =$$
$$= \sum_{m=-j}^{j} \sqrt{\binom{2j}{j-m}} \frac{\zeta^{j-m}}{(1+|\zeta|^2)^j} |j,m\rangle,$$

where

$$\xi = -\frac{\theta}{2} e^{-i\phi}, \qquad \zeta = e^{i\phi}\tan\frac{\theta}{2} \in \mathbb{C}, \qquad \theta \in [0,\pi), \quad \phi \in [0,2\pi).$$

The resolution of identity induced by the spin coherent states is given by

$$\int_{\mathbb{C}} |\zeta\rangle\langle\zeta|\, d\mu(\zeta) = \mathbf{1},$$

where

$$d\mu(\zeta) = \frac{(2j+1)}{\pi(1+|\zeta|^2)^2}\, d^2\zeta, \qquad \zeta = x + iy \in \mathbb{C}, \quad x, y \in \mathbb{R} \tag{2}$$

with $d^2\zeta = dx\,dy$ the standard Lebesgue measure on the complex plane $\mathbb{C}$. In the Fock–Bargmann representation of a spin-j system, the Hilbert space $\mathbb{C}^{2j+1}$ is identified with the analytic function space

$$H_j = \left\{ \text{analytic } f\colon \mathbb{C} \to \mathbb{C},\ \langle f|f\rangle = \int_{\mathbb{C}} f^*(\zeta) f(\zeta)\, d\mu(\zeta) < \infty \right\}.$$

We note that in the two most important sets of bases of a spin-j system, the Dicke states $|j,m\rangle$, $m = -j, -j+1, \ldots, j-1, j$, are orthonormal, while the spin coherent states $|\zeta\rangle$, $\zeta \in \mathbb{C}$, are not, because the overlap

$$\langle\zeta_1|\zeta_2\rangle = \frac{(1+\zeta_1^*\zeta_2)^{2j}}{(1+|\zeta_1|^2)^j(1+|\zeta_2|^2)^j}, \qquad \zeta_1, \zeta_2 \in \mathbb{C},$$

vanishes only for the antipodal points.

Following [23], we call any spin state ρ classical if it is a spin coherent state or can be expressed as a probabilistic mixture of spin coherent states. Otherwise, we call it nonclassical. This is a discrete analogue of the Glauber–Sudarshan scheme for optical nonclassicality of bosonic field states.

3. Quantifying spin nonclassicality

Our key idea for quantifying nonclassicality in terms of variance lies in regarding the spin state ρ as an observable and considering its average uncertainty in the spin coherent states. Thus, in the variance $V_\tau(X)$, we replace X with ρ and take τ to be any spin coherent state $|\zeta\rangle\langle\zeta|$. Consequently, we are naturally led to

$$N(\rho) = \int_{\mathbb{C}} V_{|\zeta\rangle\langle\zeta|}(\rho)\, d\mu(\zeta), \tag{3}$$

which is our quantifier of spin nonclassicality. We note that the variance is defined by Eq. (1) and $d\mu(\zeta)$ is defined by Eq. (2). It is remarkable that the above quantity has the equivalent expressions

$$N(\rho) = \int_{\mathbb{C}} \left(\langle\zeta|\rho^2|\zeta\rangle - \langle\zeta|\rho|\zeta\rangle^2\right) d\mu(\zeta) = \operatorname{tr}\rho^2 - \int_{\mathbb{C}} \tilde{\rho}^2(\zeta)\, d\mu(\zeta), \tag{4}$$

where

$$\tilde{\rho}(\zeta) = \operatorname{tr} |\zeta\rangle\langle\zeta|\rho = \langle\zeta|\rho|\zeta\rangle$$

is the Husimi function of the quantum state ρ [35], [36], which satisfies $0 \leqslant \tilde{\rho}(\zeta) \leqslant 1$ and can be regarded as a phase-space probability distribution with the measure $d\mu(\zeta)$ defined by Eq. (2), because

$$\int_{\mathbb{C}} \tilde{\rho}(\zeta)\, d\mu(\zeta) = 1.$$

There are at least three interpretations related to the operational meaning of the spin nonclassicality quantifier $N(\rho)$ defined by Eq. (3).

1. As the purity deficit. Because $\operatorname{tr} \rho^2$ is the purity of a quantum state ρ and the integral $\int_{\mathbb{C}} \tilde{\rho}^2(\zeta)\, d\mu(\zeta)$ can be formally regarded as the purity of the Husimi function of ρ, which is a kind of "classicalization" (with respect to spin coherent states), the spin nonclassicality as the difference defined by Eq. (4) is a kind of purity deficit due to the generalized measurement induced by the resolution of spin coherent states (discrete analogue of the optical heterodyne measurement).

2. As the entropy deficit. We recall that the classical Tsallis entropies are defined as [37]

$$S_r(f) = \frac{1}{1-r}\left(\int_M f^r(x)\, d\mu(x) - 1\right), \qquad r \in \mathbb{R},$$

   for a probability density function f on a measurable space M (endowed with a nonnegative measure μ), and the corresponding quantum analogues are defined as

$$S_r(\rho) = \frac{1}{1-r}(\operatorname{tr} \rho^r - 1), \qquad r \in \mathbb{R},$$

   for any quantum state ρ. In the limit $r \to 1$, we recover the quantum (von Neumann) entropy

$$S_1(\rho) = \lim_{r \to 1} S_r(\rho) = -\operatorname{tr} \rho \ln \rho.$$

   Considering $r = 2$, we can rewrite the spin nonclassicality as $N(\rho) = S_2(\tilde{\rho}) - S_2(\rho)$, which is precisely the Tsallis 2-entropy deficit.

3. As the squared norm deficit. It is interesting to note that $\operatorname{tr} \rho^2$ can be equivalently expressed as

$$\operatorname{tr} \rho^2 = \int_{\mathbb{C}} \tilde{\rho}_{\mathrm{W}}^2(\zeta)\, d\mu(\zeta),$$

   where $\tilde{\rho}_{\mathrm{W}}(\zeta)$ is the Wigner phase-space function for a spin system defined in [38]. Hence, our quantifier of spin nonclassicality can be re-expressed as

$$N(\rho) = \int_{\mathbb{C}} (\tilde{\rho}_{\mathrm{W}}^2(\zeta) - \tilde{\rho}^2(\zeta))\, d\mu(\zeta),$$

   which is precisely the difference between the squared norm of the Wigner function and that of the Husimi function.

Of course, the above interpretations are intimately related, and bear the same origin in the noncommutativity between the quantum state and the ensemble of spin coherent states.

The spin nonclassicality quantifier $N(\rho)$ has the following desirable properties.

1. $0 \leqslant N(\rho) \leqslant 1$, and $N(\rho) = 0$ if and only if ρ is the maximally mixed state $\mathbf{1}/(2j+1)$;
2. $N(\rho)$ is convex in ρ;
3. for any pure state ρ, we have
$$N(\rho) = 1 - \int_{\mathbb{C}} \tilde{\rho}^2(\zeta) d\mu(\zeta).$$
Moreover, among pure states of a spin-j system, $N(\rho)$ attains its minimal value $2j/(4j+1)$ if and only if ρ is any spin-j coherent state. Consequently, combining with convexity, we have the following criterion for spin nonclassicality: if a spin-j state ρ satisfies $N(\rho) > 2j/(4j+1)$, then it is nonclassical, i.e., it cannot be expressed as a probabilistic mixture of spin coherent states.
4. $N(\rho)$ is invariant under rotations in the sense that
$$N(e^{-i\theta \mathbf{n}\cdot\mathbf{J}} \rho e^{i\theta \mathbf{n}\cdot\mathbf{J}}) = N(\rho), \qquad \mathbf{n}\cdot\mathbf{J} = n_x J_x + n_y J_y + n_z J_z,$$
for any $\theta \in \mathbb{R}$ and any vector $\mathbf{n} = (n_x, n_y, n_z) \in \mathbb{R}^3$ with unit norm. Geometrically, $e^{-i\theta \mathbf{n}\cdot\mathbf{J}}$ is a rotation through the angle θ about the axis $\mathbf{n}$ in the Bloch sphere (as well as the Bloch ball).

All the above properties can be verified directly.

We remark that inspired by Eq. (3), we can define a quantifier characterizing the nonclassicality of a state ρ with respect to a positive-operator-valued measure (resolution of identity) $M = \{M_j\}$ with $\sum_j M_j = \mathbf{1}$, $M_j \geqslant 0$ as
$$N(\rho|M) = \sum_j V(\rho|M_j)$$
in terms of a "generalized variance" $V(\rho|X)$ as long as $V(\rho|X)$ is convex in ρ for any observable X. For example, we can take
$$V(\rho|X) = -\frac{1}{2}\,\mathrm{tr}[\sqrt{\rho}, X]^2$$
as the Wigner–Yanase skew information [39]–[42]. The case of continuous measurement outcomes can be treated similarly. The bosonic fields case is studied in [22].

4. Examples

In this section, we evaluate the spin nonclassicality for some typical spin states.

Example 1. For the spin-j coherent state
$$|z\rangle = \sum_{m=-j}^{j} \sqrt{\binom{2j}{j-m}} \frac{z^{j-m}}{(1+|z|^2)^j} |j, m\rangle, \qquad z \in \mathbb{C},$$
whose Husimi function can be readily evaluated as
$$\widetilde{|z\rangle\langle z|}(\zeta) = \frac{|1+z^*\zeta|^{4j}}{(1+|z|^2)^{2j}(1+|\zeta|^2)^{2j}}, \qquad \zeta \in \mathbb{C},$$
we have
$$N(|z\rangle\langle z|) = \frac{2j}{4j+1}.$$
This is a constant independent of the amplitude parameter $z \in \mathbb{C}$.

Example 2. For the Dicke state $|j,m\rangle$ in a spin-j system, its Husimi function can be readily evaluated as

$$\widetilde{|j,m\rangle\langle j,m|}(\zeta) = \binom{2j}{j-m} \frac{|\zeta|^{2j-2m}}{(1+|\zeta|^2)^{2j}}, \qquad \zeta \in \mathbb{C},$$

from which we obtain

$$N(|j,m\rangle\langle j,m|) = 1 - \frac{2j+1}{4j+1} \frac{\binom{2j}{j+m}^2}{\binom{4j}{2j+2m}} > \frac{2j}{4j+1}, \qquad m \neq \pm j.$$

Consequently, according to our spin nonclassicality criterion, all Dicke states except the two extremal ones (corresponding to $m = \pm j$) are nonclassical. Both $|j,-j\rangle$ and $|j,j\rangle$ are spin coherent states, and therefore classical.

Example 3. For any spin-1/2 state

$$\rho = \frac{1}{2}\left(\mathbf{1} + \sum_{i=1}^{3} r_i \sigma_i\right),$$

where $\mathbf{r} = (r_1, r_2, r_3) \in \mathbb{R}^3$, $|\mathbf{r}|^2 = \sum_{i=1}^3 r_i^2 \leqslant 1$, and σ_i are the Pauli spin matrices, we readily obtain the purity

$$\operatorname{tr} \rho^2 = \frac{1}{2}(1 + |\mathbf{r}|^2)$$

and the Husimi function

$$\tilde{\rho}(\zeta) = \frac{(1+r_3)|\zeta|^2 + (r_1 - ir_2)\zeta + (r_1 + ir_2)\zeta^* + 1 - r_3}{2(1+|\zeta|^2)}.$$

Consequently,

$$N(\rho) = \frac{|\mathbf{r}|^2}{3}.$$

In particular, $N(\rho) = 0$ for the maximally mixed state ($\mathbf{r} = \mathbf{0}$) and $N(\rho) = 1/3$ for any pure state ($|\mathbf{r}| = 1$). This is specific to the spin-1/2 system, and is consistent with the fact that any spin-1/2 pure state is a spin coherent state.

Example 4. In a spin-1 system, we consider the superposition state

$$|\psi_\lambda\rangle = \frac{1}{\sqrt{2+\lambda}}\big(|1,-1\rangle + \sqrt{\lambda}|1,0\rangle + |1,1\rangle\big), \qquad \lambda \geqslant 0,$$

whose Husimi function can be evaluated as

$$\widetilde{|\psi_\lambda\rangle\langle\psi_\lambda|}(\zeta) = \frac{|\zeta^2 + \sqrt{2\lambda}\zeta + 1|^2}{(2+\lambda)(1+|\zeta|^2)^2}.$$

Consequently, for $j = 1$,

$$N(|\psi_\lambda\rangle\langle\psi_\lambda|) = \frac{3}{5} - \frac{4\lambda}{5(2+\lambda)^2} > \frac{2j}{4j+1} = \frac{2}{5},$$

which indicates that $|\psi_\lambda\rangle$ is nonclassical.

Example 5. In a spin-j system, we consider the even cat state

$$|\psi_+\rangle = \frac{|z\rangle + |-z\rangle}{\sqrt{2\left(1 + \frac{(1-|z|^2)^{2j}}{(1+|z|^2)^{2j}}\right)}}, \qquad z \neq 0,$$

whose Husimi function can be evaluated as

$$\widetilde{|\psi_+\rangle\langle\psi_+|}(\zeta) = \frac{((1+z^*\zeta)^{2j} + (1-z^*\zeta)^{2j})((1+z\zeta^*)^{2j} + (1-z\zeta^*)^{2j})}{2((1+|z|^2)^{2j} + (1-|z|^2)^{2j})(1+|\zeta|^2)^{2j}},$$

and the nonclassicality can be calculated, but the expression is complicated. For simplicity, we consider some simple yet important cases. First, for

$$|\psi_j\rangle = \frac{1}{\sqrt{2}}(|j,-j\rangle + |j,j\rangle)$$

with the Husimi function

$$\widetilde{|\psi_j\rangle\langle\psi_j|}(\zeta) = \frac{|1+\zeta^{2j}|^2}{2(1+|\zeta|^2)^{2j}},$$

we have

$$N(|\psi_j\rangle\langle\psi_j|) = 1 - \frac{2j+1}{4j+1}\left(\frac{1}{2} + \frac{1}{\binom{4j}{2j}}\right).$$

Moreover, the spin nonclassicality of $(|j,-j\rangle - |j,j\rangle)/\sqrt{2}$ is the same as the above quantity.

Second, when $j = 1$, for the even cat states

$$|\psi_+\rangle = \frac{|z\rangle + |-z\rangle}{\sqrt{2\left(1 + \frac{(1-|z|^2)^2}{(1+|z|^2)^2}\right)}}, \qquad z \neq 0,$$

we have

$$N(|\psi_+\rangle\langle\psi_+|) = \frac{3}{5} - \frac{4|z|^4}{5(1+|z|^4)^2}.$$

In sharp contrast, for the odd cat state

$$|\psi_-\rangle = \frac{|z\rangle - |-z\rangle}{\sqrt{2\left(1 - \frac{(1-|z|^2)^2}{(1+|z|^2)^2}\right)}}, \qquad z \neq 0,$$

we have

$$N(|\psi_-\rangle\langle\psi_-|) = \frac{3}{5}$$

which is independent of the parameter $z \in \mathbb{C}$. This reveals a fundamental difference between the even and the odd cat states. It is remarkable that such kinds of state, in the name of even and odd coherent states, were introduced and studied by Dodonov, Malkin, and Man'ko as early as 1974 in the context of a quantum harmonic oscillator (single-mode bosonic field) [43].

Example 6. For the mixed spin-j state

$$\rho = p|j,m\rangle\langle j,m| + (1-p)\frac{\mathbf{1}}{2j+1}, \qquad -\frac{1}{2j} \leqslant p \leqslant 1,$$

whose purity is

$$\operatorname{tr}\rho^2 = \frac{(2jp+1)^2}{(2j+1)^2}$$

and whose Husimi function is

$$\tilde{\rho}(\zeta) = p\binom{2j}{j-m}\frac{|\zeta|^{2j-2m}}{(1+|\zeta|^2)^{2j}} + \frac{1-p}{2j+1},$$

we have

$$N(\rho) = p^2\left(1 - \frac{2j+1}{4j+1}\frac{\binom{2j}{j-m}^2}{\binom{4j}{2j-2m}}\right) + \frac{2p(1-p)}{2j+1}\left(1 - \frac{1}{\binom{2j}{j-m}}\right).$$

In particular, in the simple case $j = 1$ and $m = 0$, we have

$$N(\rho) = \frac{p}{3} + \frac{4p^2}{15} > \frac{2j}{4j+1} = \frac{2}{5}$$

when $p > 3/4$, indicating the nonclassicality in this case.

5. Conclusion

The variance, which couples the two basic ingredients (quantum state and observable) in quantum mechanics and synthesizes certain uncertainty of the observable in the state, is concave in the state variable but convex in the observable variable. In this paper, by exploiting the resolution of identity induced by the spin coherent states and the convexity of the conventional variance with respect to the observable, and regarding the spin state as an observable, we have introduced a spin nonclassicality quantifier that is easy to compute and which bears operational significance as the purity (Tsallis 2-entropy, squared norm) deficit. Its basic properties are revealed, and connections with the phase-space distribution uncertainty are established. The amounts of spin nonclassicality for a variety of important spin states are evaluated explicitly, illustrating some intrinsic features of spin nonclassicality.

Various extensions of our approach are possible. For example, by letting the spin number j tend to infinity, we can obtain a corresponding quantifier of optical nonclassicality because the spin-j system tends to the single-mode bosonic field as j tends to infinity. More generally, it is natural to generalize the concept of spin nonclassicality to the case of representations of certain Lie groups, because the construction of coherent states is available in this general case. One may also extend the approach to more complex systems such as spin-boson systems or spin chains. All these issues are worthy of further investigations.

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