

NONCOMMUTATIVE KEPLER DYNAMICS: SYMMETRY GROUPS AND BI-HAMILTONIAN STRUCTURES

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Integrals of motion are constructed from noncommutative (NC) Kepler dynamics, generating $SO(3)$, $SO(4)$, and $SO(1, 3)$ dynamical symmetry groups. The Hamiltonian vector field is derived in action–angle coordinates, and the existence of a hierarchy of bi-Hamiltonian structures is highlighted. Then, a family of Nijenhuis recursion operators is computed and discussed.

Keywords: Bi-Hamiltonian structure, noncommutative phase space, recursion operator, Kepler dynamics, dynamical symmetry groups

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1. Introduction

In “Mysterium Cosmographicum,” published in 1596, Kepler proposed a model of the Solar system by relating the five extra-terrestrial planets (Mercury, Venus, Mars, Jupiter, and Saturn) known at that time to the five Platonic solids (the tetrahedron or pyramid, cube, octahedron, dodecahedron, and icosahedron) [1], [2]. Kepler’s work attracted the attention of the Danish astronomer Brahe who recruited him in October 1600 as an assistant. Kepler then accessed Brahe’s empirical data, and published the so-called Kepler’s first two laws of planetary motion in 1609 [3], [4], and the third one in 1619 [5]. Kepler thus obtained from Brahe a detailed set of observations of the motion of the planet Mars, analyzed them, and deduced that the path of Mars is an ellipse, with the Sun located at one of its focal points, and that the radius vector from the Sun to this planet sweeps out equal areas in equal times [6]. The Kepler direct problem of determining the nature of the force required to maintain elliptical motion about a focal force center was finally solved by Newton in the 1680s. Indeed, Newton determined the functional dependence on distance of the force required to sustain such an elliptical path of Mars about the Sun as a center of force located at a focal point of the ellipse. Today, scientists still concentrate on the inverse problem that

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consists in using the combined gravitational forces of the Sun and the other planets to predict and explain perturbations in the conic paths of planets and comets.

On the other hand, although the Kepler problem was a central theme of analytical dynamics for centuries, addressed by several authors, it continues to be so in the contemporary studies as well, revealing interesting mathematical symmetries. Significant steps in understanding the symmetries underlying the Kepler dynamics were made with its quantum explorations, where the $SO(4)$, $O(4,1)$, and $O(4,2)$ symmetry groups were explored (see, e.g., [7]–[13] and the references therein). These studies led to the reinvestigation of the classical Kepler problem (see [14]–[17] and the references therein). Thus, in 1966, Bacry et al. [15] proved that the transformations generated by the angular momentum and the Runge–Lenz vector indeed form a group of canonical transformations isomorphic to $SO(4)$. In 1968, Györgyi [16] gave a formulation of the Kepler problem, manifestly invariant under the $SO(4)$ and $SO(3,1)$ symmetry groups, in terms of the Fock variables and their canonical conjugates, respectively, and introduced a new time parameter, proportional to the eccentric anomaly. In that work, a transformation of the dynamical variables was performed in order to regain the standard time t leading in a natural way to Bacry’s generators inducing the $SO(4,2)$ symmetry group. In 1970, Moser [18] regularized the Kepler problem, enlarging the phase space in such a way that the temporal evolution generates a global time flow, a situation otherwise precluded by the existence of collision orbits. Six years later, in 1976, Ligon et al. [19] completed the previous works by adding the symplectic forms and the Hamiltonian vector fields. They transformed the Kepler problem in such a way that both the time flow and the $SO(4)$ symmetry were globally realized in a simple and canonical way. This was also done for a positive energy and the group $SO(1,3)$. Since 1976, several works have focused on the classical Kepler problem, (see, e.g., [17], [20]–[24]). Further, in a remarkable book published in 2001, Vilasi [25] showed that the Lie algebra of symmetries for the Kepler dynamics is $so(3) \otimes so(3)$, or better, $su(2) \otimes su(2)$, which is locally isomorphic to $so(4)$.

In addition, in the last few decades, there was a renewed interest in the Kepler problem as one of completely integrable Hamiltonian systems (IHS), the concept of which goes back to Liouville in 1897 [26] and Poincaré in 1899 [27]. Loosely speaking, IHS are dynamical systems admitting a Hamiltonian description and possessing sufficiently many constants of motion. Many of these systems are Hamiltonian systems with respect to two compatible symplectic structures [28]–[31], permitting a geometric interpretation of the so-called recursion operator [32]. Hence, a natural approach to integrability is to try to find sufficient conditions for the eigenvalues of the recursion operator to be in involution [33]. In 1992, Marmo et al. [34] constructed two Hamiltonian structures for the Kepler problem in action–angle coordinates and proved their compatibility condition by verifying the vanishing of the Nijenhuis tensor of the corresponding recursion operator.

Over the past few years, Magri’s approach [28] to integrability through bi-Hamiltonian structures has become one of the most powerful methods relating to the integrability of evolution equations, applicable in studying both finite- and infinite-dimensional dynamical systems [35]. This approach has also been proven to be one of the classical methods of integrability of evolution equations along with, for example, the Hamilton–Jacobi method of separation of variables and the Lax representation method [32], [36]. When a completely integrable Hamiltonian system does admit a bi-Hamiltonian construction, infinite hierarchies of conserved quantities can be generated following the construction by Oevel [37] based on scaling invariances and master symmetries [38], [39]. Another generalization is due to Bogoyavlenskij [40], who proposed a complete classification of the invariant Poisson structures for nondegenerate and degenerate Hamiltonian systems. In 1997 and 1999, Smirnov [36], [39] formulated a constructive method of transforming a completely integrable Hamiltonian system, in Liouville’s sense, into Magri–Morosi–Gel’fand–Dorfman’s (MMGD) bi-Hamiltonian form. He showed that the action–angle variables can be a powerful tool in solving the problem of transforming a completely integrable Hamiltonian system into its MMGD bi-Hamiltonian

form. Smirnov’s result is applicable to the classical Kepler problem due to the possibility of transforming the action–angle coordinates connected with the spherical–polar coordinates to the Delaunay coordinates. In 2005, Rañada [41] proved the existence of a bi-Hamiltonian structure arising from a nonsymplectic symmetry as well as the existence of master symmetries and additional integrals of motion (weak superintegrability) for certain particular values of two parameters b and k . In 2015, Grigoryev et al. [35] showed that the perturbed Kepler problem is a bi-Hamiltonian system in spite of the fact that the graph of the Hamilton function is not a hypersurface of translation, which goes against a necessary condition for the existence of a bi-Hamiltonian structure according to the Fernandes theorem [42]. They explicitly presented a few nondegenerate bi-Hamiltonian formulations of the perturbed Kepler problem using the Bogoyavlenskij construction of a continuum of compatible Poisson structures for isochronous Hamiltonian systems [40]. In this paper, we focus on investigations into dynamical symmetry groups and bi-Hamiltonian structures for the Kepler dynamics in a noncommutative (NC) phase space.

The paper is organized as follows. In Sec. 2, we introduce the noncommutative phase space and give some basic notions useful in what follows. In Sec. 3, we give the Hamiltonian function, the symplectic form, and the vector field describing the Kepler dynamics in the noncommutative phase space. In Sec. 4, we study the existence of dynamical symmetry groups $SO(3)$, $SO(4)$, and $SO(1,3)$ in the described setting. In Sec. 5, we derive relevant geometric quantities in action–angle variables and obtain the corresponding Hamiltonian system. In Sec. 6, we construct bi-Hamiltonian structures and the associated recursion operators. In Sec. 7, we define the hierarchy of master symmetries and compute the conserved quantities. In Sec. 8, we end with some concluding remarks.

2. Noncommutative phase space and basic definitions

Let $\mathcal{Q} = \mathbb{R}^3 \setminus \{0\}$ be the manifold describing the configuration space of the Kepler problem, and $\mathcal{T}^*\mathcal{Q} = \mathcal{Q} \times \mathbb{R}^3$ be the cotangent bundle with the local coordinates (q, p) and a natural symplectic structure $\omega: \mathcal{T}\mathcal{Q} \rightarrow \mathcal{T}^*\mathcal{Q}$ given by

$$\omega = \sum_{i=1}^3 dp_i \wedge dq^i,$$

where $\mathcal{T}\mathcal{Q}$ is the tangent bundle. By definition, ω is nondegenerate. It induces the map $P: \mathcal{T}^*\mathcal{Q} \rightarrow \mathcal{T}\mathcal{Q}$ defined by

$$P = \sum_{i=1}^3 \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i},$$

called a bivector field, which is the inverse map of ω , i.e., $\omega \circ P = P \circ \omega = 1$ [25]. In this case, the Hamiltonian vector field X_f of a Hamiltonian function f is given by

$$X_f = P df.$$

The noncommutativity [43] between phase space variables is here understood by replacing the usual product with the $*_{\beta}$ -product, also known as the Moyal product, between two arbitrary functions of position and momentum as follows [44]–[46]:

$$(f *_{\beta} g)(q, p) = f(q_i, p_i) \exp\left(\frac{1}{2}\beta^{ab} \overleftarrow{\partial}_a \overrightarrow{\partial}_b\right) g(q_j, p_j) \Big|_{(q_i, p_i)=(q_j, p_j)},$$

where

$$\beta_{ab} = \begin{pmatrix} \alpha_{ij} & \delta_{ij} + \gamma_{ij} \\ -\delta_{ij} - \gamma_{ij} & \lambda_{ij} \end{pmatrix}.$$

The parameters α and λ are antisymmetric ($n \times n$) matrices generating the noncommutativity in coordinates and momenta, respectively; γ can depend on α and λ . The $*_\beta$ -deformed Poisson bracket is defined as

$$\{f, g\}_\beta := f *_\beta g - g *_\beta f,$$

providing the commutation relations

$$\begin{aligned} \{q_i, q_j\}_\beta &= \alpha_{ij}, & \{q_i, p_j\}_\beta &= \delta_{ij} + \gamma_{ij}, \\ \{p_i, q_j\}_\beta &= -\delta_{ij} - \gamma_{ij}, & \{p_i, p_j\}_\beta &= \lambda_{ij}. \end{aligned} \quad (1)$$

It is worth noting that the transformed coordinates

$$q'_i = q_i - \frac{1}{2} \sum_{j=1}^n \alpha_{ij} p_j, \quad p'_i = p_i + \frac{1}{2} \sum_{j=1}^n \lambda_{ij} q_j \quad (2)$$

obey the same commutation relations as in (1) with respect to the usual Poisson bracket:

$$\begin{aligned} \{q'_i, q'_j\} &= \alpha_{ij}, & \{q'_i, p'_j\} &= \delta_{ij} + \gamma_{ij}, \\ \{p'_i, q'_j\} &= -\delta_{ij} - \gamma_{ij}, & \{p'_i, p'_j\} &= \lambda_{ij}, \end{aligned}$$

while q_i and p_j satisfy the canonical commutation relations

$$\{q_i, q_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}, \quad \{p_i, p_j\} = 0.$$

A Hamiltonian system is a triple (\mathcal{Q}, ω, H) , where (\mathcal{Q}, ω) is a symplectic manifold, which in the present context is the configuration space for the Kepler problem, and H is a smooth function on \mathcal{Q} , called the *Hamiltonian* or *Hamiltonian function* [47].

Given a general dynamical system defined on the $2n$ -dimensional manifold \mathcal{Q} [36], its evolution can be described by the equation

$$\dot{x}(t) = X(x), \quad x \in \mathcal{Q}, \quad X \in \mathcal{T}\mathcal{Q}. \quad (3)$$

If system (3) admits two different Hamiltonian representations

$$\dot{x}(t) = X_{H_1, H_2} = P_1 dH_1 = P_2 dH_2,$$

its integrability as well as many other properties are subject to Magri's approach. The bi-Hamiltonian vector field X_{H_1, H_2} is defined by two pairs of Poisson bivectors P_1, P_2 and Hamiltonian functions H_1, H_2 . Such a manifold \mathcal{Q} equipped with two Poisson bivectors is called a double Poisson manifold, and the quadruple $(\mathcal{Q}, P_1, P_2, X_{H_1, H_2})$ is called a bi-Hamiltonian system. P_1 and P_2 are two compatible Poisson bivectors with the vanishing Schouten–Nijenhuis bracket [48]:

$$[P_1, P_2]_{\text{NS}} = 0.$$

3. NC Kepler Hamiltonian system

We consider the NC Kepler Hamiltonian function in the transformed phase space coordinates (2),

$$H' = \sum_{i=1}^3 \frac{p'_i p'^i}{2m} - \frac{k}{r'}.$$

The dynamical variables satisfy the NC Poisson bracket relations [49]

$$\{f, g\}_{\text{nc}} = \sum_{\nu=1}^3 \theta_{\nu}^{-1} \left(\frac{\partial f}{\partial p_{\nu}} \frac{\partial g}{\partial q^{\nu}} - \frac{\partial f}{\partial q^{\nu}} \frac{\partial g}{\partial p_{\nu}} \right)$$

with respect to the NC symplectic form

$$\omega_{\text{nc}} := \sum_{i=1}^3 dp'_i \wedge dq^i = \sum_{\nu=1}^3 \theta_{\nu} dp_{\nu} \wedge dq^{\nu},$$

with

$$\theta_{\nu} = \sum_{\mu=1}^4 \left(\delta_{\mu\nu} + \frac{1}{4} \lambda_{\mu\nu} \alpha_{\mu\nu} \right) \neq 0, \quad \delta_{\mu\nu} = \begin{cases} 0, & \mu \neq \nu, \\ 1, & \mu = \nu. \end{cases}$$

For $n = 3$, the Hamiltonian function H' takes the form

$$H' = \frac{1}{2m} \sum_{i=1}^3 \left(p_i + \frac{1}{2} \sum_{j=1}^3 \lambda_{ij} q^j \right)^2 - k \left[\sum_{i=1}^3 \left(q^i - \frac{1}{2} \sum_{j=1}^3 \alpha_{ij} p_j \right)^2 \right]^{-1/2}, \quad (4)$$

yielding the Hamilton equations:

$$\dot{q}^{\mu} := \{H', q^{\mu}\}_{\text{nc}} = \theta_{\mu}^{-1} \left(\sigma_{\mu} p_{\mu} + \sum_{s=1}^3 R_{\mu s} q^s + \frac{k}{4Y^3} \sum_{l,\nu=1}^3 \alpha_{l\mu} \alpha_{l\nu} p_{\nu} \right), \quad (5)$$

$$\dot{p}_{\mu} := \{H', p_{\mu}\}_{\text{nc}} = -\theta_{\mu}^{-1} \left(\tilde{\sigma}_{\mu} q^{\mu} - \sum_{s=1}^3 R_{\mu s} p_s + \frac{1}{4m} \sum_{l,\nu=1}^3 \lambda_{l\mu} \lambda_{l\nu} q^{\nu} \right), \quad (6)$$

where

$$Y = \left[\left(q^1 - \frac{1}{2} (\alpha_{12} p_2 + \alpha_{13} p_3) \right)^2 + \left(q^2 - \frac{1}{2} (\alpha_{21} p_1 + \alpha_{23} p_3) \right)^2 + \left(q^3 - \frac{1}{2} (\alpha_{31} p_1 + \alpha_{32} p_2) \right)^2 \right]^{1/2},$$

$$\sigma_{\mu} = \frac{1}{m} + \frac{1}{4Y^3} \sum_{i=1}^3 (\alpha_{i\mu})^2, \quad \tilde{\sigma}_{\mu} = \frac{1}{Y^3} + \frac{1}{4m} \sum_{i=1}^3 (\lambda_{i\mu})^2, \quad R_{\mu s} = \frac{\lambda_{\mu s}}{2m} - \frac{\alpha_{s\mu} k}{2Y^3}.$$

In terms of the new coordinates p'_i and q^i , Hamilton equations (5) and (6) become

$$\dot{q}^i = \sum_{j=1}^3 \theta_i^{-1} \left(\frac{1}{m} p'_i + \frac{k}{2Y^3} \alpha_{ij} q^j \right), \quad \dot{p}_i = \sum_{j=1}^3 \theta_i^{-1} \left(\frac{1}{2m} \lambda_{ji} p'_j + \frac{k}{Y^3} q^i \right),$$

where $\mu, i = 1, 2, 3, \nu \neq \mu; q'^i = q^i(q, p)$ and $p'_i = p'_i(q, p)$ are given by relations (2).

The Hamiltonian function H' in (4) can be viewed as a generalization of the Hamiltonian function obtained in our previous work [50]. The quantity Y deforming the distance between the Sun and the considered planet is responsible for the distortion of the conic path about the Sun.

The 1-form $dH' \in \mathcal{T}^* \mathcal{Q}$ is given by

$$dH' = \sum_{\mu=1}^3 \left[\left(\sigma_{\mu} p_{\mu} + \sum_{s=1}^3 R_{\mu s} q^s + \frac{k}{4Y^3} \sum_{l,\nu=1}^3 \alpha_{l\mu} \alpha_{l\nu} p_{\nu} \right) dp_{\mu} + \left(\tilde{\sigma}_{\mu} q^{\mu} - \sum_{s=1}^3 R_{\mu s} p_s + \frac{1}{4m} \sum_{l,\nu=1}^3 \lambda_{l\mu} \lambda_{l\nu} q^{\nu} \right) dq^{\mu} \right], \quad (7)$$

where $\nu \neq \mu$.

Using the NC Poisson bracket in (3), we obtain the NC Hamiltonian vector field

$$X_{H'} = \sum_{\mu=1}^3 \theta_{\mu}^{-1} \left[\left(\sigma_{\mu} p_{\mu} + \sum_{s=1}^3 R_{\mu s} q^s + \frac{k}{4Y^3} \sum_{l,\nu=1}^3 \alpha_{l\mu} \alpha_{l\nu} p_{\nu} \right) \frac{\partial}{\partial q^{\mu}} - \left(\tilde{\sigma}_{\mu} q^{\mu} - \sum_{s=1}^3 R_{\mu s} p_s + \frac{1}{4m} \sum_{l,\nu=1}^3 \lambda_{l\mu} \lambda_{l\nu} q^{\nu} \right) \frac{\partial}{\partial p_{\mu}} \right], \quad \nu \neq \mu,$$

satisfying the required condition for a Hamiltonian system, i.e.,

$$\iota_{X_{H'}} \omega' = -dH',$$

where $\iota_{X_{H'}} \omega'$ is the interior product of ω' with the Hamiltonian vector field $X_{H'}$. Hence, the triplet $(\mathcal{T}^* \mathcal{Q}, \omega', H')$ is a Hamiltonian system.

The NC coordinates q'^i and p'_i generate the noncommutative relations

$$\{p'_i, q'^j\}_{\text{nc}} = F_{ij}, \quad \{p'_i, p'_j\}_{\text{nc}} = D_{ij}, \quad \{q'^i, q'^j\}_{\text{nc}} = E_{ij},$$

where

$$F_{ii} = \theta_i^{-1} + \frac{1}{4} \sum_{j=1}^3 \lambda_{ij} \alpha_{ij} \theta_j^{-1}, \quad F_{ij} = \frac{1}{4} \lambda_{ir} \alpha_{jr} \theta_r^{-1}, \quad D_{ij} = \frac{\lambda_{ji}}{2} (\theta_i^{-1} + \theta_j^{-1}),$$

$$E_{ij} = \frac{\alpha_{ji}}{2} (\theta_i^{-1} + \theta_j^{-1}), \quad i, j, r = 1, 2, 3.$$

4. Dynamical symmetry groups

We start by defining the NC phase space angular momentum vector L' and the Laplace–Runge–Lenz (LRL) vector A' as

$$L' = q' \times p', \quad A' = p' \times L' - mk \frac{q'}{Y},$$

where p' is the momentum vector and q' is the position vector of the particle of mass m . Their components

$$\begin{cases} L'_1 = q'^2 p'_3 - q'^3 p'_2, \\ L'_2 = q'^3 p'_1 - q'^1 p'_3, \\ L'_3 = q'^1 p'_2 - q'^2 p'_1, \end{cases} \quad \begin{cases} A'_1 = p'_2 L'_3 - p'_3 L'_2 - mk \frac{q'^1}{Y}, \\ A'_2 = p'_3 L'_1 - p'_1 L'_3 - mk \frac{q'^2}{Y}, \\ A'_3 = p'_1 L'_2 - p'_2 L'_1 - mk \frac{q'^3}{Y} \end{cases}$$

do not commute with the Hamiltonian function H' :

$$\{H', L'_i\}_{\text{nc}} = \sum_{\mu,\nu,j=1}^3 \varepsilon_{\mu i \nu} \left[\left(\frac{1}{m} D_{\nu j} p'_j + \frac{k}{Y^3} F_{\nu j} q'^j \right) q'^{\mu} + \left(\frac{1}{m} F_{j\nu} p'_j + \frac{k}{Y^3} E_{j\nu} q'^j \right) p'_{\mu} \right], \quad (8)$$

$$\begin{aligned} \{H', A'_i\}_{\text{nc}} &= \sum_{\eta,\varrho=1}^3 \varepsilon_{i\eta\varrho} \left\{ \sum_{\mu,\nu,j=1}^3 \varepsilon_{\mu\varrho\nu} \left[\left(\frac{1}{m} D_{\nu j} p'_j + \frac{k}{Y^3} F_{\nu j} q'^j \right) q'^{\mu} + \left(\frac{1}{m} F_{j\nu} p'_j + \frac{k}{Y^3} E_{j\nu} q'^j \right) p'_{\mu} \right] p'_{\eta} + \sum_{j=1}^3 \left(\frac{1}{m} D_{\eta j} p'_j + \frac{k}{Y^3} F_{\eta j} q'^j \right) L'_{\eta} \right\} - \\ &- \frac{mk}{Y} \sum_{j=1}^3 \left(\frac{1}{m} F_{ji} p'_j + \frac{k}{Y^3} E_{ji} q'^j \right) + \frac{mk}{Y^3} \sum_{j,h=1}^3 \left(\frac{1}{m} F_{jh} p'_j + \frac{k}{Y^3} E_{jh} q'^j \right) q'^h q'^i, \quad (9) \end{aligned}$$

where $i = 1, 2, 3$, $\varepsilon_{\mu\rho\nu}$, $\varepsilon_{i\eta\rho}$, and $\varepsilon_{\mu i\nu}$ are Levi-Civita symbols given by

$$\varepsilon_{ijk} := \frac{1}{2}(i-j)(j-k)(k-i).$$

Proposition 1. *Under the conditions*

- 1) $\lambda_{ij}\alpha_{ij} = -[(\lambda_{i\kappa}\alpha_{i\kappa} + \lambda_{j\kappa}\alpha_{j\kappa})/2 + 4]$, $i, j, \kappa = 1, 2, 3$,
- 2) $\frac{q^i}{q^j} = \frac{\lambda_{\kappa i}\theta_j}{\lambda_{\kappa j}\theta_i} = -\frac{\alpha_{\kappa j}\theta_i}{\alpha_{\kappa i}\theta_j}$ and $\frac{p'_i}{p'_j} = \frac{\alpha_{\kappa i}\theta_j}{\alpha_{\kappa j}\theta_i} = -\frac{\lambda_{\kappa j}\theta_i}{\lambda_{\kappa i}\theta_j}$,
- 3) F is a symmetric 3×3 matrix with $F_{11} = F_{22} = F_{33}$,

the vectors L'_i and A'_i are in involution with the Hamiltonian function H' :

$$\{H', L'_i\}_{\text{nc}} = 0, \quad \{H', A'_i\}_{\text{nc}} = 0, \quad (10)$$

and hence become constants of motion or first integrals of H' on $\mathcal{T}^*\mathcal{Q}$.

Proof. Condition 1 leads to $\lambda_{ij}\theta_i = \lambda_{ji}\theta_j$ and $\alpha_{ij}\theta_i = \alpha_{ji}\theta_j$. Therefore, $D_{ij} = E_{ij} = 0$, and Eqs. (8) and (9) reduce to

$$\begin{aligned} \{H', L'_i\}_{\text{nc}} &= \sum_{\mu, \nu, j=1}^3 \varepsilon_{\mu i \nu} \left(\frac{k}{Y^3} F_{\nu j} q'^j q'^{\mu} + \frac{1}{m} F_{j\nu} p'_j p'_\mu \right), \quad i = 1, 2, 3, \\ \{H', A'_i\}_{\text{nc}} &= \sum_{\eta, \varrho=1}^3 \varepsilon_{i\eta\varrho} \left\{ \sum_{\mu, \nu, j=1}^3 \varepsilon_{\mu\varrho\nu} \left(\frac{k}{Y^3} F_{\nu j} q'^j q'^{\mu} + \frac{1}{m} F_{j\nu} p'_j p'_\mu \right) p'_\eta + \sum_{j=1}^3 \frac{k}{Y^3} F_{\eta j} q'^j L'_\eta \right\} - \\ &\quad - \frac{mk}{Y} \sum_{j=1}^3 \frac{1}{m} F_{ji} p'_j + \frac{mk}{Y^3} \sum_{j, h=1}^3 \frac{1}{m} F_{jh} p'_j q'^h q'^i, \quad i = 1, 2, 3. \end{aligned}$$

After computing, replacing the F_{ij} by their expressions, and using conditions 2 and 3, we obtain $\{H', L'_i\}_{\text{nc}} = \{H', A'_i\}_{\text{nc}} = 0$. Then L' and A' are constants of motion or first integrals of H' on $\mathcal{T}^*\mathcal{Q}$. ■

Equation (10) means that the functions L'_i and A'_i , $i = 1, 2, 3$, are also constant along the orbits of H' . Then, in $\mathcal{T}^*\mathcal{Q}$, the orbits of H' lie in the inverse image of a value of these functions [17], [19].

Proposition 1 naturally leads to the following one.

Proposition 2. *Let L' be an integral of motion on $\mathcal{T}^*\mathcal{Q}$. Then, under the NC Poisson bracket defined in (3), its components L'_i 's generate the Lie algebra $so(3)$ of the group $SO(3)$, \dots , $\{L'_i, L'_j\}_{\text{nc}} = \varepsilon_{ijh} F'_{hh} L'_h$, where $\varepsilon_{ijh} F'_{hh}$ are the structure constants of the Lie algebra.*

Proof. Under conditions 1–3 of Proposition 1, the Poisson bracket

$$\begin{aligned} \{L'_i, L'_j\}_{\text{nc}} &= \varepsilon_{ij\kappa} \left\{ \sum_{\eta, \varrho, \nu=1}^3 \left[\frac{1}{2} \varepsilon_{\eta\varrho\nu} (D_{\eta\varrho} q'^{\nu} q'^{\kappa} + E_{\eta\varrho} p'_\nu p'_\kappa) + \right. \right. \\ &\quad \left. \left. + \varepsilon_{\eta\varrho\kappa} (F_{\eta\kappa} p'_\varrho q'^{\kappa} - F_{\kappa\eta} q'^\varrho p'_\kappa - F_{\eta\varrho} q'^{\kappa} p'_\kappa) \right] - F_{\kappa\kappa} L'_\kappa \right\}, \end{aligned}$$

where $i, j, h = 1, 2, 3$, yields $\{L'_i, L'_j\}_{\text{nc}} = \varepsilon_{ijh} F'_{hh} L'_h$, where $\varepsilon_{ijh} F'_{hh}$ are the structure constants of the Lie algebra. ■

Moreover, the following relations hold:

$$\{A'_i, A'_j\}_{\text{nc}} = -2m\varepsilon_{ijh}F'_{hh}H'L'_h, \quad (11)$$

$$\{L'_i, A'_i\}_{\text{nc}} = F'_{jh}(L'_hp'_j + L'_jp'_h), \quad (12)$$

$$\{L'_i, A'_j\}_{\text{nc}} = \varepsilon_{ijh}\left(F'_{hh}A'_h - F'_{hj}\frac{mk}{Y}q'^j\right) + F'_{hj}L'_ip_h, \quad (13)$$

where $F'_{ij} = -F_{ij}$ and $i, j, h = 1, 2, 3$.

We now consider some noncommutative constant-energy hypersurfaces Π_c defined as

$$\Pi_c := \{(q, p) \in \mathcal{T}^*\mathcal{Q} \mid H(q, p) = c\},$$

where c is a constant. Then, following [19] and [51], because the 1-form dH' obtained in (7) has no zeroes on $\mathcal{T}^*\mathcal{Q}$, the noncommutative constant-energy hypersurfaces Π_c are closed submanifolds of $\mathcal{T}^*\mathcal{Q}$. Moreover, we define open submanifolds $\Pi_\tau := \bigcup_{c \geq 0} \Pi_c$, $\tau = -, +$, of $\mathcal{T}^*\mathcal{Q}$ such that

$$\begin{aligned} \mathcal{T}^*\mathcal{Q} &= \Pi_- \cup \Pi_0 \cup \Pi_+, \\ \Pi_- &= \{(q, p) \in \mathcal{T}^*\mathcal{Q} \mid H(q, p) < 0\}, \quad \Pi_+ = \{(q, p) \in \mathcal{T}^*\mathcal{Q} \mid H(q, p) > 0\}, \end{aligned}$$

with Π_0 being the common boundary of Π_- and Π_+ .

On Π_τ , we introduce

$$L_i'^\tau := L'_i|_{\Pi_\tau}, \quad \tau = +, -, \quad i = 1, 2, 3,$$

and define a scaled Runge–Lenz–Pauli vector $\widehat{\Gamma}'$ by

$$\widehat{\Gamma}' = \begin{cases} \widehat{\Gamma}'^- = \frac{1}{(-2mH')^{1/2}}A', & \tau = -, \\ \widehat{\Gamma}'^+ = \frac{1}{(2mH')^{1/2}}A', & \tau = +, \end{cases}$$

where H' is the Hamiltonian function given in (4). We then obtain $\{H', \widehat{\Gamma}'^+\}_{\text{nc}} = \{H', \widehat{\Gamma}'^-\}_{\text{nc}} = 0$, proving that $\widehat{\Gamma}'^+$ and $\widehat{\Gamma}'^-$ are also constants of motion respectively on Π_+ and Π_- . We can then rewrite relations (11), (12), and (13) as

$$\begin{aligned} \{\widehat{\Gamma}'^i_\tau, \widehat{\Gamma}'^j_\tau\}_{\text{nc}} &= -\tau\varepsilon_{ijh}F'_{hh}L_h'^\tau, \\ \{L_i'^\tau, \widehat{\Gamma}'^i_\tau\}_{\text{nc}} &= (2m\tau H')^{-1/2}F'_{jh}(L'_hp'_j + L'_jp'_h), \\ \{L_i'^\tau, \widehat{\Gamma}'^j_\tau\}_{\text{nc}} &= \varepsilon_{ijh}\left(F'_{hh}\widehat{\Gamma}'^i_\tau - (2m\tau H')^{-1/2}F'_{hj}\frac{mk}{Y}q'^j\right) + (2m\tau H')^{-1/2}F'_{hj}L_i'^\tau p_h, \end{aligned}$$

where $F'_{ij} = -F_{ij}$, $i, j, h = 1, 2, 3$, and $\tau = -, +$.

For all $i, j, h = 1, 2, 3$, setting $L_h'^\tau p'_j = -L_j'^\tau p'_h$ and $(m/Y)q'^j = \varepsilon_{ijh}L_i'^\tau p'_h$, we arrive at a Lie algebra isomorphic to the Lie algebra $so(4)$ for $\tau = -$:

$$\begin{aligned} \{L_i'^-, L_j'^-\}_{\text{nc}} &= \varepsilon_{ijh}F'_{hh}L_h'^-, \\ \{\widehat{\Gamma}'^i_-, \widehat{\Gamma}'^j_-\}_{\text{nc}} &= \varepsilon_{ijh}F'_{hh}L_h'^-, \\ \{L_i'^-, \widehat{\Gamma}'^j_-\}_{\text{nc}} &= \varepsilon_{ijh}F'_{hh}\widehat{\Gamma}'^i_- \end{aligned}$$

with the associated generators given by

$$\begin{aligned}\Phi_{hj} &= \sum_{i=1}^3 \varepsilon_{hji} F'_{ii} L_i'^-, & h, j, i &= 1, 2, 3, \\ \Phi_{h4} &= -\Phi_{h4} = F'_{hh} \widehat{\Gamma}_h'^-, & \Phi_{44} &= 0, & h &= 1, 2, 3,\end{aligned}$$

and, for $\tau = +$, at a Lie algebra isomorphic to the Lie algebra $so(1, 3)$:

$$\begin{aligned}\{L_i'^+, L_j'^+\}_{\text{nc}} &= \varepsilon_{ijh} F'_{hh} L_h'^+, \\ \{\widehat{\Gamma}_i'^+, \widehat{\Gamma}_j'^+\}_{\text{nc}} &= -\varepsilon_{ijh} F'_{hh} L_h'^+, \\ \{L_i'^+, \widehat{\Gamma}_j'^+\}_{\text{nc}} &= \varepsilon_{ijh} F'_{hh} \widehat{\Gamma}_h'^+\end{aligned}$$

with the corresponding generators given by

$$\begin{aligned}\Psi_{hj} &= \sum_{i=1}^3 \varepsilon_{hji} F'_{ii} L_i'^+, & h, j, i &= 1, 2, 3, \\ \Psi_{h4} &= \Psi_{h4} = -F'_{hh} \widehat{\Gamma}_h'^+, & \Psi_{44} &= 0, & h &= 1, 2, 3.\end{aligned}$$

5. The case of action–angle coordinates

The idea of considering a Liouville-integrable Hamiltonian system in the action–angle coordinates leads to many interesting results elucidating the general properties of Hamiltonian systems [36], [52].

Assuming the conditions $\sum_{i,j,k=1}^3 \lambda_{ij} \lambda_{ik} q^j q^k = 0$ and $\alpha_{ij} = 0$, Hamiltonian function (4) takes the form

$$H' = \frac{1}{2m} \sum_{i=1}^3 \left[p_i^2 - \frac{1}{2} \sum_{j,k=1}^3 \varepsilon_{ijk} L_k + \frac{1}{4} \sum_{j=1}^3 (\lambda_{ij} q^j)^2 \right] - \frac{k}{r}, \quad (14)$$

where $L_k = q^j p_i - q^i p_j$ ($i, j, k = 1, 2, 3$) denote the components of the angular momentum L on the cotangent bundle $\mathcal{T}^*Q = Q \times \mathbb{R}^3$. The term

$$\varpi = -\frac{1}{4m} \sum_{i,j,k=1}^3 \varepsilon_{ijk} L_k + \frac{1}{8m} \sum_{i,j=1}^3 (\lambda_{ij} q^j)^2$$

contains the deformation parameters perturbing the initial Kepler Hamiltonian function

$$H = \frac{1}{2m} \sum_{i=1}^3 p_i^2 - \frac{k}{r}.$$

Without loss of generality, we can consider the matrix λ of the form

$$\lambda = \begin{pmatrix} 0 & -\vartheta \dot{\varphi} \sin(2\varphi) & \sqrt{2} \vartheta \dot{\varphi} \cos \varphi \\ \vartheta \dot{\varphi} \sin(2\varphi) & 0 & \sqrt{2} \vartheta \dot{\varphi} \sin \varphi \\ -\sqrt{2} \vartheta \dot{\varphi} \cos \varphi & -\sqrt{2} \vartheta \dot{\varphi} \sin \varphi & 0 \end{pmatrix},$$

where we have $\varphi \in (0, 2\pi)$, $\vartheta \in (0, \pi)$, and $\vartheta \sin(2\varphi) \ll m$ and $\dot{\varphi}, \vartheta$ are constants. This yields the Hamiltonian function in spherical–polar coordinates in the form

$$H' = \frac{1}{2m} \left[p_r^2 + M^2 \frac{p_\vartheta^2}{r^2} + \left(1 + \frac{\vartheta}{m} \sin(2\varphi) \right) \frac{p_\varphi^2}{r^2 \sin^2 \vartheta} \right] - \frac{k}{r},$$

where

$$M = \left(1 + \frac{\sqrt{2}}{m}\dot{\varphi} + \frac{1}{2m}\dot{\varphi}^2\right)^{1/2}$$

is a constant of motion,

$$p_r = m\dot{r}, \quad p_\vartheta = mr^2\dot{\vartheta}, \quad p_\varphi = mr^2\dot{\varphi}\sin^2\vartheta.$$

The associated Hamiltonian vector field $X_{H'} \in \mathcal{T}^*\mathcal{Q}$ corresponding to the symplectic form

$$\omega' = dp_r \wedge dr + dp_\vartheta \wedge d\vartheta + dp_\varphi \wedge d\varphi$$

is given by

$$\begin{aligned} X_{H'} = \frac{1}{m} \left\{ p_r \frac{\partial}{\partial r} - \frac{1}{r^2} \left(mk - \frac{M^2 p_\vartheta^2 \sin^2 \vartheta + (1 + (\dot{\vartheta}/m) \sin(2\varphi)) p_\varphi^2}{r \sin^2 \vartheta} \right) \frac{\partial}{\partial p_r} + \right. \\ \left. + M^2 \frac{p_\vartheta}{r^2} \frac{\partial}{\partial \vartheta} + \left(1 + \frac{\dot{\vartheta}}{m} \sin(2\varphi) \right) \frac{p_\varphi^2 \cos \vartheta}{r^2 \sin^3 \vartheta} \frac{\partial}{\partial p_\vartheta} + \right. \\ \left. + \left(1 + \frac{\dot{\vartheta}}{m} \sin(2\varphi) \right) \frac{p_\varphi}{r^2 \sin^2 \vartheta} \frac{\partial}{\partial \varphi} - \frac{\dot{\vartheta} \cos 2\varphi}{mr^2 \sin^2 \vartheta} \frac{\partial}{\partial p_\varphi} \right\}. \end{aligned}$$

We introduce two additional integrals of motion

$$D_\varphi = \left(1 + \frac{\dot{\vartheta}}{m} \sin(2\varphi) \right)^{1/2} p_\varphi, \quad \tilde{L}^2 = M^2 p_\vartheta^2 + \frac{D_\varphi^2}{\sin^2 \vartheta}, \quad (15)$$

where \tilde{L} is the modified angular momentum vector in spherical–polar coordinates,

$$D_\varphi = \tilde{L} \cos \xi,$$

and ξ denotes the angle between the orbit plane and the equatorial plane (x, y) [25]. Because the Hamiltonian function H' does not explicitly depend on time, setting $V = W - Et$ we can find an additive separable solution

$$W = W_r(r) + W_\vartheta(\vartheta) + W_\varphi(\varphi),$$

which reduces the Hamilton-Jacobi equation [53]

$$\frac{\partial V}{\partial t} + H' \left(\frac{\partial V}{\partial q} \middle/ q/t \right) = 0$$

to a simpler form

$$E = \frac{1}{2m} \left(\frac{\partial W}{\partial r} \right)^2 + \frac{M^2}{r^2} \left(\frac{\partial W}{\partial \vartheta} \right)^2 + \frac{1}{2mr^2 \sin^2 \vartheta} \left(1 + \frac{\dot{\vartheta}}{m} \sin(2\varphi) \right) \left(\frac{\partial W}{\partial \varphi} \right)^2 - \frac{k}{r},$$

leading to the set of equations

$$\begin{aligned} \left(\frac{dW_\varphi(\varphi)}{d\varphi} \right)^2 &= p_\varphi^2, \\ \left(\frac{dW_\vartheta(\vartheta)}{d\vartheta} \right)^2 &= \frac{1}{M^2} \left(\tilde{L}^2 - \frac{D_\varphi^2}{\sin^2 \vartheta} \right), \\ -r^2 \left(\frac{dW_r(r)}{dr} \right)^2 + 2mr^2 E + 2mrk &= \tilde{L}^2. \end{aligned}$$

In the compact case characterized by $E < 0$, the action–angle variables [36], [40], [53], [54] can be expressed as

$$\begin{aligned} J_\varphi &= \frac{1}{2\pi} \oint \frac{dW_\varphi(\varphi)}{d\varphi} d\varphi = \frac{1}{2\pi} \oint p_\varphi d\varphi = p_\varphi, \\ J_\vartheta &= \frac{1}{2\pi} \oint \frac{dW_\vartheta(\vartheta)}{d\vartheta} d\vartheta = \frac{1}{2\pi M} \oint \left(\tilde{L}^2 - \frac{D_\varphi^2}{\sin^2 \vartheta} \right)^{1/2} d\vartheta, \\ J_r &= \frac{1}{2\pi} \oint \frac{dW_r(r)}{dr} dr = \frac{1}{2\pi} \oint \left(2mE + \frac{2mk}{r} - \frac{\tilde{L}^2}{r^2} \right)^{1/2} dr, \\ \varphi^i &= \frac{\partial W}{\partial J_i}, \quad \varphi^i(0) = 0, \quad i = 1, 2, 3. \end{aligned}$$

According to Eq. (15), J_φ can be rewritten in terms of D_φ and $\sin(2\varphi)$:

$$J_\varphi = D_\varphi \left(1 + \frac{\dot{\vartheta}}{m} \sin(2\varphi) \right)^{-1/2}.$$

Because $\dot{\vartheta} \sin(2\varphi) \ll m$, it follows that

$$\frac{\dot{\vartheta} \sin(2\varphi)}{m} \rightarrow 0, \quad \frac{\dot{\vartheta} \sin(2\varphi)}{2m} \ll 1.$$

Using the first-order Maclaurin expansion [55],

$$\left(1 + \frac{\dot{\vartheta}}{m} \sin(2\varphi) \right)^{-1/2} \simeq 1 - \frac{\dot{\vartheta}}{2m} \sin(2\varphi) \simeq 1$$

and the standard integration method [56], we obtain

$$\begin{aligned} J_\varphi &= J_3 = D_\varphi, \quad J_\vartheta = J_2 = \frac{1}{M}(\tilde{L} - D_\varphi), \quad J_r = J_1 = -\tilde{L} + \frac{mk}{\sqrt{-2mE}}, \\ \varphi^1 &= -\frac{1}{(J_1 + MJ_2 + J_3)^2} \sqrt{\tilde{G}} + \arcsin \left[\frac{mkr - (J_1 + MJ_2 + J_3)^2}{\tilde{Q}} \right], \\ \varphi^2 &= M\varphi^1 - M \arcsin \left[\frac{(1 - (MJ_2 + J_3)/mkr)(J_1 + MJ_2 + J_3)^{3/2}}{\tilde{Q}} \right] + M \arcsin[\tilde{U} \cos \vartheta], \\ \varphi^3 &= \frac{1}{M}\varphi^2 + \arcsin \left[\frac{J_3 \cot \vartheta}{\sqrt{(MJ_2 + J_3)^2 - J_3^2}} \right] + \varphi, \end{aligned} \tag{16}$$

where

$$\begin{aligned} \tilde{Q} &= (J_1 + MJ_2 + J_3) \sqrt{(J_1 + MJ_2 + J_3)^2 - (MJ_2 + J_3)^2}, \\ \tilde{U} &= \frac{MJ_2 + J_3}{\sqrt{(MJ_2 + J_3)^2 - J_3^2}}, \\ \tilde{G} &= -m^2 k^2 r^2 + 2mk(J_1 + MJ_2 + J_3)^2 r - (MJ_2 + J_3)^2 (J_1 + MJ_2 + J_3)^2. \end{aligned}$$

Then we obtain the Hamiltonian H' , the Poisson bivector P' , the symplectic form ω' , and the Hamiltonian vector field $X_{H'}$,

$$\begin{aligned} H' &= E = -\frac{mk^2}{2(J_1 + MJ_2 + J_3)^2}, \quad P' = \sum_{h=1}^3 \frac{\partial}{\partial J_h} \wedge \frac{\partial}{\partial \varphi^h}, \\ \omega' &= \sum_{h=1}^3 dJ_h \wedge d\varphi^h, \quad X_{H'} := \{H', \cdot\} = \frac{mk^2}{(J_1 + MJ_2 + J_3)^3} \left(\frac{\partial}{\partial \varphi^1} + M \frac{\partial}{\partial \varphi^2} + \frac{\partial}{\partial \varphi^3} \right), \end{aligned}$$

satisfying the required relation $\iota_{X_{H'}} \omega' = -dH'$. Therefore, in the action–angle coordinates (J, φ) , the triplet $(\mathcal{T}^*Q, \omega', H')$ is also a Hamiltonian system.

6. Construction of bi-Hamiltonian structures

The generalized action–angle variables of Kepler’s problem are usually denoted by $L, G, H, l = M, g = \omega, h = \Omega$, called the Delaunay variables [57]. G and H have the meaning of the respective total and azimuthal angular momentum, and L has the meaning of the total orbital action. The angles ω and Ω are the argument of periapsis and the longitude of ascending node. The angle M is just the mean anomaly containing the only real dynamics in the Kepler problem.

We consider the Delaunay-type variables

$$\begin{cases} I_1 = J_3, \\ I_2 = MJ_2 + J_3, \\ I_3 = J_1 + MJ_2 + J_3, \end{cases} \quad \begin{cases} \phi^1 = \varphi^3 - \frac{1}{M}\varphi^2, \\ \phi^2 = \varphi^2 - M\varphi^1, \\ \phi^3 = \varphi^1, \end{cases}$$

which coincide with the classical Delaunay variables for $\lambda_{ij} = 0$ [58]:

$$\begin{cases} I_1 \equiv H = \sqrt{mka(1 - \epsilon^2)} \cos \xi, \\ I_2 \equiv G = \sqrt{mka(1 - \epsilon^2)}, \\ I_3 \equiv L = \sqrt{mka}, \end{cases} \quad \begin{cases} \phi^1 \equiv h = \Omega, \\ \phi^2 \equiv g = \omega, \\ \phi^3 \equiv l = M = n(t - t_0), \end{cases}$$

where ξ is the inclination, n is the mean motion, a is the semimajor axis of the orbit, ϵ is the eccentricity, and t_0 is the time at which the satellite passes through the perigee.

Then the Hamiltonian function H' , the symplectic form ω' , the Poisson bivector P' , and the Hamiltonian vector field X'_H reduce to the expressions

$$\begin{aligned} H' &= -\frac{mk^2}{2I_3^2}, & P' &= \sum_{j=1}^3 \tilde{N}_j \frac{\partial}{\partial I_j} \wedge \frac{\partial}{\partial \phi^j}, \\ \omega' &= \sum_{j=1}^3 \frac{1}{\tilde{N}_j} dI_j \wedge d\phi^j, & X'_H &= \frac{mk^2}{I_3^3} \frac{\partial}{\partial \phi^3}, \end{aligned}$$

where $\tilde{N}_1 = 1, \tilde{N}_2 = M, \tilde{N}_3 = 1$.

We recall that a vector field X is called nondegenerate or anisochronous if the Kolmogorov condition [40] for the Hessian matrix

$$\det \left| \frac{\partial^2 H(J_1, \dots, J_n)}{\partial J_i \partial J_k} \right| \neq 0 \quad (17)$$

is satisfied almost everywhere in the given action–angle coordinates. This condition implies that the dense subsets of the invariant n -dimensional tori of X are closures of trajectories. If (17) is not satisfied, X is called a degenerate or isochronous vector field.

In our framework, X'_H is a degenerate Hamiltonian vector field because

$$\det \left| \frac{\partial^2 H'(J_1, \dots, J_n)}{\partial J_i \partial J_k} \right| = 0.$$

Our system is isochronous with a well-defined derivative for $H' = E < 0$,

$$\tilde{\mathbf{a}} = \frac{\partial H'}{\partial J_1} = \frac{1}{k\sqrt{m}}(-2E)^{3/2}.$$

and therefore, according to the Bogoyavlenskij theorem [40], we can obtain a bi-Hamiltonian formulation of this system in the domain of definition of action–angle variables (16). In particular, we here use the Hamiltonian function H' in the defined Delaunay-type variables (I, ϕ) to construct bi-Hamiltonian structures. Following the example of the generic Bogoyavlenskij construction for the isochronous Hamiltonian system proposed in [35], we can make the canonical transformations

$$\begin{aligned}\tilde{I}_1 &= I_1, & \tilde{I}_2 &= I_2, & \tilde{I}_3 &= H' = -\frac{mk^2}{2I_3^2}, \\ \tilde{\phi}^1 &= \phi^1, & \tilde{\phi}^2 &= \phi^2, & \tilde{\phi}^3 &= \frac{k\sqrt{m}}{(-2H)^{3/2}}\phi^3,\end{aligned}$$

permitting us to construct a set of Poisson bivectors for all $h \in \mathbb{N}$:

$$\tilde{P}_h = \tilde{\beta}_1(\tilde{I}_1) \frac{\partial}{\partial \tilde{I}_1} \wedge \frac{\partial}{\partial \tilde{\phi}^1} + \tilde{\beta}_2(\tilde{I}_2) \frac{\partial}{\partial \tilde{I}_2} \wedge \frac{\partial}{\partial \tilde{\phi}^2} + \left(\frac{dF_h}{d\tilde{I}_3} \right)^{-1} \frac{\partial}{\partial \tilde{I}_3} \wedge \frac{\partial}{\partial \tilde{\phi}^3}.$$

Putting

$$\begin{aligned}\tilde{\beta}_1(\tilde{I}_1) &= \tilde{I}_1^h = I_1^h, & \tilde{\beta}_2(\tilde{I}_2) &= M^{h+1} \tilde{I}_2^h = M^{h+1} I_2^h, \\ F_h &= -\frac{mk^2}{(2+h)I_3^{2+h}}, & h &\in \mathbb{N},\end{aligned}$$

simplifies the expression of the Poisson bivectors \tilde{P}_h as

$$\tilde{P}_h = \sum_{j=1}^3 \tilde{N}_j^{h+1} I_j^h \frac{\partial}{\partial I_j} \wedge \frac{\partial}{\partial \phi^j}, \quad \tilde{N}_1 = 1, \quad \tilde{N}_2 = M, \quad \tilde{N}_3 = 1.$$

Each of \tilde{P}_h is compatible with P' , i.e., $[\tilde{P}_h, P']_{\text{NS}} = 0$. In this case, the eigenvalues of the corresponding recursion operator $T := \tilde{P} \circ P'^{-1}$ are integrals of motion only [34], [39], [40]. Then, for all $h \in \mathbb{N}$, a hierarchy of Poisson bivectors \tilde{P}_h and their corresponding 2-forms $\tilde{\omega}_h$ is given by

$$\tilde{P}_h = \sum_{i,j=1}^6 (\tilde{P}_h)^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}, \quad \tilde{\omega}_h = \sum_{i,j=1}^6 (\tilde{\omega}_h)_{ij} dx^i \wedge dx^j,$$

where $n = 3$, $x^k = I_k$, $x^{k+3} = \phi^k$, $k \leq 3$,

$$\begin{cases} (\tilde{P}_h)^{14} = I_1^h, \\ (\tilde{P}_h)^{25} = M^{h+1} I_2^h, \\ (\tilde{P}_h)^{36} = I_3^h, \\ (\tilde{P}_h)^{ij} = 0 \text{ otherwise,} \end{cases} \quad \begin{cases} (\tilde{\omega}_h)_{41} = I_1^{-h}, \\ (\tilde{\omega}_h)_{52} = M^{-(h+1)} I_2^{-h}, \\ (\tilde{\omega}_h)_{63} = I_3^{-h}, \\ (\tilde{\omega}_h)_{ij} = 0 \text{ otherwise,} \end{cases}$$

$$(\tilde{P}_h)^{ij} = -(\tilde{P}_h)^{ji}, \quad (\tilde{\omega}_h)_{ij} = -(\tilde{\omega}_h)_{ji}.$$

In the previous action–angle coordinate system (J, φ) , they become

$$\left\{ \begin{array}{l} (\tilde{P}_h)^{14} = \left(k \sqrt{\frac{-m}{2H'}} \right)^h, \\ (\tilde{P}_h)^{24} = \frac{1}{M} ((\tilde{P}_h)^{14} - (\tilde{P}_h)^{25}), \\ (\tilde{P}_h)^{25} = M^h \tilde{L}^h, \\ (\tilde{P}_h)^{34} = M (\tilde{P}_h)^{24}, \\ (\tilde{P}_h)^{35} = M ((\tilde{P}_h)^{25} - (\tilde{P}_h)^{36}), \\ (\tilde{P}_h)^{36} = J_3^h, \\ (\tilde{P}_h)^{ij} = 0 \text{ otherwise,} \end{array} \right. \quad \left\{ \begin{array}{l} (\tilde{\omega}_h)_{41} = \left(\frac{1}{k} \sqrt{\frac{-2H'}{m}} \right)^h, \\ (\tilde{\omega}_h)_{42} = M ((\tilde{\omega}_h)_{41} - (\tilde{\omega}_h)_{52}), \\ (\tilde{\omega}_h)_{52} = \frac{1}{M^h \tilde{L}^h}, \\ (\tilde{\omega}_h)_{43} = \frac{1}{M} (\tilde{\omega}_h)_{42}, \\ (\tilde{\omega}_h)_{53} = \frac{1}{M} ((\tilde{\omega}_h)_{52} - (\tilde{\omega}_h)_{63}), \\ (\tilde{\omega}_h)_{63} = J_3^{-h}, \\ (\tilde{\omega}_h)_{ij} = 0 \text{ otherwise,} \end{array} \right.$$

$$(\tilde{P}_h)^{ij} = -(\tilde{P}_h)^{ji}, \quad (\tilde{\omega}_h)_{ij} = -(\tilde{\omega}_h)_{ji}, \quad H' = H'(J, \varphi), \quad \tilde{L}^h = \tilde{L}^h(J, \varphi).$$

The Poisson bracket $\{ \cdot, \cdot \}_{\tilde{\omega}_h}$ with respect to each symplectic form $\tilde{\omega}_h$ is now defined as

$$\{f, g\}_{\tilde{\omega}_h} = \sum_{i,j=1}^3 (\Lambda_h)_j^i \left(\frac{\partial f}{\partial I_i} \frac{\partial g}{\partial \phi^j} - \frac{\partial f}{\partial \phi^j} \frac{\partial g}{\partial I_i} \right), \quad (\Lambda_h) = \begin{pmatrix} (I_1)^h & 0 & 0 \\ 0 & M^{h+1} I_2^h & 0 \\ 0 & 0 & (I_3)^h \end{pmatrix}.$$

Proposition 3. For each $h \in \mathbb{N}$, the vector field X_H is a bi-Hamiltonian vector field with respect to $(\omega', \tilde{\omega}_h)$, i.e.,

$$\iota_{X_{H'}} \omega' = -dH', \quad \iota_{X_{H'}} \tilde{\omega}_h = -dF_h, \quad X_{H'} = \{H', \cdot\} = \{F_h, \cdot\}_{\tilde{\omega}_h},$$

where F_h ($F_0 \equiv H'$) are integrals of motion for $X_{H'}$.

Proof. Because

$$\iota_X (df \wedge dg) = (Xf) dg - (df) Xg,$$

we obtain

$$\iota_{X_{H'}} \omega' = -dH', \quad \iota_{X_{H'}} \tilde{\omega}_h = -dF_h, \quad X_{H'} = \{H', \cdot\} = \{F_h, \cdot\}_{\tilde{\omega}_h}. \quad \blacksquare$$

Besides, we obtain the recursion operators

$$T_h = \sum_{i,j=1}^3 (T_h)_i^j \left(\frac{\partial}{\partial I_j} \otimes dI_i + \frac{\partial}{\partial \phi^j} \otimes d\phi^i \right), \quad h \in \mathbb{N},$$

$$\begin{cases} (T_h)_1^1 = I_1^h, \\ (T_h)_2^2 = M^h I_2^h, \\ (T_h)_3^3 = I_3^h, \\ (T_h)_j^i = 0 \text{ otherwise,} \end{cases}$$

which in the action–angle coordinate system become (J, φ) :

$$T_h = \sum_{i,j=1}^3 \left((R_h)_i^j \frac{\partial}{\partial J_j} \otimes dJ_i + (S_h)_i^j \frac{\partial}{\partial \varphi^j} \otimes d\varphi^i \right),$$

with

$$\left\{ \begin{array}{l} (R_h)_1^1 = (R_h)_3^3 = (R_h)_1^3 = (\tilde{P}_h)^{14}, \\ (R_h)_1^2 = M(\tilde{P}_h)^{14}, \\ (R_h)_{1/2}^1 = \frac{1}{M}(\tilde{P}_h)^{14}, \\ (R_h)_2^2 = (\tilde{P}_h)^{14} + (\tilde{P}_h)^{25}, \\ (R_h)_2^3 = \frac{1}{M}(R_h)_2^2, \\ (R_h)_3^2 = M(R_h)_2^2, \\ (R_h)_3^3 = (R_h)_2^2 + (\tilde{P}_h)^{36}, \end{array} \right. \quad \left\{ \begin{array}{l} (S_h)_1^1 = (R_h)_2^2, \\ (S_h)_1^2 = -\frac{1}{M}(\tilde{P}_h)^{25}, \\ (S_h)_{1/2}^1 = -M(\tilde{P}_h)^{25}, \\ (S_h)_2^2 = (\tilde{P}_h)^{36} + (\tilde{P}_h)^{25}, \\ (S_h)_2^3 = -M(\tilde{P}_h)^{36}, \\ (S_h)_3^2 = -\frac{1}{M}(\tilde{P}_h)^{36}, \\ (S_h)_3^3 = (\tilde{P}_h)^{36}, \\ (S_h)_1^3 = (S_h)_{1/3}^3 = 0. \end{array} \right.$$

7. Master symmetries

In differential geometric terms, a vector field Γ on $\mathcal{T}^*\mathcal{Q}$ that satisfies

$$[X_{H'}, \Gamma] \neq 0, \quad [X_{H'}, X] = 0, \quad [X_{H'}, \Gamma] = X$$

is called a master symmetry or a generator of symmetries of degree $m = 1$ for $X_{H'}$ [38], [41], [59]–[61].

In the following, we consider the Hamiltonian system $(\mathcal{T}^*\mathcal{Q}, \omega', H')$ and the integrals of motion F_h , $h \in \mathbb{N}$. Thereby, we obtain the vector fields

$$X_h := \{F_h, \cdot\} = \frac{mk^2}{I_3^{(h+3)}} \frac{\partial}{\partial \phi^3},$$

which commute with the Hamiltonian vector field $X_0 = X_{H'}$. The X'_h s are called *dynamical symmetries* of the Hamiltonian system $(\mathcal{T}^*\mathcal{Q}, \omega', H')$, i.e., $[X_{H'}, X_h] = 0$.

For the Hamiltonian system $(\mathcal{T}^*\mathcal{Q}, \omega', H')$, we introduce the vector fields $\Gamma_{i\mu} \in \mathcal{T}^*\mathcal{Q}$,

$$\Gamma_{i\mu} = \frac{1}{(3+i)} \sum_{j=1}^3 \frac{\tilde{N}_j^\mu}{I_j^{(\mu-1)}} \left(\frac{\partial}{\partial I_j} + \frac{\partial}{\partial \phi^j} \right), \quad \tilde{N}_1 = 1, \quad \tilde{N}_2 = M, \quad \tilde{N}_3 = 1, \quad i, \mu \in \mathbb{N},$$

satisfying the relation

$$\iota_{\Gamma_{i\mu}} \omega' = -d\tilde{F}_{i\mu},$$

with

$$\tilde{F}_{i\mu} = \begin{cases} \sum_{j=1}^3 \frac{\tilde{N}_j}{3+i} \left(\ln(I_j) - \frac{\phi^j}{I_j} \right), & \mu = 2, \\ \sum_{j=1}^3 \frac{\tilde{N}_j^{\mu-1}}{3+i} \left(\frac{I_j^{2-\mu}}{2-\mu} - \frac{\phi^j}{I_j^{\mu-1}} \right), & \mu \neq 2. \end{cases}$$

Computing the Lie bracket between X_i and $\Gamma_{i\mu}$, we obtain (see Fig. 1 for their diagram representation)

$$[X_i, \Gamma_{i\mu}] = X_{i+\mu}, \quad [X_i, X_{i+\mu}] = 0, \quad X_{i+\mu} = \frac{mk^2}{I_3^{(\mu+i+3)}} \frac{\partial}{\partial I_3}. \quad (18)$$

Hence, $\Gamma_{i\mu}$ are master symmetries or generators of symmetries of degree $m = 1$ for X_i . The quantities $\tilde{F}_{i\mu}$ are called *master integrals*. Furthermore, we have

$$\begin{aligned} \mathcal{L}_{\Gamma_{i0}}(P') &= -\frac{1}{3+i}P' & \left(\tilde{\alpha} = -\frac{1}{3+i} \right); & \quad \mathcal{L}_{\Gamma_{i0}}(\tilde{P}_1) = 0 \quad (\tilde{\beta} = 0); \\ \mathcal{L}_{\Gamma_{i0}}(H') &= -\frac{2}{3+i}H' & \left(\tilde{\gamma} = -\frac{2}{3+i} \right). \end{aligned}$$

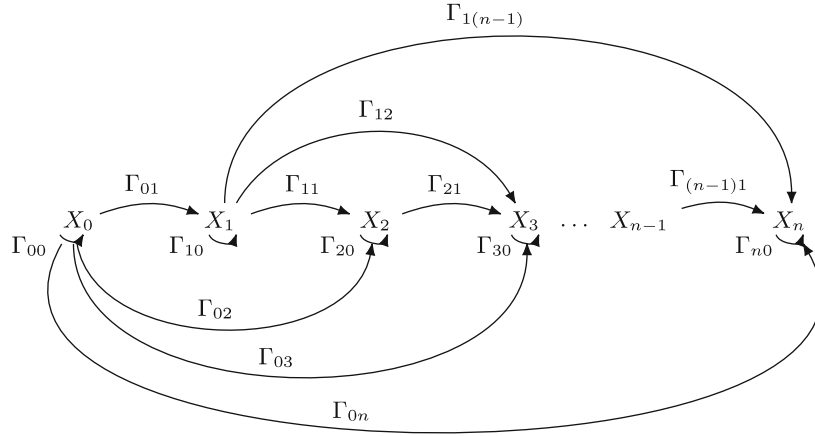


Fig. 1. Diagrammatical illustration of Eq. (18).

showing that the vector fields

$$\Gamma_{i0} = \frac{1}{(3+i)} \sum_{j=1}^3 I_j \left(\frac{\partial}{\partial I_j} + \frac{\partial}{\partial \phi^j} \right)$$

are conformal symmetries for both P' , \tilde{P}_1 and H' [38]. Now, defining the families of quantities $X'_h, \Gamma'_{ih}, P'_h, \omega'_h$ and dH'_h by

$$\begin{aligned} X'_h &:= T^h X_0, & P'_h &:= T^h P', & \omega'_h &:= (T^*)^h \omega', \\ \Gamma'_{ih} &:= T^h \Gamma_{i0}, & dH'_h &:= (T^*)^h dH', \end{aligned}$$

where $i, h \in \mathbb{N}$ and $T^* := P'^{-1} \circ \tilde{P}_1$ denotes the adjoint of $T := \tilde{P}_1 \circ P'^{-1}$, we obtain

$$\begin{aligned} P'_h &= \sum_{j=1}^3 \tilde{N}_j^{h+1} I_j^h \frac{\partial}{\partial I_j} \wedge \frac{\partial}{\partial \phi^j}, \\ \Gamma'_{ih} &= \sum_{j=1}^3 \frac{\tilde{N}_j^h I_j^{h+1}}{3+i} \left(\frac{\partial}{\partial I_j} + \frac{\partial}{\partial \phi^j} \right), & X'_h &= \frac{mk^2}{I_3^{3-h}} \frac{\partial}{\partial \phi^3}, \\ \omega'_h &= \sum_{j=1}^3 \tilde{N}_j^{h-1} I_j^h dI_j \wedge d\phi^j, & dH'_h &= \frac{mk^2}{I_3^{3-h}} dI_3, & H'_h &= \begin{cases} mk^2 \ln(I_3), & h = 2, \\ \frac{mk^2}{(2-h)I_3^{2-h}}, & h \neq 2, \end{cases} \end{aligned}$$

and, for each $i \in \mathbb{N}$, we derive the plethora of conserved quantities

$$\begin{aligned} \mathcal{L}_{\Gamma'_{ih}}(\Gamma'_{il}) &= \frac{l-h}{3+i} \Gamma'_{i(l+h)}, & \mathcal{L}_{\Gamma'_{ih}}(X'_l) &= -\frac{3-l}{3+i} X'_{l+h}, & \mathcal{L}_{\Gamma'_{ih}}(P'_l) &= \frac{l-h-1}{3+i} P'_{l+h}, \\ \mathcal{L}_{\Gamma'_{ih}}(\omega'_l) &= \frac{l+h+1}{3+i} \omega'_{l+h}, & \mathcal{L}_{\Gamma'_{ih}}(T) &= \frac{1}{3+i} T^{1+h}, & l \in \mathbb{N}, \\ \langle dH'_l, \Gamma'_{ih} \rangle &= \begin{cases} \frac{mk^2}{3+i}, & h+l=2, \\ -\frac{2-(h+l)}{3+i} H'_{l+h}, & h+l \neq 2, \end{cases} \end{aligned}$$

satisfying

$$\begin{aligned}\mathcal{L}_{\Gamma'_{ih}}(\Gamma'_{il}) &= (\tilde{\beta} - \tilde{\alpha})(l - h)\Gamma'_{i(l+h)}, & \mathcal{L}_{\Gamma'_{ih}}(X'_l) &= (\tilde{\beta} + \tilde{\gamma} + (l - 1)(\tilde{\gamma} - \tilde{\alpha}))X'_{l+h}, \\ \mathcal{L}_{\Gamma'_{ih}}(P'_l) &= (\tilde{\beta} + (l - h - 1)(\tilde{\beta} - \tilde{\alpha}))P'_{l+h}, & \mathcal{L}_{\Gamma'_{ih}}(\omega'_l) &= (\tilde{\beta} + (l + h + 1)(\tilde{\beta} - \tilde{\alpha}))\omega'_{l+h}, \\ \mathcal{L}_{\Gamma'_{ih}}(T) &= (\tilde{\beta} - \tilde{\alpha})T^{1+h}, & \langle dH'_l, \Gamma'_{ih} \rangle &= \begin{cases} \frac{mk^2}{3+i}, & h+l=2, \\ (\tilde{\gamma} + (l+h)(\tilde{\beta} - \tilde{\alpha}))H'_{l+h}, & h+l \neq 2, \end{cases}\end{aligned}$$

similarly to the Oevel formulas (see [36]–[39]).

8. Concluding remarks

In this paper, we have defined a noncommutative phase space, derived a Hamiltonian system, and proved the existence of dynamical symmetry groups $SO(3)$, $SO(4)$, and $SO(1,3)$ for the Kepler problem. Further, we have investigated the same Kepler problem in action–angle coordinates and obtained its corresponding Hamiltonian system. Then, we have constructed a hierarchy of bi-Hamiltonian structures in the considered action–angle coordinates following the example of the generic Bogoyavlenskij construction for the isochronous Hamiltonian system proposed by Grigoryev et al., and computed conserved quantities using related master symmetries.

Conflicts of interest. The authors declare no conflicts of interest.

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