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MULTI-COMPONENT TODA LATTICE IN CENTRO-AFFINE $\mathbb{R}^n$

Xiaojuan Duan*, **Chuanzhong Li†‡**, **Jing Ping Wang§**

We use the group-based discrete moving frame method to study invariant evolutions in a n-dimensional centro-affine space. We derive the induced integrable equations for invariants, which can be transformed to local and nonlocal multi-component Toda lattices under a Miura transformation, and thus establish their geometric realizations in centro-affine space.

Keywords: Discrete moving frame, multi-component Toda lattices, Hamiltonian structures

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1. Introduction

Integrable systems are closely linked to classical geometries. Geometric evolutions for curves in homogeneous spaces induce integrable flows for geometric invariants such as curvatures. The space of these invariants can be viewed as coordinates of the moduli space of curves under the group action. The moving frame approach leads to a natural description of its associated Hamiltonian structures defined on the moduli space.

A well-known example was given by Hasimoto [1]. He showed that a curve flow in Euclidean space, invariant under the Euclidean group, known as the vortex-filament flow, induces the nonlinear Schrödinger (NLS) equation for the curvature and torsion of the curve flow. The vortex filament flow is called an Euclidean realization of the NLS equation. Geometric realizations of other integrable systems such as the Korteweg–de Vries (KdV) equation, the modified KdV and sine-Gordon equations are derived in classical geometries. The method of a group-based moving frame introduced by Fels and Olver [2], [3] has played a very important role in establishing the relations. There are many papers devoted to this topic. We refer to [4], [5] and the references therein.

*Department of Mathematics and Physics, Xiamen University of Technology, Xiamen, China, e-mail: 2010111012@xmut.edu.cn.

†School of Mathematics and Statistics, Ningbo University, Ningbo, China, e-mail: lichuanzhong@nbu.edu.cn.

‡College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, China.

§School of Mathematics, Statistics and Actuarial Science, University of Kent, Canterbury, UK, e-mail: j.wang@kent.ac.uk (corresponding author).

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In 2013, the method of group-based discrete moving frames was introduced by Mansfield, Marí-Beffa and Wang [6], which essentially amounted to a sequence of moving frames with overlapping domains. It provides a powerful tool to study the link between induced completely integrable systems on discrete curvatures (or invariants) and the invariant evolutions of polygons in different geometric settings. As examples, the projective $\mathbb{RP}^2$ and 2-dimensional centro-affine $\mathbb{R}^2$ discrete realizations of the modified Volterra and Toda lattices were derived in [6]. That study was soon extended to projective polygons in $\mathbb{RP}^n$ in [7], establishing a close relation between the projective invariant evolutions and the Hamiltonian evolutions on the invariants of the flow.

The induced flows for invariants from geometric evolutions for curves in classical geometries can be viewed as a syzygy between differential and difference invariants [8], [9], which offers a great advantage in direct computation of the Euler–Lagrange equations in terms of invariants from given invariant Lagrangians.

This paper is devoted to the study of invariant evolutions in centro-affine $\mathbb{R}^n$ and induced integrable systems. In the 3-dimensional centro-affine case, the authors of [10] studied the geometric realizations of the B-Toda and C-Toda lattices. Recently, Beffa and Calini investigated the evolutions of arc length-parameterized polygons (corresponding to the case $p_s = 1$ in Section 4) in an n-dimensional centro-affine space, which can be identified with the case of projective $\mathbb{RP}^{n-1}$ [11]. They proved that the Poisson brackets derived in [7] form a bi-Hamiltonian pair.

In this paper, we derive the induced integrable equations from invariant evolutions in the n-dimensional centro-affine space and establish their geometric realizations.

The paper is organized as follows. In Sec. 2, we review basic facts on discrete moving frames and invariants of evolutions, mainly following [6], [7]. In Sec. 3, we give a brief introduction to multi-component Toda lattices for both local and nonlocal flows. Our main results are in Sec. 4. We use the approach of discrete moving frames to derive the flow of invariants for a given invariant time evolution. In the 3-dimensional centro-affine space, we construct a Hamiltonian pair that generates both local and nonlocal integrable differential–difference systems, which can be transformed into 3-component local and nonlocal Toda lattices under the same Miura transformation. In the general n-dimension case, although it is difficult to give the Hamiltonian pair explicitly, we write the integrable differential–difference systems that are multi-component local and nonlocal Toda lattices.

2. Discrete moving frames and invariant evolutions

In this section, we describe basic definitions and theorems on discrete group-based moving frames and invariant evolutions. We only state results (without proofs) for the left group action and the right discrete moving frame, which are taken from [6], [7]. We refer the readers to the original papers for the details.

2.1. Discrete moving frames. Let M be an $n-$dimensional manifold and $G \times M \to M$ be a left action of an r-dimensional Lie group G on M.

We begin with a discrete analogue of the mth-order submanifold jet bundle introduced in [12]. We assume that $x\colon \mathbb{Z} \to M$ is a discrete function. Here we use the subscript notation $x_s = x(s)$ to denote the evaluation of x at an integer point $s \in \mathbb{Z}$. The collection of $m+1$ points

$$x_s^{[m]} = (x_s, x_{s+1}, \dots, x_{s+m}), \qquad x_{s+i} \in M,\ i = 0, \dots, m,$$

is the mth-order forward discrete jet at $s \in \mathbb{Z}$ denoted by $(s, x_s^{[m]})$. Then the mth-order forward discrete jet space $J^{[m]}$ is defined as the collection of $(s, x_s^{[m]})$, that is,

$$J^{[m]} = \bigcup_{s\in\mathbb{Z}} (s, x_s^{[m]}).$$

Let $\pi^m \colon J^{[m]} \to \mathbb{Z}$ denote the projection onto the discrete index

$$\pi^m(s, x_s^{[m]}) = s.$$

Then for each $s \in \mathbb{Z}$, the fiber $J^{[m]}|_s = (\pi^m)^{-1}(s) \simeq M^{m+1}$ is a smooth manifold, when $m \geqslant 0$. Naturally, we can extend the action of G on M to $J^{[m]}$ as follows:

$$g \cdot (s, x_s^{[m]}) = (s, g \cdot x_s, g \cdot x_{s+1}, \ldots, g \cdot x_{s+m}). \tag{1}$$

Definition 1. A *discrete right* (respectively *left*) *moving frame* is a G-equivariant map $\rho \colon J^{[m]} \to G$ satisfying

$$\rho(s, g \cdot x_s^{[m]}) = \rho(s, x_s^{[m]}) g^{-1}$$

(respectively $g \cdot \rho(s, x_s^{[m]})$) for all $g \in G$.

For simplicity, we use the notation ρ_s to denote the image of the moving frame ρ at the point $(s, x_s^{[m]})$. If ρ_s is a left moving frame, then ρ_s^{-1} is a right moving frame. As in the continuous case, the construction of a discrete moving frame is based on the choice of the cross section. The cross section is not unique. One adequate cross section can simplify the computation. We use $\mathcal{K}_s$ to denote the cross sections over s on $J^{[m]}$. For a discrete moving frame, its cross section over s is replicated for all other base points $s+i$, which means the cross section over $s+i$ is represented by $\mathcal{K}_{s+i} = \mathcal{T}^i \mathcal{K}_s$ for all $i \in \mathbb{Z}$, where $\mathcal{T}$ is the shift operator. Consequently, we have that $\rho_{s+i} = \mathcal{T}^i \rho_s$.

The discrete moving frames provide a powerful approach to construct discrete invariants. We say that a function $F \colon J^{[m]} \to \mathbb{R}$ is a discrete invariant if

$$F(g \cdot x_s^{[m]}) = F(x_s^{[m]}) \qquad \text{for all } g \in G \text{ and } x_s^{[m]} \in J^{[m]}.$$

For a right moving frame, the quantities

$$I_{s,j} := \rho_s \cdot x_j \tag{2}$$

are invariants. The induced action on the coordinate functions also produces discrete invariants, that is, for any difference function $F \colon J^{[m]} \to \mathbb{R}$, the induced action on it $F(s, \rho_s \cdot x_s^{[m]})$ is a discrete invariant. We are able to describe a smaller set of generating invariants, the Maurer–Cartan invariants.

Definition 2. Let $\rho \colon J^{[m]} \to G$ be a right moving frame. The element of the group

$$K_s = \rho_{s+1} (\rho_s)^{-1} \tag{3}$$

is called the *right Maurer–Cartan matrix* for ρ.

The equivariance of ρ implies that the K_s are invariant under the group action. In addition, using (2) and (3) we have

$$K_s \cdot I_{s,j} = \rho_{s+1} \rho_s^{-1} \cdot \rho_s \cdot x_j = \rho_{s+1} \cdot x_j = I_{s+1,j}, \tag{4}$$

and iterating this, we have $K_{s+1} K_s \cdot I_{s,j} = I_{s+2,j}$, and so on. Hence, the components of K_s, together with the set of all diagonal invariants, $I_{j,j} = \rho_j \cdot x_j$, generate all other invariants [6].

2.2. Invariant evolutions. For an evolution equation

$$(x_s)_t = F_s((x_r)) \tag{5}$$

we say that it is an invariant time evolution under the action of the group G if the group action takes solutions to solutions, that is, if (x_r) is a solution, then so is $(g \cdot x_r)$ for any $g \in G$. Any invariant time evolution can be explicitly expressed in terms of the invariants and the moving frame.

Here we consider homogeneous manifolds $M = G/H$ with H a closed subgroup and assume that G acts on M via left multiplication on representatives of the class. The distinguished class of H is denoted by $o \in G/H$. Let ρ_s be the discrete right moving frame satisfying $\rho_s \cdot x_s = o$ for all s. We can describe the general formula for an invariant evolution (5) in terms of the moving frame [6], [7].

We let $\Gamma_g \colon G/H \to G/H$ denote the map defined by the action of $g \in G$, that is, $\Gamma_g(x) = g \cdot x$, and $d\Gamma_g(x)$ denote the tangent map of Γ_g at $x \in G/H$. Any G-invariant evolution of form (5) can be written as

$$(x_s)_t = d\Gamma_{\rho_s^{-1}}(o)(\mathbf{v}_s), \tag{6}$$

where $\mathbf{v}_s$ is an invariant vector in the tangent space to M at x_s.

For invariant evolution (6), there is a simple process to describe the evolution induced on the Maurer–Cartan matrices, and hence on a generating set of invariants as stated in the next theorem. Its proof for the left discrete moving frame can be found [6].

Theorem 1. *Let $\varsigma \colon G/H \to G$ be a section of G/H such that $\varsigma(o) = e \in G$, where e is the identity. Given a right moving frame ρ_s, we assume that $\rho_s \cdot x_s = o$ and $\rho_s = \rho_s^H \varsigma(x_s)^{-1}$, for some $\rho_s^H \in H$. Then the invariant evolution* (6) *leads to the structure equation*

$$(K_s)_t = N_{s+1} K_s - K_s N_s, \tag{7}$$

where K_s is the right Maurer–Cartan matrix and $N_s = (\rho_s)_t \rho_s^{-1} \in \mathfrak{g}$. Furthermore, we assume that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, where $\mathfrak{g}$ is the algebra of G, $\mathfrak{h}$ is the algebra of H, and $\mathfrak{m}$ is a linear complement (*which can be identified with the tangent to the image of the section ς*)*. Then, if $N_s = N_s^{\mathfrak{h}} + N_s^{\mathfrak{m}}$ splits accordingly, we have*

$$N_s^{\mathfrak{m}} = -d\varsigma(o)\mathbf{v}_s. \tag{8}$$

In this paper, we apply Theorem 1 to the centro-affine space. In fact, equation (7) and condition (8) completely determine the evolution of K_s [6] [7], [10]. We note that identity (7) is similar to the zero-curvature condition (without the spectral parameter) for completely integrable systems. This is a key point when we link integrable systems to invariant evolutions.

3. Multi-component Toda lattices

We link the invariant evolutions in a centro-affine space to multi-component Toda lattices. To be self-contained, we recall some facts on the Toda lattices in this section.

The well-known Toda lattice [13] is given by

$$\frac{d^2 u_s}{dt^2} = e^{u_{s-1} - u_s} - e^{u_s - u_{s+1}}. \tag{9}$$

Here, the dependent variable u is a function of the time t and a discrete variable $s \in \mathbb{Z}$. It can be viewed as a discretization of the KdV equation. Using the Flaschka coordinates [14], [15]

$$q_s = \frac{du_s}{dt}, \qquad p_s = e^{u_s - u_{s+1}},$$

we rewrite Toda lattice (9) in the form

$$\frac{dp_s}{dt} = p_s(q_s - q_{s+1}), \qquad \frac{dq_s}{dt} = p_{s-1} - p_s. \tag{10}$$

Its complete integrability was first established by Flaschka and Manakov [14]–[16]. Its Lax representation can be written as

$$\frac{\partial \mathcal{L}}{\partial t} = [B, \mathcal{L}] = B\mathcal{L} - \mathcal{L}B, \tag{11}$$

where

$$\mathcal{L} = \mathcal{T}^{-1} + q_s + p_s\mathcal{T}, \qquad B = -\mathcal{L}_{\geq 1} = -p_s\mathcal{T}.$$

The above scalar Lax representation has been generalized to higher-order difference operators involving more dependent variables [17]. From now on, we drop the subscript s without causing confusion. For instance, we simply write p for p_s and p_i for p_{s+i}. Let

$$\mathcal{L} = \mathcal{T}^{-n} + \sum_{j=1}^{n} w^j \mathcal{T}^{-n+j} + u\mathcal{T}, \qquad B = \mathcal{L}_{\geq 1} = u\mathcal{T}, \tag{12}$$

where u and w^j, $j = 1, 2, \ldots, n$ are dependent variables. It follows from Lax equation (11) that the $(n+1)$-component Toda lattice is of the form [17]

$$\begin{aligned} w_t^1 &= u - u_{-n}, \\ w_t^j &= u w_1^{j-1} - w^{j-1} u_{-n+j-1}, \quad j = 2, \ldots, n, \\ u_t &= u(w_1^n - w^n). \end{aligned} \tag{13}$$

When $n = 1$, this leads to Toda lattice (10) by letting $u = p_s$ and $w^1 = -q_s$. For any fixed n, applying an r-matrix formalism, their bi-Hamiltonian structures can be constructed. In [17], the bi-Hamiltonian structures for two and three fields are explicitly given.

From Lax operator (12), we can also derive a nonlocal multi-component Toda equation by taking its nth root [18]. Here, we write the first two terms of this Laurent series in $\mathcal{T}$:

$$\mathcal{L}^{1/n} = \mathcal{T}^{-1} + \eta + \cdots, \qquad \eta = (1 + \mathcal{T}^{-1} + \cdots + \mathcal{T}^{1-n})^{-1} w^1.$$

Then the Lax flow is given by

$$\partial_t \mathcal{L} = [(\mathcal{L}^{1/n})_{\geq 1}, \mathcal{L}] = -[(\mathcal{L}^{1/n})_{\leq 0}, \mathcal{L}] = -[\mathcal{T}^{-1} + \eta, \mathcal{L}],$$

which leads to

$$\begin{aligned} u_t &= u(\mathcal{T} - 1)\eta, \\ w_t^j &= w^{j+1} - w_{-1}^{j+1} - w^j(1 - \mathcal{T}^{-n+j})\eta, \qquad j = 1, \ldots, n-1, \\ w_t^n &= u - u_{-1}, \end{aligned} \tag{14}$$

where $\eta = (1 + \mathcal{T}^{-1} + \cdots + \mathcal{T}^{1-n})^{-1} w^1$.

In particular, if we take $n = 2$, it leads to a local 3-component Toda equation [17]

$$\begin{aligned} u_t &= u(w_1^2 - w^2), \\ w_t^1 &= u - u_{-2}, \\ w_t^2 &= u w_1^1 - u_{-1} w^1 \end{aligned} \tag{15}$$

and a nonlocal 3-component Toda equation

$$\begin{aligned} u_t &= u(\mathcal{T} - 1)\mathcal{T}(1 + \mathcal{T})^{-1} w^1, \\ w_t^1 &= w^2 - w_{-1}^2 - w^1(\mathcal{T} - 1)(1 + \mathcal{T})^{-1} w^1, \\ w_t^2 &= u - u_{-1}. \end{aligned} \tag{16}$$

4. Invariant evolutions in centro-affine $\mathbb{R}^n$

Centro-affine geometry is obtained by deleting translations from the affine geometry. Let $G = \mathrm{SL}(n, \mathbb{R})$ act linearly on $M = \mathbb{R}^n$ as $x \to g \cdot x$, where x is a n-vector, $g \in G$, and the product is the matrix multiplication. The n-dimensional centro-affine space $M = \mathbb{R}^n$ can be regarded as the homogeneous space $\mathrm{SL}(n, \mathbb{R})/H$, where H is the isotropy subgroup of $e_1 = (1, 0, \dots, 0)^{\mathrm{T}}$, where the superscript T denotes matrix transposition. We write

$$H = \begin{pmatrix} 1 & Y_{1\times(n-1)} \\ 0_{(n-1)\times 1} & A_{(n-1)\times(n-1)} \end{pmatrix}$$

and

$$G/H = \begin{pmatrix} x & 0_{1\times(n-1)} \\ y_{(n-1)\times 1} & B_{(n-1)\times(n-1)} \end{pmatrix},$$

with the matrix $B_{(n-1)\times(n-1)} = \mathrm{diag}(1/x, 1, \dots, 1)$, where (x, y^{T}) can be viewed as the n coordinates on $M = \mathbb{R}^n$.

To construct the right frame ρ_s, we take the normalization equation to be

$$\rho_s \cdot (x_s, x_{s+1}, x_{s+2}, \dots, x_{s+n-1}) = (e_1, e_2, \dots, (-1)^{n-1} p_s e_n), \tag{17}$$

where $x_{s+k} = (x^0_{s+k}, x^1_{s+k}, x^2_{s+k}, \dots, x^{n-1}_{s+k})^{\mathrm{T}}$ for $k = 0, 1, \dots, n-1$ and

$$p_s = (-1)^{n-1} \det(x_s, x_{s+1}, x_{s+2}, \dots, x_{s+n-1}).$$

This leads to the right Maurer–Cartan matrix

$$K_s = \rho_{s+1}\rho_s^{-1} = \begin{pmatrix} r_s^1 & 1 & 0 & \dots & 0 & 0 \\ r_s^2 & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ r_s^{n-2} & 0 & \dots & 0 & 1 & 0 \\ r_s^{n-1} & 0 & \dots & 0 & 0 & (-1)^{n-1}/p_s \\ p_s & 0 & \dots & 0 & 0 & 0 \end{pmatrix}, \tag{18}$$

where

$$r_s^k = -\frac{\det(x_s, \dots, x_{k-1+s}, x_{s+n}, x_{k+1+s}, \dots, x_{s+n-1})}{\mathcal{T} p_s}, \qquad k = 1, \dots, n-1.$$

We drop the subscript s without causing confusion. For example, when $n = 2$, normalization equation (17) becomes

$$\rho \cdot (x, x_1) = (e_1, -pe_2), \qquad p = -\det(x, x_1) = -(x^0 x_1^1 - x_1^0 x^1).$$

Solving it, we obtain the left moving frame

$$\rho^{-1} = \left(x, -\frac{x_1}{p}\right),$$

and thus the corresponding right Maurer–Cartan matrix is

$$K = \rho_1 \rho^{-1} = \begin{pmatrix} r^1 & -1/p \\ p & 0 \end{pmatrix}, \qquad r^1 = -\frac{\det(x, x_2)}{p_1}. \tag{19}$$

In this case, the link between invariant evolutions and integrable systems has been discussed in [6]. Next, we focus on the invariant evolutions and the related integrable systems in the case $n = 3$.

4.1. The case of centro-affine $\mathbb{R}^3$. It follows from (18) that in this case, the right Maurer–Cartan matrix (after dropping the subscript s) is

$$K = \rho_1 \rho^{-1} = \begin{pmatrix} r^1 & 1 & 0 \\ r^2 & 0 & 1/p \\ p & 0 & 0 \end{pmatrix}, \tag{20}$$

where

$$p = \det(x, x_1, x_2), \qquad r^1 = -\frac{1}{p_1}\det(x, x_3, x_2), \qquad r^2 = -\frac{1}{p_1}\det(x, x_1, x_3)$$

and

$$\rho^{-1} = \left(x, x_1, \frac{x_2}{p}\right). \tag{21}$$

From (6), the general invariant evolution is given by

$$(x)_t = \rho^{-1}\mathbf{v}, \qquad \mathbf{v} = \left(v^1, v^2, v^3\right)^{\mathrm{T}}, \tag{22}$$

where v^1, v^2, and v^3 are arbitrary functions of the invariants r^1, r^2, and p and their shifts. Using (21) and (22), we can easily see that the first column of N is $-\mathbf{v}$. From structure equation (7), we obtain

$$N = \begin{pmatrix} -v^1 & \dfrac{-\mathcal{T}v^3}{p} & \dfrac{-p_1\mathcal{T}^2v^2 + r_1^2\mathcal{T}^2v^3}{p^2} \\ -v^2 & -\mathcal{T}v^1 + \dfrac{r^1}{p}\mathcal{T}v^3 & \dfrac{r^1}{p^2}(p_1\mathcal{T}^2v^2 - r_1^2\mathcal{T}^2v^3) - \dfrac{\mathcal{T}^2v^3}{pp_1} \\ -v^3 & -p\mathcal{T}v^2 + r^2\mathcal{T}v^3 & v^1 + \mathcal{T}v^1 - \dfrac{r^1}{p}\mathcal{T}v^3 \end{pmatrix},$$

and the evolution equation

$$\frac{d}{dt}(p, r^1, r^2)^{\mathrm{T}} = \mathrm{P}\mathbf{v}, \tag{23}$$

where

$$\mathrm{P} = \begin{pmatrix} p(1+\mathcal{T}+\mathcal{T}^2) & -r^2p_1\mathcal{T}^2 & r^2r_1^2\mathcal{T}^2 - r^1\mathcal{T} - \dfrac{r_1^1p}{p_1}\mathcal{T}^2 \\ r^1(1-\mathcal{T}) & 1 - \dfrac{pp_2}{p_1^2}\mathcal{T}^3 & \dfrac{pr_2^2\mathcal{T}^3}{p_1^2} - \dfrac{r^2\mathcal{T}^2}{p_1} \\ r^2(1-\mathcal{T}^2) & \dfrac{r_1^1pp_2}{p_1^2}\mathcal{T}^3 - r^1\mathcal{T} & \dfrac{1}{p} + \dfrac{r_1^1r^2}{p_1}\mathcal{T}^2 - \dfrac{p\mathcal{T}^3}{p_1p_2} - \dfrac{pr_1^1r_2^2\mathcal{T}^3}{p_1^2} \end{pmatrix}. \tag{24}$$

Remark 1. Under the transform $p = p_k, r^1 = q_kp_k, r^2 = -r_k$, we can obtain the corresponding evolution equations in [10].

We define a matrix

$$C = \begin{pmatrix} (\mathcal{T}^2-1)^{-1}p & -(\mathcal{T}+1)^{-1}r^1 & 0 \\ 0 & -\mathcal{T}^{-1}r^2 & -\mathcal{T}^{-2}\dfrac{p}{p_1} \\ 0 & -\mathcal{T}^{-1}p & 0 \end{pmatrix}. \tag{25}$$

Then we compute operator multiplication PC and obtain the pseudo-difference antisymmetric operator

$$\begin{pmatrix} p(1+\mathcal{T}+\mathcal{T}^2)(\mathcal{T}^2-1)^{-1}p & p\mathcal{T}(\mathcal{T}+1)^{-1}r^1 & pr^2 \\ -r^1(\mathcal{T}+1)^{-1}p & \begin{matrix} r^1(\mathcal{T}-1)(\mathcal{T}+1)^{-1}r^1 \\ -\mathcal{T}^{-1}r^2 + r^2\mathcal{T} \end{matrix} & \dfrac{p}{p_1}\mathcal{T} - \mathcal{T}^{-2}\dfrac{p}{p_1} \\ -pr^2 & \dfrac{p}{p_1}\mathcal{T}^2 - \mathcal{T}^{-1}\dfrac{p}{p_1} & \dfrac{r^1}{p}\mathcal{T}^{-1}p - p\mathcal{T}\dfrac{r^1}{p} \end{pmatrix}. \tag{26}$$

In fact, this operator is a Hamiltonian operator as stated in the following theorem.

Theorem 2. *The operator $\mathcal{H} = PC$ in (26) is a Hamiltonian operator and it forms a Hamiltonian pair with*

$$\mathcal{H}^0 = \begin{pmatrix} 0 & 0 & p \\ 0 & \mathcal{T} - \mathcal{T}^{-1} & 0 \\ -p & 0 & 0 \end{pmatrix}. \tag{27}$$

Proof. We introduce the transformation

$$u = \frac{p}{p_1}, \qquad v = -r^2, \qquad w = r^1, \tag{28}$$

whose Frechet derivative is

$$D_{(u,v,w)} = \begin{pmatrix} \frac{1}{p_1} - \frac{p}{p_1^2}\mathcal{T} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{p_1} - \frac{u}{p_1}\mathcal{T} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{29}$$

Under th3transformation (28),

$$D_{(u,v,w)}\mathcal{H}^0 D^\dagger_{(u,v,w)} = \begin{pmatrix} 0 & -u(1-\mathcal{T}) & 0 \\ (1-\mathcal{T}^{-1})u & 0 & 0 \\ 0 & 0 & \mathcal{T} - \mathcal{T}^{-1} \end{pmatrix} \tag{30}$$

and

$$D_{(u,v,w)}\mathcal{H} D^\dagger_{(u,v,w)} = \begin{pmatrix} u(\mathcal{T}^{-1} - \mathcal{T}^2)(\mathcal{T}+1)^{-1}u & -u(\mathcal{T}-1)v & u(1-\mathcal{T})(\mathcal{T}^{-1}+1)^{-1}w \\ v(\mathcal{T}^{-1}-1)u & w\mathcal{T}^{-1}u - u\mathcal{T}w & -u\mathcal{T}^2 + \mathcal{T}^{-1}u \\ w(\mathcal{T}+1)^{-1}(\mathcal{T}^{-1}-1)u & \mathcal{T}^{-2}u - u\mathcal{T} & \begin{matrix} w(\mathcal{T}-1)(\mathcal{T}+1)^{-1}w \\ +\mathcal{T}^{-1}v - v\mathcal{T} \end{matrix} \end{pmatrix},$$

which form a Hamiltonian pair for a three-component Toda system with the Lax operator

$$L = \mathcal{T}^{-2} + w\mathcal{T}^{-1} + v + u\mathcal{T}.$$

Here, we used the same notation as in [17], where the Hamiltonian pair is given explicitly. The statement is proved.

Remark 2. 1. This theorem can also be proved by verifying that the operator $\mathcal{H}$ defines a Poisson bivector, as is done in [7].

2. Another method to prove this statement is by using recent results on pre-Hamiltonian operators [19], [20]. We call a difference operator pre-Hamiltonian if its image is a Lie subalgebra with respect to the Lie bracket of evolutionary vector fields. Direct computation shows that the operator P is indeed a pre-Hamiltonian operator.

Theorem 2 implies the following result on invariant evolutions (22).

Theorem 3. *The invariant evolution in the centro-affine space $\mathbb{R}^3$ described by*

$$(x)_t = \left(x, x_1, \frac{x_2}{p}\right) \begin{pmatrix} 0 \\ -\frac{p_{-2}}{p_{-1}} \\ 0 \end{pmatrix}, \tag{31}$$

induces the integrable system

$$\begin{pmatrix} p_t \\ r^1_t \\ r^2_t \end{pmatrix} = \begin{pmatrix} pr^2 \\ \dfrac{p}{p_1} - \dfrac{p_{-2}}{p_{-1}} \\ r^1\dfrac{p_{-1}}{p} - \dfrac{p}{p_1}r^1_1 \end{pmatrix} = \mathcal{H}\,\delta r^2 = \mathcal{H}_0\,\delta\left(\frac{p_{-1}}{p}r^1 + \frac{(r^2)^2}{2}\right), \tag{32}$$

where the Hamiltonian pair $\mathcal{H}$ *and* $\mathcal{H}^0$ *is given in Theorem* 2, *and it becomes* 3*-component Toda lattice* (15) *under the transformation*

$$u = \frac{p}{p_1}, \qquad w^1 = r^1, \qquad w^2 = -r^2. \tag{33}$$

Proof. Taking $\mathbf{v} = C(0,0,1)^{\mathrm{T}} = (0, -p_{-2}/p_{-1}, 0)^{\mathrm{T}}$, we obtain (31) from (22) and (32) from (23). Moreover, Eq. (32) is a bi-Hamiltonian systems because $\mathcal{H}$ and $\mathcal{H}^0$ form a Hamiltonian pair, as follows from Theorem 2.

Under the transformation, we have

$$u_t = \frac{p_t}{p_1} - \frac{p}{p_1^2}p_{1,t} = \frac{p}{p_1}r^2 - \frac{p}{p_1}r^2_1 = u(w^2_1 - w^2).$$

After a direct calculation for $w^i_t, i = 1, 2$, we obtain the 3-component Toda lattice as stated.

We note that $e_2 = (0, 1, 0)^{\mathrm{T}}$ is in the kernel of $\mathcal{H}^0$. If we take

$$\mathbf{v} = -Ce_2 = ((\mathcal{T} + 1)^{-1}r^1, \mathcal{T}^{-1}r^2, \mathcal{T}^{-1}p)^{\mathrm{T}}$$

in (23), then we obtain the integrable nonlocal equation

$$\begin{aligned} p_t &= -p(1 + \mathcal{T})^{-1}r^1_1, \\ r^1_t &= r^2_{-1} - r^2 - r^1(\mathcal{T} - 1)(1 + \mathcal{T})^{-1}r^1, \\ r^2_t &= \frac{p_{-1}}{p} - \frac{p}{p_1}, \end{aligned} \tag{34}$$

which becomes nonlocal 3-component Toda lattice (16) under transformation (33).

4.2. Towards the general centro-affine case. For an n-dimensional centro-affine space, we can in principle carry out the same study as we did for the centro-affine $\mathbb{R}^3$. However, the explicit formula for the operator P in (24) is rather bulky in this case and thus we do not write it here. We simply present the expression for the matrix N and some results.

We assume that the general invariant evolution is given by

$$(x)_t = \rho^{-1}\mathbf{v}, \qquad \mathbf{v} = (v^1, v^2, \ldots, v^n)^{\mathrm{T}}, \tag{35}$$

where v^1, v^2, and $\ldots, v^n$ are arbitrary functions of the invariants $r^k, k = 1, \ldots, n-1$ and p and their shifts. We know that $N = \rho_t\rho^{-1}$. Thus the first column of N is $-\mathbf{v}$, the kth column of N is $M^{k-1}\mathbf{v}$ for $k = 2, \ldots, n-1$ with

$$M = \begin{pmatrix} 0 & 0 & \ldots & 0 & \dfrac{1}{p}\mathcal{T} \\ \mathcal{T} & 0 & \ldots & 0 & \dfrac{-r^1}{p}\mathcal{T} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \ldots & \mathcal{T} & 0 & \dfrac{-r^{n-2}}{p}\mathcal{T} \\ 0 & \ldots & 0 & (-1)^{n-1}p\mathcal{T} & (-1)^n r^{n-1}\mathcal{T} \end{pmatrix}, \tag{36}$$

and the last column of N can be obtained from structure equation (7), namely,

$$
\begin{aligned}
N_{1,n} &= \frac{(-1)^{n-1}}{p^2}\mathcal{T}N_{n,n-1},\\
N_{j+1,n} &= \frac{(-1)^{n-1}\mathcal{T}N_{j,n-1}}{p} - r^j N_{1,n}, \qquad j=1,\ldots,n-2,\\
N_{n,n} &= -(N_{1,1}+\cdots+N_{n-1,n-1}),
\end{aligned}
$$

reflecting the fact that the matrix N is traceless.

Theorem 4. *The invariant evolution in the centro-affine space* $\mathbb{R}^{n+1}$ *given by*

$$
(x)_t = \left(x, x_1, \ldots, \frac{(-1)^n}{p}x_n\right)\left(0, -\frac{p_{-n}}{p_{1-n}}, 0, \ldots, 0\right)^{\mathrm{T}}, \tag{37}
$$

induces the integrable system

$$
\begin{aligned}
p_t &= (-1)^n r^n p,\\
r_t^1 &= -\frac{p_{-n}}{p_{1-n}} + \frac{p}{p_1},\\
r_t^j &= r^{j-1}\frac{p_{j-1-n}}{p_{j-n}} - r_1^{j-1}\frac{p}{p_1}, \qquad j=2,\ldots,n,
\end{aligned} \tag{38}
$$

which becomes $(n+1)$*-component Toda lattice* (13) *under the transformation*

$$
u = \frac{p}{p_1}, \qquad w^k = (-1)^{k+1}r^k, \qquad k=1,\ldots,n. \tag{39}
$$

Proof. From the given invariant evolution (37), we know

$$
\mathbf{v} = (0, -p_{-n}/p_{1-n}, 0, \ldots, 0)^{\mathrm{T}}.
$$

Thus, we obtain an expression for N using the formula above. Alternatively, we can determine N using structure equation (7): the nonzero entries are

$$
\begin{aligned}
N_{i+1,i} &= \mathcal{T}^{i-1}\frac{p_{-n}}{p_{1-n}}, \; i=1,2,\ldots,n-1,\\
N_{n+1,n} &= (-1)^n p_{-1},\\
N_{1,n+1} &= \frac{1}{p},\\
N_{i+1,n+1} &= -\frac{r^i}{p}, \qquad i=1,2,\ldots,n-1.
\end{aligned}
$$

We further obtain the corresponding flow of invariants as given by (38). The second part of the statement can be proved by direct calculation of changing variables.

Similarly, we can obtain the result for the invariant evolution relating it to nonlocal multi-component Toda lattices as in the centro-affine $\mathbb{R}^3$.

Theorem 5. *The invariant evolution in the centro-affine space* $\mathbb{R}^{n+1}$ *given by*

$$(x)_t = \left(x, x_1, \ldots, (-1)^n \frac{x_n}{p}\right) \mathcal{T}^{-1} \left(\sum_{k=0}^{n-2} \mathcal{T}^{-k} \eta, r^2, r^3, \ldots, r^n, p \right)^{\mathrm{T}}, \tag{40}$$

where $\eta = (1 + \mathcal{T}^{-1} + \cdots + \mathcal{T}^{1-n})^{-1} r^1$, *induces the integrable system*

$$\begin{aligned} p_t &= -p\eta, \\ r_t^j &= r_{-1}^{j+1} - r^{j+1} - r^j (1 - \mathcal{T}^{-n+j})\eta, \qquad j = 1, \ldots, n-1, \\ r_t^n &= (-1)^n \left(\frac{p_{-1}}{p} - \frac{p}{p_1} \right), \end{aligned} \tag{41}$$

which becomes the nonlocal $(n+1)$*-component Toda lattice under transformation* (39).

Proof. The second part of the statement can be proved by direct calculation of changing variables. For the proof of the first part, we set

$$\bar{\eta} = (1 + \mathcal{T}^{-1} + \cdots + \mathcal{T}^{2-n})\eta. \tag{42}$$

From the given invariant evolution (40), we see that the first column of the matrix N is

$$-(\mathcal{T}^{-1}\bar{\eta}, \mathcal{T}^{-1} r^2, \mathcal{T}^{-1} r^3, \ldots, \mathcal{T}^{-1} r^n, \mathcal{T}^{-1} p)^{\mathrm{T}}.$$

Using structure equation (7), we can determine N as

$$N = -\begin{pmatrix} \mathcal{T}^{-1}\bar{\eta} & 1 & 0 & \cdots & 0 & 0 \\ \mathcal{T}^{-1} r^2 & \bar{\eta} - r^1 & 1 & \ddots & \vdots & \vdots \\ \mathcal{T}^{-1} r^3 & 0 & \mathcal{T}(\bar{\eta} - r^1) & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ \mathcal{T}^{-1} r^n & \vdots & \ddots & 0 & \mathcal{T}^{n-2}(\bar{\eta} - r^1) & (-1)^n/p \\ \mathcal{T}^{-1} p & 0 & \cdots & 0 & 0 & \xi \end{pmatrix},$$

where $\xi = (\mathcal{T}^{n-1} - \mathcal{T}^{-1})\bar{\eta} + (1 - \mathcal{T}^{n-1}) r^1$. Moreover, we obtain the flow for invariants

$$\begin{aligned} p_t &= p(\mathcal{T}^{-1}\bar{\eta} - r^1), \\ r_t^j &= r^j(\mathcal{T}^{-1} - \mathcal{T}^{j-1})\bar{\eta} + (\mathcal{T}^{-1} - 1) r^{j+1} - r^j (1 - \mathcal{T}^{j-1}) r^1, \qquad j = 1, \ldots, n-1, \\ r_t^n &= r^n(\mathcal{T}^{-1} - \mathcal{T}^{n-1})\bar{\eta} - r^n (1 - \mathcal{T}^{n-1}) r^1 + (-1)^n \left(\frac{p_{-1}}{p} - \frac{p}{p_1} \right). \end{aligned}$$

We note that N is traceless under the given relation between η and r^1, that is,

$$r^1 = (1 + \mathcal{T}^{-1} + \cdots + \mathcal{T}^{1-n})\eta.$$

Substituting this and (42) in the above flow, we obtain the system in Eq. (41).

Conflicts of interest. The authors declare no conflicts of interest.

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