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DETERMINANTS IN QUANTUM MATRIX ALGEBRAS AND INTEGRABLE SYSTEMS

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We define quantum determinants in quantum matrix algebras related to pairs of compatible braidings. We establish relations between these determinants and the so-called column and row determinants, which are often used in the theory of integrable systems. We also generalize the quantum integrable spin systems using generalized Yangians related to pairs of compatible braidings. We demonstrate that such quantum integrable spin systems are not uniquely determined by the "quantum coordinate ring" of the basic space V. For example, the "quantum plane" $xy = qyx$ yields two different integrable systems: rational and trigonometric.

Keywords: compatible braiding, quantum matrix algebra, half-quantum algebra, generalized Yangian, quantum symmetric polynomial, quantum determinant

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1. Introduction

The quantum or q-determinant was introduced in works of L. D. Faddeev's school (see, e.g., [1]) in connection with the quantum inverse scattering method. Such determinants were initially introduced in RTT algebras associated with the $U_q(sl(N))$ R-matrices or with some current (i.e., depending on spectral parameters) R-matrices.

Nevertheless, a large family of other involutive and Hecke symmetries was constructed in [2], and quantum determinants were defined in RTT algebras associated with *even*[1] symmetries. We recall that these symmetries are particular cases of braidings, i.e., their operators $R\colon V^{\otimes 2} \to V^{\otimes 2}$ satisfy the so-called braid relation

$$(R\otimes I)(I\otimes R)(R\otimes I) = (I\otimes R)(R\otimes I)(I\otimes R).$$

Hereafter, V is a finite-dimensional vector space (called the basic space) and I denotes the identity operator or matrix.[2]

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[1] The term *even* means that the R-skew-symmetric algebra $\Lambda_R(V)$ has a finite number of nontrivial homogenous components and the highest nontrivial component $\Lambda^m_R(V)$ is one-dimensional. In this case, we say that R is of rank m.

[2] We note that the operators PR, where P is a flip, are subject to the so-called quantum Yang–Baxter equation and are usually called R-matrices.

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A braiding R is called a Hecke symmetry or an involutive symmetry if it satisfies the respective supplementary condition

$$(R - qI)(R + q^{-1}I) = 0, \quad q \neq \pm 1 \qquad \text{or} \qquad R^2 = I.$$

We assume that the nonzero parameter $q \in \mathbb{C}$ is general, i.e., such that

$$k_q = \frac{q^k - q^{-k}}{q - q^{-1}} \neq 0 \quad \text{for all } k \in \mathbb{Z}.$$

Our objective here is threefold. First, using the scheme in [2], we define quantum determinants in quantum matrix algebras (QMAs) associated with the pairs of compatible braidings introduced in [3], half-quantum algebras (HQAs) defined in [4], and generalized Yangians introduced in [5], [6]. We also show that in some RTT algebras, including those associated with the quantum group (QG) $U_q(sl(N))$, quantum determinants can be written in the form of column or row determinants, which are very popular in the literature on integrable systems.

Second, using quantum elementary symmetric polynomials closely related to quantum determinants, we present the Bethe subalgebras in all considered generalized Yangians. We thus obtain quantum integrable systems that are a far-reaching generalization of the spin systems in [7] and their rational counterparts. We also discuss different forms of the corresponding determinants.

Third, we emphasize that the "quantum coordinate ring" of the basic space V, contrary to popular belief, does not uniquely determine the corresponding quantum algebra and quantum determinant. For example, the so-called quantum plane defined by $xy - qyx = 0$ yields two completely different RTT algebras and generalized Yangians and consequently leads to two different integrable systems.

This paper is organized as follows. In Sec. 2, using the method in [2], we define the quantum determinants in the QMAs associated with pairs (R, F) of compatible braidings, where R is an even involutive or Hecke symmetry. In Sec. 3, we present some properties of the quantum determinants in different algebras. In particular, we describe symmetries that allow defining column or row determinants. We also consider the quantum determinants in the right and left HQAs. In Sec. 4, we define the quantum determinant in rational and trigonometric generalized Yangians and present integrable systems associated with such Yangians.

2. Quantum determinants in QMAs

With any Hecke symmetry[3] R, we associate the R-symmetric and R-skew-symmetric algebras of the space V defined as the respective quotients of the free tensor algebra $T(V)$ of the space V:

$$\mathrm{Sym}_R(V) = T(V)/\langle \mathrm{Im}(qI - R)\rangle, \qquad \Lambda_R(V) = T(V)/\langle \mathrm{Im}(q^{-1}I + R)\rangle.$$

Here, $\langle J\rangle$ denotes the two-sided ideal generated by a set $J \subset T(V)$. The ground field is $\mathbb{C}$.

Each homogenous component $\mathrm{Sym}_R^{(k)}(V)$ or $\Lambda_R^{(k)}(V)$ can be identified with the respective image of the R-symmetrizer $S^{(k)}(R)$ or R-skew-symmetrizer $A^{(k)}(R)$) acting on the space $V^{\otimes k}$. These projectors can be defined by the recurrence relations

$$\begin{aligned} S^{(1)} &= I, \qquad S^{(k)} = \frac{1}{k_q} S^{(k-1)}(q^{-(k-1)}I + (k-1)_q R_{k-1\,k}) S^{(k-1)}, \\ A^{(1)} &= I, \qquad A^{(k)} = \frac{1}{k_q} A^{(k-1)}(q^{k-1}I - (k-1)_q R_{k-1\,k}) A^{(k-1)}, \end{aligned} \qquad k \geq 2. \tag{2.1}$$

[3]In this section, we mainly consider Hecke symmetries. The corresponding results and formulas for involutive symmetries can be obtained by setting $q = 1$.

As usual, the subscripts indicate the positions where matrices or operators are located. These formulas can be deduced from the representation theory of the symmetric group if R is involutive or of the Hecke algebra if R is a Hecke symmetry (see [8]).

Let R and F be braidings. Following [3], we say that the ordered pair (R, F) is compatible (or the braidings R and F are compatible) if the relations

$$R_{12}F_{23}F_{12} = F_{23}F_{12}R_{23}, \qquad R_{23}F_{12}F_{23} = F_{12}F_{23}R_{12}$$

are satisfied. Below, we always assume that R to be an involutive or a Hecke symmetry.

Let $L = \|l_j^i\|_{1\le i,j\le N}$ be an $N\times N$ matrix and $L_1 = L \otimes I_{2\dots p}$, $p \ge 2$. (Hence, L_1 is an $N^p\times N^p$ matrix.) We introduce the notation

$$L_{\bar 1} = L_1, \qquad L_{\overline{k+1}} = F_{k\,k+1}L_{\bar k}F^{-1}_{k\,k+1}, \quad k \le p-1. \tag{2.2}$$

We recover the standard definition $L_{k+1} = P_{k\,k+1}L_kP_{k\,k+1}$ in the case $F = P$, where P denotes the usual flip.

Following [3] (also see the references therein), we define a QMA $\mathcal{L}(R, F)$ as a unital associative algebra generated by elements of the matrix $L = \|l_i^j\|$ subject to the system of commutation relations

$$R_{12}L_{\bar 1}L_{\bar 2} = L_{\bar 1}L_{\bar 2}R_{12}. \tag{2.3}$$

The matrix L is called the *generating matrix* of the algebra $\mathcal{L}(R, F)$. We note that the compatibility of the braidings R and F implies that the defining relations of the algebra $\mathcal{L}(R, F)$ can be pushed forward to higher positions in the sense that

$$R_{k\,k+1}L_{\bar k}L_{\overline{k+1}} = L_{\bar k}L_{\overline{k+1}}R_{k\,k+1}, \quad k < p.$$

We note that each of the pairs (R, P) and (R, R) is obviously compatible. The corresponding algebras $\mathcal{L}(R, P)$ and $\mathcal{L}(R, R)$ are respectively the RTT algebra and reflection equation (RE) algebra.[4] The defining relations of the former algebra $\mathcal{L}(R, P)$ are

$$R_{12}L_1L_2 = L_1L_2R_{12}.$$

The defining relations of the RE algebra $\mathcal{L}(R, R)$ can be written in the form

$$R_{12}L_1R_{12}L_1 = L_1R_{12}L_1R_{12}.$$

Remark. We emphasize that if a symmetry R is a deformation of the usual flip, then the corresponding RTT and RE algebras are deformations of the commutative algebra $\mathrm{Sym}(gl(N))$, and hence the dimensions of the homogenous components of these QMAs are classical (if R is a Hecke symmetry, the parameter q must be general). But if R is a braiding coming from the QGs of the series B_n, C_n, D_n, then this property is absent, and any similar deformation of the algebra $\mathrm{Sym}(\mathfrak{g})$, where $\mathfrak{g}$ is a Lie algebra belonging to one of these series, therefore does not exist. In contrast, there exist quantum deformations of the function algebra $Fun(G)$, where G is the corresponding Lie group. The corresponding quotients of the RTT algebras were presented in [10].

[4] One more example of compatible braidings (R, F) is that formed by the braidings in (2.6), where the second matrix plays the role of R. An example of such was also presented in [9].

We now assume that the symmetry R is even. Let $\Lambda_R^{(m)}(V)$, $m \geq 2$, be the highest nontrivial homogenous component[5] of the algebra $\Lambda_R(V)$. Because the dimension of this component is 1 by definition, there exist two tensors $\mathrm{u} = \|u_{i_1 \dots i_m}\|$ and $\mathrm{v} = \|v^{j_1 \dots j_m}\|$ such that

$$A^{(m)}(x_{i_1} \otimes \cdots \otimes x_{i_m}) = u_{i_1 \dots i_m} \, v^{j_1 \dots j_m} x_{j_1} \otimes \cdots \otimes x_{j_m},$$

$$\langle \mathrm{v}, \mathrm{u} \rangle := v^{i_1 \dots i_m} \, u_{i_1 \dots i_m} = 1.$$

Hereafter, $\{x_i\}_{1 \leq i \leq N}$ is a basis of the space V and summation over repeated indices is always understood. Hence, the element $v^{j_1 \dots j_m} x_{j_1} \otimes \cdots \otimes x_{j_m}$ is a generator of $\operatorname{Im} A^{(m)}$. We note that the tensors u and v are defined up to a rescaling[6]

$$\mathrm{u} \mapsto a\mathrm{u}, \qquad \mathrm{v} \mapsto a^{-1}\mathrm{v}, \quad a \in \mathbb{C}, \quad a \neq 0. \tag{2.4}$$

By analogy with [2], we introduce the following definition.

Definition. The element of the QMA $\mathcal{L}(R, F)$

$$\det{}_{\mathcal{L}(R,F)}(L) := \langle \mathrm{v} | L_{\bar{1}} \cdots L_{\overline{m}} | \mathrm{u} \rangle := v^{i_1 \dots i_m} (L_{\bar{1}} \cdots L_{\overline{m}})^{j_1 \dots j_m}_{i_1 \dots i_m} \, u_{j_1 \dots j_m} \tag{2.5}$$

is called *the quantum determinant* of the generating matrix L.

Of course, the quantum determinant $\det_{\mathcal{L}(R,F)}(L)$ can be written in other explicit forms modulo the defining relations of the QMA $\mathcal{L}(R, F)$. We present some of them below. We say that form (2.5) is *canonical*.

We now consider two examples. We fix a basis $\{x = x_1, y = x_2\}$ of the basic space V, $N = \dim V = 2$, and introduce two symmetries represented by the matrices in this basis

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \tag{2.6}$$

Each of these symmetries is a deformation of the usual flip P. The first symmetry is involutive, and the second is a Hecke symmetry coming from the QG $U_q(sl(2))$.

For the involutive symmetry, we have

$$\mathrm{u} = (u_{11}, u_{12}, u_{21}, u_{22}) = \frac{1}{2}(0, 1, -q^{-1}, 0), \qquad \mathrm{v} = (v^{11}, v^{12}, v^{21}, v^{22}) = (0, 1, -q, 0).$$

For the Hecke symmetry, we have

$$\mathrm{u} = \frac{q^{-1}}{2_q}(0, 1, -q, 0), \qquad \mathrm{v} = (0, 1, -q, 0).$$

We note that the tensors v corresponding to these symmetries coincide and the algebras

$$\operatorname{Sym}_R(V) = T(V)/\langle \mathrm{v} \rangle = T(V)/\langle xy - qyx \rangle, \tag{2.7}$$

[5]In general, m could be different from $N = \dim V$ (see [2], [11]).

[6]If the rank of a symmetry R is two, then we can recover the symmetry R by knowing u and v. All pairs (u, v) yielding such symmetries were classified in [2].

called the “quantum plane,” are therefore the same for the two symmetries in (2.6). Nevertheless, the tensors u differ. Consequently, the canonical forms of the corresponding determinants $\det_{\mathcal{L}(R,F)}(L)$ differ for all pairs (R, F).

We compute these determinants for the corresponding RTT algebras $\mathcal{L}(R, P)$ and RE algebras $\mathcal{L}(R, R)$. We let $l_1^1 = a$, $l_1^2 = b$, $l_2^1 = c$, and $l_2^2 = d$ denote elements of the generating matrix L of these algebras:

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Example 1. The defining relations of the RTT algebra $\mathcal{L}(R, P)$ corresponding to the first (involutive) matrix in (2.6),

$$R_{12}L_1L_2 = L_1L_2R_{12},$$

lead to the system of equations for the generators

$$ab = q^{-1}ba, \qquad ac = qca, \qquad ad = da, \qquad bc = q^2cb, \quad bd = qdb, \qquad cd = q^{-1}dc.$$

According to our definition, the canonical form of the quantum determinant in this algebra is

$$\det_{\mathcal{L}(R,P)}(L) = \frac{ad + da}{2} - \frac{q^{-1}bc + qcb}{2}.$$

Using the above commutation relations on the generators, we can transform the canonical form to the expressions

$$\det_{\mathcal{L}(R,P)}(L) = ad - qcb = ad - q^{-1}bc. \tag{2.8}$$

The defining relations between the generators of the algebra corresponding to the second matrix in (2.6) are (see [10])

$$ab = qba, \qquad ac = qca, \qquad ad - da = (q - q^{-1})bc, \qquad bc = cb, \qquad bd = qdb, \qquad cd = qdc.$$

The corresponding quantum determinant is

$$\det_{\mathcal{L}(R,P)}(L) = \frac{q^{-1}ad + qda}{2_q} - \frac{bc + cb}{2_q} = ad - qcb = ad - qbc. \tag{2.9}$$

The first expression here is the canonical form of the determinant. We discuss other expressions in the next section.

We also present the corresponding algebras $\Lambda_R(V)$. If R is the first symmetry in (2.6), then we have

$$\Lambda_R(V) = T(V)/\langle x^2, y^2, xy + qyx\rangle. \tag{2.10}$$

If R is the second symmetry in (2.6), then the last generator of the ideal in the above quotient should be $qxy + yx$.

Therefore, we see that quantum plane (2.7) yields two different RTT algebras and consequently two *different* determinants, although we can find a specific form $ad - qcb$ that is the same for both determinants. In the next section, we consider higher-dimensional analogues of these algebras and determinants in more detail and explain this coincidence.

Example 2. The defining relations

$$R_{12}L_1R_{12}L_1 = L_1R_{12}L_1R_{12}$$

of the RE algebra $\mathcal{L}(R,R)$ corresponding to the first (involutive) matrix in (2.6) in explicit form are

$$ab = ba, \qquad ac = ca, \qquad ad = da, \qquad bc = cb, \qquad bd = db, \qquad cd = dc.$$

This algebra is therefore commutative. This is an unsurprising result because the involutive R in (2.6) is related to the usual flip P by conjugation,

$$R_{12} = D_1P_{12}D_1^{-1}, \qquad D = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}.$$

Therefore, the matrix

$$\tilde{L} = D^{-1}LD = \begin{pmatrix} a & b/q \\ qc & d \end{pmatrix}$$

satisfies the relation

$$P_{12}\tilde{L}_1P_{12}\tilde{L}_1 = \tilde{L}_1P_{12}\tilde{L}_1P_{12} \quad \Longleftrightarrow \quad \tilde{L}_2\tilde{L}_1 = \tilde{L}_1\tilde{L}_2,$$

which means that the elements of $\tilde{L}$ generate a commutative algebra. Therefore, the elements of the matrix L also mutually commute. The canonical form of the determinant in this case is

$$\det_{\mathcal{L}(R,R)}(L) = \frac{ad+da}{2} - \frac{bc+cb}{2},$$

or with the commutativity of the RE $\mathcal{L}(R,R)$ taken into account, it is reducible to the classical expression $ad - bc$.

Finally, if we take the second (Hecke) symmetry R in (2.6), we obtain the system of equations for the generators

$$q^2ab = ba, \qquad q^2ca = ac, \qquad ad = da,$$
$$q(bc - cb) = \lambda a(d - a), \qquad q(cd - dc) = \lambda ca, \qquad q(db - bd) = \lambda ab,$$

where we set $\lambda = q - q^{-1}$. The canonical form of the determinant is

$$\det_{\mathcal{L}(R,R)}(L) = \frac{q(ad+da)}{2_q} - \frac{q(bc+q^2cb)}{2_q} - \frac{\lambda a^2}{2_q}. \tag{2.11}$$

Taking the relations between the generators into account, we can transform the canonical determinant to the equivalent forms

$$\det_{\mathcal{L}(R,R)}(L) = ad - q^2cb = q^2(ad - bc) - q\lambda a^2. \tag{2.12}$$

Another way to introduce quantum analogues of the determinant is based on the notion of quantum elementary symmetric polynomials defined via *quantum traces*. Such a quantum trace is well known in the cases related to the QG $U_q(sl(N))$. Nevertheless, the quantum trace can be associated with any skew-invertible braiding R using the Lyubashenko method [12], [13]. We say that a given braiding $R\colon V \to V$ is *skew-invertible* if there exists an operator $\Psi\colon V^{\otimes 2} \to V^{\otimes 2}$ such that

$$\mathrm{Tr}_{(2)}\, R_{12}\Psi_{23} = P_{13} = \mathrm{Tr}_{(2)}\, \Psi_{12}R_{23} \quad \Longleftrightarrow \quad R_{ij}^{kl}\Psi_{lm}^{jn} = \delta_m^k\delta_i^n = \Psi_{ij}^{kl}R_{lm}^{jn}.$$

If R is a skew-invertible braiding, then the corresponding *R-trace* Tr_R is defined by the formula

$$\mathrm{Tr}_R X = \mathrm{Tr}(C^R X), \qquad C^R := \mathrm{Tr}_{(2)} \Psi,$$

where X is an arbitrary $N\times N$ matrix (possibly with noncommutative elements).

We consider a compatible pair of braidings (R, F) and assume that the braiding R is skew-invertible. We now define a quantum version of the elementary symmetric polynomials in the algebra $\mathcal{L}(R, F)$ as

$$e_0 = 1, \qquad e_k = \mathrm{Tr}_{R(1\dots k)} A^{(k)} L_{\bar{1}} L_{\bar{2}} \cdots L_{\bar{k}}, \quad k \geq 1. \tag{2.13}$$

Hereafter, $\mathrm{Tr}_{(1\dots k)} = \mathrm{Tr}_{(1)} \cdots \mathrm{Tr}_{(k)}$. Using the equality

$$A^{(k)} L_{\bar{1}} \cdots L_{\bar{k}} = A^{(k)} L_{\bar{1}} \cdots L_{\bar{k}} A^{(k)}, \tag{2.14}$$

which holds in any QMA, we obtain the relation

$$\mathrm{Tr}_{R(1\dots m)} A^{(m)} L_{\bar{1}} \cdots L_{\overline{m}} = \mathrm{Tr}_{R(1\dots m)} A^{(m)} L_{\bar{1}} \cdots L_{\overline{m}} A^{(m)} = (\mathrm{v} \;\cdot_R\; \mathrm{u}) \langle \mathrm{v}| L_{\bar{1}} \cdots L_{\overline{m}} |\mathrm{u}\rangle, \tag{2.15}$$

where

$$(\mathrm{v} \;\cdot_R\; \mathrm{u}) = v^{j_1 \dots j_m} (C^R)^{i_1}_{j_1} (C^R)^{i_2}_{j_2} \cdots (C^R)^{i_m}_{j_m} u_{i_1 \dots i_m}. \tag{2.16}$$

Hence, the highest-degree elementary symmetric polynomial e_m differs from the quantum determinant $\det_{\mathcal{L}(R,F)}(L)$ by a numerical factor. In the particular case $F = P$, these elements are just equal to each other because $(\mathrm{v} \;\cdot_P\; \mathrm{u}) = 1$ in this case (we note that $C^P = I$).

3. Some properties of quantum determinants

In this section, we consider two questions. The first is what is the relation between the determinant $\det_{\mathcal{L}(R,F)}(L)$ and the characteristic polynomial of the matrix L. The second is whether the quantum determinant is central. We always assume that the rank of a symmetry R is m.

We say that an m-degree monic polynomial $ch(t)$ is *characteristic* if $ch(L) = 0$. By virtue of the Cayley–Hamilton theorem in the classical case $R = F = P$ (the corresponding algebra $\mathcal{L}(P, P)$ is commutative), the characteristic polynomial is

$$ch(t) = \det_{\mathcal{L}(P,P)} (L - tI).$$

Proposition 1. *In the algebras $\mathcal{L}(R, R)$, where R is a Hecke symmetry, we have the relation*

$$\det_{\mathcal{L}(R,R)} (L - tI) = \sum_{0 \leq k \leq m} (-t)^{m-k} \alpha_k e_k, \tag{3.1}$$

where $\alpha_k = q^{mk}(m!/k!\,(m-k)!)(k_q!\,(m-k)_q!/m_q!)$.

We note that similar assertions can be found in [14].

Proof. For any even Hecke symmetry of rank m, quantity (2.16) is

$$(v \;\cdot_R\; u) = q^{-m^2}.$$

This follows from the relation (see [6])

$$\mathrm{Tr}_{R(k+1\dots m)} A^{(m)}_{1\dots m} = q^{-m(m-k)} \frac{k_q!(m-k)_q!}{m_q!} A^{(k)}_{1\dots k} \tag{3.2}$$

for $k = 0$. In the expansion of the element

$$q^{m^2} \det_{\mathcal{L}(R,R)}(L - tI) = \mathrm{Tr}_{R(1\dots m)} A^{(m)}_{1\dots m}(L - tI)_{\bar{1}} \cdots (L - tI)_{\bar{m}},$$

we combine terms containing k factors $L_{\bar{i}}$ in some places and the identity matrices in other places. The number of such terms is $m!/k!\,(m-k)!$, and they are equal to each other. This property holds because[7]

$$\mathrm{Tr}_{R(1\dots m)} A^{(m)}_{1\dots m} L_{\bar{i}_1} L_{\bar{i}_2} \cdots L_{\bar{i}_k} = \mathrm{Tr}_{R(1\dots m)} A^{(m)}_{1\dots m} L_{\bar{1}} L_{\bar{2}} \cdots L_{\bar{k}}$$

for any ordered subset of indices $1 \le i_1 < i_2 < \cdots < i_k \le m$. It now suffices to apply formula (3.2). Finally, we obtain

$$\mathrm{Tr}_{R(1\dots m)} A^{(m)}_{1\dots m} L_{\bar{1}} \cdots L_{\bar{k}} = q^{-m(m-k)} \frac{k_q!(m-k)_q!}{m_q!} e_k(L).$$

The proof is complete.

If R is an involutive symmetry, then by setting $q = 1$ in (3.1), we obtain the following proposition.

Proposition 2. *If R is an involutive symmetry, then the polynomial* $\det_{\mathcal{L}(R,R)}(L - tI)$ *is characteristic.*

If R is a Hecke symmetry, the polynomial $\det_{\mathcal{L}(R,R)}(L - tI)$ is not characteristic. But we obtain the characteristic polynomial by replacing α_k with q^k in the right-hand side of (3.1), i.e.,

$$ch(t) := t^m - qe_1 t^{m-1} + q^2 e_2 t^{m-2} + \cdots + (-q)^{m-1} e_{m-1} t + (-q)^m e_m.$$

Therefore, substituting $t = L$ in this polynomial, we obtain the Cayley–Hamilton identity for the matrix L:

$$L^m - qe_1 L^{m-1} + q^2 e_2 L^{m-2} + \cdots + (-q)^{m-1} e_{m-1} L + (-q)^m e_m I = 0.$$

We note that this identity in the algebras $\mathcal{L}(R,R)$ was first proved in [15].

We now pass to the second question. It is well known that if a Hecke symmetry R comes from the QG $U_q(sl(N))$, then the quantum determinant $\det_{\mathcal{L}(R,P)}(L)$ is central (see [10]). If the quantum determinant is central in a given RTT algebra $\mathcal{L}(R,P)$, then by imposing the condition $\det_{\mathcal{L}(R,P)}(L) = 1$, we can define a Hopf algebra structure in the quotient algebra. But the quantum determinant in general is not central in the algebras $\mathcal{L}(R,P)$. As was shown in [2], the quantum determinant $\det_{\mathcal{L}(R,P)}(L)$ is central if and only if the matrix

$$M = \|M_i^j\|, \quad \text{where } M_i^j = u_{ii_2\cdots i_m} v^{i_2\cdots i_m\, j},$$

is scalar.

We study the centrality of the quantum determinants in the algebras $\mathcal{L}(R,P)$ corresponding to symmetries (2.6). By straightforward computations, we obtain the respective matrices M for the symmetries in (2.6):

$$-\frac{1}{2}\begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}, \qquad -\frac{1}{2_q}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, the quantum determinant is not central in the algebra $\mathcal{L}(R,P)$ corresponding to the involutive symmetry in (2.6) and is central in the algebra corresponding to the Hecke symmetry in (2.6). In contrast, the quantum determinant in the RE algebras $\mathcal{L}(R,R)$ is *always* central. Hence, imposing the condition $\det_{\mathcal{L}(R,R)}(L) = 1$, we obtain a braided Hopf algebra structure in the quotient algebra (see [11]).

[7]We emphasize that this property is absent if $F \neq R$.

We now discuss a way of reducing the quantum determinants to the so-called column and row determinants, which play an important role in the theory of integrable systems. We set $F = P$, i.e., we consider the RTT algebra $\mathcal{L}(R, P)$. Using relation (2.14). we obtain the equality

$$u_{i_1\dots i_m} v^{j_1\dots j_m} l^{k_1}_{j_1} \cdots l^{k_m}_{j_m} = u_{i_1\dots i_m} \langle \mathrm{v}|L_1 \cdots L_m|\mathrm{u}\rangle v^{k_1\dots k_m}.$$

Because the tensor $\mathrm{u} \not\equiv 0$, the factors $u_{i_1\dots i_m}$ can be canceled. We also assume that $m = N$ and $v^{12\dots N} = 1$. This condition can be satisfied by a proper normalization of v if $v^{12\dots N} \neq 0$. We then obtain

$$\det\nolimits_{\mathcal{L}(R,P)}(L) = v^{j_1\dots j_N} l^1_{j_1} \cdots l^N_{j_N}. \tag{3.3}$$

This form of the quantum determinant is called the *column determinant* of L. It is characterized by the property that the factors l^j_i in each of its summands are arranged in the order of columns of the matrix L enumerated by the superscripts of l^j_i. We note that if $m \neq N$, then we have no privileged component of the tensor v (like the component $v^{12\dots N}$).

Similarly, if the factors in each of the summands of the determinant are arranged in the order of rows of the matrix L, then we call it the *row determinant*. If $m = N$ and $u_{12\dots N} \neq 0$, then we can transform the canonical determinant $\det_{\mathcal{L}(R,P)}$ to the form of a row determinant:

$$\det\nolimits_{\mathcal{L}(R,P)}(L) = l^{i_1}_1 \cdots l^{i_N}_N u_{i_1\dots i_N}. \tag{3.4}$$

We note that the column determinant or row determinant depends on only the respective tensor v or u. Therefore, if two symmetries have the same tensors v but different tensors u, we have identical column determinants but different row determinants. This is the case of symmetries (2.6) considered above. We see that the column determinants (the middle expressions in (2.8) and (2.9)) are equal to each other but the row determinants (the right expressions) differ.

We now introduce the higher-dimensional counterparts of symmetries (2.6) and present the corresponding quantum determinants in the algebras $\mathcal{L}(R, P)$.

The Hecke symmetry R from the QG $U_q(sl(N))$ is

$$R^{kl}_{ij} = q^{\delta_{k,l}} \delta^k_j \delta^l_i + (q - q^{-1}) \theta_{(l>k)} \delta^k_i \delta^l_j,$$

where $\theta_{(l>k)} = 1$ if $l > k$ and $\theta_{(l>k)} = 0$ if $l \leq k$. We also introduce the involutive symmetry R by its action on the basis vectors $x_i \otimes x_j$ of the space $V^{\otimes 2}$:

$$R(x_i \otimes x_i) = x_i \otimes x_i, \qquad R(x_i \otimes x_j) = \begin{cases} q x_j \otimes x_i & \text{if } i < j, \\ q^{-1} x_j \otimes x_i & \text{if } i > j. \end{cases}$$

For both symmetries, the components of the tensors u and v are nontrivial if and only if their indices are pairwise distinct. For both symmetries, we can take the nontrivial components of v as

$$v^{j_1\dots j_N} = (-q)^{l(\sigma)},$$

where $l(\sigma)$ is the length (i.e., the minimum number of transpositions) of the permutation

$$\sigma\colon\ (1\dots N) \mapsto (j_1 \dots j_N).$$

Such a tensor v is sometimes called the q-Levi-Civita tensor.

The tensors u for these symmetries are respectively equal to

$$u_{i_1\dots i_N}=\alpha_1(-q)^{l(\sigma)},\qquad u_{i_1\dots i_N}=\alpha_2(-q^{-1})^{l(\sigma)},$$

where $\alpha_1^{-1}=q^{N(N-1)/2}N_q!$ and $\alpha_2^{-1}=N!$ are normalizing factors.

Similarly to the two-dimensional example above, we have the same "quantum coordinate ring" for both symmetries:

$$x_i x_j = q x_j x_i,\quad i<j,$$

often called the *quantum torus* (with the additional condition that the generators are invertible).

The formulas for the quantum column determinants are also the same in both RTT algebras:

$$\det{}_{\mathcal{L}(R,P)}(L)=\sum_\sigma (-q)^{l(\sigma)} l^1_{j_1}\cdots l^N_{j_N}.$$

In this form, the quantum determinant $\det_{\mathcal{L}(R,P)}(L)$ was given in [10] for the RTT algebra associated with the $U_q(sl(N))$ symmetry R. But the tensors u corresponding to the considered symmetries differ. Consequently, the row determinants in the corresponding RTT algebras differ from each other.

Concluding this section, we turn to the so-called HQAs and the corresponding quantum determinants. We again consider a compatible pair (R,F), where R is a skew-invertible Hecke symmetry. We introduce two systems of equations or the generating matrix $L=\|l^i_j\|_{1\le i,j\le N}$:

$$S^{(2)}L_{\bar 1}L_{\bar 2}A^{(2)}=0\quad\Longleftrightarrow\quad L_{\bar 1}L_{\bar 2}A^{(2)}=A^{(2)}L_{\bar 1}L_{\bar 2}A^{(2)},\tag{3.5}$$

$$A^{(2)}L_{\bar 1}L_{\bar 2}S^{(2)}=0\quad\Longleftrightarrow\quad A^{(2)}L_{\bar 1}L_{\bar 2}=A^{(2)}L_{\bar 1}L_{\bar 2}A^{(2)},\tag{3.6}$$

where the R-symmetrizer $S^{(2)}$ and the R-skew-symmetrizer $A^{(2)}$ are defined in (2.1). The matrices with barred indices have the same meaning as above (see (2.2)).

The following statement is well known and can be directly verified.

Proposition 3. *System* (2.3) *is equivalent to the union of systems* (3.5) *and* (3.6).

Imposing only *half* of the relations (only (3.5) or (3.6)) on the generators, we obtain a larger algebra than $\mathcal{L}(R,F)$. Nevertheless, even in such an algebra, we can develop some elements of a linear algebra (see [4], where these algebras were introduced and studied). Some particular cases of these algebras were also considered in [16] and [7], where they were called *Manin matrices* and q-Manin matrices.

We call a unital algebra defined by system (3.5) or (3.6)) the respective right or left HQA, denoted by $\mathcal{H}_{\rm r}(R,F)$ and $\mathcal{H}_\ell(R,F)$. If R is an even symmetry, then we define the quantum determinant in the algebra $\mathcal{H}_\epsilon(R,F)$, $\epsilon\in\{{\rm r},\ell\}$, by formula (2.5) and let $\det_{\mathcal{H}_\epsilon(R,F)}(L)$ denote it.

We can still define quantum elementary symmetric polynomials in the algebras $\mathcal{H}_\epsilon(R,F)$ by formulas (2.13), where the projectors $A^{(k)}$ can be moved to the rightmost position or placed in both positions, to the right and to the left of the chain of L-matrices.

We note that the quantum determinant in the algebra $\mathcal{H}_\epsilon(R,F)$ also differs from the highest quantum elementary symmetric polynomial by a factor.

Remark. Such algebras were first considered by Manin's monograph [17]. Their definition was motivated by the following consideration. We endow the space V with the coaction of an RTT algebra, $x_i\to t_i^j\otimes x_j$, and extend it to the space V by assuming that the generators x_i and t_k^j mutually commute. Relation (3.5) or (3.6) where we set $F=P$ then means that the respective subspace $\operatorname{Im}A^{(2)}$ or $\operatorname{Im}S^{(2)}$ is preserved under this coaction. But if $F\ne P$, then the assumption that the generators x_i and t_k^j commute is no longer applicable.

Because relation (2.14) holds in any left HQA, we can write the quantum determinant $\det_{H_\ell(R,P)}(L)$, where R is one of symmetries (2.6) or its higher-dimensional counterpart, as a column determinant. In any right HQA, we can write the quantum determinants as row determinants. Nevertheless, in any HQA the number of relations between the generators is insufficient to prove that the elementary polynomials mutually commute.

4. Generalized Yangians and integrable systems of CFRS type

We first describe the Baxterization procedure, which allows constructing current braidings via involutive and Hecke symmetries (see [18], [19]). We clarify that by a current braiding $R(u,v)$, we mean an operator depending on parameters and satisfying the braid relation

$$R_{12}(u,v)R_{23}(u,w)R_{12}(v,w) = R_{23}(v,w)R_{12}(u,w)R_{23}(u,v). \tag{4.1}$$

We associate an involutive symmetry R with a current braiding by the rule

$$R(u,v) = R - \frac{I}{u-v} \tag{4.2}$$

and a Hecke symmetry R with a current braiding by the rule

$$R(u,v) = R - \frac{(q-q^{-1})uI}{u-v}. \tag{4.3}$$

By a direct calculation, we can verify that these operators satisfy relation (4.1). Current braidings (4.2) and (4.3) (and all corresponding algebras) are respectively said to be *rational* and *trigonometric*.

We introduce a countable set of elements $l_j^i[k]$, $k \in \mathbb{Z}_\geq$, $1 \le i,j \le N$, and consider a formal power series

$$L(u) = \sum_{k\ge 0} L[k]u^{-k}, \qquad L[k] = \|l_i^j[k]\|_{1\le i,j\le N}, \tag{4.4}$$

i.e., $L(u)$ is an $N\times N$ matrix, and its elements are power series in u^{-1} with the coefficients $l_j^i[k]$.

A generalized Yangian $\mathbf{Y}(R,F)$ is an associative unital algebra generated by elements $l_j^i[k]$ subject to the system

$$R_{12}(u,v)L_{\bar 1}(u)L_{\bar 2}(v) - L_{\bar 1}(v)L_{\bar 2}(u)R_{12}(u,v) = 0, \tag{4.5}$$

where $L_{\bar 1}(u) = L_1(u)$ and $L_{\bar 2}(u) = F_{12}L_{\bar 1}(u)F_{12}^{-1}$. We note that expanding the current matrix $L(u)$ in a series as indicated in (4.4), we obtain a countable set of *polynomial* relations for the generators $l_i^j[k]$.

In the case $F = R$, the algebra $\mathbf{Y}(R,R)$ with the supplementary condition $L[0] = I$ is called a *braided or generalized Yangian of RE type* (see [5] for details). We note that the condition $L[0] = I$ is motivated by the evaluation morphism, similar to that in the Drinfeld Yangian $\mathbf{Y}(gl(N))$, which is a particular case of rational Yangians corresponding to the symmetry $R = F = P$.

We note that for a special value of the ratio $u/v = q^2$ in the trigonometric case, system (4.5) can be treated in terms of the HQA. A similar treatment is possible in the rational case if $u-v = 1$. More precisely, for the indicated relations between the parameters u and v, the current braiding $R(u,v)$ becomes equal (up to a numerical factor) to the R-skew-symmetrizer $A^{(2)}$. Therefore, in the rational and trigonometric cases, we obtain the relations

$$\begin{aligned} A^{(2)}L_{\bar 1}(u)L_{\bar 2}(u-1) &= L_{\bar 1}(u-1)L_{\bar 2}(u)A^{(2)}, \\ A^{(2)}L_{\bar 1}(u)L_{\bar 2}(q^{-2}\,u) &= L_{\bar 1}(q^{-2}\,u)L_{\bar 2}(u)A^{(2)}, \end{aligned} \tag{4.6}$$

and we consequently have

$$A^{(2)}L_{\bar{1}}(u)L_{\bar{2}}(u-1)S^{(2)}=0, \qquad S^{(2)}L_{\bar{1}}(u-1)L_{\bar{2}}(u)A^{(2)}=0,$$
$$A^{(2)}L_{\bar{1}}(u)L_{\bar{2}}(q^{-2}\,u)S^{(2)}=0, \qquad S^{(2)}L_{\bar{1}}(q^{-2}\,u)L_{\bar{2}}(u)A^{(2)}=0 \tag{4.7}$$

in the respective rational and trigonometric cases; using the formal Taylor series expansions

$$L(u-1)=e^{-\partial_u}L(u)e^{\partial_u}, \qquad L(q^{-2}u)=q^{-2u\partial_u}L(u)q^{2u\partial_u},$$

where $\partial_u = d/du$, we can rewrite relations (4.7) in the form

$$A^{(2)}(e^{-\partial_u}L_{\bar{1}}(u))(e^{-\partial_u}L_{\bar{2}}(u))S^{(2)}=0, \qquad S^{(2)}(e^{\partial_u}L_{\bar{1}}(u))(e^{\partial_u}L_{\bar{2}}(u))A^{(2)}=0,$$
$$A^{(2)}(q^{-2u\partial_u}L_{\bar{1}}(u))(q^{-2u\partial_u}L_{\bar{2}}(u))S^{(2)}=0, \qquad S^{(2)}(q^{2u\partial_u}L_{\bar{1}}(u))(q^{2u\partial_u}L_{\bar{2}}(u))A^{(2)}=0$$

in the respective rational and trigonometric cases. Hence, the operator $e^{-\partial_u}L(u)$ or $q^{-2u\partial_u}\,L(u))$ plays the role of the generating matrix of a left HQA, and the operator $e^{\partial_u}L(u)$ or $q^{2u\partial_u}L(u))$ plays the role of the generating matrix of a right HQA.

We now define quantum elementary symmetric polynomials in the respective rational and trigonometric cases as

$$\begin{aligned} e_0(u)=1, \quad e_k(u)&=\mathrm{Tr}_{R(1\ldots k)}\,A^{(k)}\,L_{\bar{1}}(u)\,L_{\bar{2}}(u-1)\cdots L_{\bar{k}}(u-k+1),\\ e_0(u)=1, \quad e_k(u)&=\mathrm{Tr}_{R(1\ldots k)}\,A^{(k)}\,L_{\bar{1}}(u)\,L_{\bar{2}}(q^{-2}u)\cdots L_{\bar{k}}(q^{-2(k-1)}u), \end{aligned} \qquad k\ge 1.$$

We note that we can place the projectors $A^{(k)}$ in these formulas after the chain of the matrices L or at two positions, to the right and to the left of the chain of the L. Moreover, we can write the parameters of the matrices in the reverse order. These transformations all lead to identical results.

Let R be an even symmetry of rank m. We define the quantum determinants in the respective rational and trigonometric generalized Yangians by setting

$$\det_{\mathbf{Y}(R,F)}(L(u))=\langle \mathrm{v}|L_{\bar{1}}(u)\,L_{\bar{2}}(u-1)\cdots L_{\overline{m}}(u-m+1)|\mathrm{u}\rangle,$$
$$\det_{\mathbf{Y}(R,F)}(L(u))=\langle \mathrm{v}|L_{\bar{1}}(u)\,L_{\bar{2}}(q^{-2}u)\cdots L_{\overline{m}}(q^{-2(m-1)}u)|\mathrm{u}\rangle.$$

We note that the quantum determinant equals the highest elementary symmetric polynomial $e_m(u)$ up to a factor in full analogy with (2.15). If $F=P$, then this equality is exact.

We consider the case $F=P$ in more detail. Assuming that $m=N$ and that $v^{1\ldots N}$ and $u_{1\ldots N}$ are nontrivial, we can write the quantum determinant as a column or row determinant:

$$\det_{\mathbf{Y}(R,P)}(L)=v^{j_1\ldots j_N}l^1_{j_1}(u)\cdots l^N_{j_N}(u-N+1)=u_{i_1\ldots i_N}l^{i_1}_1(u-N+1)\cdots l^{i_N}_N(u)$$

in the rational case and

$$\det_{\mathbf{Y}(R,P)}(L)=v^{j_1\ldots j_N}l^1_{j_1}(u)\cdots l^N_{j_N}(q^{-2(N-1)}u)=u_{i_1\ldots i_N}l^{i_1}_1(q^{-2(N-1)}u)\cdots l^{i_N}_N(u)$$

in the trigonometric case.

We now take the first (involutive) symmetry in (2.6) or its higher-dimensional counterpart as R. Because the corresponding generalized Yangian is rational, we can write the corresponding quantum determinant as

$$\det_{\mathbf{Y}(R,P)}(L) = \sum_\sigma (-q)^{l(\sigma)} l^1_{\sigma(1)}(u) \cdots l^N_{\sigma(N)}(u-N+1) =$$
$$= \sum_\sigma (-q^{-1})^{l(\sigma)} l_1^{\sigma(1)}(u-N+1) \cdots l_N^{\sigma(N)}(u).$$

In the case of the $U_q(sl(N))$ symmetries R, the corresponding generalized Yangian is trigonometric. Consequently, we have

$$\det_{\mathbf{Y}(R,P)}(L) = \sum_\sigma (-q)^{l(\sigma)} l^1_{\sigma(1)}(u) \cdots l^N_{\sigma(N)}(q^{-2(N-1)}u) =$$
$$= \sum_\sigma (-q)^{l(\sigma)} l_1^{\sigma(1)}(q^{-2(N-1)}u) \cdots l_N^{\sigma(N)}(u).$$

We note that the orders of arguments in the matrices are opposite in the expressions for the column and row determinants. We also note that similarly to the case of the algebras $\mathcal{L}(R,F)$, the quantum determinant is always central in the generalized Yangians $\mathbf{Y}(R,R)$ of RE type but this is not so in $\mathbf{Y}(R,P)$ of RTT type. More precisely, the quantum determinant $\det_{\mathbf{Y}(R,P)}(L)$ is central if and only if it is central for the quantum determinant $\det_{\mathcal{L}(R,P)}(L)$. This property was proved in [5].

The quantum versions of power sums can also be defined in all generalized Yangians, and some quantum versions of the Cayley–Hamilton identity hold in all of them. The subalgebra generated in the generalized Yangian $\mathbf{Y}(R,F)$ by the quantum elementary polynomials is called the Bethe subalgebra and is denoted by $\mathcal{B}(R,F)$.

Proposition 4 [20]. *For any compatible pair (R,F) of braidings such that R is a skew-invertible involutive or Hecke symmetry the Bethe subalgebra $\mathcal{B}(R,F) \subset \mathbf{Y}(R,F)$ is commutative.*

A particular case of this proposition, corresponding to $F = P$ and R coming from the $U_q(sl(N))$ QG was proved in [7]. (We note that the formula for the projectors $A^{(k)}$ should be taken as in (2.1))

The generalized Yangians of the RE type have a very important property: they admit evaluation morphisms. These morphisms were constructed in [5]. Similarly to the evaluation map in the Drinfeld Yangian $\mathbf{Y}(P,P)$, they have the form

$$L(u) \mapsto 1 + \frac{M}{u}$$

in both the rational and trigonometric cases. But the target algebras generated by matrix elements of M differ: it is a modified RE algebra with respect to the symmetry R in the rational case and an unmodified RE algebra in the trigonometric case.

In conclusion, we make a short remark. As already noted, it is impossible to prove that the quantum elementary polynomials in a HQA commute. Nevertheless, relations (4.6) allow establishing this property for the generalized Yangians because they are more restrictive than the defining relations in an HQA.

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Conflicts of interest. The authors declare no conflicts of interest.

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