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PROJECTORS ON INVARIANT SUBSPACES OF REPRESENTATIONS $\mathrm{ad}^{\otimes 2}$ OF LIE ALGEBRAS $so(N)$ AND $sp(2r)$ AND VOGEL PARAMETERIZATION

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Using the split Casimir operator, we find explicit formulas for the projectors onto invariant subspaces of the $\mathrm{ad}^{\otimes 2}$ *representation of the algebras* $so(N)$ *and* $sp(2r)$*. We also consider these projectors from the standpoint of the universal description of complex simple Lie algebras using the Vogel parameterization.*

Keywords: invariant subspace, projector, simple Lie algebra, split Casimir operator, Vogel parameter

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1. Introduction

The Yang–Baxter equation is one of the most important objects of study in modern theoretical and mathematical physics. This equation appeared in the works of McGuire [1] and Yang [2] and plays a key role in studying quantum integrable systems [3], [4]. In particular, in the framework of the quantum inverse scattering method [5], structures appeared that led to creating the theory of quantum groups [6], [7] (also see the references therein) describing symmetries of quantum integrable systems (see, e.g., [8]–[10]). We note that the Yang–Baxter equations are used for both formulating quantum groups and studying their properties [11]–[13]. An important class of solutions of the Yang–Baxter equations comprises solutions that are invariant under the action of a Lie group G (or its Lie algebra $\mathcal{A}$) in a particular representation T. Let V be the representation space of T. In this case, a solution of the Yang–Baxter equation (the R-matrix) is an operator in the space $V \otimes V$ of the representation $T \otimes T$. Let the representation $T \otimes T$ be completely reducible and decompose into irreducible representations T_λ as $T \otimes T = \sum_\lambda T_\lambda$, where the index λ enumerates the irreducible representations. Then the sought solutions of the Yang–Baxter equation can be expanded as a sum of the projectors onto the invariant subspaces $V_\lambda \subset V \otimes V$ of the representations T_λ with some coefficients that are functions of spectral parameters. To find such solutions, it is useful to have explicit expressions for the projectors onto the subspaces V_λ.

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In the case where $T = \mathrm{ad}$ is the adjoint representation, constructing the projectors onto invariant subspaces of the representation $T \otimes T = \mathrm{ad} \otimes \mathrm{ad}$ has one more significance. It is related to the notion of the *universal Lie algebra*, which was introduced by Vogel in [14] (also see [15], [16]). The universal Lie algebra by supposition is a model of all complex simple Lie algebras $\mathcal{A}$. For example, many quantities characterizing an algebra $\mathcal{A}$ in different representations T_λ (in this case, possibly reducible) in the decomposition $\mathrm{ad}^{\otimes k} = \sum_\lambda T_\lambda$, where $k \geq 1$, are expressed by analytic functions of three Vogel parameters (see their definition in Sec. 5). These parameters take specific values for each complex simple Lie algebra $\mathcal{A}$ (see, e.g., [17] and Sec. 5 below). In particular, it was shown that in terms of the Vogel parameters, we can express the dimensions of the representations T_λ in the cases where $k = 2, 3$ [14], the dimension of an arbitrary representation $T_{\lambda'}$ with the highest weight $\lambda' = k\lambda_{\mathrm{ad}}$, where λ_{ad} is the highest root of a Lie algebra $\mathcal{A}$ [18], and also values of the higher Casimir operators in the adjoint representation of a Lie algebra $\mathcal{A}$ [17]. Moreover, it was shown in [19] that the universal description of complex simple Lie algebras allows formulating some types of knot polynomials as a single function simultaneously for all simple Lie algebras. Here, we also demonstrate a particular example of the universal description of Lie algebras: the projectors onto invariant subspaces of the tensor product of two adjoint representations of the Lie algebra $so(N, \mathbb{C})$ for $N \geq 3$ and $sp(N, \mathbb{C})$ for $N = 2r \geq 2$ are written in a unified form, illustrating a correspondence between some structures related to $so(N, \mathbb{C})$ and $sp(N, \mathbb{C})$ Lie algebras under the substitution $N \to -N$. These results expressed in terms of the Vogel parameters completely agree with the conclusions obtained in [14], [17], [18].

2. A definition of the algebras $so(N, \mathbb{C})$ and $sp(N, \mathbb{C})$

For a unified description of the Lie algebras $so(N, \mathbb{C})$ and $sp(N, \mathbb{C})$, we let V_N denote the space of their defining representations and introduce the metric $||c_{ij}||_{i,j=1,\ldots,N}$ on V_N defined as the $N \times N$ identity matrix I_N in the $so(N, \mathbb{C})$ case and as the antisymmetric $N \times N$ matrix

$$\|c_{ij}\| = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} \tag{2.1}$$

in the $sp(N, \mathbb{C})$ case, where $N = 2r$ is even. Hence, we have $c_{ij} = \epsilon c_{ji}$, where the parameter ϵ takes the value $+1$ in the $so(N, \mathbb{C})$ case and -1 in the $sp(N, \mathbb{C})$ case (whence we obtain $\epsilon^2 = 1$). The matrix $\bar{c}^{\,ij}$ inverse to the metric c is defined as $\bar{c}^{\,ik} c_{kj} = \delta^i_j$. Using c and $\bar{c}$, we can raise and lower indices: $z_{i_1}{}^{j_2 j_3 \cdots} = c_{i_1 j_1} z^{j_1 j_2 j_3 \cdots}$ and $z^{j_1}{}_{i_2 i_3 \ldots} = \bar{c}^{\,j_1 i_1} z_{i_1 i_2 i_3 \ldots}$.

Using the matrix identities $(e_s{}^r)^i{}_k = \delta^r_k \delta^i_s$, which form a basis of the algebra of linear operators on V_N, we can write bases of $so(N, \mathbb{C})$ and $sp(N, \mathbb{C})$ in the defining representation. Lowering the index r yields $(e_{sr})^i{}_k = c_{rk} \delta^i_s$. We can express the generators of $so(N, \mathbb{C})$ and $sp(N, \mathbb{C})$ in these terms as

$$M_{ij} = e_{ij} - \epsilon e_{ji}, \tag{2.2}$$

$$(M_{ij})^k{}_l = c_{jl} \delta^k_i - \epsilon c_{il} \delta^k_j = 2 \delta^k_{[i} c_{j)l}, \tag{2.3}$$

where $[ij)$ denotes antisymmetrization in the $so(N, \mathbb{C})$ case and symmetrization in the $sp(N, \mathbb{C})$ case. The commutation relations of both algebras are given by

$$[M_{ij}, M_{kl}] = c_{jk} M_{il} - \epsilon c_{ik} M_{jl} - \epsilon c_{jl} M_{ik} + c_{il} M_{jk} = X_{ij,kl}{}^{mn} M_{mn}, \tag{2.4}$$

whence we obtain the structure constants of the considered algebras in basis (2.2) in the form

$$X_{ij,kl}{}^{mn} = c_{jk} \delta^{[m}_i \delta^{n)}_l - \epsilon c_{ik} \delta^{[m}_j \delta^{n)}_l - \epsilon c_{jl} \delta^{[m}_i \delta^{n)}_k + c_{il} \delta^{[m}_j \delta^{n)}_k. \tag{2.5}$$

We can write them concisely in the form

$$X_{i_1 i_2, j_1 j_2}{}^{k_1 k_2} = 4\mathrm{Sym}^{\epsilon}_{1\leftrightarrow 2}(c_{i_2 j_1}\delta^{k_1}_{i_1}\delta^{k_2}_{j_2}), \tag{2.6}$$

where $\mathrm{Sym}^{\epsilon}_{1\leftrightarrow 2}$ denotes (anti)symmetrization over the index pairs (i_1, i_2), (j_1, j_2), and (k_1, k_2). For example, $\mathrm{Sym}^{\epsilon}_{1\leftrightarrow 2}(x_{i_1 i_2}) \equiv x_{i_1 i_2} - \epsilon x_{i_2 i_1}$. In what follows, we also use the notation $\mathfrak{g}^{\epsilon}_N$ as a unified description of $so(N, \mathbb{C})$ (if $\epsilon = +1$) and $sp(N, \mathbb{C})$ (if $\epsilon = -1$).

We note that (anti)symmetrized index pairs (i_1, i_2), (j_1, j_2), and (k_1, k_2), as indices of the basis vectors $M_{i_1 i_2}$ of the algebra $\mathfrak{g}^{\epsilon}_N$, can also serve as coordinate indices in the space of the adjoint representation of $\mathfrak{g}^{\epsilon}_N$.

3. The split Casimir operator

In this section, we describe a general procedure for constructing the projectors onto invariant subspaces of the representation $\mathrm{ad}^{\otimes 2}(\mathcal{A})$ of a complex simple Lie algebra $\mathcal{A}$ using the split Casimir operator $\widehat{C}$ of the algebra. We also find explicit formulas for the operator $\widehat{C}$ in the representation $\mathrm{ad}^{\otimes 2}(\mathcal{A})$ in the cases where $\mathcal{A} = so(N, \mathbb{C})$ and $\mathcal{A} = sp(N, \mathbb{C})$.

3.1. The split Casimir operator in highest-weight representations. Let $\mathcal{A}$ be a simple Lie algebra with the basis elements X_a and the structure relations

$$[X_a, X_b] = C^d_{ab} X_d, \tag{3.1}$$

where C^d_{ab} are the structure constants. The Cartan–Killing metric is standardly defined as

$$\mathsf{g}_{ab} \equiv C^d_{ac}\, C^c_{bd} = \mathrm{tr}(\mathrm{ad}(X_a)\cdot \mathrm{ad}(X_b)), \tag{3.2}$$

and the structure constants $C_{abc} \equiv C^d_{ab}\, \mathsf{g}_{dc}$ are antisymmetric under permutations of the indices (a, b, c). The universal enveloping algebra of $\mathcal{A}$ is denoted by $\mathcal{U}(\mathcal{A})$. We consider the operator

$$\widehat{C} = \mathsf{g}^{ab} X_a \otimes X_b \in \mathcal{A}\otimes\mathcal{A} \subset \mathcal{U}(\mathcal{A})\otimes\mathcal{U}(\mathcal{A}), \tag{3.3}$$

where the matrix $\|\mathsf{g}^{ab}\|$ is inverse to $\|\mathsf{g}_{ab}\|$, which in turn defines Cartan–Killing metric (3.2):

$$\mathsf{g}^{ab}\mathsf{g}_{bc} = \delta^a_c. \tag{3.4}$$

The operator $\widehat{C}$ is called the *split* (or *polarized*) *Casimir operator of the Lie algebra* $\mathcal{A}$. This operator is related to the usual quadratic Casimir operator

$$C_{(2)} = \mathsf{g}^{ab} X_a \cdot X_b \tag{3.5}$$

by the formula

$$\Delta(C_{(2)}) = C_{(2)}\otimes I + I \otimes C_{(2)} + 2\widehat{C}, \tag{3.6}$$

where Δ is the comultiplication

$$\Delta(X_a) = (X_a \otimes I + I \otimes X_a). \tag{3.7}$$

Let T_1 and T_2 be two irreducible representations whose highest weights are λ_1 and λ_2 and which act on the spaces $\mathcal{V}_1$ and $\mathcal{V}_2$. Using formula (3.6) and the decomposition

$$\mathcal{V}_1\otimes\mathcal{V}_2 = \sum_{\lambda} \mathcal{V}_{\lambda} \tag{3.8}$$

where $\mathcal{V}_\lambda$ are the spaces of the irreducible representations T_λ with the highest weights λ, we can then derive the relations

$$T(\widehat{C})\cdot(\mathcal{V}_1\otimes\mathcal{V}_2)=\frac{1}{2}\sum_\lambda(c_2^{(\lambda)}-c_2^{(\lambda_1)}-c_2^{(\lambda_2)})\mathcal{V}_\lambda \quad\Longleftrightarrow\quad T(\widehat{C})\cdot\mathcal{V}_\lambda=\frac{1}{2}(c_2^{(\lambda)}-c_2^{(\lambda_1)}-c_2^{(\lambda_2)})\mathcal{V}_\lambda, \tag{3.9}$$

where we use the notation $T(\widehat{C}):=(T_1\otimes T_2)\widehat{C}$ and the equalities[1]

$$\Delta(C_{(2)})\sum_\lambda\mathcal{V}_\lambda=\sum_\lambda c_2^{(\lambda)}\mathcal{V}_\lambda, \qquad (C_{(2)}\otimes I+I\otimes C_{(2)})\mathcal{V}_1\otimes\mathcal{V}_2=(c_2^{(\lambda_1)}+c_2^{(\lambda_2)})\mathcal{V}_1\otimes\mathcal{V}_2.$$

Here, $c_2^{(\lambda)}$ is the value of the quadratic Casimir operator in the representation with the highest weight λ,

$$c_2^{(\lambda)}=(\lambda,\lambda+2\delta), \qquad \delta=\sum_{f=1}^r\lambda_{(f)}=\frac{1}{2}\sum_{\alpha>0}\alpha, \tag{3.10}$$

where $\lambda_{(f)}$ are the fundamental weights of the rank-r Lie algebra $\mathcal{A}$, α are the roots of $\mathcal{A}$, and the summation ranges the positive roots ($\alpha>0$). The metric on the root space is given by matrix (3.4). We also note that formula (3.9) can be used to derive the characteristic identity

$$\prod_\lambda{}'\Big(T(\widehat{C})-\frac{1}{2}(c_2^{(\lambda)}-c_2^{(\lambda_1)}-c_2^{(\lambda_2)})\Big)=0, \tag{3.11}$$

where the product $\prod'$ is only over the weights λ in (3.8) corresponding to different eigenvalues c_2^λ.

In the following sections, we find the explicit form of the split Casimir operator $T(\widehat{C})=(T_1\otimes T_2)(\widehat{C})$ for orthogonal and symplectic Lie algebras in the case where T_1 and T_2 are adjoint representations: $T_1=T_2=\mathrm{ad}$. In this case, characteristic identity (3.11) is rewritten in the form

$$\prod_\lambda{}'\Big(\mathrm{ad}^{\otimes 2}(\widehat{C})-\frac{1}{2}(c_2^{(\lambda)}-2c_2^{(\lambda_{\mathrm{ad}})})\Big)=0, \tag{3.12}$$

where λ_{ad} is the highest weight of the adjoint representation of $\mathcal{A}$. Further, using characteristic identity (3.12), we explicitly construct the projectors (for orthogonal and symplectic Lie algebras) onto the invariant subspaces $\mathcal{V}_\lambda$ of the irreducible representations T_λ in the decomposition of the representation $\mathrm{ad}^{\otimes 2}$.

3.2. The split Casimir operator of the algebras $so(N,\mathbb{C})$ and $sp(N,\mathbb{C})$ in the tensor product of adjoint representations. We show that the components of split Casimir operator (3.3) in the adjoint representation ad of the algebras $so(N,\mathbb{C})$ and $sp(N,\mathbb{C})$ is written in terms of the components of the same operator in the defining representation.

The Cartan–Killing metric of algebra (2.4) with structure constants (2.5) is given by the expression [20]

$$\mathsf{g}_{i_1i_2,j_1j_2}=X_{i_1i_2,\ell_1\ell_2}{}^{k_1k_2}X_{j_1j_2,k_1k_2}{}^{\ell_1\ell_2}=2(N-2\epsilon)(c_{i_2j_1}c_{j_2i_1}-\epsilon c_{i_1j_1}c_{j_2i_2}). \tag{3.13}$$

Correspondingly, the inverse Cartan–Killing metric has the form

$$\mathsf{g}^{i_1i_2,j_1j_2}=\frac{1}{8(N-2\epsilon)}(\epsilon\bar{c}^{\,i_1j_2}\bar{c}^{\,i_2j_1}-\bar{c}^{\,i_1j_1}\bar{c}^{\,i_2j_2}). \tag{3.14}$$

[1]The first of those equalities is a consequence of the fact that for each $U\in\mathcal{U}(\mathcal{A})$, $(T_1\otimes T_2)\Delta(U)$ commutes with the projectors P_λ, which distinguish the irreducible subrepresentations of $(T_1\otimes T_2)$ and are hence ad-invariant operators.

In the adjoint representations of $so(N,\mathbb{C})$ and $sp(N,\mathbb{C})$, the operator $\widehat{C}$ given by (3.3) is written as $\mathrm{ad}^{\otimes 2}(\widehat{C}) \equiv \widehat{C}_{\mathrm{ad}}$, and in basis (2.4) with structure constants (2.5), we have

$$(\widehat{C}_{\mathrm{ad}})^{k_1k_2k_3k_4}_{j_1j_2j_3j_4} = \mathsf{g}^{i_1i_2i_3i_4}\, X_{i_1i_2,j_1j_2}{}^{k_1k_2} X_{i_3i_4,j_3j_4}{}^{k_3k_4} = \\ = 16\mathsf{g}^{i_1i_2i_3i_4}\mathrm{Sym}^{\epsilon}_{1\leftrightarrow 2}(c_{i_2j_1}\delta^{k_1}_{i_1}\delta^{k_2}_{j_2})\mathrm{Sym}^{\epsilon}_{3\leftrightarrow 4}(c_{i_4j_3}\delta^{k_3}_{i_3}\delta^{k_4}_{j_4}), \tag{3.15}$$

where $\mathrm{Sym}_{\alpha\leftrightarrow\beta}$ as before denotes the (anti)symmetrizator over the index pairs (i_α, i_β), (j_α, j_β), and (k_α, k_β). In these expressions, the first two indices correspond to the first copy of the algebra $\mathfrak{g}^{\epsilon}_N$, and the third and fourth indices correspond to the second copy of $\mathfrak{g}^{\epsilon}_N$ (in accordance with the fact that the basis elements of $so(N,\mathbb{C})$ and $sp(N,\mathbb{C})$ each have two indices and are embedded into $V_N \otimes V_N$ as vector spaces).

The split Casimir operator $\widehat{C}$ in the defining representation $T_f^{\otimes 2}(\widehat{C}) \equiv \widehat{C}_f \colon V_N \otimes V_N \to V_N \otimes V_N$ in basis (2.2) with structure constants (2.5) has the form

$$(\widehat{C}_f)^{k_1k_3}_{j_1j_3} = \mathsf{g}^{i_1i_2i_3i_4}(M_{i_1i_2})^{k_1}{}_{j_1}(M_{i_3i_4})^{k_3}{}_{j_3} = 4\mathsf{g}^{i_1i_2i_3i_4}(\delta^{k_1}_{[i_1}c_{i_2)j_1})(\delta^{k_3}_{[i_3}c_{i_4)j_3}). \tag{3.16}$$

From expressions (3.15) and (3.16), we see that

$$(\widehat{C}_{\mathrm{ad}})^{k_1k_2k_3k_4}_{j_1j_2j_3j_4} = 4\mathrm{Sym}^{\epsilon}_{1\leftrightarrow 2,3\leftrightarrow 4}\big((\widehat{C}_f)^{k_1k_3}_{j_1j_3}\delta^{k_2}_{j_2}\delta^{k_4}_{j_4}\big), \tag{3.17}$$

where $\mathrm{Sym}^{\epsilon}_{1\leftrightarrow 2,3\leftrightarrow 4}$ denotes (anti)symmetrization over the four index pairs (j_1, j_2), (k_1, k_2), (j_3, j_4), and (k_3, k_4). Here, the indices are interpreted as indices of vectors in the space V_N of the defining representation of $so(N,\mathbb{C})$ or $sp(N,\mathbb{C})$. Correspondingly, the split Casimir operator acts on the space $V_N^{\otimes 2}$ in the defining representation and on the space $V_N^{\otimes 4}$ in the adjoint representation. Here, the first and the last pair of spaces V_N in $V_N^{\otimes 4}$ are antisymmetrized in the $so(N,\mathbb{C})$ case and symmetrized in the $sp(N,\mathbb{C})$ case. We introduce the operator of (anti)symmetrization of the ath and bth spaces:

$$\mathcal{P}^{\epsilon}_{ab} = \frac{1}{2}(I - \epsilon P_{ab}), \tag{3.18}$$

where $a \neq b$, $a, b = 1, \ldots, 4$, and $P_{ab} \colon V_N^{\otimes 4} \to V_N^{\otimes 4}$ is the permutation operator acting on the ath and the bth spaces (e.g., P_{13} has the components $(P_{13})^{i_1i_2i_3i_4}{}_{j_1j_2j_3j_4} = \delta^{i_1}_{j_3}\delta^{i_3}_{j_1}\delta^{i_2}_{j_2}\delta^{i_4}_{j_4}$). For the space V_{ad} of the adjoint representation and also for $V_{\mathrm{ad}}^{\otimes 2}$, we then obtain

$$V_{\mathrm{ad}} = \mathcal{P}^{\epsilon}_{12}V_N^{\otimes 2}, \qquad V_{\mathrm{ad}} \otimes V_{\mathrm{ad}} = \mathcal{P}^{\epsilon}_{12}\mathcal{P}^{\epsilon}_{34}V_N^{\otimes 4}.$$

Moreover, we can write relation (3.17) between the Casimir operators in the defining and the adjoint representations in the form

$$\widehat{C}_{\mathrm{ad}} = 4\,\mathcal{P}^{\epsilon}_{12}\mathcal{P}^{\epsilon}_{34}(\widehat{C}_f)_{13}\mathcal{P}^{\epsilon}_{12}\mathcal{P}^{\epsilon}_{34}. \tag{3.19}$$

In (3.19), we use the known expression for the split Casimir operator in the defining representation (see, e.g., [20])

$$(\widehat{C}_f)_{13} = \frac{1}{2(N-2\epsilon)}(P_{13} - \epsilon K_{13}), \tag{3.20}$$

which is derived from (3.16). Here, the operators $(\widehat{C}_f)_{13}$, K_{13}, and P_{13} act nontrivially only on the first and third spaces V_N in $V_N^{\otimes 4}$, the operator K_{13} has the components $K^{i_1i_2i_3i_4}{}_{j_1j_2j_3j_4} = \bar{c}^{\,i_1i_3}c_{j_1j_3}\delta^{i_2}_{j_2}\delta^{i_4}_{j_4}$, and the operator P_{13}, as previously noted, exchanges the first and the third spaces in $V_N^{\otimes 4}$. For $\widehat{C}_{\mathrm{ad}}$, we finally obtain

$$\widehat{C}_{\mathrm{ad}} = \frac{2}{N-2\epsilon}\mathcal{P}^{\epsilon}_{12}\mathcal{P}^{\epsilon}_{34}(P_{13} - \epsilon K_{13})\mathcal{P}^{\epsilon}_{12}\mathcal{P}^{\epsilon}_{34}. \tag{3.21}$$

4. Decomposition of $\mathrm{ad}^{\otimes 2}(\mathfrak{g}_N^\epsilon)$ into irreducible representations

In this section, we use the split Casimir operator in the adjoint representation to construct projectors onto invariant subspaces of the representation $\mathrm{ad}^{\otimes 2}(\mathfrak{g}_N^\epsilon)$. Taking the presence of the "accidental" isomorphisms $so(3) \cong sl(2) \cong sp(2)$, $so(4) \cong sl(2) + sl(2)$, $so(5) \cong sp(4)$, and $so(6) \cong sl(4)$ into account, we consider the cases of those algebras except $so(5) \cong sp(4)$ at the end of the section (the case of $so(5) \cong sp(4)$ fits the general picture and does not need additional discussion).

4.1. Characteristic identity for the operator $\widehat{C}$ in the adjoint representation of $so(N,\mathbb{C})$ and $sp(N,\mathbb{C})$. We define the spaces V_{ad}^ϵ of the adjoint representation of $so(N)$ and $sp(N)$ as $V_{\mathrm{ad}}^\epsilon = \mathcal{P}_{12}^{(\epsilon)} V_N^{\otimes 2}$. The algebra $\mathfrak{g}_N^\epsilon$ coincides with V_{ad}^ϵ as a vector space. We introduce the operators acting in $V_{\mathrm{ad}}^\epsilon \otimes V_{\mathrm{ad}}^\epsilon \subset V_N^{\otimes 4}$:

$$\begin{aligned} &\mathbf{I} = \mathcal{P}_{12}^{(\epsilon)} \mathcal{P}_{34}^{(\epsilon)} \equiv \mathcal{P}_{12,34}^{(\epsilon)}, \qquad \mathbf{P} = \mathcal{P}_{12,34}^{(\epsilon)} P_{13} P_{24} \mathcal{P}_{12,34}^{(\epsilon)}, \\ &\mathbf{K} = \mathcal{P}_{12,34}^{(\epsilon)} K_{13} K_{24} \mathcal{P}_{12,34}^{(\epsilon)}, \end{aligned} \tag{4.1}$$

for which we have the relations

$$\begin{aligned} &\mathbf{I} = \mathbf{I} P_{12} P_{34} = P_{12} P_{34} \mathbf{I}, \qquad \mathbf{P} = P_{12} P_{34} \mathbf{I} = \mathbf{I} P_{13} P_{24}, \\ &\mathbf{P}^2 = \mathbf{I}, \qquad \mathbf{K}\mathbf{P} = \mathbf{P}\mathbf{K} = \mathbf{K}, \qquad \mathbf{K}^2 = \frac{N(N-\epsilon)}{2}\mathbf{K}, \end{aligned} \tag{4.2}$$

and

$$\widehat{C}_{\mathrm{ad}} \mathbf{P} = \mathbf{P} \widehat{C}_{\mathrm{ad}}, \qquad \widehat{C}_{\mathrm{ad}} \mathbf{K} = \mathbf{K} \widehat{C}_{\mathrm{ad}} = -\mathbf{K}. \tag{4.3}$$

Operators (4.1) are obviously ad-invariant under the adjoint action of $\mathfrak{g}_N^\epsilon$. In what follows, we use the notation $M \equiv \epsilon N$, which allows rewriting the last relation in (4.2) in the form

$$\mathbf{K}^2 = \frac{M(M-1)}{2}\mathbf{K}. \tag{4.4}$$

To find the characteristic identity, it is convenient to introduce the symmetric and antisymmetric projectors $\widehat{C}_+$ and $\widehat{C}_-$ of the operator $\widehat{C}_{\mathrm{ad}}$:

$$\widehat{C}_\pm = \frac{1}{2}(\mathbf{I} \pm \mathbf{P}) \widehat{C}_{\mathrm{ad}}, \tag{4.5}$$

which satisfy the relations

$$\widehat{C}_\pm \widehat{C}_\mp = 0, \qquad \mathbf{P}\widehat{C}_\pm = \pm \widehat{C}_\pm, \qquad \mathbf{K}\widehat{C}_+ = \widehat{C}_+ \mathbf{K} = -\mathbf{K}. \tag{4.6}$$

Substituting (3.21) in (4.5) yields explicit formulas for the antisymmetric and the symmetric parts of $\widehat{C}_{\mathrm{ad}}$:

$$(\widehat{C}_-)_{12,34} = \frac{1}{N-2\epsilon} \mathcal{P}_{12,34}^{(\epsilon)} (P_{24} - \epsilon) K_{13} \mathcal{P}_{12,34}^{(\epsilon)}, \tag{4.7}$$

$$(\widehat{C}_+)_{12,34} = \frac{1}{N-2\epsilon} \mathcal{P}_{12,34}^{(\epsilon)} (2P_{24} - P_{24}K_{13} - \epsilon K_{13}) \mathcal{P}_{12,34}^{(\epsilon)}. \tag{4.8}$$

By direct calculations using formula (4.7), we can find the characteristic identity for $\widehat{C}_-$:

$$\widehat{C}_-^2 = -\frac{1}{2}\widehat{C}_- \quad \Longleftrightarrow \quad \widehat{C}_-\left(\widehat{C}_- + \frac{1}{2}\right) = 0. \tag{4.9}$$

We also note a useful consequence of relation (4.9):

$$\widehat{C}_-^k = \left(-\frac{1}{2}\right)^{k-1} \widehat{C}_-, \quad k \geq 1. \tag{4.10}$$

Analogously, using the explicit formula (4.8) for $\widehat{C}_+$, we find an expression for $\widehat{C}_+^2$:

$$\widehat{C}_+^2 = \frac{1}{(M-2)^2}(\mathbf{I}+\mathbf{P}+\mathbf{K}) - \frac{1}{M-2}\widehat{C}_+ + \frac{M-8}{2(M-2)^2}\mathcal{P}^{(\epsilon)}_{12,34}K_{13}(1+\epsilon P_{24})\mathcal{P}^{(\epsilon)}_{12,34}. \tag{4.11}$$

If $M = 8$, then the last term in (4.11) is zero, and the identity for $\widehat{C}_+$ becomes

$$\widehat{C}_+^2 = -\frac{1}{6}\widehat{C}_+ + \frac{1}{36}(\mathbf{I}+\mathbf{P}+\mathbf{K}). \tag{4.12}$$

If $M \neq 8$, then multiplying (4.11) by $\widehat{C}_+$, we obtain a third-degree identity:

$$\widehat{C}_+^3 = -\frac{1}{2}\widehat{C}_+^2 - \frac{M-8}{2(M-2)^2}\widehat{C}_+ + \frac{M-4}{2(M-2)^3}(\mathbf{I}+\mathbf{P}-2\mathbf{K}). \tag{4.13}$$

Because identity (4.12) for $\widehat{C}_+$ with $M = 8$ is of the second and not the third degree, the case $M = 8$ is exceptional and is considered separately in Sec. 4.2. We also note that if $M = 4$, then the last term in (4.13) vanishes, and identity (4.13) in the $so(4,\mathbb{C})$ case is hence characteristic for the operator $\widehat{C}_+$ and has the explicit form

$$\widehat{C}_+^3 = -\frac{1}{2}\widehat{C}_+^2 + \frac{1}{2}\widehat{C}_+. \tag{4.14}$$

To obtain characteristic identities for $\widehat{C}_+$ with $M \neq 4, 8$, we eliminate the operators $\mathbf{P}$ and $\mathbf{K}$ in formula (4.13). Multiplying it by C_+ and using (4.6), we express $\mathbf{K}$ in terms of $\widehat{C}_+$:

$$\frac{M-4}{(M-2)^3}\mathbf{K} = \widehat{C}_+^4 + \frac{1}{2}\widehat{C}_+^3 + \frac{M-8}{2(M-2)^2}\widehat{C}_+^2 - \frac{M-4}{(M-2)^3}\widehat{C}_+. \tag{4.15}$$

We substitute $\mathbf{K}$ given by (4.15) in (4.13) and obtain the identity (which turns out to also hold in the case $M = 4$, as can be verified by direct calculation)

$$\widehat{C}_+^4 + \frac{3}{2}\widehat{C}_+^3 + \frac{(M+1)(M-4)}{2(M-2)^2}\widehat{C}_+^2 + \frac{M^2-12M+24}{2(M-2)^3}\widehat{C}_+ - \frac{M-4}{2(M-2)^3}(\mathbf{I}+\mathbf{P}) = 0. \tag{4.16}$$

Multiplying (4.16) by $\widehat{C}_+$ once more, we obtain the characteristic identity for $\widehat{C}_+$

$$\widehat{C}_+^5 + \frac{3}{2}\widehat{C}_+^4 + \frac{(M+1)(M-4)}{2(M-2)^2}\widehat{C}_+^3 + \frac{M^2-12M+24}{2(M-2)^3}\widehat{C}_+^2 - \frac{M-4}{(M-2)^3}\widehat{C}_+ = 0, \tag{4.17}$$

which we can also rewrite in the form

$$\widehat{C}_+^5 = -\frac{3}{2}\widehat{C}_+^4 - \frac{(M+1)(M-4)}{2(M-2)^2}\widehat{C}_+^3 - \frac{M^2-12M+24}{2(M-2)^3}\widehat{C}_+^2 + \frac{M-4}{(M-2)^3}\widehat{C}_+. \tag{4.18}$$

For subsequent calculations, we also need an expression for $\widehat{C}_+^6$, which can be obtained by multiplying identity (4.18) by $\widehat{C}_+$ and substituting the known polynomial for $\widehat{C}_+^5$ given by (4.18):

$$\widehat{C}_+^6 = \frac{7M^2-30M+44}{4(M-2)^2}\widehat{C}_+^4 + \frac{3M^3-17M^2+30M-24}{4(M-2)^3}\widehat{C}_+^3 + \\ + \frac{3M^2-32M+56}{4(M-2)^3}\widehat{C}_+^2 - \frac{3(M-4)}{2(M-2)^3}\widehat{C}_+. \tag{4.19}$$

We now use the obtained expressions to find the characteristic polynomial for the split Casimir operator $\widehat{C}_{\rm ad} = \widehat{C}_+ + \widehat{C}_-$. We seek this expression in the form of a sixth-degree polynomial in $\widehat{C}_{\rm ad}$ with arbitrary coefficients α_i:

$$\widehat{C}_{\rm ad}^6 + \alpha_5 \widehat{C}_{\rm ad}^5 + \alpha_4 \widehat{C}_{\rm ad}^4 + \alpha_3 \widehat{C}_{\rm ad}^3 + \alpha_2 \widehat{C}_{\rm ad}^2 + \alpha_1 \widehat{C}_{\rm ad} + \alpha_0. \tag{4.20}$$

We find the α_i for which expression (4.20) vanishes. It follows from formula (4.6) that $\widehat{C}^k = \widehat{C}_+^k + \widehat{C}_-^k$. Hence, the vanishing of polynomial (4.20) yields the equation

$$\widehat{C}_+^6 + \alpha_5 \widehat{C}_+^5 + \alpha_4 \widehat{C}_+^4 + \alpha_3 \widehat{C}_+^3 + \alpha_2 \widehat{C}_+^2 + \alpha_1 \widehat{C}_+ + \\ + \widehat{C}_-^6 + \alpha_5 \widehat{C}_-^5 + \alpha_4 \widehat{C}_-^4 + \alpha_3 \widehat{C}_-^3 + \alpha_2 \widehat{C}_-^2 + \alpha_1 \widehat{C}_- + \alpha_0 = 0.$$

Substituting the expressions for $\widehat{C}_+^{5,6}$ in terms of $\widehat{C}_+^{4,3,2,1}$ according to formulas (4.18) and (4.19) and the expressions for $\widehat{C}_-^{6,5,4,3,2}$ in terms of $\widehat{C}_-$ according to formula (4.10) and then setting the coefficients of those operators to zero, we obtain the values of α_i:

$$\alpha_0 = 0, \qquad \alpha_1 = -\frac{M-4}{2(M-2)^3},$$

$$\alpha_2 = \frac{M^2 - 16M + 40}{4(M-2)^3}, \qquad \alpha_3 = \frac{M^3 - 3M^2 - 22M + 56}{4(M-2)^3},$$

$$\alpha_4 = \frac{5M^2 - 18M + 4}{4(M-2)^2}, \qquad \alpha_5 = 2.$$

The characteristic identity for $\widehat{C}_{\rm ad}$ hence has the form

$$\widehat{C}_{\rm ad}^6 + 2\widehat{C}_{\rm ad}^5 + \frac{5M^2 - 18M + 4}{4(M-2)^2}\widehat{C}_{\rm ad}^4 + \frac{M^3 - 3M^2 - 22M + 56}{4(M-2)^3}\widehat{C}_{\rm ad}^3 + \\ + \frac{M^2 - 16M + 40}{4(M-2)^3}\widehat{C}_{\rm ad}^2 - \frac{M-4}{2(M-2)^3}\widehat{C}_{\rm ad} = 0. \tag{4.21}$$

We can find the roots of Eq. (4.21) explicitly:

$$a_1 = 0, \qquad a_2 = -\frac{1}{2}, \qquad a_3 = -1, \qquad a_4 = \frac{1}{M-2}, \\ a_5 = -\frac{2}{M-2}, \qquad a_6 = \frac{4-M}{2(M-2)}. \tag{4.22}$$

We note that for the following values of M, we have degenerate roots (we immediately discard $M = 0, 1, 2$ because they do not correspond to semisimple Lie algebras):

$$M = 4 \quad \Longrightarrow \quad a_1 = a_6 = 0, \quad a_3 = a_5 = -1, \tag{4.23}$$

$$M = 6 \quad \Longrightarrow \quad a_2 = a_5 = -\frac{1}{2}, \tag{4.24}$$

$$M = 8 \quad \Longrightarrow \quad a_5 = a_6 = -\frac{1}{3}. \tag{4.25}$$

Hence, we see that if $M = 4, 6, 8$, then the characteristic polynomials are not of degree six, as in the general case, but of the respective degrees $4, 5, 5$ (this is manifested in the differences between identities (4.16) and (4.13) for $M = 4$ and identities (4.12) and (4.13) for $M = 8$). We consider the cases $M = 4, 6$ in more detail at the end of this section and the case $M = 8$ in Sec. 4.2, as noted above.

Taking values (4.22) of the roots of the polynomial in the left-hand side of (4.21) into account, we can write characteristic identity (4.21) in the factored form

$$\widehat{C}_{\rm ad}\Big(\widehat{C}_{\rm ad}+\frac{1}{2}\Big)(\widehat{C}_{\rm ad}+1)\Big(\widehat{C}_{\rm ad}-\frac{1}{M-2}\Big)\Big(\widehat{C}_{\rm ad}+\frac{2}{M-2}\Big)\Big(\widehat{C}_{\rm ad}+\frac{M-4}{2(M-2)}\Big)=0. \tag{4.26}$$

Form (4.26) of the characteristic identity allows constructing the projectors onto the invariant subspaces in $V_{\rm ad}\otimes V_{\rm ad}$. These projectors are given by the standard formula

$$\mathrm{P}_j \equiv \mathrm{P}_{a_j} = \prod_{\substack{i=1,\\ i\neq j}}^{6} \frac{\widehat{C}-a_i\mathbf{I}}{a_j-a_i}, \tag{4.27}$$

where a_i are roots (4.22) of characteristic equation (4.26). Using identities (4.6) and also formulas (4.10), (4.13), (4.16), and (4.18), we obtain expressions for projectors (4.27) in terms of the operators $\mathbf{I}$, $\mathbf{P}$, $\mathbf{K}$, $\widehat{C}_+$, and $\widehat{C}_-$:

$$\begin{aligned}
\mathrm{P}_1 &= \frac{1}{2}(\mathbf{I}-\mathbf{P})+2\widehat{C}_-,\\
\mathrm{P}_2 &= -2\widehat{C}_-,\\
\mathrm{P}_3 &= \frac{2\mathbf{K}}{(M-1)M},\\
\mathrm{P}_4 &= \frac{2}{3}(M-2)\widehat{C}_+^2+\frac{M}{3}\,\widehat{C}_+ +\frac{(M-4)(\mathbf{I}+\mathbf{P})}{3(M-2)}-\frac{2(M-4)\mathbf{K}}{3(M-2)(M-1)},\\
\mathrm{P}_5 &= -\frac{2(M-2)^2}{3(M-8)}\widehat{C}_+^2-\frac{(M-2)(M-6)}{3(M-8)}\widehat{C}_+ +\frac{(M-4)(\mathbf{I}+\mathbf{P})}{6(M-8)}+\frac{2\mathbf{K}}{3(M-8)},\\
\mathrm{P}_6 &= \frac{4(M-2)}{M-8}\widehat{C}_+^2+\frac{4}{M-8}\widehat{C}_+ -\frac{4(\mathbf{I}+\mathbf{P})}{(M-2)(M-8)}-\frac{8(M-4)\mathbf{K}}{M(M-2)(M-8)}.
\end{aligned} \tag{4.28}$$

We note that the projectors P_5 and P_6 are formally undefined for $M = 8$. This is a consequence of the factor $a_5 - a_6 = (M-8)/2(M-2)$ in the denominators of these projectors in (4.27). Nevertheless, if we substitute formula (4.11) for $\widehat{C}_+^2$ in the expressions for P_5 and P_6, then the pole at $M = 8$ cancels. As a result, for P_5, we have the expression

$$\mathrm{P}_5 = \frac{1}{6}(1-\epsilon(P_{14}+P_{23}+P_{13}+P_{24})+P_{13}P_{24})\mathcal{P}^{(\epsilon)}_{12,34}, \tag{4.29}$$

which is independent of M and coincides with the total antisymmetrizer on $V_N^{\otimes 4}$ in the $so(N,\mathbb{C})$ case ($\epsilon = +1$) or with the total symmetrizer on $V_N^{\otimes 4}$ in the $sp(N,\mathbb{C})$ case ($\epsilon = -1$). For the projector P_6, we have

$$\mathrm{P}_6 = \frac{4}{M-2}\mathcal{P}^{(\epsilon)}_{12,34}K_{13}\Big[\frac{1}{2}(1+\epsilon P_{24})-\frac{1}{M}\;K_{24}\Big]\mathcal{P}^{(\epsilon)}_{12,34}. \tag{4.30}$$

Both operators P_5 and P_6 are well defined at $M = 8$. Nevertheless, because identity (4.12) differs from (4.13), we consider the $so(8,\mathbb{C})$ case separately (see Sec. 4.2).

The dimensions of the eigensubspaces V_{a_i} of $\widehat{C}_{\rm ad}$ in the space $V_{\rm ad}\otimes V_{\rm ad}$ are equal to the traces of the corresponding projectors. To find them, we calculate the auxiliary traces

$$\begin{aligned}
&\operatorname{tr}\mathbf{I} = \frac{M^2}{4}(M-1)^2, \qquad \operatorname{tr}\mathbf{P} = \frac{M}{2}(M-1), \qquad \operatorname{tr}\mathbf{K} = \frac{M}{2}(M-1),\\
&\operatorname{tr}\widehat{C}_+ = \frac{M}{4}(M-1), \qquad \operatorname{tr}\widehat{C}_+^2 = \frac{3M}{8}(M-1), \qquad \operatorname{tr}\widehat{C}_- = -\frac{M}{4}(M-1).
\end{aligned} \tag{4.31}$$

We hence obtain

$$\begin{aligned}
\dim(V_{a_1}) &= \operatorname{tr} \mathrm{P}_1 = \frac{1}{8} M(M-1)(M+2)(M-3), \\
\dim(V_{a_2}) &= \operatorname{tr} \mathrm{P}_2 = \frac{1}{2} M(M-1), \\
\dim(V_{a_3}) &= \operatorname{tr} \mathrm{P}_3 = 1, \\
\dim(V_{a_4}) &= \operatorname{tr} \mathrm{P}_4 = \frac{1}{12} M(M+1)(M+2)(M-3), \\
\dim(V_{a_5}) &= \operatorname{tr} \mathrm{P}_5 = \frac{1}{24} M(M-1)(M-2)(M-3), \\
\dim(V_{a_6}) &= \operatorname{tr} \mathrm{P}_6 = \frac{1}{2} (M-1)(M+2).
\end{aligned} \tag{4.32}$$

We note that the sum of these traces

$$\sum_{i=1}^{6} \operatorname{tr} \mathrm{P}_i = \frac{M^2}{4}(M-1)^2 = \operatorname{tr} \mathbf{I} \tag{4.33}$$

coincides with the dimension of $\mathfrak{g}_N^\epsilon \otimes \mathfrak{g}_N^\epsilon$, as it should.

For $M = 4, 6$, the characteristic identities, as already noted in the discussion after (4.13) and also after formulas (4.23)–(4.25), have the corresponding degrees four and five. Nevertheless, all operators (4.28) are well defined for $M = 4, 6$, are constructed from the ad-invariant operators $\mathbf{I}$, $\mathbf{P}$, $\mathbf{K}$, $\widehat{C}_+$, $\widehat{C}_+^2$, and $\widehat{C}_-$, and satisfy the relations

$$\mathrm{P}_i \mathrm{P}_j = \delta_{ij} \mathrm{P}_i, \qquad \sum_{i=1}^{6} \mathrm{P}_i = \mathbf{I}, \quad i, j = 1, \ldots, 6, \tag{4.34}$$

i.e., they form a full system of projectors onto the invariant subspaces of the representations $\mathrm{ad}^{\otimes 2}(so(4))$ and $\mathrm{ad}^{\otimes 2}(so(6))$. We can easily find a decomposition of the representation $\mathrm{ad}^{\otimes 2}(so(4))$ into irreducible subrepresentations if we use the isomorphism $so(4) \cong sl(2) + sl(2)$ and the known decomposition $\mathrm{ad}^{\otimes 2}(sl(2)) = 1 + 3 + 5$, and we can use, for example, the program `LieART` [21] for a decomposition of $\mathrm{ad}^{\otimes 2}(so(6))$:

$$\mathrm{ad}^{\otimes 2}(so(4)) \equiv 6 \otimes 6 = 1 + 1 + 3 + 3 + 5 + 5 + 9 + 9, \tag{4.35}$$

$$\mathrm{ad}^{\otimes 2}(so(6)) \equiv 15 \otimes 15 = 1 + 15 + 15 + 20 + 45 + 45' + 84. \tag{4.36}$$

Comparing representation dimensions (4.35) and invariant subspace dimensions (4.32), we see that in the $so(4, \mathbb{C})$ case, of projectors (4.28), only P_1, P_6, P_3, and P_5 are primitive, of which the first two project onto ten-dimensional subspaces and the second two project onto one-dimensional subspaces. The operator P_2 projects onto the sum of two three-dimensional subspaces, and P_4 projects onto the sum of two five-dimensional subspaces. In the $so(6, \mathbb{C})$ case, comparing formulas (4.36) and (4.32) shows that all the projectors except P_1 are primitive and P_1 projects onto the sum of two invariant subspaces, each of which is 45-dimensional.

We also consider the case of the algebra $so(3, \mathbb{C}) \cong sl(2, \mathbb{C}) \cong sp(2, \mathbb{C})$ here, i.e., where $M = 3$ (the algebra $so(3, \mathbb{C})$) or $M = -2$ (the algebra $sp(2, \mathbb{C})$). Although roots (4.22) of the characteristic polynomial in the left-hand side of (4.21) differ, identity (4.26) is not in fact characteristic for the operator $\widehat{C}_{\mathrm{ad}}$ if $M = 3$ or $M = -2$. This is related to the fact that the projectors P_1, P_4, and P_5 vanish for $M = 3$ and P_1, P_4, and P_6 vanish for $M = -2$ (because the quantities in (4.22) corresponding to the vanishing projectors are not eigenvalues of $\widehat{C}_{\rm ad}$ in the considered representations). The remaining three projectors form a full system of projectors and are primitive, as can be seen from the decomposition

$$\mathrm{ad}^{\otimes 2}(sl(2)) \equiv 3 \otimes 3 = 1 + 3 + 5. \tag{4.37}$$

Traces of those projectors in the cases $M = 3$ and $M = -2$ are equal to dimensions of the corresponding invariant subspaces and agree with the isomorphisms $so(3,\mathbb{C}) \cong sl(2,\mathbb{C}) \cong sp(2,\mathbb{C})$.

For illustration, we present dimensions (4.32) of irreducible representations in the decomposition of $\mathrm{ad} \otimes \mathrm{ad}$ for some of the simple Lie algebras $so(N)$ and $sp(N)$ and for several values of N in Tables 1 and 2. The dimensions in these tables agree with the data presented in [22]. We also note that characteristic identities (4.26) and dimensions (4.32) for $so(N,\mathbb{C})$ and $sp(N,\mathbb{C})$ turn into each other under the change $N \to -N$. This manifests a duality between the algebras $so(N)$ and $sp(N)$ (see [23] for more details).

Table 1

N	$\dim_1$	$\dim_2$	$\dim_3$	$\dim_4$	$\dim_5$	$\dim_6$
5	35	10	1	35	5	14
7	189	21	1	168	35	27
9	594	36	1	495	126	44
10	945	45	1	770	210	54
11	1434	55	1	1144	330	65

Dimensions of irreducible representations for the algebra $so(N)$.

Table 2

N	$\dim_1$	$\dim_2$	$\dim_3$	$\dim_4$	$\dim_5$	$\dim_6$
4	35	10	1	14	35	5
6	189	21	1	90	126	14
8	594	36	1	308	330	27
10	1430	55	1	780	715	44
12	2925	78	1	1650	1365	65

Dimensions of irreducible representations for the algebra $sp(N)$.

4.2. The case of the algebra *so*(8). To derive the characteristic identity for $\widehat{C}_{\rm ad}$ in the $so(8)$ case, we use identities (4.9) and (4.12) for the operators $\widehat{C}_-$ and $\widehat{C}_+$:

$$\widehat{C}_-^2 = -\frac{1}{2}\widehat{C}_-, \qquad \widehat{C}_+^2 = -\frac{1}{6}\widehat{C}_+ + \frac{1}{36}(\mathbf{I} + \mathbf{P} + \mathbf{K}). \tag{4.38}$$

We multiply the second equality by $\widehat{C}_+$ and then multiply the obtained identity by $(\widehat{C}_+ + 1)$ and use the condition $\mathbf{K}(\widehat{C}_+ + 1) = 0$. As a result, we obtain

$$\widehat{C}_+(\widehat{C}_+ + 1)\left(\widehat{C}_+ + \frac{1}{3}\right)\left(\widehat{C}_+ - \frac{1}{6}\right) = 0.$$

The characteristic identity for the split Casimir operator $\widehat{C}_{\rm ad} = \widehat{C}_+ + \widehat{C}_-$ is now constructed analogously to the general case $so(M)$ by introducing indeterminate coefficients:

$$\widehat{C}_{\rm ad}\left(\widehat{C}_{\rm ad} + \frac{1}{2}\right)(\widehat{C}_{\rm ad} + 1)\left(\widehat{C}_{\rm ad} + \frac{1}{3}\right)\left(\widehat{C}_{\rm ad} - \frac{1}{6}\right) = 0. \tag{4.39}$$

Hence, the operator $\widehat{C}_{\rm ad}$ has the eigenvalues $a_1 = 0$, $a_2 = -1/2$, $a_3 = -1$, $a_4 = -1/3$, and $a_5 = 1/6$. We write all the projectors onto the eigensubspaces of $\widehat{C}_{\rm ad}$ (which follow from (4.39)) in terms of $\widehat{C}_+$, $\widehat{C}_-$, $\mathbf{I}$, $\mathbf{P}$, and $\mathbf{K}$:

$$\begin{aligned}
&{\rm P}'_1 = \frac{1}{2}(\mathbf{I} - \mathbf{P}) + 2\widehat{C}_- \equiv {\rm P}_1\Big|_{M=8}, && \dim = 350,\\
&{\rm P}'_2 = -2\widehat{C}_- \equiv {\rm P}_2\big|_{M=8}, && \dim = 28,\\
&{\rm P}'_3 = \frac{1}{28}\mathbf{K} \equiv {\rm P}_3\Big|_{M=8}, && \dim = 1,\\
&{\rm P}'_4 = \frac{1}{6}(\mathbf{I} + \mathbf{P}) - 2\widehat{C}_+ - \frac{1}{12}\mathbf{K} \equiv ({\rm P}_5 + {\rm P}_6)\Big|_{M=8}, && \dim = 105,\\
&{\rm P}'_5 = \frac{1}{3}(\mathbf{I} + \mathbf{P}) + 2\widehat{C}_+ + \frac{1}{21}\mathbf{K} \equiv {\rm P}_4\Big|_{M=8}, && \dim = 300,
\end{aligned} \tag{4.40}$$

where ${\rm P}'_k \equiv {\rm P}'_{a_k}$ and the projectors ${\rm P}_i$ are defined in (4.28) and are written for $M = 8$ with identity (4.12) taken into account. On the right in (4.40), we give the dimensions of the eigensubspaces calculated using formulas (4.31).

It is known that ${\rm ad}^{\otimes 2}(so(8))$ decomposes into irreducible representations as [21]

$${\rm ad}^{\otimes 2}(so(8)) = 1 + 28 + 35 + 35' + 35'' + 300 + 350. \tag{4.41}$$

The subspaces on which each of the irreducible representations in (4.41) act are denoted by V_1, V_{28}, V_{35}, etc. Comparing the dimensions of the subrepresentations in (4.40) and (4.41) implies that ${\rm P}'_4$ projects $V_{\rm ad}^{\otimes 2}$ onto $V_{35} + V_{35'} + V_{35''}$. We note that for the projector ${\rm P}'_4$, we have the decomposition

$${\rm P}'_4 = {\rm P}_5|_{M=8} + {\rm P}_6|_{M=8}, \tag{4.42}$$

where ${\rm P}_5$ and ${\rm P}_6$ are defined in (4.29) and (4.30). Moreover, in accordance with (4.32), the dimension of the space $V_{a_5} = {\rm P}_5|_{M=8}(V_{\rm ad} \otimes V_{\rm ad})$ is 70, and the dimension of $V_{a_6} = {\rm P}_6|_{M=8}(V_{\rm ad} \otimes V_{\rm ad})$ is 35. It hence follows that V_{a_6} is the space of one of the 35-dimensional irreducible representations of $so(8)$ (e.g., $V_{35''}$) and the space V_{a_5} obtained by the action of total antisymmetrizer (4.29) must decompose into the sum $V_{35} + V_{35'}$ of the two remaining 35-dimensional irreducible representations.

To construct the projectors onto the invariant subspaces V_{35} and $V_{35'}$, we introduce an operator $E_8 \colon V_8^{\otimes 4} \to V_8^{\otimes 4}$ with the components

$$(E_8)^{i_i \dots i_4}{}_{j_1 \dots j_4} \equiv \frac{1}{4!}\varepsilon^{i_i \dots i_4}{}_{j_1 \dots j_4} = \frac{1}{4!} c_{j_1 i_5} \cdots c_{j_1 i_8} \varepsilon^{i_1 \dots i_8}. \tag{4.43}$$

Here, $\varepsilon^{i_1 \dots i_8}$ is the totally antisymmetric invariant tensor: $\varepsilon^{12345678} = 1$. This operator, constructed from the invariant tensors $\varepsilon^{i_1 \dots i_8}$ and $c_{i_1 i_2}$, is ad-invariant. Taking into account that we have $c_{i_1 i_2} = \delta_{i_1 i_2}$ in the $so(8)$ case, we do not distinguish upper and lower indices.

To construct the projectors onto the eigenspaces of the operator E_8, we find the characteristic identity for this operator. It is convenient to calculate in the more general case of the algebra $so(2r)$, i.e., for the

$(2r)^r$-dimensional space $V_{2r}^{\otimes r}$ and the totally antisymmetric rank-$2r$ tensor ε. The operator E_r in this case is defined as

$$(E_r)^{i_i\ldots i_r}{}_{j_1\ldots j_r} = \frac{(-1)^{r/2}}{r!}\varepsilon^{i_i\ldots i_r}{}_{j_1\ldots j_r}. \tag{4.44}$$

We note that we have the relations

$$\left.\begin{aligned} &A_rE_r = E_rA_r = E_r,\\ &E_r^2 = A_r \end{aligned}\right\} \quad \Longrightarrow \quad E_r^3 = E_r, \tag{4.45}$$

where A_r is the total antisymmetrizer in the space of all rank-r tensors. The components of A_r are

$$(A_r)^{i_1\ldots i_r}{}_{k_1\ldots k_r} = \frac{1}{r!}\sum_{\sigma\in S_r}(-1)^{p(\sigma)}\delta^{i_{\sigma(1)}}_{k_1}\cdots\delta^{i_{\sigma(r)}}_{k_r} = \frac{1}{(r!)^2}\varepsilon^{i_1\ldots i_r j_1\ldots j_r}\varepsilon_{k_1\ldots k_r j_1\ldots j_r}, \tag{4.46}$$

where the summation ranges all permutations $\sigma \in S_r$ and $p(\sigma)$ is the parity of σ. We can write characteristic identity (4.45) for E_r as

$$E_r(E_r - \bar{\mathbf{I}})(E_r + \bar{\mathbf{I}}) = 0, \tag{4.47}$$

where $\bar{\mathbf{I}}$ denotes the identity operator in $V_{2r}^{\otimes r}$.

Using (4.47), we can construct the projectors $\overline{P}_1$, $\overline{P}_2$, and $\overline{P}_3$ onto the eigenspaces of E_r:

$$\overline{P}_{(b_i)} \equiv \overline{P}_j = \prod_{\substack{i=1,\\ i\neq j}}^{3}\frac{E_r - b_i\bar{\mathbf{I}}}{b_j - b_i}, \tag{4.48}$$

where $b_1 = 0$, $b_2 = 1$, and $b_3 = (-1)$ are the eigenvalues of E_r. The concrete formulas are

$$\begin{aligned} \overline{P}_1 &= -(E_r - \bar{\mathbf{I}})(E_r + \bar{\mathbf{I}}) = \bar{\mathbf{I}} - A_r,\\ \overline{P}_2 &= \frac{1}{2}E_r(E_r + \bar{\mathbf{I}}) = \frac{1}{2}(A_r + E_r),\\ \overline{P}_3 &= \frac{1}{2}E_r(E_r - \bar{\mathbf{I}}) = \frac{1}{2}(A_r - E_r), \end{aligned} \tag{4.49}$$

where we use the second formula in the braces in (4.45) to simplify the expressions.

In accordance with (4.49), we have $\overline{P}_2 + \overline{P}_3 = A_r$, and the images of the operators $\overline{P}_2$ and $\overline{P}_3$ belong to the space $V_{2r}^{\wedge r}$ of totally antisymmetric rank-r tensors. The operators $\overline{P}_2$ and $\overline{P}_3$ split the space $V_{2r}^{\wedge r}$ into two parts, called the spaces of self-dual and anti-self-dual tensors. The operator $\overline{P}_1 = \bar{\mathbf{I}} - A_r$ acts trivially on the space $V_{2r}^{\wedge r}$, and we therefore do not use $\overline{P}_1$ in what follows.

To calculate the traces of the second and the third projectors in (4.49) and find the dimension of the corresponding subspaces, we use the auxiliary traces

$$\operatorname{tr} A_r = \binom{2r}{r} = \frac{(2r)!}{(r!)^2}, \qquad \operatorname{tr} E_r = 0. \tag{4.50}$$

As a result, the traces of $\overline{P}_2$ and $\overline{P}_3$ have the form

$$\operatorname{tr}\overline{P}_2 = \frac{1}{2}\frac{(2r)!}{(r!)^2}, \qquad \operatorname{tr}\overline{P}_3 = \frac{1}{2}\frac{(2r)!}{(r!)^2}. \tag{4.51}$$

We now return to the algebra $so(8)$ and substitute the value $r = 4$ in the expressions for traces (4.51):

$$\mathrm{tr}\, \overline{P}_2|_{r=4} = 35, \qquad \mathrm{tr}\, \overline{P}_3|_{r=4} = 35 \tag{4.52}$$

(in what follows, we use $\overline{P}_i$ instead of $\overline{P}_i|_{r=4}$ for simplicity). The dimensions of the subspaces $\overline{P}_2 \cdot V_8^{\wedge 4}$ and $\overline{P}_3 \cdot V_8^{\wedge 4}$ thus coincide and are equal to 35. We recall that the projector P_5 given by (4.29) is equal to the total antisymmetrizer A_4 for every M and, in particular $\mathrm{P}_5|_{M=8} = A_4$. Hence,

$$\mathrm{P}_5|_{M=8} = \overline{P}_2 + \overline{P}_3, \tag{4.53}$$

and the images of $\overline{P}_2$ and $\overline{P}_3$ belong to the subspace extracted by the projector $\mathrm{P}_5|_{M=8} = A_4$. The values of traces (4.52) allow stating that $\overline{P}_2$ and $\overline{P}_3$ are the projectors onto the two 35-dimensional invariant subspaces of the representation $\mathrm{ad}^{\otimes 2}(so(8))$.

To write the projector $\mathrm{P}_6|_{M=8}$ in terms of the operators E_4 and A_4, introduced in (4.44) and (4.46) and in terms of the operators $\mathbf{I}$, $\mathbf{P}$, $\mathbf{K}$, and $\widehat{C}_+$, we use formula (4.42), the explicit form of the projector P'_4 given by (4.40), and also the fact that $\mathrm{P}_5|_{M=8} = A_4$. As a result, we obtain

$$\mathrm{P}_6|_{M=8} = \mathrm{P}'_4 - \mathrm{P}_5|_{M=8} = \frac{1}{6}(\mathbf{I} + \mathbf{P}) - 2\widehat{C}_+ - \frac{1}{12}\mathbf{K} - A_4. \tag{4.54}$$

Finally, we have the full system of mutually orthogonal and primitive projectors onto the spaces of the irreducible subrepresentations in $\mathrm{ad}^2(so(8))$:

$$\begin{aligned}
&\mathrm{P}'_1 = \frac{1}{2}(\mathbf{I} - \mathbf{P}) + 2\widehat{C}_-, && \dim = 350,\\
&\mathrm{P}'_2 = -2\widehat{C}_-, && \dim = 28,\\
&\mathrm{P}'_3 = \frac{1}{28}\mathbf{K}, && \dim = 1,\\
&\mathrm{P}_6|_{M=8} = \frac{1}{6}(\mathbf{I} + \mathbf{P}) - 2\widehat{C}_+ - \frac{1}{12}\mathbf{K} - A_4, && \dim = 35,\\
&\overline{P}_2 = \frac{1}{2}(A_4 + E_4), && \dim = 35,\\
&\overline{P}_3 = \frac{1}{2}(A_4 - E_4), && \dim = 35,\\
&\mathrm{P}'_5 = \frac{1}{3}(\mathbf{I} + \mathbf{P}) + 2\widehat{C}_+ + \frac{1}{21}\mathbf{K}, && \dim = 300.
\end{aligned} \tag{4.55}$$

5. A connection between the eigenvalues of the operator $\widehat{C}$ in the adjoint representation of the Lie algebras $so(N, \mathbb{C})$ and $sp(N, \mathbb{C})$ and the Vogel parameters

Many results in this paper, namely, construction of the projectors onto invariant subspaces of the representation $\mathrm{ad}^{\otimes 2}(\mathfrak{g}_N^\epsilon)$ in terms of the split Casimir operator and the calculation of the dimensions of these subspaces, can also be obtained by using the so-called Vogel parameters [14], [18]. These parameters are defined as three numbers (α, β, γ) modulo a common multiplier and an arbitrary permutation from the symmetric group S_3 (and can hence be interpreted as coordinates in the space $\mathbb{P}^2/S_3$). It is known [14], [15], [18] that certain values of the Vogel parameters (or, equivalently, a certain point in the space $\mathbb{P}^2/S_3$) correspond to each simple Lie algebra. For the simple Lie algebras of the classical series, the values of these parameters are given in Table 3 [17], where $t \equiv \alpha + \beta + \gamma$ is the dual Coxeter number.

Table 3

Type	Lie Algebra	α	β	γ	t
A_r	$sl(r+1)$	-2	2	$r+1$	$r+1$
B_r	$so(2r+1)$	-2	4	$2r-3$	$2r-1$
C_r	$sp(2r)$	-2	1	$r+2$	$r+1$
D_r	$so(2r)$	-2	4	$2r-4$	$2r-2$

Vogel parameters for simple Lie algebras.

The representation $\mathrm{ad}^{\otimes 2}\mathcal{A}$ of an arbitrary simple Lie algebra $\mathcal{A}$ decomposes into symmetric and antisymmetric parts. According to [14]–[16], the symmetric part (for all simple Lie algebras of classical series except $sl(2,\mathbb{C}) \cong sp(2,\mathbb{C}) \cong so(3,\mathbb{C})$, $sl(3,\mathbb{C})$, and $so(8,\mathbb{C})$) in turn decomposes into four irreducible subrepresentations: a singlet T_0 and three irreducible representations denoted by $Y_2^{(\alpha)}$, $Y_2^{(\beta)}$, and $Y_2^{(\gamma)}$, where α, β, and γ are the Vogel parameters of the algebra $\mathcal{A}$ in Table 3. The dimensions of these four representations and the eigenvalues of the quadratic Casimir operator $C_{(2)}$ (defined in (3.5)) acting on the spaces of these representations are [14], [18]

$$\begin{aligned}
&\dim T_0(\mathcal{A}) = 1, && c_2^0 = 0,\\
&\dim Y_2^{(\alpha)}(\mathcal{A}) = -\frac{(3\alpha-2t)(\beta-2t)(\gamma-2t)t(\beta+t)(\gamma+t)}{\alpha^2(\alpha-\beta)\beta(\alpha-\gamma)\gamma}, && c_2^\alpha = 2-\frac{\alpha}{t},\\
&\dim Y_2^{(\beta)}(\mathcal{A}) = -\frac{(3\beta-2t)(\alpha-2t)(\gamma-2t)t(\alpha+t)(\gamma+t)}{\beta^2(\beta-\alpha)\alpha(\beta-\gamma)\gamma}, && c_2^\beta = 2-\frac{\beta}{t},\\
&\dim Y_2^{(\gamma)}(\mathcal{A}) = -\frac{(3\gamma-2t)(\beta-2t)(\alpha-2t)t(\beta+t)(\alpha+t)}{\gamma^2(\gamma-\beta)(\beta(\gamma-\alpha)\alpha}, && c_2^\gamma = 2-\frac{\gamma}{t},
\end{aligned} \tag{5.1}$$

where c_2^0, c_2^α, c_2^β, and c_2^γ are the eigenvalues of the operator $C_{(2)}$ acting in the respective representations T_0, $Y_2^{(\alpha)}$, $Y_2^{(\beta)}$, and $Y_2^{(\gamma)}$ (here the upper index of c_2 is chosen in accordance with the notation for the corresponding representation and is not the highest weight, as in formula (3.10)).

The antisymmetric part decomposes into the direct sum of two representations, one of which is the adjoint representation ad. The dimension of this representation and the value c_2^{ad} of the quadratic Casimir operator $C_{(2)}$ in it are expressed by the formulas

$$\dim(\mathrm{ad}) \equiv \dim\mathcal{A} = \frac{(\alpha-2t)(\beta-2t)(\gamma-2t)}{\alpha\beta\gamma}, \qquad c_2^{\mathrm{ad}} = 1. \tag{5.2}$$

The second antisymmetric subrepresentation is denoted by X_2. Its dimension and the value c_2^X of the operator $C_{(2)}$ in it are given by the formulas

$$\dim(X_2) = \frac{1}{2}\dim\mathcal{A}(\dim\mathcal{A}-3), \qquad c_2^X = 2. \tag{5.3}$$

Comparison of the dimensions [21] of the irreducible representations in $\mathrm{ad}^{\otimes 2}$ with dimensions (4.32) implies that the representation X_2 for the Lie algebras of the types B_r, C_r, and D_r is irreducible. We note that the representation X_2 for Lie algebras of the type A_r, $r > 1$, is reducible and decomposes into the sum of two irreducible representations with the same dimensions.

The final decomposition of the representation $\mathrm{ad}^{\otimes 2}(\mathcal{A})$, where $\mathcal{A}$ is a classical simple Lie algebra (except the algebras $sl(2,\mathbb{C}) \cong sp(2,\mathbb{C}) \cong so(3,\mathbb{C})$, $sl(3,\mathbb{C})$ and $so(8,\mathbb{C})$), has the form

$$\mathrm{ad}^{\otimes 2}(\mathcal{A}) = T_0(\mathcal{A}) + Y_2^{(\alpha)}(\mathcal{A}) + Y_2^{(\beta)}(\mathcal{A}) + Y_2^{(\gamma)}(\mathcal{A}) + \mathrm{ad}(\mathcal{A}) + X_2(\mathcal{A}). \tag{5.4}$$

Knowing values (5.1)–(5.3) for the quadratic Casimir operator $C_{(2)}$ in the six representations (in the case where those representations are irreducible), we can calculate the values of the split Casimir operator $\widehat{C}$ (defined in (3.3)) in these representations using formula (3.9). The relation between the eigenvalues of $\widehat{C}$ and $C_{(2)}$ in any irreducible subrepresentation T_λ in the decomposition $\mathrm{ad} \otimes \mathrm{ad}$ can be found using formula (3.9), where we must set λ_1 and λ_2 equal to the highest weight of an irreducible representation of $\mathcal{A}$. Because the value of the quadratic Casimir operator of any simple Lie Algebra $\mathcal{A}$ in the adjoint representation is equal to unity (see, e.g., [20]), $c_2^{(\lambda_1)} = c_2^{(\lambda_2)} = 1$, and in accordance with (3.9), we have

$$\hat{c}^{(\lambda)} = \frac{1}{2} c_2^{(\lambda)} - 1. \tag{5.5}$$

Further, we consider only the cases of the algebras $so(N,\mathbb{C})$ and $sp(N,\mathbb{C})$, to which our paper is devoted. Substituting the values of α, β, γ, and t for $so(N,\mathbb{C})$ and $sp(N,\mathbb{C})$ from Table 3 in formulas (5.1)–(5.3), we find that the dimensions of the six representations T_0, $Y_2^{(\alpha)}$, $Y_2^{(\beta)}$, $Y_2^{(\gamma)}$, $\mathrm{ad}(\mathfrak{g}_N^\epsilon)$, and X_2 coincide with dimensions (4.32) of the invariant subspaces of the representations $\mathrm{ad}^{\otimes 2}(\mathfrak{g}_N^\epsilon)$. Taking formula (5.5) into account, we can show that the values of the split Casimir operator in each of those representations coincide with roots (4.22) of the polynomial in the left-hand side of (4.26), which is the characteristic polynomial of $\widehat{C}$ in the representation $\mathrm{ad}^{\otimes 2}(\mathfrak{g}_N^\epsilon)$. These coincidences allow concluding that the spaces of the representations in the right-hand side of (5.4) uniquely correspond to the invariant subspaces obtained using projectors (4.28). This correspondence is shown in Table 4, where all six projectors (4.28) are given in the first row and the subrepresentations whose subspaces are extracted in $V_{\mathrm{ad}} \otimes V_{\mathrm{ad}}$ by the projectors in the first row are given in the second, third, and fourth rows.

Table 4

	P_1	P_2	P_3	P_4	P_5	P_6
B_r	X_2	$\mathrm{ad}(so(2r+1))$	T_0	$Y_2^{(\alpha)}$	$Y_2^{(\beta)}$	$Y_2^{(\gamma)}$
C_r	X_2	$\mathrm{ad}(sp(2r))$	T_0	$Y_2^{(\beta)}$	$Y_2^{(\alpha)}$	$Y_2^{(\gamma)}$
D_r	X_2	$\mathrm{ad}(so(2r))$	T_0	$Y_2^{(\alpha)}$	$Y_2^{(\beta)}$	$Y_2^{(\gamma)}$

Correspondence between the representations and the projectors.

We have thus shown that the values of the split Casimir operator in the irreducible subrepresentations in $\mathrm{ad}^{\otimes 2}$ calculated in the preceding sections for the Lie algebras $so(N,\mathbb{C})$ and $sp(N,\mathbb{C})$ and also the dimensions of the corresponding invariant subspaces can be written in terms of the Vogel parameters in complete accordance with [14]–[16].

6. Conclusion

We have found explicit formulas for the projectors onto the spaces of the irreducible subrepresentations of the tensor product of two adjoint representations for all complex simple Lie algebras $so(N,\mathbb{C})$ for $N \geq 3$,

$N \neq 6$ and $sp(N, \mathbb{C})$ for $N \geq 2$. In all the cases except the $so(8)$ case, the construction was performed by finding the characteristic identity for the split Casimir operator. In the separately considered $so(8, \mathbb{C})$ case, we used an additional invariant operator to construct the sought projectors. This operator allows decomposing the totally antisymmetric rank-4 representation into self-dual and the anti-self-dual parts. Further, we showed that dimensions of the irreducible representations obtained here and also the values of the quadratic Casimir operator in these representations completely agree with the results in [14]–[16], where these values were written in terms of the Vogel parameters.

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