INTEGRABLE SYSTEM OF GENERALIZED RELATIVISTIC INTERACTING TOPS

I. A. Sechin*† **and A. V. Zotov***

We describe a family of integrable $GL(NM)$ models generalizing classical spin Ruijsenaars–Schneider systems (the case $N = 1$) on one hand and relativistic integrable tops on the $GL(N)$ Lie group (the case $M = 1$) on the other hand. We obtain the described models using the Lax pair with a spectral parameter and derive the equations of motion. To construct the Lax representation, we use the $GL(N)$ R-matrix in the fundamental representation of $GL(N)$.

1. Introduction

This paper continues a series of articles [1]–[3] extending known integrable systems and their related structures using quantum R-matrices (in the fundamental representation of $GL(N)$ Lie groups) interpreted as matrix generalizations of the Kronecker function. At the very beginning, it is convenient to give its explicit form in the rational, trigonometric, and elliptic cases because the identities that we use hold separately in each of these cases:

$$
\phi(z,q) = \begin{cases} \dfrac{1}{z} + \dfrac{1}{q}, \\ \coth z + \coth q, \\ \dfrac{\vartheta'(0)\vartheta(z+q)}{\vartheta(z)\vartheta(q)}. \end{cases} \qquad E_1(z) = \begin{cases} \dfrac{1}{z}, \\ \coth(z), \\ \dfrac{\vartheta'(z)}{\vartheta(z)} \end{cases} \qquad \wp(z) = \begin{cases} \dfrac{1}{z^2}, \\ \dfrac{1}{\sinh^2 z}, \\ -E_1'(z) + \dfrac{1}{3}\dfrac{\vartheta'''(0)}{\vartheta'(0)}. \end{cases} \tag{1}
$$

All the variables and functions are complex valued. Therefore, the trigonometric and hyperbolic cases are essentially the same. In (1) in all three cases, we give definitions of the first Eisenstein function $E_1(z)$ and the Weierstrass $\wp$-function. They appear in the expansion of $\phi(z,q)$ near its simple pole (with the residue equal to unity) at $z = 0$:

$$
\phi(z,q) = z^{-1} + E_1(q) + \frac{z(E_1^2(q) - \wp(q))}{2} + O(z^2). \tag{2}
$$

*Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia, e-mail: shnbuz@gmail.com, zotov@mi-ras.ru (corresponding author).

†Skolkovo Institute of Science and Technology, Moscow, Russia.

The definitions and properties of elliptic functions can be found in [4] (also see the appendix in [3]).

In each of the three cases, the Kronecker function satisfies the summation formula, the genus-1 Fay identity,

$$\phi(z_1, q_1)\phi(z_2, q_2) = \phi(z_1 - z_2, q_1)\phi(z_2, q_1 + q_2) + \phi(z_2 - z_1, q_2)\phi(z_1, q_1 + q_2), \tag{3}$$

and also its degenerations corresponding to equal arguments,

$$\phi(z, q_1)\phi(z, q_2) = \phi(z, q_1 + q_2)(E_1(z) + E_1(q_1) + E_1(q_2) - E_1(q_1 + q_2 + z)), \tag{4}$$

$$\phi(z, q)\phi(z, -q) = \wp(z) - \wp(q). \tag{5}$$

Fay identity (3) can be regarded as a particular scalar case of the associative Yang–Baxter equation [5]:

$$R_{12}^z(q_{12})R_{23}^w(q_{23}) = R_{13}^w(q_{13})R_{12}^{z-w}(q_{12}) + R_{23}^{w-z}(q_{23})R_{13}^z(q_{13}), \qquad q_{ab} = q_a - q_b. \tag{6}$$

Here, we use R-matrix notation of the quantum inverse scattering method, for example,

$$R_{12}^z(q) = \sum_{i,j,k,l=1}^{N} R_{ij,kl}(z, q)E_{ij} \otimes E_{kl} \otimes 1_N,$$

$$\tag{7}$$

$$R_{13}^z(q) = \sum_{i,j,k,l=1}^{N} R_{ij,kl}(z, q)E_{ij} \otimes 1_N \otimes E_{kl},$$

where E_{ij} is the standard matrix basis in $\mathrm{Mat}(N, \mathbb{C})$, 1_N is the identity matrix, and $R_{ij,kl}(z, q)$ is a set of functions of z and q. The normalization of the matrix operator $R_{ab}^z(q_{ab})$ is chosen such that with $N = 1$, it reduces to the scalar function $\phi(z, q)$ given by (1). In this respect, Eq. (6) is a noncommutative generalization of (3), and the operator R is a noncommutative generalization of the Kronecker function.

In addition to (6), we can require the properties of antisymmetry and unitarity (the latter is a matrix analogue of (5)):

$$R_{12}^z(q) = -R_{21}^{-z}(-q), \qquad R_{12}^z(q)R_{21}^z(-q) = 1_N \otimes 1_N(\wp(z) - \wp(q)). \tag{8}$$

Then such an R-operator satisfies the quantum Yang–Baxter equation

$$R_{12}^\hbar(z_{12})R_{13}^\hbar(z_{13})R_{23}^\hbar(z_{23}) = R_{23}^\hbar(z_{23})R_{13}^\hbar(z_{13})R_{12}^\hbar(z_{12}). \tag{9}$$

In other words, a solution of (6) satisfying conditions (8) is a quantum R-matrix. We note that even in the scalar case, condition (6) or (3) is very restrictive. At the same time, Eq. (9) is not restrictive at all because the quantum Yang–Baxter equation holds identically in the scalar case. The class of R-matrices with the listed properties includes the elliptic Baxter–Belavin R-matrix and also its trigonometric and rational degenerations, which are equal to the function $\phi(z, q)$ in the scalar case. A more detailed description of these R-matrices can be found in [6]–[8], where an application of this class of R-matrices to an integrable system was given, a construction of integrable tops. The main idea goes back to Sklyanin's paper [9], where he suggested a Hamiltonian description of the classical Euler top using quadratic Poisson algebras obtained in the classical limit of RLL relations. That is, the classical Euler top was described as the classical limit of a spin chain with one site. This approach can be developed to obtain an explicit description of the Lax pairs with spectral parameters constructed using the data of R-matrices satisfying (6) and (8). A detailed derivation of the equations of motion together with the Hamiltonian description using the R-matrix data was given in [6] and [7] in the respective nonrelativistic and relativistic cases.

1.1. Relativistic integrable GL_N-top. In the general case, the phase space of a GL_N top is given by the set of coordinate functions S_{ij}, $i, j = 1, \ldots, N$, on the Lie group GL_N. They are unified into the $N \times N$ matrix $S = \sum_{ij} S_{ij} E_{ij}$. The equations of motion then become the Euler–Arnold equations

$$\dot{S} = [S, J(S)], \tag{10}$$

where $J(S)$ is a linear functional on S. It can be written as

$$J(S) = \sum_{i,j,k,l=1}^{N} J_{i,j,k,l} E_{ij} S_{lk} \in \mathrm{Mat}(N, \mathbb{C}) \tag{11}$$

or, using the standard notation $S_1 = S \otimes 1_N$ and $S_2 = 1_N \otimes S$,

$$J(S) = \mathrm{tr}_2(J_{12} S_2), \qquad J_{12} = \sum_{i,j,k,l=1}^{N} J_{i,j,k,l} E_{ij} \otimes E_{kl}, \tag{12}$$

where tr_2 is the trace over the second space in the tensor product. Below, we give the Lax pair of the relativistic integrable top using the above notation (of course, Eq. (10) is not integrable in the general case). For this, we consider the classical limit of the R-matrix

$$R_{12}^{\hbar}(z) = \frac{1}{\hbar} 1_N \otimes 1_N + r_{12}(z) + \frac{\hbar}{2} \left(r_{12}(z)^2 - 1 \otimes 1 \, \wp(z) \right) + O(\hbar^2), \tag{13}$$

where $r_{12}(z) = -r_{21}(-z)$ is the classical r-matrix and the $\hbar$-order term follows from (8). Comparing this expression with (2), we conclude that while the quantum R-matrix is a matrix analogue of the Kronecker function, the classical r-matrix is a matrix analogue of the first Eisenstein function $E_1(z)$ given in (1).

We consider the expansions

$$R_{12}^{z}(q) = \frac{1}{q} P_{12} + R_{12}^{z,(0)} + O(q), \qquad r_{12}(z) = \frac{1}{z} P_{12} + r_{12}^{(0)} + O(z),$$

$$P_{12} = \sum_{i,j=1}^{N} E_{ij} \otimes E_{ji}, \tag{14}$$

where P_{12} is the matrix permutation operator. Generally speaking, the existence of expansions of types (13) and (14) is an additional nontrivial requirement for the R-matrix. Finally, we impose one more condition on the R-matrix:

$$R_{12}^{z}(q) = R_{12}^{q}(z) P_{12}. \tag{15}$$

In the scalar case, it leads to the obvious equality $\phi(z, q) = \phi(q, z)$. Using (15) and comparing (13) and (14), we easily obtain

$$r_{12}(z) = R_{12}^{z,(0)} P_{12}. \tag{16}$$

We can now formulate the statement about the Lax pair of the relativistic top. Namely, for a pair of matrices

$$L(z) = \mathrm{tr}_2(R_{12}^{\eta}(z) S_2) = \mathrm{tr}_2(R_{12}^{z}(\eta) P_{12} S_2),$$

$$M(z) = -\mathrm{tr}_2(r_{12}(z) S_2) = -\mathrm{tr}_2(R_{12}^{z,(0)} P_{12} S_2), \tag{17}$$

the Lax equation

$$\dot{L}(z) = [L(z), M(z)] \tag{18}$$

is equivalent to the equation of motion of form (10), where

$$J_{12} = R_{12}^{\eta,(0)} - r_{12}^{(0)}. \tag{19}$$

1.2. Spin generalization of the Ruijsenaars–Schneider model. In integrable many-body systems, relativistic generalizations are known as Ruijsenaars–Schneider models [10]. We are interested in their spin extensions. The dynamical variables comprise a set of coordinates and velocities of M particles and also classical spin variables described in the matrix $S \in \mathrm{Mat}(M, \mathbb{C})$. The equations of motion have the forms (for the diagonal and off-diagonal parts of S)

$$\dot{S}_{ii} = -\sum_{k:\,k\neq i}^{M} S_{ik}S_{ki}(E_1(q_{ik}+\eta) + E_1(q_{ik}-\eta) - 2E_1(q_{ik})),$$

$$\dot{S}_{ij} = \sum_{k:\,k\neq j}^{M} S_{ik}S_{kj}(E_1(q_{kj}+\eta) - E_1(q_{kj})) - \sum_{k:\,k\neq i}^{M} S_{ik}S_{kj}(E_1(q_{ik}+\eta) - E_1(q_{ik})) \tag{20}$$

and

$$\ddot{q}_i = \dot{S}_{ii}, \tag{21}$$

where $i \neq j$ and $q_{ij} = q_i - q_j$. The Lax pair with a spectral parameter

$$L_{ij}(z) = S_{ij}\phi(z, q_{ij}+\eta), \quad i,j = 1,\ldots,M,$$

$$\mathop{\mathrm{Res}}_{z=0} L(z) = S \in \mathrm{Mat}(M, \mathbb{C}), \tag{22}$$

$$M_{ij}(z) = -\delta_{ij}(E_1(z) + E_1(\eta))S_{ii} - (1 - \delta_{ij})S_{ij}\phi(z, q_{ij}),$$

satisfies the Lax equation with an additional term (here $\mu_i = \dot{q}_i - S_{ii}$)

$$\dot{L}(z) = [L(z), M(z)] + \sum_{i,j=1}^{M} E_{ij}(\mu_i - \mu_j)S_{ij}f(z, q_{ij}+\eta), \qquad f(z,q) = \partial_q\phi(z,q), \tag{23}$$

which vanishes with the on-shell constraints

$$\mu_i = 0 \qquad \text{or} \qquad S_{ii} = \dot{q}_i, \quad i = 1,\ldots,M. \tag{24}$$

More precisely, Eq. (23) is equivalent to (20), and under condition (24), Lax equations (23) with an additional term become the ordinary Lax equations (18), and (21) is satisfied. A detailed derivation of (23) can be found in [3] in addition to the original paper [11]. This derivation is convenient below in considering a more general system where the functions in (20)–(22) are replaced with their R-matrix analogues. Although the Hamiltonian structure is not used in the description indicated above, we note that it is known for the rational and trigonometric systems (see [12]–[15]).

Our main result here is the following generalization of simultaneously both relativistic top (17)–(19) and spin Ruijsenaars–Schneider model (20)–(22). We consider a $\mathrm{Mat}(NM, \mathbb{C})$-valued Lax pair subdivided into $M \times M$ block-matrices $\mathcal{L}^{ij}(z) = \mathcal{L}^{ij}(\mathcal{S}^{ij}, z)$ each of size $N \times N$:

$$\mathcal{L}(z) = \sum_{i,j=1}^{M} E_{ij} \otimes \mathcal{L}^{ij}(z) \in \mathrm{Mat}(NM, \mathbb{C}), \quad \mathcal{L}^{ij}(z) \in \mathrm{Mat}(N, \mathbb{C}),$$

$$\mathcal{L}^{ij}(z) = \mathrm{tr}_2(R_{12}^z(q_{ij}+\eta)P_{12}\mathcal{S}_2^{ij}), \quad \mathcal{S}^{ij} = \mathop{\mathrm{Res}}_{z=0} \mathcal{L}^{ij}(z) \in \mathrm{Mat}(N, \mathbb{C}),$$

$$\mathcal{M}(z) = \sum_{i,j=1}^{M} E_{ij} \otimes \mathcal{M}^{ij}(z) \in \mathrm{Mat}(NM, \mathbb{C}), \quad \mathcal{M}^{ij}(z) \in \mathrm{Mat}(N, \mathbb{C}), \tag{25}$$

$$\mathcal{M}^{ij}(z) = -\delta^{ij}\,\mathrm{tr}_2\big(R_{12}^{(0),z}P_{12}\mathcal{S}_2^{ii}\big) - (1 - \delta^{ij})\,\mathrm{tr}_2\big(R_{12}^z(q_{ij})P_{12}\mathcal{S}_2^{ij}\big).$$

The R-matrix in this definition satisfies associative Yang–Baxter equation (6) and also conditions (8) and (15) and expansions (13) and (14). The Lax equation with an additional term

$$\dot{\mathcal{L}}(z) = [\mathcal{L}(z), \mathcal{M}(z)] + \sum_{i,j=1}^{M} (\mu_0^i - \mu_0^j)\, E_{ij} \otimes \mathrm{tr}_2(F_{12}^z(q_{ij} + \eta)P_{12}\mathcal{S}_2^{ij}), \tag{26}$$

where by analogy with (23)

$$F_{12}^z(q) = \partial_q R_{12}^z(q) \tag{27}$$

and $\mu_0^i = \dot{q}_i - \mathrm{tr}(\mathcal{S}^{ii})$, $i = 1, \ldots, M$, is then equivalent to the equations of motion (we assume that $i \neq j$ in (26))

$$\dot{\mathcal{S}}^{ii} = [\mathcal{S}^{ii}, J^\eta(\mathcal{S}^{ii})] + \sum_{k:\,k\neq i}^{M} \left(\mathcal{S}^{ik} J^{\eta,q_{ki}}(S^{ki}) - J^{\eta,q_{ik}}(\mathcal{S}^{ik})S^{ki}\right), \tag{28}$$

$$\dot{\mathcal{S}}^{ij} = \mathcal{S}^{ij} J^\eta(\mathcal{S}^{jj}) - J^\eta(\mathcal{S}^{ii})\mathcal{S}^{ij} + \sum_{k:\,k\neq j}^{M} \mathcal{S}^{ik} J^{\eta,q_{kj}}(\mathcal{S}^{kj}) - \sum_{k:\,k\neq i}^{M} J^{\eta,q_{ik}}(\mathcal{S}^{ik})\mathcal{S}^{kj}. \tag{29}$$

With the on-shell constraints $\mu_0^i = 0$ or $\dot{q}_i = \mathrm{tr}(\mathcal{S}^{ii})$, $i = 1, \ldots, M$, Eqs. (26) reduce to the Lax equations, and we have the equations

$$\ddot{q}_i = \mathrm{tr}(\dot{\mathcal{S}}^{ii}) = \sum_{k:\,k\neq i}^{M} \mathrm{tr}\left(\mathcal{S}^{ik} J^{\eta,q_{ki}}(\mathcal{S}^{ki}) - J^{\eta,q_{ik}}(\mathcal{S}^{ik})\mathcal{S}^{ki}\right). \tag{30}$$

The linear functionals J^η and $J^{\eta,q}$ in the equations of motion are given by

$$J^\eta(\mathcal{S}^{ii}) = \mathrm{tr}_2\left((R_{12}^{(0),\eta} - r_{12}^{(0)})\mathcal{S}_2^{ii}\right), \qquad J^{\eta,q}(\mathcal{S}^{ij}) = \mathrm{tr}_2\left((R_{12}^{(0),q+\eta} - R_{12}^{(0),q})\mathcal{S}_2^{ij}\right). \tag{31}$$

The presented Lax pairs and equations of motion reproduce the results in the elliptic case in our previous paper [3][1] and the results in the nonrelativistic limit in [1]. With $N = 1$, the used R-matrix operators become the scalar functions in (1), thus reproducing spin Ruijsenaars–Schneider model (20)–(24). With $M = 1$, the Lax matrices have a single block. We thus obtain relativistic top (17)–(19). In the nonrelativistic elliptic case, models of the described type were first obtained in [16] and were later described as Hitchin systems on bundles with nontrivial characteristic classes [17]. Explicit examples of the systems can be easily obtained using R-matrices used in [1] in the same normalization as here.

2. Derivation of the equations of motion

2.1. R-matrix identities. To derive the equations of motion in the spin Ruijsenaars–Schneider model, we should use identity (4). We rewrite it differently:

$$\phi(z, q_1)\phi(z, q_2) = \phi(z, q_1 + q_2)(E_1(q_1) + E_1(q_2)) - \partial_z\phi(z, q_1 + q_2), \tag{32}$$

where we use the fact that (1) implies that $\partial_z\phi(z, q) = \phi(z, q)(E_1(z + q) - E_1(z))$. In this form, identity (4) is generalized to the matrix case:

$$R_{12}^z(x)R_{23}^z(y) = R_{13}^z(x + y)r_{12}(x) + r_{23}(y)R_{13}^z(x + y) - \frac{\partial}{\partial z}R_{13}^z(x + y). \tag{33}$$

---

[1]In [3], the elliptic case was described in a slightly different normalization. It differs from the one used here by $q_j \to q_j/N$, which leads to the additional factor $1/N$ in the equations of motion in [3].

Applications of this identity can be found in [8]. We also write its corollary:

$$R_{12}^z(q_{ik})R_{23}^z(q_{kj} + \eta) - R_{12}^z(q_{ik} + \eta)R_{23}^z(q_{kj}) =$$

$$= R_{13}^z(q_{ij} + \eta)(r_{12}(q_{ik}) - r_{12}(q_{ik} + \eta)) + (r_{23}(q_{kj} + \eta) - r_{23}(q_{kj}))R_{13}^z(q_{ij} + \eta). \qquad (34)$$

Moreover, we need degenerations of (33). We expand both its sides in a neighborhood of $x = 0$:

$$\left(\frac{1}{x}P_{12} + R_{12}^{(0),z} + \cdots\right)R_{23}^z(y) = \left(R_{13}^z(y) + xF_{13}^z(y) + \cdots\right)\left(\frac{1}{x}P_{12} + r_{12}^{(0)} + \cdots\right) +$$

$$+ r_{23}(y)\left(R_{13}^z(y) + xF_{13}^z(y) + \cdots\right) - \frac{\partial}{\partial z}\left(R_{13}^z(y) + xF_{13}^z(y) + \ldots\right), \quad (35)$$

where $F_{ab}^z(y)$ is defined as in (27). In the zeroth order in x, we obtain

$$R_{12}^{(0),z}R_{23}^z(y) = F_{13}^z(y)P_{12} + R_{13}^z(y)r_{12}^{(0)} + r_{23}(y)R_{13}^z(y) - \frac{\partial}{\partial z}R_{13}^z(y). \qquad (36)$$

Expanding (33) with small y, we similarly obtain

$$R_{12}^z(x)\left(\frac{1}{y}P_{23} + R_{23}^{(0),z} + \cdots\right) = \left(R_{13}^z(x) + yF_{13}^z(x) + \cdots\right)r_{12}(x) +$$

$$+ \left(\frac{1}{y}P_{23} + r_{23}^{(0)} + \cdots\right)\left(R_{13}^z(x) + yF_{13}^z(x) + \cdots\right) -$$

$$- \frac{\partial}{\partial z}\left(R_{13}^z(x) + yF_{13}^z(x) + \ldots\right), \qquad (37)$$

$$R_{12}^z(x)R_{23}^{(0),z} = R_{13}^z(x)r_{12}(x) + r_{23}^{(0)}R_{13}^z(x) + P_{23}F_{13}^z(x) - \frac{\partial}{\partial z}R_{13}^z(x). \qquad (38)$$

It follows from (36) and (38) that

$$R_{12}^{(0),z}R_{23}^z(q_{ij} + \eta) - R_{12}^z(\eta)R_{23}^z(q_{ij}) = F_{13}^z(q_{ij} + \eta)P_{12} + R_{13}^z(q_{ij} + \eta)(r_{12}^{(0)} - r_{12}(\eta)) +$$

$$+ (r_{23}(q_{ij} + \eta) - r_{23}(q_{ij}))R_{13}^z(q_{ij} + \eta). \qquad (39)$$

2.2. Lax equation. We write Lax equation (26) with an additional term explicitly in terms of $N \times N$ blocks. For the diagonal blocks, we obtain

$$\dot{\mathcal{L}}_{ii}(z) = \mathcal{L}^{ii}(z)\mathcal{M}^{ii}(z) - \mathcal{M}^{ii}(z)\mathcal{L}^{ii}(z) + \sum_{k \neq i}\left(\mathcal{L}^{ik}(z)\mathcal{M}^{ki}(z) - \mathcal{M}^{ik}(z)\mathcal{L}^{ki}(z)\right). \qquad (40)$$

Similarly, for the off-diagonal part, we have

$$\dot{\mathcal{L}}_{ij}(z) = \mathcal{L}^{ii}(z)\mathcal{M}^{ij}(z) - \mathcal{M}^{ii}(z)\mathcal{L}^{ij}(z) + \mathcal{L}^{ij}(z)\mathcal{M}^{jj}(z) - \mathcal{M}^{ij}(z)\mathcal{L}^{jj}(z) +$$

$$+ \sum_{k \neq i,j}\left(\mathcal{L}^{ik}(z)\mathcal{M}^{kj}(z) - \mathcal{M}^{ik}(z)\mathcal{L}^{kj}(z)\right) +$$

$$+ (\mu_0^i - \mu_0^j)\,\mathrm{tr}_2\left(F_{12}^z(q_{ij} + \eta)P_{12}\mathcal{S}_2^{ij}\right). \qquad (41)$$

The problem is to show that (40) and (41) are equivalent to the respective equations of motion (28) and (29). We note that

$$\mathop{\mathrm{Res}}_{z=0}\mathcal{L}(z) = S = -\mathop{\mathrm{Res}}_{z=0}\mathcal{M}(z) \in \mathrm{Mat}(NM, \mathbb{C}), \qquad (42)$$

i.e., the second-order pole in z cancels in the commutator $[\mathcal{L}(z), \mathcal{M}(z)]$.

2.2.1. Off-diagonal part. In the left-hand side of (41), we have

$$\dot{\mathcal{L}}^{ij}(z)_1 = \text{tr}_2(F_{12}^z(q_{ij}+\eta)P_{12}\mathcal{S}_2^{ij}\dot{q}_{ij}) + \text{tr}_2(R_{12}^z(q_{ij}+\eta)P_{12}\dot{\mathcal{S}}_2^{ij}). \tag{43}$$

The subscript 1 in the left-hand side means that the Lax equation is in the first tensor component. We consider expression in the right-hand side of (41):

$$(\mathcal{L}^{ik}(z)\mathcal{M}^{kj}(z) - \mathcal{M}^{ik}(z)\mathcal{L}^{kj}(z))_1 =$$

$$= \text{tr}_{23}\big(-R_{12}^z(q_{ik}+\eta)P_{12}\mathcal{S}_2^{ik}R_{13}^z(q_{kj})P_{13}\mathcal{S}_3^{kj} +$$

$$+ R_{12}^z(q_{ik})P_{12}\mathcal{S}_2^{ik}R_{13}^z(q_{kj}+\eta)P_{13}\mathcal{S}_3^{kj}\big) =$$

$$= \text{tr}_{23}\big((R_{12}^z(q_{ik})R_{23}^z(q_{kj}+\eta) - R_{12}^z(q_{ik}+\eta)R_{23}^z(q_{kj}))P_{12}\mathcal{S}_2^{ik}P_{13}\mathcal{S}_3^{kj}\big) \overset{(34)}{=}$$

$$\overset{(34)}{=} \text{tr}_{23}\big(R_{13}^z(q_{ij}+\eta)(r_{12}(q_{ik}) - r_{12}(q_{ik}+\eta))P_{12}\mathcal{S}_2^{ik}P_{13}\mathcal{S}_3^{kj}\big) +$$

$$+ \text{tr}_{23}\big((r_{23}(q_{kj}+\eta) - r_{23}(q_{kj}))R_{13}^z(q_{ij}+\eta)P_{12}\mathcal{S}_2^{ik}P_{13}\mathcal{S}_3^{kj}\big). \tag{44}$$

We transform the two obtained terms using (15) and (16) and the respective permutation operator properties $P_{12}U_{12} = U_{21}P_{12}$ and $P_{12}U_{23} = U_{13}P_{12}$. We transform the first term in the right-hand side of (44):

$$\text{tr}_{23}\big(R_{13}^z(q_{ij}+\eta)(r_{12}(q_{ik}) - r_{12}(q_{ik}+\eta))P_{12}\mathcal{S}_2^{ik}P_{13}\mathcal{S}_3^{kj}\big) =$$

$$= -\text{tr}_3\big(R_{13}^z(q_{ij}+\eta)P_{13}P_{13}\,\text{tr}_2((R_{12}^{(0),q_{ik}+\eta} - R_{12}^{(0),q_{ik}})\mathcal{S}_2^{ik})P_{13}\mathcal{S}_3^{kj}\big) =$$

$$= -\text{tr}_3\big(R_{13}^z(q_{ij}+\eta)P_{13}\,\text{tr}_2((R_{32}^{(0),q_{ik}+\eta} - R_{32}^{(0),q_{ik}})\mathcal{S}_2^{ik})\mathcal{S}_3^{kj}\big) =$$

$$= -\text{tr}_2\big(R_{12}^z(q_{ij}+\eta)P_{12}\,\text{tr}_3((R_{23}^{(0),q_{ik}+\eta} - R_{23}^{(0),q_{ik}})\mathcal{S}_3^{ik})\mathcal{S}_2^{kj}\big). \tag{45}$$

Using definition (31), we obtain

$$\text{tr}_{23}\big(R_{13}^z(q_{ij}+\eta)(r_{12}(q_{ik}) - r_{12}(q_{ik}+\eta))P_{12}\mathcal{S}_2^{ik}P_{13}\mathcal{S}_3^{kj}\big) = -\text{tr}_2\big(R_{12}^z(q_{ij}+\eta)P_{12}J^{\eta,q_{ik}}(\mathcal{S}^{ik})_2\mathcal{S}_2^{kj}\big). \tag{46}$$

We similarly transform the second term in the right-hand side of (44):

$$\text{tr}_{23}\big((r_{23}(q_{kj}+\eta) - r_{23}(q_{kj}))R_{13}^z(q_{ij}+\eta)P_{12}\mathcal{S}_2^{ik}P_{13}\mathcal{S}_3^{kj}\big) =$$

$$= \text{tr}_{23}\big(R_{13}^z(q_{ij}+\eta)P_{12}\mathcal{S}_2^{ik}P_{13}\mathcal{S}_3^{kj}(r_{23}(q_{kj}+\eta) - r_{23}(q_{kj}))\big) =$$

$$= \text{tr}_{23}\big(R_{13}^z(q_{ij}+\eta)P_{13}P_{23}\mathcal{S}_2^{ik}\mathcal{S}_3^{kj}(R_{23}^{(0),q_{kj}+\eta} - R_{23}^{(0),q_{kj}})P_{23}\big) =$$

$$= \text{tr}_{23}\big(P_{23}R_{13}^z(q_{ij}+\eta)P_{13}P_{23}\mathcal{S}_2^{ik}\mathcal{S}_3^{kj}(R_{23}^{(0),q_{kj}+\eta} - R_{23}^{(0),q_{kj}})\big) =$$

$$= \text{tr}_{23}\big(R_{12}^z(q_{ij}+\eta)P_{12}\mathcal{S}_2^{ik}\mathcal{S}_3^{kj}(R_{23}^{(0),q_{kj}+\eta} - R_{23}^{(0),q_{kj}})\big) =$$

$$= \text{tr}_2\big(R_{12}^z(q_{ij}+\eta)P_{12}\mathcal{S}_2^{ik}\,\text{tr}_3(\mathcal{S}_3^{kj}(R_{23}^{(0),q_{kj}+\eta} - R_{23}^{(0),q_{kj}}))\big) =$$

$$= \text{tr}_2\big(R_{12}^z(q_{ij}+\eta)P_{12}\mathcal{S}_2^{ik}J^{\eta,q_{kj}}(\mathcal{S}^{kj})_2\big). \tag{47}$$

From (46) and (47), for the initial expression (44), we finally obtain

$$\big(\mathcal{L}^{ik}(z)\mathcal{M}^{kj}(z) - \mathcal{M}^{ik}(z)\mathcal{L}^{kj}(z)\big)_1 = \text{tr}_2\big(R_{12}^z(q_{ij}+\eta)P_{12}(\mathcal{S}^{ik}J^{\eta,q_{kj}}(\mathcal{S}^{kj}) - J^{\eta,q_{ik}}(\mathcal{S}^{ik})\mathcal{S}^{kj})_2\big). \tag{48}$$

1297

We next consider the expression from (41)

$$\left(\mathcal{L}^{ii}(z)\mathcal{M}^{ij}(z) - \mathcal{M}^{ii}(z)\mathcal{L}^{ij}(z)\right)_1 =$$

$$= \mathrm{tr}_{23}\left(-R_{12}^z(\eta)P_{12}\mathcal{S}_2^{ii}R_{13}^z(q_{ij})P_{13}\mathcal{S}_3^{ij} + R_{12}^{(0),z}P_{12}\mathcal{S}_2^{ii}R_{13}^z(q_{ij}+\eta)P_{13}\mathcal{S}_3^{ij}\right) =$$

$$= \mathrm{tr}_{23}\left((R_{12}^{(0),z}R_{23}^z(q_{ij}+\eta) - R_{12}^z(\eta)R_{23}^z(q_{ij}))P_{12}\mathcal{S}_2^{ii}P_{13}\mathcal{S}_3^{ij}\right). \tag{49}$$

We apply relation (39),

$$\left(\mathcal{L}^{ii}(z)\mathcal{M}^{ij}(z) - \mathcal{M}^{ii}(z)\mathcal{L}^{ij}(z)\right)_1 = \mathrm{tr}_{23}\left(F_{13}^z(q_{ij}+\eta)P_{12}P_{12}\mathcal{S}_2^{ii}P_{13}\mathcal{S}_3^{ij}\right) +$$

$$+ \mathrm{tr}_{23}\left(R_{13}^z(q_{ij}+\eta)(r_{12}^{(0)} - r_{12}(\eta))P_{12}\mathcal{S}_2^{ii}P_{13}\mathcal{S}_3^{ij}\right) +$$

$$+ \mathrm{tr}_{23}\left((r_{23}(q_{ij}+\eta) - r_{23}(q_{ij}))R_{13}^z(q_{ij}+\eta)P_{12}\mathcal{S}_2^{ii}P_{13}\mathcal{S}_3^{ij}\right). \tag{50}$$

We simplify all three terms in the right-hand side of (50). We transform the first term,

$$\mathrm{tr}_{23}\left(F_{13}^z(q_{ij}+\eta)P_{12}P_{12}\mathcal{S}_2^{ii}P_{13}\mathcal{S}_3^{ij}\right) = \mathrm{tr}_2(\mathcal{S}_2^{ii})\,\mathrm{tr}_3\left(F_{13}^z(q_{ij}+\eta)P_{13}\mathcal{S}_3^{ij}\right) =$$

$$= \mathrm{tr}\,\mathcal{S}^{ii} \cdot \mathrm{tr}_2\left(F_{12}^z(q_{ij}+\eta)P_{12}\mathcal{S}_2^{ij}\right). \tag{51}$$

The third term is already known:

$$\mathrm{tr}_{23}\left((r_{23}(q_{ij}+\eta) - r_{23}(q_{ij}))R_{13}^z(q_{ij}+\eta)P_{12}\mathcal{S}_2^{ii}P_{13}\mathcal{S}_3^{ij}\right) = \mathrm{tr}_2\left(R_{12}^z(q_{ij}+\eta)P_{12}\mathcal{S}_2^{ii}J^{\eta,q_{ij}}(\mathcal{S}^{ij})_2\right). \tag{52}$$

For the second term in the right-hand side of (50), we obtain

$$\mathrm{tr}_{23}\left(R_{13}^z(q_{ij}+\eta)(r_{12}^{(0)} - r_{12}(\eta))P_{12}\mathcal{S}_2^{ii}P_{13}\mathcal{S}_3^{ij}\right) =$$

$$= \mathrm{tr}_{23}\left(R_{13}^z(q_{ij}+\eta)P_{13}(r_{32}^{(0)} - r_{32}(\eta))P_{32}\mathcal{S}_2^{ii}\mathcal{S}_3^{ij}\right) =$$

$$= \mathrm{tr}_2\left(R_{12}^z(q_{ij}+\eta)P_{12}\,\mathrm{tr}_3\left((r_{23}^{(0)} - r_{23}(\eta))P_{23}\mathcal{S}_3^{ii}\right)\mathcal{S}_2^{ij}\right) =$$

$$= -\mathrm{tr}_2\left(R_{12}^z(q_{ij}+\eta)P_{12}J^{\eta}(\mathcal{S}^{ii})_2\mathcal{S}_2^{ij}\right). \tag{53}$$

Expression (50) thus becomes

$$\left(\mathcal{L}^{ii}(z)\mathcal{M}^{ij}(z) - \mathcal{M}^{ii}(z)\mathcal{L}^{ij}(z)\right)_1 = \mathrm{tr}_2\left(R_{12}^z(q_{ij}+\eta)P_{12}\left(\mathcal{S}^{ii}J^{\eta,q_{ij}}(\mathcal{S}^{ij}) - J^{\eta}(\mathcal{S}^{ii})\mathcal{S}^{ij}\right)_2\right) +$$

$$+ \mathrm{tr}\,\mathcal{S}^{ii} \cdot \mathrm{tr}_2\left(F_{12}^z(q_{ij}+\eta)P_{12}\mathcal{S}_2^{ij}\right). \tag{54}$$

We transform one more expression in (41), $\mathcal{L}^{ij}(z)\mathcal{M}^{jj}(z) - \mathcal{M}^{ij}(z)\mathcal{L}^{jj}(z)$, similarly to (50). This yields

$$\left(\mathcal{L}^{ij}(z)\mathcal{M}^{jj}(z) - \mathcal{M}^{ij}(z)\mathcal{L}^{jj}(z)\right)_1 = \mathrm{tr}_2\left(R_{12}^z(q_{ij}+\eta)P_{12}(\mathcal{S}^{ij}J^{\eta}(\mathcal{S}^{jj}) - J^{\eta,q_{ij}}(\mathcal{S}^{ij})\mathcal{S}^{jj})_2\right) -$$

$$- \mathrm{tr}\,\mathcal{S}^{jj} \cdot \mathrm{tr}_2\left(F_{12}^z(q_{ij}+\eta)P_{12}\mathcal{S}_2^{ij}\right). \tag{55}$$

Collecting terms (48), (54), and (55), we obtain the ijth block of the commutator:

$$\left([\mathcal{L}(z),\mathcal{M}(z)]^{ij}\right)_1 = (\mathrm{tr}\,\mathcal{S}^{ii} - \mathrm{tr}\,\mathcal{S}^{jj})\,\mathrm{tr}_2\left(F_{12}^z(q_{ij}+\eta)P_{12}\mathcal{S}_2^{ij}\right) + \mathrm{tr}_2\left(R_{12}^z(q_{ij}+\eta)P_{12}A_2\right),$$

$$A = \mathcal{S}^{ii}J^{\eta,q_{ij}}(\mathcal{S}^{ij}) - J^{\eta}(\mathcal{S}^{ii})\mathcal{S}^{ij} + \mathcal{S}^{ij}J^{\eta}(\mathcal{S}^{jj}) - J^{\eta,q_{ij}}(\mathcal{S}^{ij})\mathcal{S}^{jj} +$$

$$+ \sum_{k:\,k\neq i,j}\left(\mathcal{S}^{ik}J^{\eta,q_{kj}}(\mathcal{S}^{kj}) - J^{\eta,q_{ik}}(\mathcal{S}^{ik})\mathcal{S}^{kj}\right). \tag{56}$$

Also taking the last term (with μ_i^0) in the right-hand side of (41) into account, we obtain the second equation in (20) in the form

$$\dot{\mathcal{S}}^{ij} = \mathcal{S}^{ii} J^{\eta, q_{ij}}(\mathcal{S}^{ij}) - J^{\eta}(\mathcal{S}^{ii})\mathcal{S}^{ij} + \mathcal{S}^{ij} J^{\eta}(\mathcal{S}^{jj}) - J^{\eta, q_{ij}}(\mathcal{S}^{ij})\mathcal{S}^{jj} +$$

$$+ \sum_{k:\, k \neq i,j}^{M} \left(\mathcal{S}^{ik} J^{\eta, q_{kj}}(\mathcal{S}^{kj}) - J^{\eta, q_{ik}}(\mathcal{S}^{ik})\mathcal{S}^{kj} \right). \tag{57}$$

Here, we must clarify the transition from (56) to (57). Strictly speaking, we proved that the Lax equations hold on the equations of motion, but we did not prove the converse. To prove the converse, we must verify that all components of matrix equation (57) are independently contained in (56) taking into account that R_{12} is a linear operator that somehow mixes these components in linear combinations. In other words, we must show that it follows from $\mathrm{tr}_2(R_{12}^z(q_{ij} + \eta) P_{12} C_2) = 0$ that $C = 0$. For this, we consider the Lax equation near $z = 0$. It follows from (13)–(15) that $R_{12}^z(q_{ij} + \eta) P_{12}$ has a simple pole at $z = 0$ with the residue equal to P_{12}. The needed statement then follows from the fact that $\mathrm{tr}_2(P_{12} A_2) = A$.

2.2.2. Diagonal part. We now consider equation (40), whose left-hand side has the form

$$\dot{\mathcal{L}}_{ii}(z)_1 = \mathrm{tr}_2\left(R_{12}^z(\eta) P_{12} \dot{\mathcal{S}}_2^{ii}\right). \tag{58}$$

Using (34), we transform the expression in the summation in the right-hand side of (40):

$$\left(\mathcal{L}^{ik}(z)\mathcal{M}^{ki}(z) - \mathcal{M}^{ik}(z)\mathcal{L}^{ki}(z)\right)_1 =$$

$$= \mathrm{tr}_{23}\left[-R_{12}^z(q_{ik} + \eta)P_{12}\mathcal{S}_2^{ik} R_{13}^z(q_{ki})P_{13}\mathcal{S}_3^{ki} + R_{12}^z(q_{ik} + \eta)P_{12}\mathcal{S}_2^{ik} R_{23}^z(q_{ki} + \eta)P_{13}\mathcal{S}_3^{ki}\right] =$$

$$= \mathrm{tr}_{23}\left[(R_{12}^z(q_{ik})R_{23}(q_{ki} + \eta) - R_{12}^z(q_{ik} + \eta)R_{23}^z(q_{ki}))P_{12}\mathcal{S}_2^{ik} P_{13}\mathcal{S}_3^{ki}\right] \stackrel{(34)}{=}$$

$$\stackrel{(34)}{=} \mathrm{tr}_{23}\left[R_{13}^z(\eta)(r_{12}(q_{ik}) - r_{12}(q_{ik} + \eta))P_{12}\mathcal{S}_2^{ik} P_{13}\mathcal{S}_3^{ki}\right] +$$

$$+ \mathrm{tr}_{23}\left[(r_{23}(q_{ki} + \eta) - r_{23}(q_{ki}))R_{13}^z(\eta)P_{12}\mathcal{S}_2^{ik} P_{13}\mathcal{S}_3^{ki}\right] =$$

$$= \mathrm{tr}_2\left[R_{12}^z(\eta)P_{12}(\mathcal{S}^{ik} J^{\eta, q_{ki}}(\mathcal{S}^{ki}) - J^{\eta, q_{ik}}(\mathcal{S}^{ik})\mathcal{S}^{ki})_2\right]. \tag{59}$$

Using (39), we simplify the rest of the right-hand side of (40):

$$\left(\mathcal{L}^{ii}(z)\mathcal{M}^{ii}(z) - \mathcal{M}^{ii}(z)\mathcal{L}^{ii}(z)\right)_1 =$$

$$= \mathrm{tr}_{23}\left[-R_{12}^z(\eta)P_{12}\mathcal{S}_2^{ii} R_{13}^{(0),z} P_{13}\mathcal{S}_3^{ii} + R_{12}^{(0),z} P_{12}\mathcal{S}_2^{ii} R_{13}^z(\eta)P_{13}\mathcal{S}_3^{ii}\right] =$$

$$= \mathrm{tr}_{23}\left[(R_{12}^{(0),z} R_{23}^z(\eta) - R_{12}^z(\eta)R_{23}^{(0),z})P_{12}\mathcal{S}_2^{ii} P_{13}\mathcal{S}_3^{ii}\right] \stackrel{(39)}{=}$$

$$\stackrel{(39)}{=} \mathrm{tr}_{23}\left[R_{13}^z(\eta)(r_{12}^{(0)} - r_{12}(\eta))P_{12}\mathcal{S}_2^{ii} P_{13}\mathcal{S}_3^{ii}\right] +$$

$$+ \mathrm{tr}_{23}\left[(r_{23}(\eta) - r_{23}^{(0)})R_{13}^z(\eta)P_{12}\mathcal{S}_2^{ii} P_{13}\mathcal{S}_3^{ii}\right] +$$

$$+ \mathrm{tr}_{23}\left[F_{13}^z(\eta)P_{12}P_{12}\mathcal{S}_2^{ii} P_{13}\mathcal{S}_3^{ii}\right] - \mathrm{tr}_{23}\left[P_{23}F_{13}^z(\eta)P_{12}\mathcal{S}_2^{ii} P_{13}\mathcal{S}_3^{ii}\right]. \tag{60}$$

We note that the two last terms are equal and therefore cancel:

$$\mathrm{tr}_{23}\left(F_{13}^z(\eta)P_{12}P_{12}\mathcal{S}_2^{ii} P_{13}\mathcal{S}_3^{ii}\right) = \mathrm{tr}_{23}\left(P_{23}F_{13}^z(\eta)P_{12}\mathcal{S}_2^{ii} P_{13}\mathcal{S}_3^{ii}\right) = \mathrm{tr}\,\mathcal{S}^{ii}\,\mathrm{tr}_2\left(F_{12}^z(\eta)P_{12}\mathcal{S}_2^{ii}\right). \tag{61}$$

The first and the second terms in (60) have the form

$$\mathrm{tr}_{23}\big[R^z_{13}(\eta)\big(r^{(0)}_{12}-r_{12}(\eta)\big)P_{12}\mathcal{S}^{ii}_2 P_{13}\mathcal{S}^{ii}_3\big]+\mathrm{tr}_{23}\big[\big(r_{23}(\eta)-r^{(0)}_{23}\big)R^z_{13}(\eta)P_{12}\mathcal{S}^{ii}_2 P_{13}\mathcal{S}^{ii}_3\big]=$$

$$=\mathrm{tr}_2\big[R^z_{12}(\eta)P_{12}\big(J^\eta(\mathcal{S}^{ii})\mathcal{S}^{ii}-\mathcal{S}^{ii}J^\eta(\mathcal{S}^{ii})\big)_2\big]. \quad (62)$$

From (59) and (62), we finally obtain

$$\big([L(z),M(z)]_{ii}\big)_1=\mathrm{tr}_2(R^z_{12}(\eta)P_{12}B_2), \quad (63)$$

where

$$B=J^\eta(\mathcal{S}^{ii})\mathcal{S}^{ii}-\mathcal{S}^{ii}J^\eta(\mathcal{S}^{ii})+\sum_{k\neq i}\big(\mathcal{S}^{ik}J^{\eta,q_{ki}}(\mathcal{S}^{ki})-J^{\eta,q_{ik}}(\mathcal{S}^{ik})\mathcal{S}^{ki}\big). \quad (64)$$

Here, we should also use the argument given after (57). We have thus verified the equations of motion for the diagonal blocks.

2.3. Interacting tops. As explained in [3], in the particular case $\mathrm{rk}(\mathcal{S})=1$, we can write the equations of motion in terms of only the diagonal blocks. We recall the main idea. The additional property $\mathrm{rk}(\mathcal{S})=1$ yields

$$\mathcal{S}^{ik}_1 P_{12}\mathcal{S}^{ki}_1=\mathcal{S}^{ii}_1\mathcal{S}^{kk}_2. \quad (65)$$

Further, for an arbitrary $J(S)=\mathrm{tr}_2(J_{12}S_2)$ of form (12) and $\breve{J}_{12}=J_{12}P_{12}$, we have

$$J(S)=\mathrm{tr}_2(J_{12}S_2)=\mathrm{tr}_2(\breve{J}_{12}P_{12}S_2)=\mathrm{tr}_2(S_2\breve{J}_{12}P_{12})=\mathrm{tr}_2(S_2 P_{12}\breve{J}_{21})=\mathrm{tr}_2(P_{12}S_1\breve{J}_{21}). \quad (66)$$

Therefore,

$$\mathcal{S}^{ik}J(\mathcal{S}^{ki})=\mathrm{tr}_2(\mathcal{S}^{ik}_1 P_{12}\mathcal{S}^{ki}_1\breve{J}_{21})=\mathcal{S}^{ii}\,\mathrm{tr}_2(\breve{J}_{21}\mathcal{S}^{kk}_2), \quad (67)$$

where $\breve{J}_{21}=P_{12}\breve{J}_{21}P_{12}=P_{12}J_{12}$. Similarly,

$$J(\mathcal{S}^{ik})\mathcal{S}^{ki}=\mathrm{tr}_2(\breve{J}_{12}\mathcal{S}^{ik}_1 P_{12}\mathcal{S}^{ki}_1)=\mathrm{tr}_2(\breve{J}_{12}\mathcal{S}^{kk}_2)\mathcal{S}^{ii}. \quad (68)$$

We finally write Eqs. (28) and (30) in the forms

$$\dot{\mathcal{S}}^{ii}=[\mathcal{S}^{ii},J^\eta(\mathcal{S}^{ii})]+\sum_{k:\,k\neq i}^{M}\big(\mathcal{S}^{ii}\widetilde{J}^{\eta,q_{ki}}(\mathcal{S}^{kk})-\breve{J}^{\eta,q_{ik}}(\mathcal{S}^{kk})\mathcal{S}^{ii}\big),$$

$$\ddot{q}_i=\mathrm{tr}(\dot{\mathcal{S}}^{ii})=\sum_{k:\,k\neq i}^{M}\mathrm{tr}\big(\mathcal{S}^{ii}\widetilde{J}^{\eta,q_{ki}}(\mathcal{S}^{kk})-\breve{J}^{\eta,q_{ik}}(\mathcal{S}^{kk})\mathcal{S}^{ii}\big),$$

$$(69)$$

where

$$\widetilde{J}^{\eta,q_{ki}}(\mathcal{S}^{kk})=\mathrm{tr}_2(\breve{J}^{\eta,q_{ki}}_{21}\mathcal{S}^{kk}_2)=\mathrm{tr}_2(P_{12}J^{\eta,q_{ki}}_{12}\mathcal{S}^{kk}_2),$$

$$\breve{J}^{\eta,q_{ik}}(\mathcal{S}^{kk})=\mathrm{tr}_2(\breve{J}^{\eta,q_{ik}}_{12}\mathcal{S}^{kk}_2)=\mathrm{tr}_2(J^{\eta,q_{ik}}_{12}P_{12}\mathcal{S}^{kk}_2).$$

$$(70)$$

Written in form (69), the equations of motion are interpreted as the dynamical equations for M particles with additional "spin" degrees of freedom, i.e., the particles can be identified with tops that also have coordinates and velocities in addition to their own internal degrees of freedom. The interaction between the tops depends on both the distance and the spin dynamical variables.

Conflicts of interest. The authors declare no conflicts of interest.

REFERENCES

1. A. Grekov, I. Sechin, and A. Zotov, "Generalized model of interacting integrable tops," *JHEP*, **1910**, 081 (2019); arXiv:1905.07820v2 [math-ph] (2019).

2. I. Sechin and A. Zotov, "R-matrix-valued Lax pairs and long-range spin chains," *Phys. Lett. B*, **781**, 1–7 (2018); arXiv:1801.08908v3 [math-ph] (2018); A. Grekov and A. Zotov, "On R-matrix valued Lax pairs for Calogero–Moser models," *J. Phys. A*: *Math. Theor.*, **51**, 315202 (2018); arXiv:1801.00245v2 [math-ph] (2018); I. A. Sechin and A. V. Zotov, "GL_{NM} quantum dynamical R-matrix based on solution of the associative Yang–Baxter equation," *Russian Math. Surveys*, **74**, 767–769 (2019); arXiv:1905.08724v2 [math.QA] (2019).

3. A. V. Zotov, "Relativistic interacting integrable elliptic tops," *Theor. Math. Phys.*, **201**, 1565–1580 (2019); arXiv:1910.08246v1 [math-ph] (2019).

4. A. Weil, *Elliptic Functions According to Eisenstein and Kronecker*, Springer, Berlin (1976); D. Mumford, *Tata Lectures on Theta I, II* (Progr. Math., Vol. 43), Birkhäuser, Boston (1984).

5. S. Fomin and A. N. Kirillov, "Quadratic algebras, Dunkl elements, and Schubert calculus," in: *Advances in Geometry* (Progr. Math., Vol. 172, J.-L. Brylinski, R. Brylinski, V. Nistor, B. Tsygan, and P. Xu, eds.), Birkhäuser, Boston, Mass. (1999), pp. 147–182; A. Polishchuk, "Classical Yang–Baxter equation and the A^{∞}-constraint," *Adv. Math.*, **168**, 56–95 (2002); A. M. Levin, M. A. Olshanetsky, and A. V. Zotov, "Quantum Baxter–Belavin R-matrices and multidimensional Lax pairs for Painlevé VI," *Theor. Math. Phys.*, **184**, 924–939 (2015); arXiv:1501.07351v3 [math-ph] (2015).

6. G. Aminov, S. Arthamonov, A. Smirnov, and A. Zotov, "Rational top and its classical R-matrix," *J. Phys. A*: *Math. Theor.*, **47**, 305207 (2014); arXiv:1402.3189v3 [hep-th] (2014); A. Levin, M. Olshanetsky, and A. Zotov, "Noncommutative extensions of elliptic integrable Euler–Arnold tops and Painlevé VI equation," *J. Phys. A*: *Math. Theor.*, **49**, 395202 (2016); arXiv:1603.06101v2 [math-ph] (2016).

7. A. Levin, M. Olshanetsky, and A. Zotov, "Relativistic classical integrable tops and quantum R-matrices," *JHEP*, **1407**, 012 (2014); arXiv:1405.7523v3 [hep-th] (2014); T. Krasnov and A. Zotov, "Trigonometric integrable tops from solutions of associative Yang–Baxter equation," *Ann. Henri Poincaré*, **20**, 2671–2697 (2019); arXiv:1812.04209v3 [math-ph] (2018).

8. A. V. Zotov, "Calogero–Moser model and R-matrix identities," *Theor. Math. Phys.*, **197**, 1755–1770 (2018); "Higher-order analogues of the unitarity condition for quantum R-matrices," *Theor. Math. Phys.*, **189**, 1554–1562 (2016); A. M. Levin, M. A. Olshanetsky, and A. V. Zotov, "Quantum Baxter–Belavin R-matrices and multidimensional Lax pairs for Painlevé VI," *Theor. Math. Phys.*, **184**, 924–939 (2015); arXiv:1501.07351v3 [math-ph] (2015).

9. E. K. Sklyanin, "Some algebraic structures connected with the Yang–Baxter equation," *Funct. Anal. Appl.*, **16**, 263–270 (1982).

10. S. N. M. Ruijsenaars, "Complete integrability of relativistic Calogero–Moser systems and elliptic function identities," *Commun. Math. Phys.*, **110**, 191–213 (1987).

11. I. M. Krichever and A. V. Zabrodin, "Spin generalization of the Ruijsenaars–Schneider model, the non-Abelian Toda chain, and representations of the Sklyanin algebra," *Russian Math. Surveys*, **50**, 1101–1150 (1995); arXiv:hep-th/9505039v1 (1995).

12. G. E. Arutyunov and S. A. Frolov, "On Hamiltonian structure of the spin Ruijsenaars–Schneider model," *J. Phys. A*: *Math. Gen.*, **31**, 4203–4216 (1998); arXiv:hep-th/9703119v2 (1997); G. E. Arutyunov and E. Olivucci, "Hyperbolic spin Ruijsenaars–Schneider model from Poisson reduction," *Proc. Steklov Inst. Math.*, **309**, 31–45 (2020); arXiv:1906.02619v2 [hep-th] (2019).

13. N. Reshetikhin, "Degenerately integrable systems," *J. Math. Sci.* (N. Y.), **213**, 769–785 (2016); arXiv:1509.00730v1 [math-ph] (2015).

14. L. Fehér, "Poisson–Lie analogues of spin Sutherland models," *Nucl. Phys. B*, **949**, 114807 (2019); arXiv:1809.01529v3 [math-ph] (2018); "Bi-Hamiltonian structure of a dynamical system introduced by Braden and Hone," *Nonlinearity*, **32**, 4377–4394 (2019); arXiv:1901.03558v2 [math-ph] (2019).

15. O. Chalykh and M. Fairon, "On the Hamiltonian formulation of the trigonometric spin Ruijsenaars–Schneider system," arXiv:1811.08727v3 [math-ph] (2018); M. Fairon, "Spin versions of the complex trigonometric Ruijsenaars–Schneider model from cyclic quivers," *J. Integrable Syst.*, **4**, xyz008 (2019); arXiv:1811.08717v2 [math-ph] (2018).

16. A. P. Polychronakos, "Calogero–Moser models with noncommutative spin interactions," *Phys. Rev. Lett.*, **89**, 126403 (2002); arXiv:hep-th/0112141v3 (2001); "The physics and mathematics of Calogero particles," *J. Phys. A*: *Math. Gen.*, **39**, 12793–12945 (2006); arXiv:hep-th/0607033v2 (2006).

17. A. Levin, M. Olshanetsky, A. Smirnov, and A. Zotov, "Characteristic classes of $SL(N)$-bundles and quantum dynamical elliptic R-matrices," *J. Phys. A*: *Math. Theor.*, **46**, 035201 (2013); arXiv:1208.5750v1 [math-ph] (2012); A. V. Zotov and A. M. Levin, "Integrable model of interacting elliptic tops," *Theor. Math. Phys.*, **146**, 45–52 (2006); A. V. Zotov and A. V. Smirnov, "Modifications of bundles, elliptic integrable systems, and related problems," *Theor. Math. Phys.*, **177**, 1281–1338 (2013); A. Levin, M. Olshanetsky, A. Smirnov, and A. Zotov, "Characteristic classes and Hitchin systems: General construction," *Commun. Math. Phys.*, **316**, 1–44 (2012); "Calogero–Moser systems for simple Lie groups and characteristic classes of bundles," *J. Geom. Phys.*, **62**, 1810–1850 (2012).