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# MAGNETIC HELICITY FLUX FOR MEAN MAGNETIC FIELD EQUATIONS

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*The mean magnetic field equation describes the process of generating a magnetic field on a large scale as a result the EMF arising on a small scale. We consider the case where the large-scale magnetic field is also random and determine the density function of magnetic helicity. This function is invariant under gauge transformations of the magnetic vector potential. We study the equation for the magnetic helicity flux of a large-scale field and introduce a correction term related to the quadratic magnetic helicity invariant.*

**Keywords:** magnetic helicity, magnetic energy, current helicity, highest moment of magnetic helicity

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## 1. Introduction

Studying a magnetic field in a liquid conducting medium [1] is relevant in modern mathematical physics. If the density of the magnetic liquid conducting medium is low, then the approach discovered in the framework of classical hydrodynamics in the middle of the last century by Steinbeck, Krause, and Redler, whose principles were described in [2], works. The equation of the mean magnetic field also arises in other problems, for example, when taking the interaction of a magnetic field with elementary particles into account [3].

The concept of the $\alpha$-effect, which underlies the mean field theory, allows obtaining a simple equation for the magnetic helicity flux. We study this equation.

Here, we apply the Kolmogorov theory of turbulence (see, e.g., [4] for the details of this theory) to the mean field equation. In the framework of the theory of turbulence, taking the influence of the magnetic field on the hydrodynamic velocity vector (Lorentz force) into account leads to cascade MHD models [5]. Magnetic helicity plays an important role in the theory of cascade models. The mean magnetic field equations are significantly simpler than the complete system of MHD equations because they are linear in the magnetic field. We assume that the hydrodynamic velocity is known (but complex and multiscale) and the magnetic field is random with the Kolmogorov distribution.

Together with magnetic helicity, which should be regarded as the zeroth moment of the magnetic field distribution, we can determine and consider higher-order helicity moments. The need to study the distribution function of the link indices of magnetic lines was explicitly indicated in [6] (the remark after Example 5.2). The second magnetic helicity moments, called quadratic helicities, were constructed and studied in [7]; they are denoted by $\chi_{\mathbf{B}}^{(2)}$ and $\chi_{\mathbf{B}}^{[2]}$.

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The possibility of using the concept of higher moments of helicity as an obstruction to the increase of magnetic energy was analyzed in [8] by analogy with magnetic helicity. The main conclusion in that work was that the density of the quadratic magnetic helicity is unknown and this invariant hence cannot be used in practical calculations. Moreover, the expression for the density of magnetic helicity, which is typically used in MHD problems, depends on the gauge of the magnetic potential and is not invariant under coordinate transformations. Therefore, even magnetic helicity is difficult to use in the theory of turbulence.

In [6], the magnetic field was regarded as a dynamical system with a magnetic flux (with a dynamical flux defined by magnetic field vectors in which particles of a liquid conducting medium move along magnetic lines). The density of the magnetic helicity was defined invariantly as the result of averaging the function $(\mathbf{A}(x), \mathbf{B}(x))$ in the phase volume $\Omega$ over the magnetic flux. The invariance of the density of magnetic helicity means that this density function does not change under the action of noncontracting diffeomorphisms of the phase space that transform the magnetic field (see [9] for more about this standard definition). Nevertheless, the density of magnetic helicity in [6] was not defined explicitly but was obtained from general considerations of ergodic theory as some measurable function. It was verified in [7] (see formula (3) there) that the density of magnetic helicity is integrable together with its square and it therefore belongs to the same Hilbert space as the density of magnetic energy.

Using an idea in [10], where a local formula for the density $\chi_{\mathbf{B}}^{(2)}$ of magnetic helicity was written, we here write local formula (6) for the gauge-invariant density of magnetic helicity. Many physicists have variously attempted to make the density of magnetic helicity gauge-invariant and hence practically calculable [11]–[13].

Our gauge transformation of the expression $(\mathbf{A}(x), \mathbf{B}(x))$ coincides, for example, with a particular case of the gauge transformation in [12], where $\mathbf{v} = \mathbf{B}$ should be set in formula (7), i.e., the case where the hydrodynamic velocity is proportional to the magnetic field and directed along the magnetic lines should be considered.

If we assume that in the problem posed in [13], the magnetic field is everywhere tangent to the boundary of the region with the magnetic field, then the concept of "helicity of magnetic lines" studied there also coincides with our concept of the density of magnetic helicity. But in this case, we must additionally perform a suitable limit transition where the lengths of the magnetic lines tend to infinity.

A gauge-invariant density of magnetic helicity for random small-scale magnetic fields was proposed in [11]. On one hand, such an assumption does not allow using the visual representation that the density of magnetic helicity is a function frozen in the phase volume. On the other hand, this assumption is more convenient for subsequently developing the theory. For higher moments of magnetic helicity (not only for the quadratic magnetic helicity $\chi_{\mathbf{B}}^{(2)}$), local formulas exist and can possibly be applied practically. For moments of magnetic helicity of a sufficiently high order, the formulas for a gauge-invariant density similar to (6), even if they exist, can be inapplicable in practice.

The main idea, which is justified by the proposed formula, can be stated as follows. Because the density function of magnetic helicity is frozen in a phase volume, it is transported by the hydrodynamic velocity vector, and we assume that the density function is fractalized to an arbitrarily small scale. For simplicity, we assume that this happens instantly. If this happens with some delay, then the parameter $\varepsilon^{-1}$ is introduced in the theory of turbulence, and the transfer rate is observed over the spectrum (depending on the scale) in the stationary mode. We merely outline the construction of the density of magnetic helicity with a finite dissipation rate because it requires a more detailed study of the mathematical side of the theory of moments of magnetic helicity and is beyond the scope of this paper (some results were obtained in [14]).

We assume that the velocity vector distributes the magnetic helicity generated by the $\alpha$-term according to Eqs. (1) and (2) instantly over the entire phase volume. The magnetic helicity flux equation then has a main term that determines the exponential growth of magnetic helicity. For example, it was noted in [15] that the magnetic helicity can vanish while the magnetic energy does not vanish and the magnetic helicity flux is nonzero. This means that the magnetic helicity flux equation cannot be satisfied even approximately if the magnetic helicity is zero at the initial instant.

The main correction to the magnetic helicity flux equation is given by the quadratic magnetic helicity. This correction does not reduce to a correction for the square of magnetic helicity, because the square of the magnetic helicity also vanishes if the magnetic helicity is zero.

This paper has the following structure. In Sec. 2, we recall what the mean field equation is and consider a two-scale approximation. On the small scale, EMF is generated in the form of an $\alpha$-term and a $\beta$-term (we assume that $\eta$-term in the equation vanishes, which does not lessen the mathematical problem). On the large scale, we also assume that the magnetic field is random. Formally speaking, we consider the mean field equation in the two-scale approximation [1]. We give the magnetic helicity flux equation (without details of the proof). A complete proof is simple and can be found, for example, in [16].

In Sec. 3, we give well-known formulas for the eigenvector functions of the vorticity operator for magnetic fields with a three-dimensional continuous spectrum. In Sec. 4, we write local formula (6), which is invariant under gauge transformations, for the density of magnetic helicity. We prove that the corresponding integral over the phase volume converges to the helicity invariant. Based on this local formula, we prove the preliminary result in Sec. 6 and the main result in Sec. 8. To prove the preliminary result, we must study the magnetic helicity density spectrum (we call it a Kolmogorov spectrum; Kolmogorov considered similar spectra for hydrodynamic equations [4]). To prove the main result, we use the simple arguments presented in Sec. 7.

## 2. The mean magnetic field equation

As in [1], [2], we consider a region $\Omega$ in $\mathbb{R}^3$ filled with a conductive fluid. For simplicity, we assume that this region is compact. The vector $\mathbf{u}$ of the hydrodynamic velocity of the liquid medium and the magnetic field vector $\mathbf{B}$ are defined in $\Omega$. We assume that both the velocity vector and the magnetic field vector are represented as the sum of large-scale and small-scale components:

$$\mathbf{B} = \overline{\mathbf{B}} + \mathbf{B}', \qquad \mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}'.$$

In this decomposition, we assume that $\overline{\mathbf{B}}$ is the mean of the random vector $\mathbf{B}$. We additionally assume that the component $\overline{\mathbf{B}}$ is itself a random field. The mean $\bar{\mathbf{u}}$ is not assumed to be random but is assumed to be fractal up to the boundary of the turbulence region.

We assume that the mean velocity field $\bar{\mathbf{u}}(t)$ is given. In this case, the equation for the mean magnetic field has the form

$$\begin{gathered}\operatorname{curl}(\eta \operatorname{curl}\overline{\mathbf{B}}) - \operatorname{curl}(\bar{\mathbf{u}} \times \overline{\mathbf{B}} + \mathbf{E}) + \frac{\partial \overline{\mathbf{B}}}{\partial t} = 0, \\ \mathbf{E} = \overline{\mathbf{B}' \times \mathbf{u}'}, \qquad \operatorname{div}\overline{\mathbf{B}} = 0.\end{gathered} \tag{1}$$

Equation (1) is called the kinematic dynamo equation of the mean magnetic field. The conditions $\eta = 0$ and $\mathbf{E} = 0$ are called the freezing conditions for the magnetic field.

We assume that the equation

$$\mathbf{E} = \alpha\overline{\mathbf{B}} - \beta \operatorname{curl}\overline{\mathbf{B}} \tag{2}$$

is satisfied. Under this assumption, we take the scalar product of both sides of Eq. (2) with the vector $\overline{\mathbf{B}}$, additionally assuming that $\eta = 0$ (molecular diffusion is absent), and integrate the result over the phase volume $\Omega$. Using $\mathbf{E} = \partial\overline{\mathbf{A}}/\partial t$, we obtain an equation describing the transfer of the magnetic helicity $\chi_{\overline{\mathbf{B}}} = \int(\overline{\mathbf{A}}, \overline{\mathbf{B}})\, d\Omega$:

$$\frac{d\chi_{\overline{\mathbf{B}}}}{dt} = 2\alpha \int (\overline{\mathbf{B}}, \overline{\mathbf{B}})\, d\Omega - 2\beta \int (\overline{\mathbf{B}}, \operatorname{curl} \overline{\mathbf{B}})\, d\Omega. \tag{3}$$

The integral $U_{\overline{\mathbf{B}}} = 2\int(\overline{\mathbf{B}}, \overline{\mathbf{B}})\, d\Omega$ is called the magnetic energy (of the mean magnetic field), and the integral $\chi_{\operatorname{curl}\overline{\mathbf{B}}} = 2\int(\overline{\mathbf{B}}, \operatorname{curl}\overline{\mathbf{B}})\, d\Omega$ is called the current helicity (of the mean magnetic field). The turbulent diffusion coefficient $\beta$ always plays a role comparable to or even greater than the $\alpha$-effect because of the presence of the first term in (2). The magnetic helicity flux is due to the fact that these two related characteristics behave differently, which is the focus of our investigation.

## 3. State space of a system of magnetic fields

Let the magnetic field $\mathbf{B}$ be distributed throughout the whole space $\mathbb{R}^3$. We assume that the energy and helicity are normalized to the volume $vol(\Omega)$ of the conventional region $\Omega \subset \mathbb{R}^3$. This allows assuming that the spectrum of the vorticity operator is continuous.

We understand the spectrum of a magnetic field as the expression

$$\mathbf{B}(\vec{\mathbf{x}}) = \int_{\mathbf{k}} \mathbf{B}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k}, \tag{4}$$

where $\mathbf{k}\cdot\mathbf{B}(\mathbf{k}) = 0$ (the condition for the absence of magnetic charges), $\mathbf{B}(-\mathbf{k}) = \mathbf{B}^*(\mathbf{k})$ (real vector condition for $\mathbf{B}$; the asterisk indicates complex conjugation). Right-hand wave vectors with a positive eigenvalue $k = \|\mathbf{k}\|$ and left-hand wave vectors with a negative eigenvalue $-k = -\|\mathbf{k}\|$ appear in expression (4). An identical amplitude of the indicated pair of wave vectors corresponds to the case of a right–left mirror pair of wave vectors.

We understand an observable quantity (or an observable random vector) as a random distribution of vectors in accordance with formula (4). Moreover, we assume that the distribution density is such that the mean strength $\|\mathbf{B}(\mathbf{k})\|$ of a random vector $\mathbf{B}(\mathbf{k})$ with the wavenumber $k$ satisfies the power law $\|\mathbf{B}(\mathbf{k})\| = Ck^{\aleph}$ (we briefly write ${}_{\mathbf{k}}\mathbf{B} \sim k^{\aleph}$), where $C$ is a positive parameter of dimension $\mathrm{G\,cm}^{\aleph}$ and $\aleph$ is a dimensionless parameter. Magnetic fields corresponding to the same wave vector $\pm\mathbf{k}$ differ in the choice of $\mathbf{B}(\mathbf{k})$. The distribution of the measure on such a family is invariant under translations along the direction of the wave vector. Therefore, in calculating the measure of (right-hand) wave vectors with a given wavenumber $k > 0$, we must include not only the factor $4\pi k^2$ (the area of a sphere of radius $k$) but also the additional factor $k^{-1}$, responsible for the rate of rotation in the frontal plane of the vector $\mathbf{B}(\vec{\mathbf{x}})$ with a parallel transport of the magnetic field along the direction of the wave vector.

We understand the turbulence interval as an arbitrary finite closed interval $(\delta_0, \delta_1) \subset \mathbb{R}_+ = (0, +\infty)$ on the half-line of positive wavenumbers. The turbulence interval is

$$\Delta = \delta_1 - \delta_0, \tag{5}$$

and the length of this interval is measured in $\mathrm{cm}^{-1}$. It determines the region of turbulence, i.e., the region of wave vectors with the prescribed eigenvalues. This region is finite and separated from zero. We assume that $0 < \delta_0 \ll 1$ and $1 \ll \delta_1$. The turbulence region comprises right-hand (+) (for $k > 0$) and left-hand (−) (for $k < 0$) wave vectors with prescribed absolute values of their eigenvalues in the turbulence interval. Therefore, the region of turbulence is bounded and does not contain an eigenvalue zero.

## 4. Magnetic helicity density

The function $\chi\colon \Omega \to \mathbb{R}$, $\chi \in L_1(\Omega)$ is called the density of magnetic helicity in the phase region $\Omega$ with the magnetic field defined as the result of averaging the function $(\mathbf{A}(x), \mathbf{B}(x))$ over the magnetic flux (see [6] and Chap. III, Lemma 4.12, in [9]).

According to the ergodic theorem, we have $\chi \in L_1(\Omega)$ [17]. It is easy to prove (see Eq. (11) in [7]) that $\chi \in L_1(\Omega) \cap L_2(\Omega)$ (i.e., the function $\chi$ is not only measurable and absolutely integrable but also integrable together with its square). We write the local formula for $\chi$ as a series. In this formula, all terms (except the term with $i = 0$) have a zero mean along a magnetic line starting at an arbitrary point $x \in \Omega$.

**Definition 1.** We define the density of magnetic helicity by the formula

$$\chi(x) = \lim_{a\to+\infty} \sum_{i=0}^{\infty} \frac{a^i}{i!}[(\nabla_{\mathbf{B}})^{2i}(\mathbf{A}(x), \mathbf{B}(x))]. \tag{6}$$

We let $\nabla_{\mathbf{B}}(\dots)$ denote the derivative of the corresponding function along the vectors $\mathbf{B}$: $\nabla_{\mathbf{B}}(g(x)) = (\mathbf{B}, \operatorname{grad} g(x))$. We should understand the expression in the right-hand side of (6) as a function in $L_2(\Omega)$ that is the limit of trigonometric polynomials in an arbitrary Fourier basis in $\Omega$. This follows because $\chi(x) \in L_1(\Omega)$ in the right-hand side of (6) everywhere (perhaps except a subdomain in $\Omega$ of arbitrarily small measure) uniformly tends to the limit of its means along magnetic lines with the magnetic length $T$ as the magnetic length tends to infinity. Therefore, without loss of accuracy, it suffices to consider the Fourier series in the space of magnetic lines of fixed length $T$; the limit $T \to +\infty$ leads to the exact value.

We verify that formula (6) is invariant under transformations of $\Omega$ by an noncontracting diffeomorphism. We obtain the gauge transformation $\mathbf{A} \mapsto \mathbf{A} + \operatorname{grad} f$,

$$\chi(x) \mapsto \chi(x) + \lim_{a\to+\infty} \sum_{i=0}^{\infty} \frac{a^i}{i!}[(\nabla^2_{\mathbf{B}})^i(\operatorname{grad} f, \mathbf{B}(x))].$$

On each magnetic line $l$ of the field $\mathbf{B}$, we have $(\operatorname{grad} f(x), \mathbf{B}(x)) = g(x)$, $x \in l$, where the mean $\bar{g} = 0$ along $l$. We expand $g(x)$, $x \in l$, in a Fourier series and verify the invariance in the case $g = \sin(kx)$ because the right-hand side of (6) is linear in $f$:

$$\chi(x) \mapsto \chi(x) + \lim_{a\to+\infty} \sum_{i=0}^{\infty} \frac{a^i}{i!}[(\nabla^2_{\mathbf{B}})^i(\operatorname{grad} f(x), \mathbf{B}(x))].$$

We rewrite the gauge term using the coordinate $x$ on the line $l$ and taking into account that $\mathbf{B}(x)$ is a coordinate vector field on the magnetic line $l$ with an invariant coordinate in the magnetic flux. We obtain

$$\lim_{a\to+\infty} \sum_{i=0}^{\infty} \frac{a^i}{i!} \frac{d^{2i}\sin(kx)}{(dx)^{2i}} = \lim_{a\to+\infty} \sum_{i=0}^{\infty} \frac{(-1)^i(k^2a)^i}{i!} \sin(kx) = e^{-\infty}\sin(kx) = 0.$$

In the case of the spectral distribution of the magnetic field in the whole space $\mathbb{R}^3$, we must pass from a Fourier series (for a bounded domain) to a Fourier integral, where the spectral density is supported in the turbulence interval, and the magnetic field itself is the Fourier transform of the density function. It is convenient to introduce the dimensionless parameter $\delta_1/\delta_0$, which is assumed to be large because $\delta_0 \to 0+$ and $\delta_1 \to +\infty$. Because the parameter $a$ in (6) is large and has the dimension $\mathrm{G}^{-2}\,\mathrm{cm}^2$, we define

$$a\|B_0^2\|k^2 = \frac{\delta_1^2}{\delta_0^2}, \tag{7}$$

where $\|B_0^2\|$ is the normalization coefficient of dimension $\mathrm{G}^2$ and is equal to the mean magnetic energy on the sphere of wave vectors of normalized radius and $k$ is the wavenumber of the magnetic field vector for $\nabla_{\mathbf{B}}$ of dimension $\mathrm{cm}^{-1}$. The coefficient $a$ depends on the eigenvalues of the basic magnetic fields in the space of wave vectors.

## 5. Kolmogorov magnetic helicity spectrum

To study the magnetic helicity spectrum, it is convenient to assume that all magnetic harmonics are right-hand with positive wavenumbers, which does not lead to a loss of generality.

We let ${}_{\mathbf{k}}\cdots\sim$ denote the distribution density of a quantity in the space of wave vectors and ${}_{k}\cdots\sim$ denote the distribution density of a quantity in the space of wavenumbers. We assume that ${}_{\mathbf{k}}\mathbf{B}\sim k^{\aleph}$. We recall that $\mathbf{A}$ is the vector potential for $\mathbf{B}$: $\operatorname{curl}\mathbf{A}=\mathbf{B}$. In the case where $\mathbf{B}$ is an eigenvector of curl, we obtain $\mathbf{A}=k^{-1}\mathbf{B}$, where $k$ is the eigenvalue of $\mathbf{B}$. We now obtain ${}_{\mathbf{k}}\mathbf{A}\sim k^{\aleph-1}$, and therefore ${}_{\mathbf{k}}\overline{(\mathbf{A},\mathbf{B})}\sim k^{2\aleph-1}$. Because the distribution $\mathbf{B}$ is assumed to be independent of the wave vectors and the distribution has a zero mean for each wave vector, the mean $\overline{(\mathbf{A},\mathbf{B})}$ contains only harmonics with the same wave vectors. We assume that $vol(\Omega)=1$. Then we have ${}_{k}\int(\mathbf{A},\mathbf{B})\,d\Omega\sim k^{2\aleph-1}$. The magnetic helicity density is uniformly distributed along a straight line $k$ if $\aleph=1/2$.

We can draw the same conclusion starting from the dimension $\mathrm{G}^2\,\mathrm{cm}$ of the magnetic helicity density function $\chi(x)$. Changing the scale by a factor of $\lambda$ ($x\mapsto\lambda x$) and the magnetic field by a factor of $\lambda^{-1/2}$ ($\mathbf{B}\mapsto\lambda^{-1/2}\mathbf{B}$), we find that the magnetic helicity density $\chi$ transforms invariantly. The parameter $a$ in (6) has the dimension $\mathrm{G}^{-2}\,\mathrm{cm}^2$, and all terms of the series hence have the same dimension. Under the given assumption, for the transformations to be invariant, we must have $\aleph=1/2$. The same conclusion holds for the function $(\mathbf{A},\mathbf{B})$ itself, which serves as the leading term in series (6).

To apply the theory of turbulence in our case, it suffices to show that expression (6) with the obtained exponent $\aleph$ is uniformly distributed over the space of wave vectors. Each term of the series is distributed in the space of ordered pairs of wave vectors $\{\mathbf{k}_1\times\mathbf{k}_2\}$, where the first wave vector $\mathbf{k}_1$ determines the distribution ${}_{\mathbf{k}_1}(\mathbf{A},\mathbf{B})$ and the second wave vector $\mathbf{k}_2$ determines the distribution of the vector along which the iterated derivative $\nabla_{\mathbf{B}}$ is calculated.

We pass to the dimensionless wavenumber $k_2'=k_2/4\pi\delta_0$, $1\le k_2'\le\delta_1/4\pi\delta_0$. We prove that at each point $x\in\Omega$ with a fixed $k_2'$, series (6) is distributed proportionally to the leading term with the coefficient $\sqrt{\pi}/2$ independently of $k_2'$. We note that the terms of the series are distributed identically at different points $x\in\Omega$.

We assume that the expression ${}_{k_1}(\mathbf{A},\mathbf{B})$ is distributed with the value $\chi_0$. The $i$th term, $i\ge1$, of series (6) is distributed with the coefficient

$$\chi_0\frac{(ak_1^2\|B_0^2\|)^i(-1)^i}{i!}k_2'\int_{S^2}\cos^{2i}(\theta)\sin(\theta)\,\mathbf{d}\theta=\chi_0\frac{4\pi(-1)^ik_2'(ak_1^2\|B_0^2\|^{-1})^i}{i!(2i+1)},$$

where $\theta$ is the angle between the vectors $\mathbf{B}_2(x)$ and $\operatorname{grad}(\mathbf{A}(x),\mathbf{B}(x))$, which is uniformly distributed over the sphere $S^2$ of directions, and $\sin(\theta)\,\mathbf{d}\theta$ is area element on the sphere of directions. The factor $k_2'$ is required because the angle element of the sphere of directions is taken into account proportionally to $k_2$ and is dimensionless. The parameter $ak_1^2\|B_0^2\|$ is dimensionless. The limit $\|B_0^2\|\delta_0^{-2}a\to+\infty$ as $\delta_0\to0+$ and $\delta_1\to+\infty$ is reached according to formula (7), and we can therefore assume that the Laplace integral with $x=k_1^{-2}\delta_0^{-2}$ satisfies the relation

$$\int_0^{\sqrt{xk_1^2\|B_0^2\|a}}e^{-t^2}\,dt=\frac{\sqrt{\pi}}{2}.$$

Because

$$\sum_{i=0}^{\infty}\frac{(-1)^i\left(\sqrt{xk_1^2\|B_0^2\|a}\right)^{2i}}{i!(2i+1)}=\left(\sqrt{xk_1^2\|B_0^2\|a}\right)^{-1}\int_0^{\sqrt{xk_1^2\|B_0^2\|a}}e^{-t^2}\,dt,$$

we obtain

$$k_2'\sum_{i=0}^{\infty}\frac{(-1)^i\left(\sqrt{xk_1^2\|B_0^2\|a}\right)^{2i}}{i!(2i+1)}=\frac{\sqrt{\pi}}{2\sqrt{xk_1^2k_2^{-2}\|B_0^2\|a\delta_0^2}}=\frac{\sqrt{\pi}\delta_0}{2\delta_1}.$$

In this formula, we substitute $a = \delta_1^2/\|B_0^2\|\delta_0^2 k_2^{-2}$ from (7) for $k = k_2$ and the value $x = k_1^{-2}\delta_0^{-2}$. Because

$$k_2' \sum_{i=0}^{\infty} \frac{(-1)^i \big(\sqrt{x k_1^2 \|B_0^2\| a}\big)^{2i}}{i!(2i+1)} = k_2' \sum_{i=0}^{\infty} \frac{(-1)^i (k_1^2 \|B_0^2\| a)^i}{i!(2i+1)},$$

we obtain

$$\chi_0 k_2' \sum_{i=0}^{\infty} \frac{(-1)^i (k_1^2 \|B_0^2\| a)^i}{i!(2i+1)} = \chi_0 \frac{\sqrt{\pi}\delta_0}{2\delta_1}, \tag{8}$$

where $a \to +\infty$. The terms are defined for each wave number $k_2$, but the sum is independent of this number.

To calculate the total distribution of series (6), we use formula (8) and again pass to the distribution over the turbulence interval $\delta_0 \le k_2 \le \delta_1$. We find that series (6) is distributed in proportion to the leading term (except the distribution of means, which vanishes) with the proportionality coefficient $\sqrt{\pi}/2$.

## 6. Equation for the magnetic field flux of the mean field

We assume that $\eta = 0$ in Eq. (1). For physical reasons, the case $\alpha \ge 0$ is interesting. We assume that $\alpha$ and $\beta$ are strictly positive constants that are independent of both time and the coordinates of a point in $\Omega$.

We assume that the random magnetic field $\mathbf{B}$ is right-polarized. We determine the spectral density $\chi(k)$ of the magnetic helicity invariant $\chi_{\overline{\mathbf{B}}}$. The spectral density is related to the magnetic helicity invariant by the formula (see (6))

$$\chi_{\overline{\mathbf{B}}} = \int_{\delta_0}^{\delta_1} \chi(k)\, dk. \tag{9}$$

We obtain an equation that is the spectral form of Eq. (3) with respect to the wavenumbers $k$ in the turbulence interval:

$$\frac{d\chi(k)(t)}{dt} = 2\alpha\chi(k)(t)k - 2\beta\chi(k)(t)k^2. \tag{10}$$

We assume that magnetic helicity density (6) is evenly distributed over the wave frequencies in the turbulence interval. In the formula (5), we take $\delta_1 = \Delta$ because $\delta_0$ is small and the magnetic spectrum is regular at zero. Under this assumption, we obtain $\chi(k)(t) = \chi(t)$, $\chi_{\overline{\mathbf{B}}}(t) = \Delta\chi(t)$, and

$$\frac{d\chi_{\overline{\mathbf{B}}}(t)}{dt} = \alpha\Delta\chi_{\overline{\mathbf{B}}}(t) - \frac{2\beta\Delta^2}{3}\chi_{\overline{\mathbf{B}}}(t). \tag{11}$$

We see that the increase of the right-hand helicity is substantially suppressed by turbulent diffusion for a sufficiently large turbulence interval $\Delta$.

We pass to the case where the transport of magnetic helicity in the spectrum does not occur instantaneously but is controlled by a large parameter $\varepsilon^{-1}$. Our reasoning until the end of the section is not rigorous and is based only on dimensional considerations. In the limit $\varepsilon \to 0$, we obtain the case already investigated, which is characterized by a uniform spectral distribution of the magnetic helicity density.

In the case $\varepsilon > 0$, the magnetic helicity spectrum must contain a small mode that is linearly distributed in the turbulence interval (the value of the magnetic helicity flux of this mode over the spectrum should be independent of the scale). This is possible if ${}_{\mathbf{k}}\mathbf{B} \sim Ck^{\aleph}$, where $\aleph = 0$. In the table of cubic helicities in [14], there is a third moment, there are two variants of such moments ( $\chi^{(3,1)}$ and $\chi^{(3,2)}$), and we choose one of them and let $\chi_{\mathbf{B}}^{((3))}$ denote it. The density $\chi^{((3))}$ of this moment has the dimension $\mathrm{G}^6$ and does not involve the dimension cm. We can therefore assume that the small term $\alpha U_{\mathbf{B}^{((3))}}$ is present in the right-hand side of

Eq. (12). The found hypothetical term corresponds to the magnetic mode $\mathbf{B}^{((3))}$, which is an eigenmode for the invariant $\chi_{\mathbf{B}}^{((3))}$ and has the corresponding value $U_{\mathbf{B}^{((3))}}$ of magnetic energy. Of course, we now already have another term corresponding to the value of the current helicity of the found mode.

## 7. Quadratic magnetic helicity

The quadratic helicity invariant was defined in [7], and its local formula was given in [10]. This invariant is denoted by $\chi_{\mathbf{B}}^{(2)}$ and has the dimension $\mathrm{G}^4\,\mathrm{cm}^5$. The value of the quadratic helicity invariant is defined as the result of integrating the square $\chi^2(x)$ of the local magnetic helicity density over the region $\Omega$. Therefore, for right-polarized magnetic fields, the distribution of the quadratic helicity density coincides with the distribution of the square of the magnetic helicity density, which is given by formula (6). With $\aleph = 1/2$, we obtain a uniform distribution of the quadratic helicity density with respect to the scale in the region $\Omega$.

The quadratic helicity invariant is used for a random field $\mathbf{B}$ without left–right polarization. For general fields, it should be modified,

$$\chi_0^{(2)}(\mathbf{B}) = \chi_{\mathbf{B}}^{(2)} - \frac{\chi_{\mathbf{B}}^2}{vol(\Omega)}, \tag{12}$$

and we again let $\chi_{\mathbf{B}}^{(2)}$ denote the new invariant $\chi_0^{(2)}$. For right-polarized magnetic fields with the Kolmogorov spectrum at $\aleph = 1/2$, quadratic helicity calculated by formula (12) is zero.

## 8. Main result

For simplicity, as in Sec. 6, we assume that $\eta = 0$ in Eq. (1). By the sense of the problem, we have $\alpha \geq 0$ and $\beta \geq 0$. We consider the case where the magnetic field is not spiral and is given by identical distributions of the left- and right-hand wave vectors: $\overline{\mathbf{B}} = \overline{\mathbf{B}}^{(2)}$.

The density function $\chi^{(2)}(k)$ is nonnegative. The distribution of the quadratic helicity density is given by the squared function for $\aleph = 1/2$ and is independent of $k$. Therefore, in this case, we have

$$\chi_{\overline{\mathbf{B}}^{(2)}}^{(2)} = \int_0^{\delta_1} \chi^{(2)}(k)\,dk = \Delta\chi^{(2)}. \tag{13}$$

The magnetic energy (for dimensional reasons) is related to the helicity density by the formula

$$U_{\overline{\mathbf{B}}}^{(2)} = \sqrt{\chi^{(2)}}. \tag{14}$$

The formula for the helicity flux becomes

$$\frac{d\chi_{\overline{\mathbf{B}}}(t)}{dt} = \alpha\sqrt{\chi_{\overline{\mathbf{B}}^{(2)}}^{(2)}\Delta}. \tag{15}$$

Interestingly, the term that suppresses the growth of the emerging right-hand helicity is absent. This follows because the current helicity vanishes as a result of our assumption that the right- and left-hand magnetic harmonics are distributed identically.

We assume that a random magnetic field $\mathbf{B}$ is represented by orthogonal components of spiral and nonspiral fields: $\overline{\mathbf{B}} = \overline{\mathbf{B}}^{(1)} + \overline{\mathbf{B}}^{(2)}$. We combine Eqs. (11) and (15) into one equation. Under the given assumptions, it is obvious that $U_{\overline{\mathbf{B}}} = U_{\overline{\mathbf{B}}^{(1)}} + U_{\overline{\mathbf{B}}^{(2)}}$ and $\chi_{\overline{\mathbf{B}}^{(2)}} = 0$. In Sec. 7, we proved that $\chi_{\overline{\mathbf{B}}^{(1)}}^{(2)} = 0$, where the quadratic helicity is calculated by formula (12).

According to the preceding calculations, ${}_k\overline{\mathbf{B}}^{(1)}, {}_k\overline{\mathbf{B}}^{(2)} \sim k^{1/2}$. we assume that the magnetic helicity $\chi_{\overline{\mathbf{B}}}(t) = \chi_{\overline{\mathbf{B}}^{(1)}}(t)$ is known at the initial instant $t = t_0$ and the quadratic magnetic helicity $\chi^{(2)}_{\overline{\mathbf{B}}^{(2)}}$ does not change with time. Under the assumption that $\varepsilon = 0$, Eq. (11) contains a new additional term:

$$\frac{d\chi_{\overline{\mathbf{B}}}(t)}{dt} = \alpha\Delta\chi_{\overline{\mathbf{B}}}(t) - \frac{2\beta\Delta^2}{3}\chi_{\overline{\mathbf{B}}}(t) + \alpha\sqrt{\chi^{(2)}_{\overline{\mathbf{B}}^{(2)}}\Delta^{-1}}. \tag{16}$$

As can be seen, the new correction term containing the quadratic helicity is proportional to $\sqrt{\Delta^{-1}}$. Assuming that the turbulence interval $\Delta$ is sufficiently large, the correction term is substantially smaller than the leading terms containing the magnetic helicity, which are proportional to $\Delta$ and $\Delta^2$.

## 9. Conclusions

Equation (3) for the helicity flux of the mean field can be studied using methods of the theory of turbulence under the assumption that the mean field itself is determined by a probability distribution and the fast hydrodynamic velocity field fractalizes the magnetic helicity density. In the framework of such a study, it is assumed that not only the magnetic helicity but also the quadratic magnetic helicity regulate the increase in the magnetic energy of the mean field. It turns out that if the right-hand magnetic helicity is only just emerging, then turbulent diffusion does not prevent this at the initial stage. It can be assumed that all moments of the magnetic helicity regulate the increase of the magnetic energy of the mean field in a hierarchical structure. From the mathematical standpoint, this leads to a special section of the analytic theory of differential equations, which was called power-law geometry in [18]. We hope that the theoretical study of the moments of magnetic helicity, whose first steps were taken in [14], can be motivated by experiment.

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