

# THE SELF-CONSISTENT FIELD METHOD AND MACROSCOPIC EINSTEIN EQUATIONS FOR THE EARLY UNIVERSE

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*Using a self-consistent field, we construct a complete theory of the macroscopic description of cosmological evolution, including a subsystem of linear equations for the evolution of perturbations and nonlinear macroscopic Einstein equations and a scalar field. We present example solutions of this system showing the principal difference between cosmological models of the early universe constructed on homogenous and locally fluctuating scalar fields.*

**Keywords:** macroscopic gravitation, self-consistent field, cosmological model, scalar field, averaging local fluctuations, asymptotic behavior, cosmological singularity

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## 1. Introduction

In [1]–[3], the main statements of the macroscopic theory of gravitation were formulated based on a self-consistent field method, and the macroscopic Einstein equations were obtained and integrated for the cosmological model with a  $\Lambda$  term in the second order of the perturbation theory in transverse gravitational perturbations. A practically empty universe filled with gravitational radiation was considered in [1], [2]. Scalar gravitational perturbations, for which the existence of matter is required, cannot exist in such a universe. More precisely, scalar gravitational perturbations can appear only in the second order of a perturbation theory as perturbations of the average energy density of gravitational radiation. A cosmological model with a scalar field was considered in [3], but only transverse perturbations of the metric corresponding to gravitational waves were taken into account. Such perturbations do not lead to perturbations of the scalar field in a linear approximation, and such a model is therefore almost identical to the model of an empty universe filled with gravitational radiation. Here, we consider a complete macroscopic model of the universe taking both longitudinal (scalar) perturbations and vector and tensor perturbations of the metric into account. We regard a classical Higgs scalar field as matter.

## 2. Self-consistent statistical approach for describing local metric fluctuations

**2.1. Self-consistent field method.** The statistical theory for obtaining the *macroscopic Einstein equations* can be developed similarly to many-body theory in the framework of the *self-consistent field*

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approach, which originated in celestial mechanics and was then applied in many-particle theory (see the well-known classical papers Weiss 1907, Langmuir 1913, Thomas 1927, Fermi 1928, Hartree 1928, and Fock 1930). Vlasov had a special role in developing the self-consistent field method; in his seminal paper [4], the physical properties of charged plasma particles were first analyzed in depth, the inapplicability of the Boltzmann gas-kinetic equation for describing plasma was shown, and a new kinetic plasma equation (the Vlasov equation) describing the collective interaction of plasma particles via a self-consistent field was proposed (also see [5], [6]). Vlasov's theory was subsequently refined by Landau [7], then rigorously proved and generalized by Bogoliubov [8], and brilliantly applied to quantum statistics and superfluidity theory [9]. We note Chandrasekhar's classic monograph [10], where the principles of stellar dynamics were formulated based on the self-consistent field method and the theory of galactic structures was essentially constructed.

According to the self-consistent field method, the motion of an individual particle can be described as the motion in an aggregate averaged field of particles of the rest of the system neglecting the back reaction of this single particle on the system dynamics. The applicability conditions for the self-consistent field method are that particle interactions are long range and the number of interacting particles is large,  $N \gg 1$ . We can similarly consider the method applied to a nonlinear system of pure fields. In this case, an individual small field mode characterized by certain field degrees of freedom plays the role of a single particle, and the applicability conditions for the self-consistent field method are that the field is long range and the number of its degrees of freedom is large. The gravitational interaction corresponds perfectly to these conditions: the law of conservation of total energy–mass and the absence of negative “gravitational charges” ensure that it is long range while a large number of degrees of freedom is fundamental to the field nature of the interaction itself. Therefore, we can consider systems with a gravitational interaction using the self-consistent field method, where each microscopic mode of gravitational perturbation is small and the macroscopic self-consistent gravitational field is large.

We note that a more complete macroscopic model of the early universe would be a model based on quantum field statistics. Quantum gravitational effects in the early universe are essential for wavelengths comparable to the curvature radius  $r_k = \sigma^{-1/4}$  of the universe:

$$\sigma = \sqrt{R_{ijkl}R^{ijkl}} = 6H^4(1 + w^2),$$

where  $H$  is the Hubble constant and  $w$  is the invariant cosmological acceleration. Hence, quantum effects are important for field modes with the wave vector

$$k^4 \sim 6H^4(1 + w^2).$$

For the inflationary stage of the Universe  $H = \Lambda$ ,  $w = 1$ , where  $\Lambda$  is the cosmological constant, this condition becomes  $k \sim \Lambda$ . Because  $k \sim n/a(t) = ne^{-\Lambda t}$ , the last condition becomes  $n \sim \Lambda e^{\Lambda t}$ . As a result, massless particles with a fixed mode  $n(t)$  are created at each instant, which ultimately leads to the known flat spectrum of generated gravitons [11] (also see [12]).

The back reaction of the creation of massless particles on the background geometry of the early universe was discussed in [13]–[16]. We note that these works were based on a semiphenomenological approach in which particle creation is described by a thermal spectrum with a temperature determined by the curvature. In these works, the stability of the classical background metric with respect to the back reaction factor was considered, and different authors reached directly contradictory conclusions. In particular, it was noted in [15] that conclusions about the stability of the background metric were based on a scheme without an averaging operation and are therefore incorrect. We also note that back reaction of the perturbed metric on the particle creation process was not taken into account in these papers; hence, the models studied in them are not self-consistent and take only a one-sided relation into account.

Therefore, we can assert that the problem of a quantum statistical description of the macroscopic early universe in quantum cosmology<sup>1</sup> has not been investigated sufficiently seriously. Here, we develop a classical theory of the macroscopic universe. We note that there is a “semiclassical” bridge relating the classical statistical description of the early universe to quantum theory. Perturbations of the metric are random and are determined by arbitrary initial conditions in the classical case; these perturbations are deterministic, including a sharply defined energy spectrum, and are determined by the instability of the vacuum state in the quantum case. The energy spectrum should complement the classical theory, eliminating its arbitrariness and thus playing a role of the abovementioned relation between the quantum and classical theories.

**2.2. Procedure for averaging local metric fluctuations.** We assume that the exact microscopic metric of a Riemannian space–time  $V_4$  can be written in the form [17], [18]

$$g_{ik}(x) = \overline{g_{ik}(x)} + \delta g_{ik}(x), \quad (1)$$

where  $\overline{g_{ik}(x)}$  is a certain average macroscopic metric corresponding to the macroscopic space–time  $\overline{V}_4$  and  $\delta g_{ik}(x)$  are microscopic fluctuations of the metric, and hence

$$\overline{\delta g_{ik} \delta g^{jk}} \ll 1 \quad (2)$$

and

$$\overline{\delta g_{ik}} = 0. \quad (3)$$

Here and hereafter, an overline denotes a certain operation of averaging the metric and the corresponding quantities, which we do not specify yet. We merely note that this operation is a quite delicate procedure and depends significantly on the measurement method (see [17], [18] for details). We also note that “pioneer” methods for statistically averaging metrics that are used by some researchers and essentially represent a transfer of classical averaging methods by integrating the metric quantities over a spatial volume are obviously unsuitable in relativistic gravitation. First, as is known from Riemannian geometry, integration of a tensor over a volume is an ambiguous operation, and the result is not a tensor quantity. Second, because synchronization of observations is only possible in a synchronized reference frame, the physical meaning of the result of integrating the metric tensor over a volume cannot be determined. Third, a macroscopic instrument measuring the metric of the universe simply cannot be realized. Therefore, following the methods developed in [17], [18], we average the metric over certain arbitrary quantities, for instance, wave vectors, oscillation phases, etc. For example, in the cited papers, the metric was generated by massive particles, and the averaging was performed over the coordinates of these particles, which are not arguments of tensor fields. Generally speaking, correct averaging of metrics and also other tensor fields can be performed as follows. A general theorem, well known in functional analysis, states that a function in a metric space can be expanded over a *complete* orthogonal set of eigenfunctions of a linear self-adjoint operator in this space, such as the d’Alembertian. Therefore, we can define this set by eigenfunctions of the d’Alembertian, which is a linear self-adjoint operator in the Riemannian space  $V_4$  corresponding to the eigenvalues  $\lambda$  as

$$\overline{\Delta} \psi_\lambda = \lambda \psi_\lambda, \quad (4)$$

where

$$\overline{\Delta} = \overline{\nabla}_i \overline{\nabla}^i = \frac{1}{\sqrt{\overline{g}}} \frac{\partial}{\partial x^i} \sqrt{\overline{g}} \overline{g}^{ik} \frac{\partial}{\partial x^k}$$

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<sup>1</sup>We note that such a description should be based on a density matrix but the corresponding mathematical tools have not been developed for quantum gravitation.

is a d'Alembertian defined with the metric  $\bar{g}_{ik}$ . We note that the eigenfunctions  $\psi_\lambda$  are defined by a certain *state vector*  $\vec{\lambda} = [\ell_1, \ell_2, \dots, \ell_n]$ , over which the statistical averaging should be performed. We note that such an averaging procedure is totally identical to averaging in quantum field theory. Here, we should impose certain requirements on the *distribution function*  $f(x^i, \vec{\lambda})$  reflecting the macroscopic symmetry properties of the tensor fields being averaged.

It must be understood that together with metric fluctuations, there are also associated small fluctuations  $\delta\phi_a$  of physical fields,<sup>2</sup>

$$\phi_a = \bar{\phi}_a + \delta\phi_a, \quad (5)$$

and hence

$$\overline{\delta\phi_a \delta\phi_a} \ll \overline{\phi_a^2} \quad (6)$$

and

$$\overline{\delta\phi_a} = 0. \quad (7)$$

Averaging the Einstein equations with a similar procedure, we obtain

$$\overline{G_k^i(g)} = 8\pi \overline{T_k^i(g, \phi_a)} + \Lambda \delta_k^i. \quad (8)$$

Because the Einstein and energy–momentum tensors are nonlinear, we have

$$\overline{G_k^i(g)} \neq G_k^i(\bar{g}), \quad \overline{T_k^i(g, \phi_a)} \neq T_k^i(\bar{g}, \bar{\phi}_a), \quad (9)$$

i.e., the a macroscopic source does not correspond to a macroscopic metric, as happens in electrodynamics.

How do we obtain equations for the macroscopic metric in this case? The general principles for averaging microscopic metrics in a WKB approximation and obtaining macroscopic Einstein equations were presented in Isaacson's papers [19], [20], and we later used them to construct a theory of relativistic statistical systems with gravitational interaction in [17], [21], [22] (also see [18], [23]). In [24]–[26], this theory was used to derive the kinetic equation for photons propagating in a gravitationally fluctuating Friedmann universe where massive particles generate local fluctuations. In particular, it was shown in [25], [26] and conclusively in [27] that taking such spherically symmetric local fluctuations of the gravitational field into account is equivalent to adding a fluid with the inflationary state equation ( $\varepsilon + p = 0$ ) to the Friedmann dust or adding the  $\Lambda$  term to the Einstein equations. Also close to this problem, we mention Zakharov's work [28] devoted to deriving the kinetic equations for massive particles with pairwise gravitational interactions in the Friedmann universe but without the back reaction of massive particles on the background metric. Exactly this influence was taken into account in [24]–[26].

We average the Einstein equations and the corresponding equations of physical fields. We pay attention to the following circumstance. To start the averaging, we must have a certain “primeval” macroscopic metric  $g_{ik}^{(0)}(x)$  and certain primeval physical fields  $\phi_a^{(0)}$  to serve as a zeroth-order approximation in the process of consecutive averaging iterations. These primeval fields satisfy a self-consistent set of field equations:

$$\begin{aligned} G_k^i(g_{jm}^{(0)}) &= 8\pi T_k^i(g_{jm}^{(0)}, \phi_a^{(0)}) + \Lambda \delta_k^i, \\ \nabla_k^{(0)} T_i^k(g^{(0)}, \phi_a^{(0)}) &= 0. \end{aligned} \quad (10)$$

In accordance with the usual approach for averaging gravitational fields, we consider a certain macroscopic Riemannian space and now let  $g_{ik}^{(0)}(x) \equiv \overline{g_{ik}(x)}$  and  $\phi_a^{(0)}(x) \equiv \overline{\phi_a(x)}$  be certain not yet known macroscopic averages of the field values.

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<sup>2</sup>In the considered case, this is a single field  $\phi_a = \Phi(x)$ , but the statement holds for any number of physical fields of any nature.

**Assumption 1.** In what follows, we assume that the operations of differentiation or integration with respect to the coordinates commute with the averaging operation:

$$\begin{aligned}\overline{\partial_i \phi(x)} &= \partial_i \overline{\phi(x)}, \\ \overline{\int \phi(x) dx} &= \int \overline{\phi(x)} dx.\end{aligned}\tag{11}$$

We note that the averaging operation defined above completely satisfies Assumption 1. Let

$$\delta g_{ik} = g_{ik} - g_{ik}^{(0)}, \quad \delta \phi_a = \phi_a - \phi_a^{(0)}\tag{12}$$

be small local deviations of the metric and physical fields from the average values. Hence,

$$\begin{aligned}\overline{\delta g_{ik}} &= 0, & \overline{\partial_j \delta g_{ik}} &= 0, \\ \overline{\delta \phi} &= 0, & \overline{\partial_j \delta \phi} &= 0.\end{aligned}\tag{13}$$

We expand the field equations in Taylor series in the deviations of the metric and physical fields from their average values up to the second order in perturbations:

$$G^{(0) i}_k + G^{(1) i}_k + G^{(2) i}_k = 8\pi(T^{(0) i}_k + T^{(1) i}_k + T^{(2) i}_k) + \Lambda \delta_k^i,\tag{14}$$

and we average these equations with (3) and (7) and also the linearity of the operators  $G_{ik}^{(1)}$  and  $T_{ik}^{(1)}$  with respect to fluctuations taken into account,

$$\begin{aligned}\overline{G^{(1) i}_k(\delta g)} &= G^{(1) i}_k(\overline{\delta g}) = 0, \\ \overline{T^{(1) i}_k(\delta g, \delta \phi)} &= T^{(1) i}_k(\overline{\delta g}, \overline{\delta \phi}) = 0.\end{aligned}\tag{15}$$

In the first order of the perturbation theory, we thus obtain microscopic linear equations of the first order in the perturbations of the metric and physical fields,

$$\begin{aligned}G^{(1) i}_k(\delta g) &= 8\pi T^{(1) i}_k(\delta g, \delta \phi) + \Lambda \delta_k^i, \\ \nabla_k T^{(1) i}_k(\delta g, \delta \phi) &= 0,\end{aligned}\tag{16}$$

which we call the *microscopic evolution equations for perturbations*. Setting (we note that substitution (17) is a formal metric renormalization)

$$g_{ik}^{(0)} = \overline{g_{ik}},\tag{17}$$

in the second order of the perturbation theory after averaging, we obtain the macroscopic Einstein equations for the macroscopic metric,

$$G^{(0) i}_k(\overline{g}) = -\overline{G^{(2) i}_k(\delta g)} + 8\pi T^{(0) i}_k(\overline{g}, \overline{\phi}) + 8\pi \overline{T^{(2) i}_k(\delta g, \delta \phi)} + \Lambda \delta_k^i,\tag{18}$$

according to which the macroscopic metric in the second order of the perturbation theory is determined by the Einstein equations with a cosmological constant and the summed effective energy–momentum tensor,

$$\widetilde{T}_k^i = T^{(0) i}_k(\overline{g}, \overline{\phi}) + \mathcal{T}^{(2) i}_k,\tag{19}$$

where

$$\mathcal{T}^{(2) i}_k \equiv \overline{T^{(2) i}_k} - \frac{1}{8\pi} \overline{G^{(2) i}_k}.\tag{20}$$

Hence, the macroscopic Einstein equations of the second order in perturbations take the standard form

$$G_k^i(\overline{g}) = 8\pi \widetilde{T}_k^i + \Lambda \delta_k^i.\tag{21}$$

To close the macroscopic Einstein equations, we must calculate the macroscopic averages that are quadratic in the local fluctuations of the metric and physical fields. These averages are determined by evolution equations (16).

**2.3. Macroscopic symmetries.** The Lie derivative of  $\Omega_k^i$  in the direction of a vector field  $\xi^i$  is defined as [29]

$$\mathbb{L}_\xi U_k^i = \lim_{dt \rightarrow 0} \frac{\Omega_k^i(x^i + \xi^i dt) - \Omega_k^i(x^i)}{dt}, \quad (22)$$

and hence

$$\mathbb{L}_\xi U_k^i = \xi^j \partial_j U_k^i - U_k^j \partial_j \xi^i + U_j^i \partial_k \xi^j. \quad (23)$$

In particular, if  $U_k^i$  is a tensor and we can replace the partial derivatives in (23) with covariant derivatives, then

$$\mathbb{L}_\xi U_k^i = \xi^j \nabla_j U_k^i - U_k^j \nabla_j \xi^i + U_j^i \nabla_k \xi^j, \quad (24)$$

which is a tensor of the same valency as the initial tensor.

We regard a pseudo-Riemannian space  $V_4$  as the primeval space admitting a certain group  $G^r$  of motions that have Killing vectors  $\xi$  and determine macroscopic symmetries:

$$\mathbb{L}_\alpha g_{ik}^{(0)} = 0, \quad \mathbb{L}_\alpha \phi_a^{(0)} = 0, \quad (\mathbb{L}_\alpha \equiv \mathbb{L}_{\xi_\alpha}). \quad (25)$$

In what follows, we assume the following.

**Assumption 2.** *The macroscopic average of the Einstein tensor inherits the symmetry properties of the macroscopic metric:*

$$\mathbb{L}_\xi \overline{g_{ik}} = \sigma \overline{g_{ik}} \quad \Rightarrow \quad \mathbb{L}_\xi \overline{G_{ik}} = \sigma_1 \overline{g_{ik}}, \quad (26)$$

where  $\mathbb{L}_\xi$  is the Lie derivative (see, e.g., [29]) and  $\sigma(x)$  and  $\sigma_1(x)$  are certain scalar functions.

We note that we obtain the group of motions of  $\overline{V}_4$  at  $\sigma = 0$  and  $\sigma_1 = 0$  and a group of conformal transformations at nonzero values of these scalars. We explain Assumption 2. As is known from Riemannian geometry (see [18]), all geometric objects inherit the symmetry properties of the metric tensor. For example,

$$\mathbb{L}_\xi g_{ik} = 0 \quad \Rightarrow \quad \mathbb{L}_\xi \Gamma_{ik}^j = 0, \quad \mathbb{L}_\xi R_{ijkl} = 0, \quad \mathbb{L}_\xi T_{ik} = 0. \quad (27)$$

It hence follows, for example, that all symmetric covariant tensors of valency two have the same algebraic structure as the metric tensor. It is therefore logical to assume that the algebraic structure of macroscopic tensors is also the same. We note that because the equalities

$$\mathbb{L}_\alpha G^{(0)i}_k = 0, \quad \mathbb{L}_\alpha \delta_k^i = 0, \quad \mathbb{L}_\alpha T^{(0)i}_k(\overline{g}, \overline{\phi}) = 0 \quad (28)$$

are satisfied, the total energy–momentum tensor of the second order in the perturbations as a result of Assumption 2 must also inherit the symmetries of the macroscopic metric under averaging:

$$\mathbb{L}_\alpha \overline{\mathcal{T}^{(2)i}_k} = 0 \quad \Rightarrow \quad \mathbb{L}_\alpha \widetilde{T}_k^i = 0. \quad (29)$$

We note that Assumption 2 imposes certain necessary conditions on the symmetry of a scalar distribution function  $f(x^i, \vec{\lambda})$  of random tensor fields, in particular,

$$\mathbb{L}_\alpha f(x^i, \vec{\lambda}) = 0. \quad (30)$$

**2.4. Macroscopic second-order Einstein equations.** We finally write a set of equations defining the macroscopic metric in the second order of the perturbation theory. These equations consist of a system of linear equations for local perturbations of the metric and physical fields

$$G_k^{(1)i}(\delta g) = 8\pi T_k^{(1)i}(\delta g, \delta\phi), \quad \nabla_k T_k^{(1)i}(\delta g, \delta\phi) = 0 \quad (31)$$

and the macroscopic Einstein equations

$$G_k^i(\bar{g}) - \Lambda\delta_k^i = -\overline{G_k^{(2)i}(\delta g)} + 8\pi(T_k^{(0)i}(\bar{g}, \bar{\phi}) + \overline{T_k^{(2)i}(\delta g, \delta\phi)}), \quad (32)$$

where the values  $G_k^i(\bar{g})$  are calculated with respect to the macroscopic metric  $\bar{g}_{ik}$ .

We make an important remark. Obtaining the macroscopic Einstein equations, we renormalized the metric  $g_{ik}^{(0)} \rightarrow \bar{g}_{ik}$ , which has some consequences. First, according to the self-consistent field method, the macroscopic averages  $\overline{G_{ik}^{(2)}(\delta g)}$  are not necessarily small compared with  $G_{ik}(\bar{g})$ , because they are not a result of summing an infinite number of degrees of freedom of gravitational perturbations. Second, the indicated metric renormalization leads to the change  $G_{ik}(g^{(0)}) \rightarrow G_{ik}(\bar{g})$ , which in turn means that macroscopic Einstein equations (32) are immediately solved for the macroscopic metric and not by the method of consecutive iterations:

$$\overline{g_{ik}} = g_{ik}^{(0)} + \overline{\delta g_{ik}} + \dots$$

This is one of the main advantages of the self-consistent field method of Hartree–Fock–Vlasov–Bogoliubov. The method of consecutive iterations would have pulled us in a completely different direction. To understand that, it suffices to imagine the Einstein equations for the Friedmann universe, where the energy–momentum tensor would have been defined by a gas of an infinite number of “small photons.” Following the method of consecutive approximations, we would take the Minkowski tensor as the zeroth-order approximation. It is clear that we would never obtain a cosmological singularity with that approach, given the smallness of gravitational perturbations brought by “small photons” compared with units of the Minkowski metric. This example clearly illustrates that in the theory of relativistic gravity, all researchers unconsciously use the self-consistent field method without attempting to justify it.

### 3. Basic relations of the model

**3.1. Field equations.** To realize the averaging method described above, we take the self-consistent system of Einstein equations of a classical scalar field  $\Phi$  with a Higgs potential as a concrete field model. The Lagrangian function

$$L_s = \frac{1}{8\pi} \left( \frac{1}{2} g^{ik} \Phi_{,i} \Phi_{,k} - V(\Phi) \right) \quad (33)$$

corresponds to the scalar field  $\Phi$ , where  $V(\Phi)$  is the potential energy of the scalar field and for the Higgs potential becomes

$$V(\Phi) = -\frac{\alpha}{4} \left( \Phi^2 - \frac{m^2}{\alpha} \right)^2, \quad (34)$$

where  $\alpha$  is a constant of self-action and  $m$  is the scalar boson mass.

Using the standard procedure, we obtain the equation for the scalar field from Lagrangian function (33):

$$\square\Phi + V'_\Phi = 0. \quad (35)$$

Then

$$T_k^i = \frac{1}{8\pi} \left( \Phi^{,i} \Phi_{,k} - \frac{1}{2} \delta_k^i \Phi_{,j} \Phi^{,j} + \delta_k^i m^2 V(\Phi) \right) \quad (36)$$

is the energy–momentum tensor of the scalar field. We omit the constant term from Higgs potential (34) because it leads to a simple redefinition of the cosmological constant  $\Lambda$ . We use the system of units with  $G = \hbar = c = 1$ , the metric signature is  $(-, -, -, +)$ , and the Ricci tensor is obtained by contracting the first and third indices.

The corresponding Einstein equations of the studied system have the form

$$G_k^i \equiv R_k^i - \frac{1}{2}R\delta_k^i - \Lambda\delta_k^i = 8\pi T_k^i. \quad (37)$$

For simplicity in what follows, we call Eqs. (35) and (37) the *field equations*.

**3.2. Gravitational perturbations of the isotropic universe.** We write the metric with gravitational perturbations in the form (see, e.g., [30])

$$\begin{aligned} ds_0^2 &= a^2(\eta)(d\eta^2 - dx^2 - dy^2 - dz^2), \\ ds^2 &= ds_0^2 - a^2(\eta)h_{\alpha\beta} dx^\alpha dx^\beta. \end{aligned} \quad (38)$$

We focus on the conformal factor  $-a^2(\eta)$  of the covariant amplitudes of perturbations, which disappears for mixed perturbation components  $h_\beta^\alpha$ . In this case, the covariant metric perturbations have the form

$$\delta g_{\alpha\beta} = -a^2(\eta)h_{\alpha\beta}. \quad (39)$$

We then have

$$h_\beta^\alpha = h_{\gamma\beta}g_0^{\alpha\gamma} \equiv -\frac{1}{a^2}h_{\alpha\beta}, \quad (40)$$

$$h \equiv h_\alpha^\alpha \equiv g_0^{\alpha\beta}h_{\alpha\beta} = -\frac{1}{a^2}(h_{11} + h_{22} + h_{33}). \quad (41)$$

We consider a Laplacian  $\Delta$  on the three-dimensional conformally Euclidian space  $E_3$  with the metric

$$d\ell^2 = a^2(\eta)(dx^2 + dy^2 + dz^2) \quad (42)$$

and find the eigenfunctions of the self-adjoint operator

$$\Delta\psi \equiv \frac{1}{a^2}\Delta_0\psi = k^2\psi, \quad (43)$$

where  $\Delta_0$  is a Laplacian defined on the three-dimensional Euclidian space and  $k^2(\eta)$  is a certain scalar function. Therefore, the eigenfunctions of Laplacian  $\Delta$  (43) are

$$\psi_{\mathbf{n}}(\mathbf{r}, \eta) = e^{i\mathbf{n}\mathbf{r}}, \quad (44)$$

where  $n_\alpha = \text{const}$  and  $\mathbf{k} = \mathbf{n}/a(\eta)$  is a wave vector. Hence,

$$k^2 = -g_0^{\alpha\beta}n_\alpha n_\beta \equiv \frac{\mathbf{n}^2}{a^2(\eta)}. \quad (45)$$

Further, because each perturbation mode must be real, we rewrite it as

$$h_{\alpha\beta}(\mathbf{r}, \eta) = h_{\alpha\beta}(\mathbf{n}, \eta)e^{i\mathbf{n}\mathbf{r} + i\psi_0} + h_{\alpha\beta}^*(\mathbf{n}, \eta)e^{-i\mathbf{n}\mathbf{r} - i\psi_0}, \quad (46)$$



where the asterisk denotes complex conjugation and  $\psi_0$  is an arbitrary constant phase for each harmonic  $\mathbf{n}$ . The introduced arbitrary phase of the perturbation mode reflects the fact that as a result of the homogeneity of the three-dimensional space, each metric perturbation mode must be independent of the choice of the origin of the Cartesian coordinate system. Hence, as a result of the translation symmetry, we must have written a separate harmonic in the form  $e^{-i\mathbf{n}(\mathbf{r}-\mathbf{r}_0)}$ , which we in fact did when we set  $\psi_0 = -\mathbf{r}_0\mathbf{n}$ . Therefore, at each instant  $\eta_0$ , a single independent mode of metric perturbations is fully described by the three-dimensional tensor amplitudes  $h_{\alpha\beta}(\mathbf{n}, \eta_0)$ , whose classification along the direction of the spacelike vector  $\mathbf{n}$  and the three-dimensional Kronecker tensor  $\delta_{\alpha\beta}$  was given by Lifshitz in [31].

For transverse perturbations, we have

$$h_{\alpha\beta} = e_{\alpha\beta}(S e^{i\mathbf{n}\mathbf{r} + i\psi_0} + \mathbb{C}\mathbb{C}), \quad (47)$$

where  $\mathbb{C}\mathbb{C}$  denotes a complex-conjugate quantity and  $S(\mathbf{n}, \eta)$ ,  $S^*(\mathbf{n}, \eta)$  is an amplitude of perturbations, and hence

$$h_{\beta}^{\alpha} n_{\alpha} = 0, \quad (48)$$

$$h = 0. \quad (49)$$

In an arbitrary Cartesian coordinate system of the three-dimensional Euclidian space  $E_3$ , the polarization tensor  $e_{\alpha\beta}$  in formula (47) becomes

$$e_{\alpha\beta} = 2s_{\alpha}s_{\beta} + \frac{n_{\alpha}n_{\beta}}{n^2} - \delta_{\alpha\beta}, \quad (50)$$

$$\mathbf{s}^2 = 1, \quad \mathbf{s}\mathbf{n} = 0, \quad \mathbf{n}^2 = n^2. \quad (51)$$

It is easy to verify that gauge condition (48) is automatically satisfied. Hence, each mode of transverse gravitational perturbations is described by the numbers  $\mathbf{n} = (n_1, n_2, n_3)$ ,  $\mathbf{s} = (s_1, s_2, s_3)$ , and  $\psi_0$ . Conditions (51) impose two constraints on these numbers. Therefore, each mode of transverse gravitational perturbations has five degrees of freedom over which we must average.

For vector perturbations of the metric, we have

$$h_{\alpha\beta} = V_{\alpha}n_{\beta} + V_{\beta}n_{\alpha}, \quad (\mathbf{n}\mathbf{V}) = 0, \quad (52)$$

$$\mathbf{V} = \mathbf{v}e^{i\mathbf{n}\mathbf{r} + i\psi_0} + \mathbb{C}\mathbb{C}, \quad (53)$$

where  $\mathbf{v}(\mathbf{n}, \eta)$  is the amplitude of vector perturbations. Each mode of vector perturbations is described by five degrees of freedom.

Similarly, for the longitudinal perturbation of the metric, we have

$$h_{\alpha\beta} = e^{i\mathbf{n}\mathbf{r} + i\psi_0}(\lambda P_{\alpha\beta} + \mu Q_{\alpha\beta}) + \mathbb{C}\mathbb{C}, \quad (54)$$

where  $\lambda(\mathbf{n}, \eta)$ ,  $\lambda^*(\mathbf{n}, \eta)$ ,  $\mu(\mathbf{n}, \eta)$ , and  $\mu^*(\mathbf{n}, \eta)$  are amplitudes of scalar perturbations

$$P_{\alpha\beta} = \frac{1}{3}\delta_{\alpha\beta} - \frac{n_{\alpha}n_{\beta}}{n^2}, \quad Q_{\alpha\beta} = \frac{1}{3}\delta_{\alpha\beta}. \quad (55)$$

We note the series of papers [32]–[36] (also see [37]) devoted to developing a gauge-invariant theory of scalar, vector, and tensor perturbations of second and higher orders in the Friedmann universe. The

main idea of this theory is to construct canonical field degrees of freedom of perturbations such that it would be possible to separate the indicated perturbation modes in higher orders of the perturbation theory and to construct physical observables such as energy–momentum, etc., corresponding to these modes. The appearance of these papers was motivated by requirements of the quantum theory of the early universe. We emphasize that the *theory of macroscopic Einstein equations of the second order in perturbations* developed here, on one hand, is a classical theory and does not require selecting such pure field modes for calculating physical observables corresponding to them. On the other hand, because it is a second-order macroscopic theory (macroscopic averages of perturbations vanish), it requires satisfaction of the gauge conditions only in the first order of the perturbation theory, for which the Lifshitz perturbation expansion presented above holds.

### 3.3. Averaging the Einstein equations over the states of a mode of metric perturbations.

As previously noted, each independent perturbation mode on the background of a Friedmann space is described by five independent numbers. For a given wave vector  $\mathbf{n}$ , the polarization vector  $\mathbf{s}$  of a gravitational wave has only one remaining degree of freedom. Practically, this is the angle of rotation of a unit vector in a plane orthogonal to the wave vector. The wave vector  $\mathbf{n}$  itself has three degrees of freedom. One other degree of freedom is related to the oscillation phase, which can be defined on the interval  $[0, 2\pi]$ . Longitudinal perturbations are completely determined by four numbers  $\mathbf{n}$  and  $\psi_0$ , i.e., they have four degrees of freedom. In connection with this, we average of the metric over four degrees of freedom, leaving the polarization vector of the transverse gravitational perturbations fixed.

We also do not average over the length of the wave vector because this requires setting the perturbation spectrum (we can average with a given perturbation spectrum in the final stage). As a result, we can write the average of a certain quantity  $f(n, \mathbf{r})$  over arbitrary perturbations as

$$\overline{f(n, \mathbf{r})} = \frac{1}{8\pi^2} \int_0^{2\pi} d\psi_0 \int_{\Omega_n} f(\mathbf{n}, \mathbf{r}) d\Omega_n, \quad (56)$$

where averaging over wave-vector directions reduces to calculating the integral over a two-dimensional sphere of radius  $n$ . Because the phase space is isotropic and homogeneous, we have the relations

$$\overline{\mathbf{n}} = 0, \quad \overline{n^\alpha n^\beta} = \frac{1}{3} \mathbf{n}^2. \quad (57)$$

In averaging, we must take into account that the perturbations of the metric  $h_{\alpha\beta}$  and the scalar field  $\varphi$  are random. As a result, we have

$$\overline{h_{\alpha\beta}} = 0, \quad \overline{\varphi} = 0. \quad (58)$$

As a consequence of the isotropy of the macroscopic metric and relations (58), the thus averaged total energy–momentum tensor must be isotropic, i.e., it must have the structure of the energy–momentum tensor of an ideal fluid (see [2] for the notation),

$$\overline{T_{ik}^{(2)}} = a^2 (\overline{\mathcal{E}} + \overline{\mathcal{P}}) \delta_i^4 \delta_k^4 - a^2 \eta_{ik} \overline{\mathcal{P}}, \quad (59)$$

where  $\eta_{ik} = \text{diag}(-1, -1, -1, +1)$  and

$$\overline{\mathcal{E}} = \frac{1}{a^2} \overline{T_{44}^{(2)}}, \quad \overline{\mathcal{P}} = \frac{1}{3a^2} \overline{\sum_{\alpha=1}^3 T_{\alpha\alpha}^{(2)}} \quad (60)$$

are the macroscopic energy density and pressure of the field gravitationally scalar matter.

We note that because the perturbations of the gravitational and scalar fields and their first derivatives enter the total energy–momentum tensor in degree two, the average of the product must be understood as

$$\overline{ab} = \overline{ab^* + a^*b}. \quad (61)$$

Because the macroscopic metric is isotropic and homogeneous after such averaging, we can find macroscopic Einstein equations (18) of the second order in perturbations for it:

$$\begin{aligned} 3\frac{a'^2}{a^2} &= \mathcal{E}_0 + \overline{\mathcal{E}}, \\ \frac{a'^2}{a^2} - 2\frac{a''}{a} &= \mathcal{P}_0 + \overline{\mathcal{P}}. \end{aligned} \quad (62)$$

We note that according to the self-consistent field method, the scale factor in these equations is a macroscopic value.

We represent the potential of the scalar field in the form

$$\Phi = \Phi_0(\eta) + \phi(\eta)e^{i\mathbf{nr}+i\psi_0} + \phi^*(\eta)e^{-i\mathbf{nr}-i\psi_0}. \quad (63)$$

We further expand the Einstein tensor and the energy–momentum tensor in series in the amplitude of perturbations of gravitational and matter fields  $S(\eta)$ ,  $V(\eta)$ ,  $\lambda(\eta)$ ,  $\mu(\eta)$ ,  $\phi(\eta)$ ,  $\delta\varepsilon$ , and  $\delta u^\alpha$ . Using the isotropy of the unperturbed metric and also orthogonality conditions (51) and (52), we find it convenient to introduce a local system of coordinates in which

$$\mathbf{n} = n(0, 0, 1), \quad \mathbf{s} = (1, 0, 0), \quad \mathbf{v} = (0, v(\eta), 0), \quad (64)$$

where  $\mathbf{s}$  is the unit vector of the transverse polarization perturbation. In this system of coordinates, we have

$$\begin{aligned} h_{11} &= \left[ -S(\eta) + \frac{1}{3}\lambda(\eta) + \frac{1}{3}\mu(\eta) \right] e^{inz} + \mathbb{C}\mathbb{C}, \\ h_{22} &= \left[ S(\eta) + \frac{1}{3}\lambda(\eta) + \frac{1}{3}\mu(\eta) \right] e^{inz} + \mathbb{C}\mathbb{C}, \\ h_{12} = h_{13} &= 0, \quad h_{23} = v(\eta)e^{inz} + v^*(\eta)e^{-inz}, \\ h_{33} &= \left[ -\frac{2}{3}\lambda(\eta) + \frac{1}{3}\mu(\eta) \right] e^{inz} + \mathbb{C}\mathbb{C}, \\ h &= \mu(\eta)e^{inz} + \mu^*(\eta)e^{-inz}. \end{aligned}$$

## 4. Expanding the field equations in perturbation orders

**4.1. Zeroth approximation.** Expanding the Einstein tensor in the metric perturbations, we find the known expressions in the zeroth approximation

$$G^{(0)1}_1 = G^{(0)2}_2 = G^{(0)3}_3 = 2\frac{a''}{a^3} - \frac{a'^2}{a^4}, \quad G^{(0)4}_4 = 3\frac{a'^2}{a^2}. \quad (65)$$

For scalar field equation (35), we obtain

$$\Phi_0'' + 2\frac{a'}{a}\Phi_0' + a^2(m^2\Phi_0 - \alpha\Phi_0^3) = 0. \quad (66)$$

The nonzero components of the energy–momentum tensor of the scalar field are

$$\begin{aligned} T^{(0)1}_1 = T^{(0)2}_2 = T^{(0)3}_3 &\equiv -\mathcal{P}_0 = -\frac{\Phi_0'^2}{16\pi a^2} + \frac{m^2\Phi_0^2}{16\pi} - \frac{\alpha\Phi_0^4}{32\pi}, \\ T^{(0)4}_4 &\equiv \mathcal{E}_0 = \frac{\Phi_0'^2}{16\pi a^2} + \frac{m^2\Phi_0^2}{16\pi} - \frac{\alpha\Phi_0^4}{32\pi}. \end{aligned} \quad (67)$$

In the zeroth approximation in gravitational perturbations, we thus obtain two independent Einstein equations

$$3\frac{a'^2}{a^4} - \frac{\Phi_0'^2}{2a^2} - \frac{m^2\Phi_0^2}{2} + \frac{\alpha\Phi_0^4}{4} - \Lambda = 0, \quad (68)$$

$$2\frac{a''}{a^3} - \frac{a'^2}{a^4} + \frac{\Phi_0'^2}{2a^2} - \frac{m^2\Phi_0^2}{2} + \frac{\alpha\Phi_0^4}{4} - \Lambda = 0. \quad (69)$$

These two equations comprise a system of equations of the basic cosmological model for a scalar field with a Higgs potential. Differentiating (68) with respect to time and substituting the expression for  $a''/a^3$  obtained from (69) in the resulting equation, we obtain field equation (82) with  $\Phi_0 \neq 0$  (see below). As previously noted, in accordance with the self-consistent field method, we do not solve *background equations* (82), (68), and (69). Instead, we add the averaged quadratic corrections, which are already not small quantities, to them.

**4.2. First-order equations: Evolution equations for perturbations.** In the approximation linear in  $S - V - \lambda - \mu - \phi$ , we obtain, first, the equation for the perturbation of the scalar field (the equations for the complex-conjugate amplitudes do not differ from it)

$$\phi'' + 2\frac{a'}{a}\phi' + a^2\phi\left(m^2 + \frac{n^2}{a^2} - 3\alpha\Phi_0^2\right) + \frac{\mu'}{2}\Phi_0' = 0, \quad (70)$$

second, the nonzero Einstein tensor components  $G^{(1)1}_1$ ,  $G^{(1)2}_2$ ,  $G^{(1)3}_3$ ,  $G^{(1)3}_2 = G^{(1)2}_3$ ,  $G^{(1)1}_4 = -G^{(1)4}_1$ ,  $G^{(1)2}_4 = -G^{(1)4}_2$ ,  $G^{(1)3}_4 = -G^{(1)4}_3$ , and  $G^{(1)4}_4$ , and, third, the components of the energy–momentum tensor corresponding to them. Separating the harmonics  $e^{\pm ikz}$  in the obtained equations, we write the independent combinations of the corresponding nontrivial equations:

$$\binom{4}{2} : \quad v' = 0, \quad (71)$$

$$\binom{2}{2} - \binom{1}{1} : \quad S'' + 2\frac{a'}{a}S' + n^2S = 0, \quad (72)$$

$$\binom{4}{3} : \quad \frac{1}{3}(\lambda + \mu)' + \phi\Phi_0' = 0, \quad (73)$$

$$2\binom{3}{3} - \binom{2}{2} - \binom{1}{1} : \quad \lambda'' + 2\frac{a'}{a}\lambda' - \frac{n^2}{3}(\lambda + \mu) = 0, \quad (74)$$

$$\binom{1}{1} + \binom{2}{2} + \binom{3}{3} : \quad \mu'' + 2\frac{a'}{a}\mu' + \frac{n^2}{3}(\lambda + \mu) + 3\phi'\Phi_0' - 3a^2\Phi_0\phi(m^2 - \alpha\Phi_0^2) = 0, \quad (75)$$

$$\binom{4}{4} : \quad \frac{a'}{a}\mu' + \frac{n^2}{3}(\lambda + \mu) - \Phi_0'\phi' - a^2\phi\Phi_0(m^2 - \alpha\Phi_0^2) = 0. \quad (76)$$

The equation  $\binom{2}{3}$  is a differential consequence of Eq. (71). We write the equation  $\binom{3}{3}$ , which is an algebraic consequence of Eqs. (74) and (75) (one third of their sum)

$$\binom{3}{3} : \quad (\lambda + \mu)'' + 2\frac{a'}{a}(\lambda + \mu)' + 3\Phi_0'\phi' - 3a^2\Phi_0\phi(m^2 - \alpha\Phi_0^2) = 0. \quad (77)$$

We note that the equations for the scalar  $(\lambda, \mu, \phi)$ , vector  $(v)$ , and tensor  $(S)$  perturbations are uncoupled. Hence, Eqs. (71) and (72) define the amplitudes of vector and tensor perturbations. The remaining five Eqs. (70) and (73)–(76) define three functions: the amplitudes of the scalar metric perturbations  $\lambda$  and  $\mu$  and the amplitude of the perturbation of the scalar field  $\phi$ . Moreover, Eq. (74) is independent of the scalar field and completely coincides with the corresponding equation in the Lifshitz theory. We note that we can use the consequences of the zeroth-approximation equations in the evolution equations because the subsequent terms already have the third order of the perturbation theory. We also note that among all the first-order conservation laws  $\nabla_k T_k^i = 0$ , only two are nontrivial,  $\nabla_k T_3^i = 0$  and  $\nabla_k T_4^k = 0$ : the first is equivalent to Eq. (82) for the field of the zeroth approximation, and the second is equivalent to field equation (70) of the first-order perturbation theory. Therefore, with Eq. (70) for the scalar field perturbations taken into account, only two of the four Eqs. (73)–(76) are independent. Hence, as the basic equations for the three functions  $\mu$ ,  $\lambda$ , and  $\phi$ , we can choose, for example, the subsystem of Eqs. (70), (74), and (75).

In what follows, we call system of linear equations (70)–(72), (74), (75) the *evolution equations* for perturbations of the gravitational and scalar fields. The solutions of these equations determine the quadratic corrections to the energy–momentum tensor and hence the macroscopic metric of the universe.

**4.3. Second-order approximation.** We now calculate the second-order corrections to the Einstein tensor and the energy–momentum tensor in accordance with (20):

$$\mathcal{G}^{(2)i}_k = G^{(2)i}_k - 8\pi T^{(2)i}_k \equiv -8\pi \mathcal{T}^{(2)i}_k. \quad (78)$$

Calculating these corrections, we must take equations of the zeroth and first orders in perturbations into account. We immediately obtain

$$\mathcal{G}^{(2)1}_2 = \mathcal{G}^{(2)1}_3 = \mathcal{G}^{(2)1}_4 = \mathcal{G}^{(2)2}_4 = 0. \quad (79)$$

The remaining nonzero component is  $\mathcal{G}^{(2)4}_3 \sim n_3(\mathbf{n})$ . Therefore, as a consequence of (57), we have

$$\overline{\mathcal{G}^{(2)4}_\alpha} = 0. \quad (80)$$

Moreover, in the calculation of the mean-square corrections to the energy–momentum tensor, terms containing  $e^{\pm 2i\mathbf{k}\mathbf{r}}$  vanish because the three-dimensional space is isotropic. Therefore, only terms that do not contain such factors are essential in formulas for corrections to the energy and pressure given by (60). Taking these comments and evolution equations (70)–(77) for the perturbation amplitudes into account and

averaging, we obtain corrections to the background pressure  $\mathcal{P}_0$  and the energy density  $\mathcal{E}_0$  given by (67):

$$\begin{aligned} \overline{\mathcal{P}} &= \frac{1}{24\pi} \overline{\sum_{\alpha=1}^3 \mathcal{G}^{(2)\alpha}} = \frac{1}{8\pi} \left[ n^2 \frac{7}{6a^2} \overline{SS^*} - \frac{5}{6a^2} \overline{S'S'^*} - \right. \\ &\quad - \frac{n^2}{9} \left( \frac{7}{6} \overline{\lambda\lambda^*} + \frac{2}{3} \overline{(\lambda\mu^* + \lambda^*\mu)} - \frac{1}{6} \overline{\mu\mu^*} \right) - \frac{5}{18} \overline{\lambda'\lambda'^*} + \frac{1}{18} \overline{\mu'\mu'^*} - \frac{\overline{\phi'\phi'^*}}{a^2} - \\ &\quad \left. - \overline{\phi\phi^*} \left( \frac{n^2}{3a^2} + m^2 - 3\alpha\Phi_0^2 \right) \right], \end{aligned} \quad (81)$$

$$\begin{aligned} \overline{\mathcal{E}} &= -\frac{\overline{\mathcal{G}^{(2)4}}}{8\pi} = \frac{1}{8\pi} \left[ n^2 \left( \frac{\overline{SS^*}}{2a^2} + \frac{\overline{\lambda\mu^* + \lambda^*\mu}}{9} - \frac{\overline{\lambda\lambda^*}}{18} \right) + \frac{\overline{S'S'^*}}{2a^2} + \frac{\overline{\lambda'\lambda'^*}}{6} - \frac{\overline{\mu'\mu'^*}}{6} + \right. \\ &\quad \left. + 2\frac{a'}{a^3} \overline{(SS^*)'} + \frac{1}{3} \frac{a'}{a} \overline{(\mu\mu^*)'} + \frac{\overline{\phi'\phi'^*}}{a^2} + \overline{\phi\phi^*} \left( \frac{n^2}{a^2} + m^2 - 3\alpha\Phi_0^2 \right) \right]. \end{aligned}$$

Here and hereafter,  $\overline{fF^*}$  denotes the average over the spectrum of oscillations  $f(n) \geq 0$  if there is no single isotropic mode of oscillations with the wave vector  $\mathbf{n}$ ,

$$\int_0^\infty f(n) dn = 1.$$

In this case, we must also introduce averaging over the ‘‘frequencies’’  $n$  in the right-hand sides of macroscopic equations (81),

$$\int_0^\infty F(n, \eta) f(n) dn.$$

**4.4. Second-order equations for the macroscopic scalar field in perturbations and the macroscopic Einstein equations.** To construct a closed theory of a macroscopic universe, we must obtain the equations of the second order in perturbations for the scalar field  $\Phi$  based on field equation (35). Calculating the quadratic correction to the field equation, we obtain

$$\begin{aligned} \delta^{(2)}(\square\Phi + V'_\Phi) &= -\Phi'_0 \left[ (SS^*)' + \frac{1}{3}(\lambda\lambda^*)' + \frac{1}{6}(\mu\mu^*)' \right] + \\ &\quad + \frac{1}{2} \frac{\phi'\mu'^* + \phi'^*\mu'}{a^2} - \frac{1}{2} \frac{n^2}{a^2} (\phi\mu^* + \phi^*\mu) - 6\alpha\Phi_0\phi\phi^*. \end{aligned} \quad (82)$$

According to the self-consistent field method, to determine the macroscopic scalar field  $\Phi_0(\eta)$ , instead of (82), we have

$$\begin{aligned} \Phi_0'' + 2\frac{a'}{a}\Phi_0' + a^2(m^2\Phi_0 - \alpha\Phi_0^3) - \Phi_0'a^2 \overline{\left( SS^* + \frac{1}{3}\lambda\lambda^* + \frac{1}{6}\mu\mu^* \right)'} + \\ + \frac{1}{2} \overline{\phi'\mu'^* + \phi'^*\mu'} - \frac{1}{2} n^2 \overline{(\phi\mu^* + \phi^*\mu)} - 6a^2\alpha\Phi_0\overline{\phi\phi^*} = 0. \end{aligned} \quad (83)$$

Proceeding similarly with Einstein equations (68) with (81) taken into account, we replace them with the macroscopic Einstein equations

$$\begin{aligned} 3\frac{a'^2}{a^4} - \frac{\Phi_0'^2}{2a^2} - \frac{m^2\Phi_0^2}{2} + \frac{\alpha\Phi_0^4}{4} - \Lambda = \\ = n^2 \left( \frac{\overline{SS^*}}{2a^2} + \frac{\overline{\lambda\mu^* + \lambda^*\mu}}{9} - \frac{\overline{\lambda\lambda^*}}{18} \right) + \frac{\overline{S'S'^*}}{2a^2} + \frac{\overline{\lambda'\lambda'^*}}{6} - \frac{\overline{\mu'\mu'^*}}{6} + \\ + 2\frac{a'}{a^3} \overline{(SS^*)'} + \frac{1}{3} \frac{a'}{a} \overline{(\mu\mu^*)'} + \frac{\overline{\phi'\phi'^*}}{a^2} + \overline{\phi\phi^*} \left( \frac{n^2}{a^2} + m^2 - 3\alpha\Phi_0^2 \right), \end{aligned} \quad (84)$$

$$\begin{aligned}
& 2\frac{a''}{a^3} - \frac{a'^2}{a^4} + \frac{\Phi_0'^2}{2a^2} - \frac{m^2\Phi_0^2}{2} + \frac{\alpha\Phi_0^4}{4} - \Lambda + n^2\frac{7}{6a^2}\overline{SS^*} - \frac{5}{6a^2}\overline{S'S'^*} - \\
& - \frac{n^2}{9}\left(\frac{7}{6}\overline{\lambda\lambda^*} + \frac{2}{3}\overline{(\lambda\mu^* + \lambda^*\mu)} - \frac{1}{6}\overline{\mu\mu^*}\right) - \\
& - \frac{5}{18}\overline{\lambda'\lambda'^*} + \frac{1}{18}\overline{\mu'\mu'^*} - \frac{\overline{\phi\phi'}}{a^2} - \overline{\phi\phi^*}\left(\frac{n^2}{3a^2} + m^2 - 3\alpha\Phi_0^2\right) = 0. \tag{85}
\end{aligned}$$

System of linear evolution equations (71)–(74), (76) and system of macroscopic equations (83)–(85) together comprise a complete closed system of equations defining the macroscopic Friedmann universe.

**4.5. Some considerations about macroscopic equations.** We first note that vector perturbations do not influence the macroscopic equation of the scalar field and the macroscopic Einstein equations. In what follows, we therefore assume that local vector perturbations are equal to zero, taking into account that their linear equations (71) are autonomous.

Second, adding Eqs. (74) and (75) and substituting the expression for  $(\lambda + \mu)'$  obtained from (73) in the result, we obtain

$$\phi \left[ \Phi'' + 2\frac{a'}{a}\Phi' + a^2\Phi(m^2 - \alpha\Phi^2) \right] = 0. \tag{86}$$

The expression in square brackets coincides with the left-hand side of the unperturbed equation for the background scalar field,

$$\Phi_0'' + 2\frac{a'}{a}\Phi_0' + a^2(m^2\Phi_0 - \alpha\Phi_0^3) = 0. \tag{87}$$

The difference between this equation and the macroscopic equation for the background scalar field is of the second order of smallness in perturbations (see (83)). Therefore, the left-hand side of (86) is of the third order in perturbations, and we can consequently assume that the expression in square brackets in (86) is equal to zero in the second-order perturbation theory. Because we obtained (86) as a differential-algebraic consequence of Eqs. (73)–(75), we can discard one of these equations in the framework of the approximation quadratic in perturbations. Therefore, we consider the system of the four evolution equations (70), (72), (74), and (75) for the four functions  $\phi(\eta)$ ,  $S(\eta)$ ,  $\lambda(\eta)$ , and  $\mu(\eta)$ .

Finally, we also note a feature of the equations of the macroscopic model mentioned in Sec. 1: linear evolution equations (70), (72), (74), and (75) for the perturbations  $\phi(\eta)$ ,  $S(\eta)$ ,  $\lambda(\eta)$ , and  $\mu(\eta)$  of the scalar field and the Friedmann metric are defined on the background of the macroscopic quantities  $\Phi(\eta)$  and  $a(\eta)$ , which are in turn defined by nonlinear equations (83)–(85) using quadratic fluctuations of perturbations. This feature of the equations of the macroscopic model does not allow solving them by direct methods. The corresponding evolution equations in the standard perturbation theory are solved with a *given* metric  $a(\eta)$ . In our case, we must solve these equations with undetermined background functions. Therefore, we must solve both the evolution equations and the macroscopic equations simultaneously, making assumptions regarding the background functions  $a(\eta)$  and  $\Phi(\eta)$  and the perturbations  $\phi(\eta)$ ,  $S(\eta)$ ,  $\lambda(\eta)$ , and  $\mu(\eta)$ . One such assumption concerns the characteristic scales of change of these functions, which are defined by the wave number  $n$  and the characteristic dimension of the background inhomogeneity  $f$ ,  $\ell^{-1} = f'/f$ .

**4.6. Consistency of the system of equations of the macroscopic model.** The system of macroscopic equations comprises the three second-order differential equations (83)–(85) for the two macroscopic functions  $a(\eta)$  and  $\Phi(\eta)$ . Therefore, we should first investigate the consistency of this system. From the standpoint of the Einstein equations, the problem of the consistency of this system of equations is related to satisfaction of the Bianchi identities for the Einstein tensor. In the case of an isotropic homogenous metric, the differential Bianchi identities reduce to one relation. We prove the consistency of the obtained equations.

To avoid using cumbersome formulas, we write system (83)–(85) in the form

$$\Phi'' + 2\frac{a'}{a}\Phi' + a^2\Phi(m^2 - \alpha\Phi^2) = F, \quad (88)$$

$$3\frac{a'^2}{a^4} - \frac{\Phi'^2}{2a^2} - \frac{m^2\Phi^2}{2} + \frac{\alpha\Phi^4}{4} - \Lambda = G, \quad (89)$$

$$2\frac{a''}{a^3} - \frac{a'^2}{a^4} + \frac{\Phi'^2}{2a^2} - \frac{m^2\Phi^2}{2} + \frac{\alpha\Phi^4}{4} - \Lambda = H, \quad (90)$$

where the functions  $F$ ,  $G$ , and  $H$  are determined by mean-square perturbations of the metric and scalar field:

$$\begin{aligned} F &= \Phi' \left[ \overline{(SS^*)'} + \frac{1}{3}\overline{(\lambda\lambda^*)'} + \frac{1}{6}\overline{(\mu\mu^*)'} \right] + 6a^2\alpha\overline{\phi\phi^*} - \frac{1}{2}\overline{\phi'\mu'^* + \phi'^*\mu'} + \frac{1}{2}n^2\overline{(\phi\mu^* + \phi^*\mu)}, \\ G &= n^2 \left( \frac{\overline{SS^*}}{2a^2} + \frac{\overline{\lambda\mu^* + \lambda^*\mu}}{9} - \frac{\overline{\lambda\lambda^*}}{18} \right) + \frac{\overline{S'S'^*}}{2a^2} + \frac{\overline{\lambda'\lambda'^*}}{6} - \frac{\overline{\mu'\mu'^*}}{6} + \\ &\quad + 2\frac{a'}{a^3}\overline{(SS^*)'} + \frac{1}{3}\frac{a'}{a}\overline{(\mu\mu^*)'} + \frac{\overline{\phi'\phi'^*}}{a^2} + \overline{\phi\phi^*} \left( \frac{n^2}{a^2} + m^2 - 3\alpha\Phi^2 \right), \\ H &= n^2 \frac{7}{6a^2}\overline{SS^*} - \frac{5}{6a^2}\overline{S'S'^*} - \frac{n^2}{9} \left( \frac{7}{6}\overline{\lambda\lambda^*} - \frac{1}{6}\overline{\mu\mu^*} + \frac{2}{3}\overline{(\lambda\mu^* + \lambda^*\mu)} \right) - \\ &\quad - \frac{5}{18}\overline{\lambda'\lambda'^*} + \frac{1}{18}\overline{\mu'\mu'^*} - \frac{\overline{\phi'\phi'^*}}{a^2} - \overline{\phi\phi^*} \left( \frac{n^2}{3a^2} + m^2 - 3\alpha\Phi^2 \right). \end{aligned}$$

Differentiating (89) and using (88) and (90) in the resulting expression, we obtain the relation

$$3\frac{a'}{a}(H - G) - \frac{\Phi'}{a^2}F = G', \quad (91)$$

which is the necessary and sufficient consistency condition (the proof of sufficiency is elementary) for macroscopic equations (83)–(85). In the absence of local fluctuations ( $F = G = H = 0$ ), this condition is satisfied identically. With local perturbations, we must calculate directly to verify relation (91). For this, we must first take the commutativity of differentiation, complex conjugation, and averaging into account. Second, we must use evolution equations (70)–(75) to determine the second derivatives with respect to fluctuations, which appear in the right-hand side of (91). We do not present the quite cumbersome calculations here and merely show the main techniques for calculating such values in a simple example.

As an example, we calculate  $\overline{(S'S'^*)}'$ : we have  $\overline{(S'S'^*)}' = \overline{S''S'^*} + \overline{S'S''^*}$ . From evolution equation (72), we obtain

$$S'' = -2\frac{a'}{a}S' - n^2S.$$

We hence have

$$\overline{(S'S'^*)}' = -4\frac{a'}{a}\overline{S'S'^*} - n^2\overline{SS^*} - n^2\overline{S'S^*} \quad \Rightarrow \quad \overline{(S'S'^*)}' = -4\frac{a'}{a}\overline{S'S'^*} - n^2\overline{(SS^*)}'.$$

Proceeding in the same way with the second derivatives of  $\phi$ ,  $\lambda$ , and  $\mu$ , we prove that condition (91) is satisfied.

In what follows, we define the macroscopic cosmological model by evolution equations (70) and (72)–(75) and macroscopic equations (83) and (84).



## 5. Examples of constructing macroscopic models of the universe

**5.1. The case of purely transverse perturbations.** The obtained system of equations for macroscopic cosmology is extremely complicated to analyze in a single paper. We intend to return to its study in the nearest future. Meanwhile, as an example of studying this system, we consider the case where scalar and vector perturbations of the gravitational field are absent,  $\mu = \lambda = v = 0$ . In this case, we obtain  $\phi = 0$  from Eq. (73), and from the system of evolution equations, there remains only Eq. (72) for transverse perturbations. Equation (83) thus reduces to the simpler equation

$$\Phi_0'' + \Phi_0' \left[ 2\frac{a'}{a} - \overline{(SS^*)'} \right] + a^2\Phi_0(m^2 - \alpha\Phi_0^2) = 0. \quad (92)$$

The mean-square correction to the macroscopic equation for the scalar field in this case hence influences only the value of the Hubble constant  $H = a'/a$  via the energy of gravitational waves.

From (81), we get expressions for the corrections to the effective pressure and energy density:

$$\overline{\mathcal{P}} = \frac{1}{8\pi a^2} \left( \frac{7}{6} n^2 \overline{SS^*} - \frac{5}{6} \overline{S'S'^*} \right), \quad \overline{\mathcal{E}} = \frac{1}{8\pi a^2} \left[ n^2 \frac{\overline{SS^*}}{2} + \frac{\overline{S'S'^*}}{2} + 2\frac{a'}{a} \overline{(SS^*)'} \right]. \quad (93)$$

**5.2. The WKB approximation for transverse perturbations.** We now consider the WKB approximation of evolution equation (72)

$$n \gg \frac{a'}{a}, \quad S' \gg \frac{a'}{a},$$

representing the solution in the form

$$S = \tilde{S}(\eta) \exp \left[ i \int u(\eta) d\eta \right],$$

where  $\tilde{S}(\eta)$  and  $u(\eta)$  are functions slowly changing with the scale factor such that

$$a' \ll a\{n, u\}, \quad \tilde{S}' \ll \tilde{S}\{n, u\}, \quad u' \ll \{n, u\}. \quad (94)$$

Hence, in the WKB approximation we obtain

$$S = \frac{1}{a} S_+^0 e^{in\eta} + \frac{1}{a} S_-^0 e^{-in\eta}, \quad (95)$$

where  $S_{\pm}^0$  are constant amplitudes and hence  $S_+^0 S_-^0 = |S^0|^2$ . Therefore, in the WKB approximation, we find that

$$\begin{aligned} SS^* &\simeq \frac{|S^0|^2}{a^2} \Rightarrow \overline{SS^*} \simeq \frac{|S^0|^2}{a^2}, \\ S'S'^* &\simeq n^2 \frac{|S^0|^2}{a^2} \Rightarrow \overline{S'S'^*} \simeq n^2 \frac{|S^0|^2}{a^2}, \quad (SS^*)' \simeq 0. \end{aligned}$$

Using these relations in formulas (93), we find that in the WKB approximation,

$$\overline{\mathcal{P}} \simeq \frac{1}{24\pi} n^2 \frac{|S^0|^2}{a^4}, \quad \overline{\mathcal{E}} \simeq \frac{1}{8\pi} n^2 \frac{|S^0|^2}{a^4} \Rightarrow \overline{\mathcal{P}} \simeq \frac{1}{3} \overline{\mathcal{E}}, \quad (96)$$

i.e., taking high-frequency transverse gravitational perturbations into account in the WKB approximation is equivalent to adding a component of an ultrarelativistic fluid to the macroscopic Einstein equations [2].

Substituting WKB solution (95) in the macroscopic equation for the scalar field, we reduce it in the WKB approximation to the explicit form

$$\Phi_0'' + 2\frac{a'}{a}\Phi_0' \left( 1 + \frac{|S^0|^2}{a^2} \right) + a^2\Phi_0(m^2 - \alpha\Phi_0^2) = 0. \quad (97)$$

Similarly, we reduce the independent Einstein equation to the explicit form

$$3\frac{a'^2}{a^4} - \frac{\Phi_0'^2}{2a^2} - \frac{m^2\Phi_0^2}{2} + \frac{\alpha\Phi_0^4}{4} - \Lambda = n^2 \frac{|S^0|^2}{a^4}. \quad (98)$$

**5.3. A particular solution of the macroscopic Einstein equations.** To solve the equations for the macroscopic gravitation, we must first solve macroscopic equation (97) for the scalar field, which itself is challenging with an unknown scale factor. Examples of investigations that allow bypassing this major difficulty were presented in [38], [39]. Below, we consider a simple example of a particular exact solution of field equation (97) that allows obtaining a solution of the problem as a final result.

Indeed, assuming that  $\alpha > 0$ , we set  $\Phi_0 = \pm m/\sqrt{\alpha}$  in (97). In this case, Eq. (97) becomes an identity, and Einstein equation (98) reduces to the explicit form

$$3\frac{a'^2}{a^4} - \lambda = n^2\frac{|S^0|^2}{a^4}, \quad (99)$$

where

$$\lambda = \Lambda + \frac{m^4}{4\alpha}. \quad (100)$$

We integrate (99) in elementary functions,

$$a(t) = \left(\frac{|S^0|^2 n^2}{\lambda}\right)^{1/4} \sqrt{\sinh\left(2\sqrt{\frac{\lambda}{3}}t\right)}. \quad (101)$$

We found a similar solution for the simpler macroscopic model of the universe with a cosmological term where the scalar field is absent [2]. The difference of the particular solution of the cosmological model with scalar field (101) from that case practically reduces to a renormalization of cosmological constant (100).

## 6. Conclusion

We have obtained a closed system of macroscopic Einstein–Higgs equations describing the evolution of a macroscopically homogenous and isotropic universe filled with a fluctuating scalar field with a Higgs potential. This system contains a subsystem of ordinary linear differential equations describing the evolution of perturbations of the gravitational and scalar fields and a system of nonlinear macroscopic equations describing the macroscopic dynamics of the cosmological model.

We note an important property of such macroscopic models that distinguish them from standard cosmological models with homogenous scalar fields  $\Phi(t)$ : self-consistent solution (101) always contains a cosmological singularity. Indeed, solution (101) behaves as a solution  $a \simeq (4n^2|S^0|^2/3)^{1/4}\sqrt{t} \rightarrow 0$  for an ultrarelativistic universe as  $\eta \rightarrow 0$  and as an inflationary solution  $a \simeq (n^2|S^0|^2/(4\lambda))^{1/4}e^{\sqrt{\lambda/3}t}$  as  $t \rightarrow \infty$ . A self-consistent description of the macroscopic model thus changes the real cosmological scenario radically, removing the infinite past of the universe from the standard scenario. On one hand, the universe is a macroscopic classical object. On the other hand, its local properties in the early stages of its evolution are determined by quantum processes. But these quantum processes in turn occur on a classical (massive) gravitational background governed by macroscopic laws.<sup>3</sup>

**Conflicts of interest.** The author declare no conflicts of interest.

## REFERENCES

1. Yu. G. Ignat'ev, "Physical instability of the Universe model with an inflationary (de Sitter) beginning [in Russian]," *Prostranstvo, vremya i fundamental'nye vzaimodeistviya* [Space, time, and fundamental interactions], No. 3(12), 16–22 (2015); English version: "Instability model of the Universe with de Sitter beginning," arXiv:1508.05375v1 [gr-qc] (2015).

<sup>3</sup>The prevailing classical, inert character of the gravitational background ensured by the Birkhoff theorem was noted in our cited works.

2. Yu. G. Ignat'ev, "Macroscopic Einstein equations for a cosmological model with a  $\lambda$  term," *Grav. Cosmol.*, **22**, 264–269 (2016); arXiv:1509.01235v1 [gr-qc] (2015).
3. Yu. G. Ignat'ev, "Macroscopic Einstein equations and cosmology of the early Universe: I. A mathematical model for transverse fluctuations," *Prostranstvo, vremya i fundamental'nye vzaimodeistviya* [Space, time, and fundamental interactions], No. 2(15), 47–61 (2016).
4. A. A. Vlasov, "On vibrational properties of electronic gas [in Russian]," *ZhÉTF*, **8**, No. 3, 291–318 (1938).
5. A. A. Vlasov, *Theory of Vibrational Properties of Electron Gas and Its Applications* [in Russian] (Uch. Zapiski MGU. Fizika, Vol. 75, Book 2, Part 1), Moscow State Univ., Moscow (1945); *Many-Particle Theory* [in Russian], GITTL, Moscow (1950); English transl.: *Many-Particle Theory and Its Application to Plasma*, Gordon and Breach, New York (1961).
6. A. Vlasov, *Statistical Distribution Functions* [in Russian], Nauka, Moscow (1966).
7. L. Landau, "On the vibrations of the electronic plasma [in Russian]," *ZhÉTF*, **16**, 574–586 (1946).
8. N. N. Bogolyubov, "Kinetic equations," *J. Phys. USSR*, **10**, 265–274 (1946).
9. N. N. Bogoliubov, "On the theory of superfluidity," *J. Phys. USSR*, **11**, 23–32 (1947).
10. S. Chandrasekhar, *Principles of Stellar Dynamics*, Dover Publ., New York (1942).
11. A. A. Starobinsky, "Inflaton field potential producing an exactly flat spectrum of adiabatic perturbations," *JETP Lett.*, **82**, 169–173 (2005); arXiv:astro-ph/0507193v1 (2005).
12. A. A. Starobinsky, "Lectures of modern problem of cosmology," in: *Proc. Intl. School on Gravitation and Cosmology "GRACOS-2014"* [in Russian] (Kazan, 30 June–5 July 2014, Yu. G. Ignat'ev, ed.), Kazan Federal University, Kazan (2014), pp. 48–59.
13. S. R. Green and R. M. Wald, "How well is our Universe described by an FLRW model?" *Class. Quantum Grav.*, **31**, 234003 (2014); arXiv:1407.8084v2 [gr-qc] (2014).
14. C. M. Ho and S. D. H. Hsu, "Instability of quantum de Sitter spacetime," *JHEP*, **1504**, 086 (2015); arXiv:1501.00708v2 [hep-th] (2015).
15. T. Buchert et al., "Is there proof that backreaction of inhomogeneities is irrelevant in cosmology?" *Class. Quantum Grav.*, **32**, 215021 (2015); arXiv:1505.07800v2 [gr-qc] (2015).
16. S. R. Green and R. M. Wald, "Comments on backreaction," arXiv:1506.06452v2 [gr-qc] (2015).
17. Yu. G. Ignat'ev, "Statistical dynamics of a classical particle ensemble in the gravitational field," *Grav. Cosmol.*, **13**, 59–79 (2007); arXiv:1012.5578v1 [gr-qc] (2010).
18. Yu. G. Ignatyev, *Relativistic Kinetic Theory of Nonequilibrium Processes in Gravitational Fields* [in Russian], Foliant, Kazan (2010).
19. R. A. Isaakson, "Gravitational radiation in the limit of high frequency: I. The linear approximation and geometrical optics," *Phys. Rev.*, **166**, 1263–1271 (1968).
20. R. A. Isaakson, "Gravitational radiation in the limit of high frequency: II. Nonlinear terms and the effective stress tensor," *Phys. Rev.*, **166**, 1272–1280 (1968).
21. Yu. G. Ignat'ev, "Statistical dynamics of a particle ensemble in GTR [in Russian]," in: *Gravitation and the Theory of Relativity* (No. 14–15, V. R. Kaigorodov, ed.), Kazan Univ. Press, Kazan (1978), pp. 90–107.
22. Yu. G. Ignat'ev, "Statistical dynamics of a classical particle ensemble in the gravitational field [in Russian]," in: *Gravitation and the Theory of Relativity* (No. 20, V. R. Kaigorodov, ed.), Kazan Univ. Press, Kazan (1983), pp. 50–109.
23. Yu. G. Ignat'ev, *Nonequilibrium Universe: Kinetic Models of Cosmological Evolution* [in Russian], Kazan Univ. Press, Kazan (2013).
24. Yu. G. Ignat'ev and A. A. Popov, "Kinetic equations for ultrarelativistic particles in Friedman's world and isotropization of relict radiation by gravitational interactions [in Russian]," in: *Problems in the Theory of Gravity, Relativistic Kinetics, and Evolution of the Universe*, Kazan Univ. Press, Kazan, pp. 5–16.
25. Yu. G. Ignat'ev and A. A. Popov, "Statistical description of an ensemble of ultrarelativistic particles in a spatially flat universe," *Soviet Phys. J.*, **32**, 391–395 (1989).
26. Yu. G. Ignat'ev and A. A. Popov, "Kinetic equations for ultrarelativistic particles in a Robertson–Walker universe and isotropization of relict radiation by gravitational interactions," *Astrophys. Space Sci.*, **163**, 153–174 (1990).

27. Yu. G. Ignat'ev, "The self-consistent field method and the macroscopic universe consisting of a fluid and black holes," *Grav. Cosmol.*, **25**, 354–361 (2019).
28. A. V. Zakharov, "The relativistic kinetic equation for gravitationally interacting particles," *Sov. Phys. JETP*, **69**, 437–443 (1989).
29. A. Z. Petrov, *New Methods in the General Theory of Relativity* [in Russian], Nauka, Moscow (1966).
30. L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics* [in Russian], Vol. 2, *The Classical Theory of Fields*, Nauka, Moscow (1967); English transl., Pergamon, Oxford (1971).
31. E. M. Lifshitz, "On the gravitational stability of the expanding universe," *Sov. Phys. JETP*, **16**, 697–716 (1946).
32. M. Bruni, S. Matarrese, S. Mollerach, and S. Sonego, "Perturbations of spacetime: Gauge transformations and gauge invariance at second order and beyond," *Class. Quantum Grav.*, **14**, 2585–2606 (1997).
33. K. Nakamura, "Gauge invariant variables in two-parameter nonlinear perturbations," *Prog. Theor. Phys.*, **110**, 723–755 (2003); arXiv:gr-qc/0303090v4 (2003).
34. D. Langlois and F. Vernizzi, "Evolution of nonlinear cosmological perturbations," *Phys. Rev. Lett.*, **95**, 091303 (2005); arXiv:astro-ph/0503416v2 (2005).
35. K. Nakamura, "Gauge-invariant formulation of second-order cosmological perturbations," *Phys. Rev. D*, **74**, 101301 (2006); arXiv:gr-qc/0605107v2 (2006).
36. F. Finelli, G. Marozzi, G. P. Vacca, and G. Venturi, "Second-order gauge-invariant perturbations during inflation," *Phys. Rev. D*, **74**, 083522 (2006); arXiv:gr-qc/0604081v3 (2006).
37. G. Domènech and M. Sasaki, "Hamiltonian approach to second order gauge invariant cosmological perturbations," *Phys. Rev. D*, **97**, 023521 (2018); arXiv:1709.09804v2 [gr-qc] (2017).
38. Yu. G. Ignat'ev and I. A. Kokh, "Peculiarities of cosmological models based on a nonlinear asymmetric scalar doublet with minimal interaction: I. Qualitative analysis," *Grav. Cosmol.*, **25**, 24–36 (2019).
39. Yu. G. Ignat'ev and I. A. Kokh, "Peculiarities of cosmological models based on a nonlinear asymmetric scalar doublet with minimal interaction: II. Numerical analysis," *Grav. Cosmol.*, **25**, 37–43 (2019).