Theoretical and Mathematical Physics, **202**(3): 428–436 (2020)

INTEGRABLE EVOLUTION SYSTEMS OF GEOMETRIC TYPE

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We present necessary conditions for the integrability of multicomponent third-order evolution systems of geometric type. For the considered examples, the affine connected space determining the system turns out to be symmetric in the case of zero torsion. In the case of the connection with nonzero torsion, the space is generated by a Bol loop.

Keywords: integrable system, symmetry, affine connected space

DOI: 10.1134/S0040577920030149

1. Introduction

The relations of integrable systems theory to various branches of mathematics are extremely deep and diverse (see, e.g., [1]). Modern integrability theory was inspired by the discovery of the inverse scattering method [2], [3], which allows expressing solutions of integrable differential equations in terms of a solution of the Riemann–Hilbert problem and relates integrable systems to the theory of functions of a complex variable and to functional analysis.

Methods of algebraic geometry were used to find periodic and quasiperiodic solutions. Several classical problems of algebraic geometry were later solved using integrable systems theory. It turns out that self-similar solutions of integrable partial differential equations (PDEs) have the Painlevé property. This relates the integrability of PDEs to isomonodromic deformations of linear operators and, in particular, led to the concept of Frobenius manifolds. The notions of a Hopf algebra and a W-algebra and also the elliptic Poisson bracket came from integrability. The connections between the coefficients of special solutions of some integrable equations and modern combinatorics are amazing.

The relations between various classes of polynomial integrable systems and nonassociative algebraic structures such as Jordan algebras and triple systems are closest to our subject here. We state the first of the results of this type.

The Korteweg–de Vries (KdV) equation

$$u_t = u_{xxx} + 6uu_x$$

for a function $u(x,t)$ is one of the most celebrated integrable models. We consider the multicomponent generalization of this equation

$$u^i_t = u^i_{xxx} + \sum_{j,k} C^i_{jk} u^k u^j_x, \quad i,j,k = 1,\dots,N. \tag{1.1}$$

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This research was supported in part by the Russian state assignment No. 0033-2019-0006 and the State Program of the Ministry of Education and Science of the Russian Federation (Project No. 1.12873.2018/12.1).

Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 202, No. 3, pp. 492–501, March, 2020. Received August 18, 2019. Revised September 27, 2019. Accepted October 8, 2019.

Here and hereafter, repeated indices imply summation. Let $\mathbf{e}_1, \ldots, \mathbf{e}_N$ be a basis of an N-dimensional vector space over $\mathbb{C}$. We define the structure of an algebra $\mathcal{A}$ on this space by

$$\mathbf{e}_j \circ \mathbf{e}_k = \sum_i C^i_{jk} \mathbf{e}_i,$$

where $\circ$ is the product in $\mathcal{A}$. Let

$$U = \sum_k u^k \mathbf{e}_k.$$

We can then write system (1.1) as

$$U_t = U_{xxx} + U \circ U_x. \qquad (1.2)$$

An invariant description of the collection of constants C^i_{jk} for which the system is integrable is given by the observation of Svinolupov [4]: the algebra $\mathcal{A}$ must be Jordan [5].

Example 1. The multiplication $X \circ Y = XY + YX$ defines the structure of a simple Jordan algebra on the vector space of all $m \times m$ matrices. The corresponding system (1.1) is the matrix KdV equation for a matrix $\mathbf{U}(t, x)$ of arbitrary size m

$$\mathbf{U}_t = \mathbf{U}_{xxx} + 3\mathbf{U}\mathbf{U}_x + 3\mathbf{U}_x\mathbf{U}.$$

If the studied class of systems admits arbitrary coordinate changes, then various relations appear between integrability and differential geometry. Well-known structures of this kind arise in the bi-Hamiltonian formalism and in the theory of integrable systems of hydrodynamic type.

Here, we establish relations between integrable evolution systems of geometric type and spaces of affine connection that admit covariantly constant deformations of algebraic structures such as triple Jordan systems [6]. We began an investigation of such systems in the case of torsion-free connections in 1996, but it was interrupted by the unexpected death of S. Svinolupov. Some results were formulated in [7], dedicated to his memory. Unfortunately, that survey was written in a hurry and contains both misprints and significant inaccuracies, and we therefore recently decided to return to this subject by repeating and refining our results from 1996 by a slightly different method. In addition, examples of integrable vector systems of geometric type with a nonsymmetric connection were found in [8]. For all these examples, the curvature tensor is zero. Here, we present the main integrability condition for systems with zero curvature and nonzero torsion, which establishes relations between such systems and Bol loops.

2. Systems of geometric type

We consider N-component systems of the form

$$u^i_t = u^i_{xxx} + A^i_{jk}(\vec{u}) u^j_x u^k_{xx} + B^i_{jks}(\vec{u}) u^j_x u^k_x u^s_x, \quad i = 1, \ldots, N. \qquad (2.1)$$

In contrast to (1.1), the coefficients in this system are not constant numbers but some functions of the variables $u^1, \ldots, u^N$. The right-hand side of system (2.1) is a homogeneous polynomial in derivatives if we assume that the derivative $\partial^k u^i / \partial x^k$ has the weight $k + 1$.

It is easy to see that the class of systems (2.1) is closed under arbitrary point transformations

$$\vec{v} = \vec{\Psi}(\vec{u}). \qquad (2.2)$$

It can be verified that under transformations (2.2), the collection of functions $A^i_{jk}(\vec{u})$ changes in just the same way as the components of some *affine connection*.

Our final goal is an invariant description of integrable systems of form (2.1). It is clear that the answer must be formulated in terms of differential geometry.

Remark 1. Another class of systems of geometric type is given by

$$u^i_{xy} = \sigma^i_{jk}(\vec{u})u^j_x u^k_y. \tag{2.3}$$

Here, the functions $\sigma^i_{jk}(\vec{u})$ can also be interpreted as components of some affine connection, and the classification problem for integrable cases is formulated very simply: For which affine connections is such a system integrable?

Although systems of form (2.3) are more interesting than (2.1) from the standpoint of applications, we consider systems (2.1) here because there is an extremely efficient method for classifying integrable evolution systems based on the existence of infinitesimal higher symmetries [9], [10]. This symmetry approach is much less effective for hyperbolic systems. But if we first describe integrable systems (2.1), then it is easy to construct systems (2.3) for which (2.1) are third-order symmetries (cf. [11]).

Integrable matrix equations. The equations [12], [13]

$$\mathbf{U}_t = \mathbf{U}_{xxx} - 3\mathbf{U}_x\mathbf{U}^{-1}\mathbf{U}_{xx}, \tag{2.4}$$

$$\mathbf{U}_t = \mathbf{U}_{xxx} - \frac{3}{2}\mathbf{U}_x\mathbf{U}^{-1}\mathbf{U}_{xx} - \frac{3}{2}\mathbf{U}_{xx}\mathbf{U}^{-1}\mathbf{U}_x + \frac{3}{2}\mathbf{U}_x\mathbf{U}^{-1}\mathbf{U}_x\mathbf{U}^{-1}\mathbf{U}_x \tag{2.5}$$

are integrable for any size m of the matrix $\mathbf{U}$. It is clear that these equations written in the component form belong to the class of systems (2.1) with $N = m^2$.

We rewrite (2.1) as

$$u^i_t = u^i_3 + 3\alpha^i_{jk}(\boldsymbol{u})u^j_x u^k_{xx} + \left(\frac{\partial \alpha^i_{jk}}{\partial u^l} + 2\alpha^i_{ls}\alpha^s_{jk} - \alpha^i_{sl}\alpha^s_{jk} + \beta^i_{jkl}\right)u^j_x u^k_x u^l_x. \tag{2.6}$$

It turns out that the collection of functions β^i_{jkm} transforms under change of variables (2.2) as components of a *tensor*. Without loss of generality, we assume that $\beta^i_{jkm} = \beta^i_{kjm} = \beta^i_{mkj}$, i.e., for any vectors X, Y, and Z, we have

$$\beta(X,Y,Z) = \beta(Y,X,Z) = \beta(X,Z,Y).$$

System (2.6) is thus defined by the connection Γ with the components α^i_{jk} and by the symmetric tensor β.

Example 2. In the case $N = 1$, Eq. (2.6) can be written as

$$u_t = u_{xxx} + 3\alpha(u)u_x u_{xx} + \big(\alpha'(u) + \alpha(u)^2 + \beta(u)\big)u^3_x. \tag{2.7}$$

It can be shown that it admits an infinite sequence of higher symmetries of the form

$$u_\tau = G(u, u_x, \dots, u_k), \quad \text{where } u_j = \frac{\partial^j u}{\partial x^j},$$

iff $\beta' = 2\alpha\beta$. By a point transformation, we can turn α into zero (every connection is flat for $N = 1$). Therefore, any integrable equation (2.7) is pointwise equivalent to the equation $u_t = u_{xxx} + \mathrm{const}\cdot u^3_x$.

To formulate some general results, we let R and T denote the curvature and torsion tensors of the connection Γ and introduce the tensor

$$\sigma(X,Y,Z) \stackrel{\text{def}}{=} \beta(X,Y,Z) - \frac{1}{3}\delta(X,Y,Z) + \frac{1}{3}\delta(Z,X,Y), \tag{2.8}$$

where

$$\delta(X,Y,Z) \stackrel{\text{def}}{=} T(X,T(Y,Z)) + R(X,Y,Z) - \nabla_X(T(Y,Z)). \tag{2.9}$$

It follows from the identity $\delta(X,Y,Z) = -\delta(X,Z,Y)$ that

$$\sigma(X,Y,Z) = \sigma(Z,Y,X). \tag{2.10}$$

It is easy to verify that

$$\delta(X,Y,Z) = \sigma(X,Z,Y) - \sigma(X,Y,Z). \tag{2.11}$$

Because $\beta(X,X,X) = \sigma(X,X,X)$, system (2.6) is defined by the connection Γ and the tensor $\sigma(X,X,X)$.

Theorem 1. *If system* (2.6) *has an infinite sequence of higher symmetries that are polynomial, are homogeneous with respect to the derivatives,*[1] *and have the form*

$$u^i_\tau = u^i_k + G^i(\boldsymbol{u}, \boldsymbol{u}_x, \dots, \boldsymbol{u}_{k-1}), \tag{2.12}$$

then the conditions

$$(\nabla_X)R(Y,Z,V) = R(Y,X,T(Z,V)), \tag{2.13}$$

$$(\nabla_X)\sigma(Y,Z,V) = 0 \tag{2.14}$$

are satisfied.

Remark 2. It follows from formulas (2.8), (2.9), (2.11), and (2.14) that the tensors δ and β are covariantly constant.

The case of symmetric connection. If $T = 0$, then condition (2.13) has the form

$$(\nabla_X)R(Y,Z,V) = 0. \tag{2.15}$$

This means that the affine connected space is *symmetric*. Such spaces were described by E. Cartan. In this case, further calculations are simplified, and we can prove the following statement.

Theorem 2. *If* $T = 0$ *and the system of equations has an infinite sequence of symmetries* (2.12), *then the condition*

$$\sigma(X,\sigma(Y,Z,V),W) - \sigma(W,V,\sigma(X,Y,Z)) + \sigma(Z,Y,\sigma(X,V,W)) - \sigma(X,V,\sigma(Z,Y,W)) = 0 \tag{2.16}$$

is satisfied.

Remark 3. Identities (2.10) and (2.16) mean that the functions $\sigma^i_{jkm}(\boldsymbol{u})$ are structural constants of a *triple Jordan system* for any fixed value of $\boldsymbol{u}$. Identity (2.14) shows that we are dealing with a covariantly constant deformation of a triple Jordan system, which is related (see formula (2.11)) to the curvature tensor by the identity

$$R(X,Y,Z) = \sigma(X,Z,Y) - \sigma(X,Y,Z). \tag{2.17}$$

Conjecture 1. *If* $T = 0$ *and conditions* (2.10) *and* (2.14)–(2.17) *are satisfied, then the corresponding system* (2.6) *has an infinite sequence of symmetries* (2.12).

Matrix equation (2.5) belongs to the class of systems (2.6) with a symmetric connection constructed in [13]. These systems are described by special deformations of triple Jordan systems.

In contrast to the case of Eq. (2.5), the connection that corresponds to matrix equation (2.4) is not symmetric. To construct other examples of integrable systems with a nonsymmetric connection, the problem of classifying integrable nontriangular vector systems of geometric type was considered in [8].

[1] In fact, these conditions are inessential.

3. Vector examples

We consider the class of the so-called vector isotropic equations of the form

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} + f_2\boldsymbol{u}_{xx} + f_1\boldsymbol{u}_x + f_0\boldsymbol{u}, \tag{3.1}$$

where $\boldsymbol{u}(x,t)$ is a function with values in an N-dimensional Euclidean vector space. The coefficients f_i are (scalar) functions that depend on the six scalar products

$$(\boldsymbol{u},\boldsymbol{u}), \qquad (\boldsymbol{u},\boldsymbol{u}_x), \qquad (\boldsymbol{u}_x,\boldsymbol{u}_x), \qquad (\boldsymbol{u},\boldsymbol{u}_{xx}), \qquad (\boldsymbol{u}_x,\boldsymbol{u}_{xx}), \qquad (\boldsymbol{u}_{xx},\boldsymbol{u}_{xx}).$$

Obviously, Eq. (3.1) is invariant under the group O_N.

Equations (3.1), whose component form belongs to the class of systems (2.1), have the structure

$$\begin{aligned} \boldsymbol{u}_t = \boldsymbol{u}_{xxx} + a_1 u_{[0,1]}\,\boldsymbol{u}_{xx} + (a_2u_{[0,2]} + a_3u_{[1,1]} + a_4u_{[0,1]}^2)\,\boldsymbol{u}_x + \\ + (a_5u_{[1,2]} + a_6u_{[0,2]}u_{[0,1]} + a_7u_{[1,1]}u_{[0,1]} + a_8u_{[0,1]}^3)\boldsymbol{u}, \end{aligned} \tag{3.2}$$

where

$$u_{[i,j]} \overset{\text{def}}{=} (\partial_x^i\boldsymbol{u}, \partial_x^j\boldsymbol{u}), \quad i \le j,$$

and the coefficients a_i are functions of one variable, $a_i = a_i(u_{[0,0]})$.

Some of Eqs. (3.2) have a triangular form in spherical coordinates given by

$$\boldsymbol{u} = R\boldsymbol{v}, \qquad |\boldsymbol{v}| = 1, \quad \text{where } R = |\boldsymbol{u}|.$$

Let

$$v_{[i,j]} = (\partial_x^i\boldsymbol{v}, \partial_x^j\boldsymbol{v}), \quad i \le j.$$

Because $v_{[0,0]} = 1$, we have $D_x(v_{[0,0]}) = 2v_{[0,1]} = 0$. In addition, the equation $D_xv_{[0,1]} = v_{[0,2]} + v_{[1,1]} = 0$ holds, i.e., $v_{[0,2]} = -v_{[1,1]}$ and so on. It is clear that all the variables $v_{[0,k]}$ can be expressed in terms of $v_{[i,k]}$, $1 \le i \le k < \infty$. We say that Eq. (3.1) is *triangular* if it can be written in spherical coordinates as

$$\boldsymbol{v}_t = \boldsymbol{v}_{xxx} + g_2\boldsymbol{v}_{xx} + g_1\boldsymbol{v}_x + g_0\boldsymbol{v}, \qquad R_t = R_{xxx} + S(v_{[1,1]}, v_{[1,2]}, v_{[2,2]}, R, R_x, R_{xx}),$$

where the coefficients g_i depend on $v_{[1,1]}$, $v_{[1,2]}$, and $v_{[2,2]}$.

The class of Eqs. (3.2) is invariant under point transformations of the form

$$\tilde{\boldsymbol{u}} = f(u_{[0,0]})\boldsymbol{u}. \tag{3.3}$$

It can be verified that in the case where $a_1 = -3u_{[0,0]}^{-1}$, we obtain an equation with $\tilde{a}_1 = -3v_{[0,0]}^{-1}$ as a result of any transformation (3.3). For any other coefficient a_1, there exists a function f such that $\tilde{a}_1 = 0$. We thus obtain two classes of Eqs. (3.2) with

$$a_1 = 0 \quad \text{or} \quad a_1 = -\frac{3}{u_{[0,0]}}$$

that are nonequivalent under transformations (3.3).

The following statements were proved in [8].

Theorem 3. *Any nontriangular Eq.* (3.2) *with* $a_1 = 0$ *that has infinitely many higher vector symmetries is reducible by a scaling of the form* $\boldsymbol{u} \to \mathrm{const} \cdot \boldsymbol{u}$ *to an equation in the list*

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} + 3\lambda\left(\frac{(\boldsymbol{u}, \boldsymbol{u}_x)^2}{1 + \boldsymbol{u}^2} - \boldsymbol{u}_x^2\right)\boldsymbol{u}_x + 3F\boldsymbol{u}, \tag{3.4a}$$

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} - 3\left(\frac{(\boldsymbol{u}, \boldsymbol{u}_{xx})}{1 + \boldsymbol{u}^2} - \frac{\boldsymbol{u}^2\boldsymbol{u}_x^2}{1 + \boldsymbol{u}^2} + \frac{\boldsymbol{u}^2(\boldsymbol{u}, \boldsymbol{u}_x)^2}{(1 + \boldsymbol{u}^2)^2}\right)\boldsymbol{u}_x + 3F\boldsymbol{u}, \tag{3.4b}$$

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} - 3\left(\frac{(\boldsymbol{u}, \boldsymbol{u}_{xx})}{1 + \boldsymbol{u}^2} + \frac{(1 - \boldsymbol{u}^2)\boldsymbol{u}_x^2}{2(1 + \boldsymbol{u}^2)} - \frac{(2 - \boldsymbol{u}^2)(\boldsymbol{u}, \boldsymbol{u}_x)^2}{2(1 + \boldsymbol{u}^2)^2}\right)\boldsymbol{u}_x + 3F\boldsymbol{u}, \tag{3.4c}$$

where $\lambda = 1$ *or* $\lambda = 1/2$, $\boldsymbol{u}^2 = u_{[0,0]}$, *and*

$$F = (\boldsymbol{u}, \boldsymbol{u}_x)\frac{(\boldsymbol{u}, \boldsymbol{u}_{xx}) + \boldsymbol{u}_x^2}{1 + \boldsymbol{u}^2} - (\boldsymbol{u}_x, \boldsymbol{u}_{xx}) - \frac{(\boldsymbol{u}, \boldsymbol{u}_x)^3}{(1 + \boldsymbol{u}^2)^2}.$$

Theorem 4. *Any nontriangular Eq.* (3.2) *with* $a_1 = -3/u_{[0,0]}$ *that has infinitely many higher vector symmetries is reducible to an equation in the list*

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} - 3\frac{(\boldsymbol{u}, \boldsymbol{u}_x)}{\boldsymbol{u}^2}\boldsymbol{u}_{xx} - 3\left(\frac{\boldsymbol{u}_x^2}{\boldsymbol{u}^2} - \frac{(\boldsymbol{u}, \boldsymbol{u}_x)^2}{\boldsymbol{u}^4}\right)\boldsymbol{u}_x, \tag{3.5a}$$

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} - 3\frac{(\boldsymbol{u}, \boldsymbol{u}_x)}{\boldsymbol{u}^2}\boldsymbol{u}_{xx} - \frac{3}{2}\left(\frac{\boldsymbol{u}_x^2}{\boldsymbol{u}^2} - 2\frac{(\boldsymbol{u}, \boldsymbol{u}_x)^2}{\boldsymbol{u}^4}\right)\boldsymbol{u}_x, \tag{3.5b}$$

$$\begin{aligned}\boldsymbol{u}_t = \boldsymbol{u}_{xxx} - 3\frac{(\boldsymbol{u}, \boldsymbol{u}_x)}{\boldsymbol{u}^2}\boldsymbol{u}_{xx} - \frac{3}{2}\left(2\frac{(\boldsymbol{u}, \boldsymbol{u}_{xx})}{\boldsymbol{u}^2} + \frac{\boldsymbol{u}_x^2}{\boldsymbol{u}^2}\right)\boldsymbol{u}_x + \\ + 3\left(\frac{(\boldsymbol{u}_x, \boldsymbol{u}_{xx})}{\boldsymbol{u}^2} - \frac{(\boldsymbol{u}, \boldsymbol{u}_x)\boldsymbol{u}_x^2}{\boldsymbol{u}^4} + \frac{4}{3}\frac{(\boldsymbol{u}, \boldsymbol{u}_x)^3}{\boldsymbol{u}^6}\right)\boldsymbol{u}\end{aligned} \tag{3.5c}$$

by a point transformation of form (3.3).

On the proof of Theorems 3 and 4. It was shown in [14] that if an equation of form (3.1) has infinitely many vector symmetries

$$\boldsymbol{u}_\tau = f_n\boldsymbol{u}_n + f_{n-1}\boldsymbol{u}_{n-1} + \cdots + f_0\boldsymbol{u}, \quad \text{where } \boldsymbol{u}_k = \frac{\partial^k\boldsymbol{u}}{\partial x^k}, \tag{3.6}$$

then the functions ρ_i defined below are densities of local conservation laws

$$D_t\rho_n = D_x\theta_n, \quad n = 0, 1, 2, \ldots. \tag{3.7}$$

Here, D_x and D_t denote the total derivatives with respect to x and t (by virtue of Eq. (3.1)). The first two densities have the forms

$$\rho_0 = -\frac{1}{3}f_2, \qquad \rho_1 = \frac{1}{9}f_2^2 - \frac{1}{3}f_1 + \frac{1}{3}D_xf_2, \tag{3.8}$$

and the remaining densities can be found using the recurrence relation

$$\begin{aligned}\rho_{n+2} = \frac{1}{3}(\theta_n - f_0\delta_{n,0} - 2f_2\rho_{n+1} - f_2\,D_x\rho_n - f_1\rho_n) - \\ - \frac{1}{3}\left(f_2\sum_{s=0}^{n}\rho_s\rho_{n-s} + \sum_{0\le s+k\le n}\rho_s\rho_k\rho_{n-s-k} + 3\sum_{s=0}^{n+1}\rho_s\rho_{n-s+1}\right) - \\ - D_x\left(\rho_{n+1} + \frac{1}{2}\sum_{s=0}^{n}\rho_s\rho_{n-s} + \frac{1}{3}D_x\rho_n\right), \quad n \ge 0,\end{aligned} \tag{3.9}$$

where $\delta_{i,j}$ is the Kronecker symbol.

Using relations (3.8) and (3.9), we find the next density

$$\rho_2 = -\frac{1}{3}f_0 + \frac{1}{3}\theta_0 - \frac{2}{81}f_2^3 + \frac{1}{9}f_1 f_2 - D_x\left(\frac{1}{9}f_2^2 + \frac{2}{9}\,D_x f_2 - \frac{1}{3}f_1\right), \tag{3.10}$$

and so on. We note that the density ρ_n, $n \geq 2$, is expressed in terms of the coefficients of Eq. (3.1) and the flows $\{\theta_0, \theta_1, \ldots, \theta_{n-2}\}$, which are to be found from the previous relations (3.7).

To eliminate the function θ_n from (3.7), we can apply the variational derivative

$$\frac{\delta}{\delta \boldsymbol{u}} = \sum_{0\leq i\leq j}\left[(-D_x)^i\left(\boldsymbol{u}_j \frac{\partial}{\partial u_{[i,j]}}\right) + (-D_x)^j\left(\boldsymbol{u}_i \frac{\partial}{\partial u_{[i,j]}}\right)\right]$$

to both sides of (3.7) and use the fact that $\delta(D_x g)/\delta\boldsymbol{u} = 0$ for any function g. As a result, we obtain the necessary conditions for integrability:

$$\frac{\delta}{\delta \boldsymbol{u}}(D_t\rho_n) = 0, \quad n = 1, 2, \ldots. \tag{3.11}$$

Conditions (3.11) are especially efficient in the cases where $n = 1, 2$ because expressions (3.8) are independent of the fluxes θ_i.

Splitting relations (3.11) with respect to all scalar products except $u_{[0,0]}$, we obtain a system of ordinary differential equations for the coefficients $a_i(u_{[0,0]})$, $i = 1, \ldots, 8$. The system equivalent to the first six conditions (3.11) suffices for finding equations from lists (3.4) and (3.5) and rejecting all the other equations. To verify the integrability of the obtained equations, auto-Bäcklund transformations containing an arbitrary parameter were found for them in [8].

4. Discussion of the results

It can be verified that the curvature tensor for Eq. (3.2) has the form

$$\begin{aligned}
T(X,Y) &= \frac{1}{3}(a_1 - a_2)\big((\boldsymbol{u},X)Y - (\boldsymbol{u},Y)X\big),\\
R(X,Y,Z) &= \frac{1}{9}\big(q(\boldsymbol{u},X)(\boldsymbol{u},Z) + p(X,Z)\big)Y - \frac{1}{9}\big(q(\boldsymbol{u},X)(\boldsymbol{u},Y) + p(X,Y)\big)Z + \\
&\quad + \frac{r}{9}\big((\boldsymbol{u},Y)(X,Z) - (\boldsymbol{u},Z)(X,Y)\big)\boldsymbol{u},
\end{aligned}$$

where

$$p = a_2a_5\boldsymbol{u}^2 - 3a_2 + 3a_5, \qquad q = a_2a_6\boldsymbol{u}^2 + a_2^2 + 3a_6 - 6a_2', \qquad r = a_5a_6\boldsymbol{u}^2 + a_5^2 - 3a_6 + 6a_5'.$$

Substituting the coefficients of equations in lists (3.4) and (3.5) in these formulas, we find that the connection is symmetric. It can be verified that these equations satisfy conditions (2.10) and (2.14)–(2.17). It turns out that for the remaining equations, the tensor R is equal to zero. Matrix equation (2.4) has the same property.

Conjecture 2. *For any integrable system* (2.6), *either* $T = 0$ *or* $R = 0$.

For equations with $R = 0$, the condition that the tensor δ is covariantly constant (see Remark 2) leads to the relation

$$\nabla_X\big((\nabla_Y)T(Z,V) - T(Y,T(Z,V))\big) = 0.$$

This identity together with the condition $R = 0$ means that the space of affine connection is generated by some Bol loop [15] and that the binary and ternary operations

$$X \circ Y = T(X,Y), \qquad (X,Y,Z) = \nabla_Z(T(X,Y)) - T(Z,T(X,Y))$$

satisfy the identities of the left Sabinin algebra [15]

$$X \circ X = 0, \qquad (Y,X,X) = 0, \qquad (X,Y,Z) + (Y,Z,X) + (Z,X,Y) = 0,$$
$$(X,Y,(Z,U,V)) = (Z,U,(X,Y,V)) + ((X,Y,Z),U,V) + (Z,(X,Y,U),V),$$
$$(X,Y,Z) \circ U - (X,Y,U) \circ Z + (Z,U,X \circ Y) - (X,Y,Z \circ U) + (X \circ Y) \circ (Z \circ U) = 0.$$

In addition, the ternary operation is related (see (2.9) and (2.11)) to the tensor σ by $(X,Y,Z) = \sigma(X,Y,Z) - \sigma(X,Z,Y)$. For matrix equation (2.4), these two operations and the triple system σ are generated by the associative matrix multiplication

$$X \circ Y = XY - YX, \qquad (X,Y,Z) = XYZ - XZY + ZYX - YZX,$$
$$\sigma(X,Y,Z) = XYZ + ZYX.$$

In this example, the tensor σ defines a triple Jordan system. It is still unknown whether this is always the case for integrable systems with a nonzero torsion.

Acknowledgments. The authors are grateful to E. Ferapontov, A. Meshkov, and P. Leal da Silva for the useful discussions.

Conflicts of interest. The author declares no conflicts of interest.

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