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HAMILTONIAN DESCRIPTION OF VORTEX SYSTEMS

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In the framework of two-dimensional ideal hydrodynamics, we define a vortex system as a smooth vorticity function with a few local positive maximums and negative minimums separated by curves of zero vorticity. We discuss the invariants of such structures that follow from the vorticity conservation law and the invertibility of Lagrangian motion. Introducing new functional variables diagonalizing the original noncanonical Poisson bracket, we develop a Hamiltonian formalism for vortex systems.

Keywords: vortex, continuum Hamiltonian system, Poisson bracket, vorticity, two-dimensional hydrodynamics

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1. Introduction

The fundamentals of applying the Hamiltonian formalism to hydrodynamic systems were developed in [1], [2] and proved an efficient tool for treating a variety of problems in fluid mechanics. Here, we apply that approach to a relatively simple object, a vortex system in the framework of ideal two-dimensional hydrodynamics. Our starting point is the conservation equation for the vorticity Ω,

$$\frac{\partial\Omega}{\partial t} + J(\psi,\Omega) = 0, \tag{1}$$

written in the Hamiltonian form as [3]

$$\frac{\partial\Omega}{\partial t} = \{H,\Omega\},$$

where the noncanonical Poisson bracket can be written as

$$\{F,G\}_\Omega = \int \Omega(\mathbf{r}) J_{x,y}\left(\frac{\delta F}{\delta\Omega(\mathbf{r})},\frac{\delta G}{\delta\Omega(\mathbf{r})}\right)d\mathbf{r}, \qquad \mathbf{r}=(x,y). \tag{2}$$

Here, $F=F(\Omega)$ and $G=G(\Omega)$ are smooth functionals, and $J_{x,y}(f,g) = f_x g_y - f_y g_x$ is the Jacobian. Hereafter, the integration region is the whole plane $\mathbb{R}^2$ unless other indicated. The Hamiltonian is given by

$$H = \frac{1}{2}\int |\nabla\psi|^2\,d\mathbf{r} = -\frac{1}{2}\int \psi\Omega\,d\mathbf{r}, \tag{3}$$

which implies that

$$\frac{\delta H}{\delta\Omega} = -\psi. \tag{4}$$

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Combining (2) and (3), we extend (1) to any functional:

$$\frac{\partial F(\Omega)}{\partial t} = \int \frac{\delta F}{\delta \Omega(\mathbf{r})} J_{x,y}(\psi, \Omega(\mathbf{r}))\, d\mathbf{r}. \tag{5}$$

We next assume that $\Omega(\mathbf{r})$ decays sufficiently fast as $|\mathbf{r}| \to \infty$, and we hence express the stream function in terms of the vorticity as

$$\psi(\mathbf{r}) = \frac{1}{2\pi} \int \log(\mathbf{r} - \mathbf{r}')\Omega(\mathbf{r}')\, d\mathbf{r}'. \tag{6}$$

From (4) and (6), we obtain

$$H = \frac{1}{4\pi} \iint \log(\mathbf{r}_1 - \mathbf{r}_2)\Omega(\mathbf{r}_1)\Omega(\mathbf{r}_2)\, d\mathbf{r}_1\, d\mathbf{r}_2. \tag{7}$$

We list our main objectives:

- To formulate and rigorously prove conservation laws for the topography of the vorticity Ω, such as the number of critical points (points where $\nabla\Omega = \mathbf{0}$), the vorticity values at the critical points, and the number of distinct level curves defined by $\Omega(\mathbf{r}) = w$ corresponding to a fixed value w, which we call contour lines.
- To derive translation equations for the critical points.
- To derive and discuss equations for contour lines in both a Hamiltonian form and in the form of closed integro-differential equations. In this regard, our efforts can be viewed as an extension of contour dynamics [4], [5] to smooth vorticity functions.
- To show how some well-known models such as point vortex systems and FAVOR [6] can be derived from the underlying vorticity class.

We now specify the class of functions $\Omega(\mathbf{r})$ called an N-vortex system. Let

$$\mathbf{H}_\Omega(\mathbf{r}) = \begin{pmatrix} \Omega_{xx} & \Omega_{xy} \\ \Omega_{xy} & \Omega_{yy} \end{pmatrix}$$

be the Hessian of vorticity. We assume that the function Ω satisfies the following conditions.

Condition 1. The function Ω has exactly N extremums (maximum or minimum) at the points $\mathbf{z}_k = (\xi_k, \eta_k)$, $k = 1, \ldots, N$, i.e.,

$$\nabla\Omega(\mathbf{z}_k) = \mathbf{0}, \qquad \det(\mathbf{H}_\Omega(\mathbf{z}_k)) > 0.$$

Condition 2. The set $\Gamma_0 = \{\mathbf{r} \in \mathbb{R}^2 \mid \Omega(\mathbf{r}) = 0\}$ is either empty or divides the plane $\mathbb{R}^2$ into $N \geq 2$ distinct regions G_k, $k = 1, \ldots, N$, such that each G_k contains exactly one extremum and the vorticity has the same sign for all points in G_k. In other words,

$$\mathbb{R}^2 = \bigcup_k G_k, \qquad G_k \cap G_j = \varnothing, \qquad \Omega(\partial G_k) = 0, \quad \mathbf{z}_k \in G_k.$$

Condition 3. For any two adjacent regions G_k and G_j, the signs in G_k and G_j are opposite.

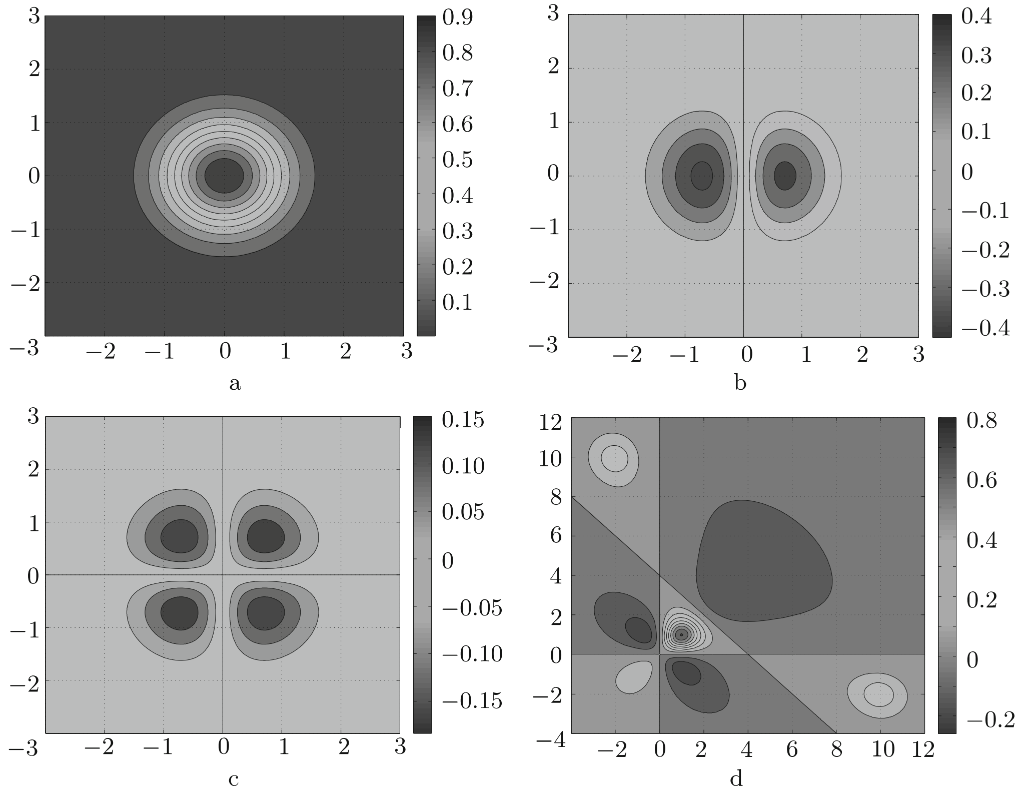

Fig. 1. Examples of N-vortex systems: (a) monopole, (b) dipole, (c) quadrupole, and (d) multipole.

We can ensure the last condition by assuming that the set Γ_0 of zero vorticity lines is an Euler graph [7]. For such graphs, the corresponding regions G_k can be painted with only two colors such that any two adjacent regions have opposite colors. Our "colors" mean positive and negative vorticity. Examples of some important vortex systems satisfying Conditions 1–3 are shown in Fig. 1.

Below, we prove the remarkable fact that Conditions 1–3 and, in particular, the value of N are preserved by Eq. (1). In other words, critical points are neither created nor annihilated during the system evolution. Moreover, the vorticity values at the critical points $\omega_k = \Omega(\mathbf{z}_k)$ are also preserved.

We do not specifically consider saddles of $w = \Omega(\mathbf{r})$ (hyperbolic critical points), because they are unimportant in the context of our goals. But we note that their number and vorticity values are also preserved. We can say even more if we assume that the Euler graph representing Γ_0 has exactly four edges incident to each vertex as in Figs. 1c and 1d. Namely, in this case, each vertex (an intersection of two zero vorticity lines) is a saddle. Conversely, each saddle is a vertex of the graph. Therefore, the vorticity at each saddle is zero.

There is one more important invariant that is a consequence of two fundamental laws: vorticity conservation in Lagrangian particles and incompressibility. For a fixed $w > 0$, let $n(w)$ be the number of disjoint connected regions where $\Omega > w$ and the number of disjoint connected regions where $\Omega < w$ if $w < 0$. We show that $n(w)$ is also preserved by (1). In other words, no merging of vorticity patches is possible in ideal two-dimensional hydrodynamics. We here clearly state this obvious claim because the problem of vortex merger has recently been considered in many works (see, e.g., [8]), but the initial equations were not always formulated explicitly.

An important role of the extreme points is that they serve as natural poles for local polar coordinates in the parameterization of vorticity lines in each particular region G_k. Let $r = \rho_k(\varphi, w)$ be the polar equation of the single contour line corresponding to the vorticity level w in a certain region

$$\Gamma_{w,k} = \{\mathbf{r} \in G_k \mid \Omega(\mathbf{r}) = w)\}. \tag{8}$$

We derive closed evolution equations for ρ_k, $k = 1, \dots, N$, that explicitly show the interaction between vorticity contours corresponding to different regions G_k and different vorticity levels w. An essential drawback of the polar parameterization is that the assumption that a curve can be represented in polar form with a single valued $\rho(\varphi)$ is not preserved by the system. Therefore, such a consideration is applicable only for small integration times or if a small vicinity of an extreme point is considered or, equivalently, small values of ρ_k where curve (8) is well approximated by an ellipse. Therefore, we first consider a general parameterization (still pinned to G_k), $\mathbf{r} = \hat{\mathbf{r}}(p, w)$ of $\Gamma_{w,k}$, where p is a positive parameter, for example, the arc length (natural parameterization). Such a parameterization is free from the above drawback. In both cases, polar and natural parameterization, the resulting equations are too sophisticated (nonlinear, integro-differential, nondecoupling) to work on them efficiently with the exception of monopoles and dipoles ($N = 1$ and $N = 2$). Hence, the equations describing the evolution of contours (8) for $N > 2$ are for now only of purely theoretical interest. Our goal is to clarify conservation laws concerning the vorticity topography and to elucidate the underlying Hamiltonian structure rewritten in the new phase variables $\{\rho_k(\varphi, w)\}$ or $\{\hat{\mathbf{r}}_k(p, w)\}$, $k = 1, \dots, N$.

We note that Condition 3 is not needed for deriving the mentioned equations. Its purpose is the permanent exclusion of unstable vortex structures. For the same reason, we do not consider bifurcation points where the determinant of the Hessian is zero.

This paper is organized as follows. In Sec. 2, we state and sketch the proof of the most important conservation laws for the considered vortex systems. In Sec. 3, we prove the main result on diagonalizing the Poisson bracket. As a consequence, we derive equations for vorticity lines and extremum translation for a general parameterization. In Sec. 4, we specify these equations for the polar parameterization and discuss them. In Sec. 5, we give examples of applying the equations to monopoles and dipoles. In Sec. 6, we briefly discuss the results and draw conclusions. Some details are presented in the appendix.

2. Invariants

Proposition 1. *The number of critical points and their type is preserved by Eq.* (1).

Proof. Let $\mathbf{r}(t, \mathbf{r}_0) = \mathbf{r}$ be the position of a Lagrangian particle starting from $\mathbf{r}_0$ at the instant t and $\mathbf{J}(\mathbf{r}, \mathbf{r}_0) = \partial\mathbf{r}/\partial\mathbf{r}_0$ be the Jacobi matrix of the diffeomorphism $T\colon \mathbf{r}_0 \to \mathbf{r}$. Direct computations based on the vorticity conservation equation give $\nabla\Omega(\mathbf{r}) = \mathbf{J}\,\nabla\Omega(\mathbf{r}_0)$ and $\mathbf{H}_\Omega(\mathbf{r}) = \mathbf{J}\mathbf{H}_\Omega(\mathbf{r}_0)\mathbf{J}^*$ if $\nabla\Omega(\mathbf{r}_0) = \mathbf{0}$, where the asterisk denotes transposition. The statement now follows from $\det(\mathbf{J}) = 1$, which is a consequence of incompressibility [9].

Under Conditions 1–3, the set $\Gamma_w = \{\mathbf{r} \mid \Omega(\mathbf{r}) = w\}$ for any $w \neq 0$ consists of a finite number of closed curves, and the number $n(w)$ of such curves obviously does not exceed N. The next statement shows that $n(w)$ is also preserved.

Proposition 2. *For definiteness, let $w > 0$, and let the region $D = \{\mathbf{r} \mid \Omega(\mathbf{r}) > w\}$ at the initial instant consist of two disjoint subregions, $D = D_1 \cup D_2$. Then D consists of two disjoint subregions at all instants.*

Proof. We suppose that the regions have merged at some instant t. Let $\mathbf{r}_1 \in D_1$ and $\mathbf{r}_2 \in D_2$. We consider a continuous curve C joining $T(\mathbf{r}_1)$ and $T(\mathbf{r}_2)$ completely belonging to the merger, i.e., $C \in$

$T(D_1) \cup T(D_2)$. This means the vorticity at each point of C is greater than w, $\Omega(C) > w$. Therefore, we also have $\Omega(T^{-1}(C)) > w$, but the curve $T^{-1}(C)$ is certainly partly outside both D_1 and D_2, and we hence have $\Omega < w$ at some point of the curve, which is a contradiction. By the invertibility of T, a connected region also cannot be broken down into two distinct subregions.

Proposition 3. *The local extremums are preserved. More exactly, the equations*

$$\frac{\partial \xi_k}{\partial t} = -\psi_y(\xi_k, \eta_k), \qquad \frac{\partial \eta_k}{\partial t} = \psi_x(\xi_k, \eta_k), \quad k = 1, \dots, N, \tag{9}$$

are satisfied for any $k = 1, \dots, N$.

Proof. We fix a certain G_k and for simplicity omit the subscript k in the following computations. Assuming that there is a single critical point of Ω in G, we can represent its coordinates as functionals of Ω [10]:

$$\xi(\Omega) = \int_G x\, \delta(\Omega_x)\delta(\Omega_y) S(\Omega)\, d\mathbf{r}, \qquad \eta(\Omega) = \int_G y\, \delta(\Omega_x)\delta(\Omega_y) S(\Omega)\, d\mathbf{r}, \tag{10}$$

where $S(\Omega) = \Omega_{xx}\Omega_{yy} - \Omega_{xy}^2$ is the determinant of the Hessian. The idea behind such a representation is that we can write a unique solution of $\mathbf{f}(\mathbf{r}) = \mathbf{0}$, where $\mathbf{r} \in \mathbb{R}^n$ and $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ as

$$\mathbf{r}_0 = \int_{\mathbb{R}^n} \mathbf{r}\, \delta(\mathbf{f}(\mathbf{r})) \left| \det\left(\frac{\partial \mathbf{f}}{\partial \mathbf{r}}\right) \right| d\mathbf{r},$$

where $\partial\mathbf{f}/\partial\mathbf{r}$ is the Jacobi matrix of the map $\mathbf{f}$.

Taking variational derivatives in (9), we obtain [10]

$$\frac{\delta \xi}{\delta \Omega(x,y)} = \delta'(\Omega_x)\delta(\Omega_y)S(\Omega), \qquad \frac{\delta \eta}{\delta \Omega(x,y)} = \delta'(\Omega_y)\delta(\Omega_x)S(\Omega)$$

and substitute each expression in (5). After changing the variables $u = \Omega_x(x,y)$ and $v = \Omega_y(x,y)$ in the integrals, we obtain (9).

3. Poisson bracket diagonalization: Evolution and translation equations for vorticity lines

We assume that the vorticity values at the extremums are ordered as $\omega_1 > \omega_2 > \dots > \omega_N$. Let

$$\begin{gathered} x = \xi_k + \hat{x}_k(p, w), \qquad y = \eta_k + \hat{y}_k(p, w), \\ (p, w) \in D_k = \{0 \le p < L_k(w),\ 0 < w < \omega_k\}, \end{gathered} \tag{11}$$

be an arbitrary parameterization of the vorticity line $\Gamma_{w,k}$ (see (8)) corresponding to the level w in the region G_k, where p is a positive parameter with the upper limit $L_k(w)$ depending on w in the general case. For example, for the natural parameterization, p is the length of a particular contour. According to (11), the origin of the local rectangular coordinate system $(\hat{x}, \hat{y})$ is at $\mathbf{z}_k = (\xi_k, \eta_k)$.

We assume that for a fixed (ξ_k, η_k), the map $(p, w) \to (x, y)$ defined by (11) is a one-to-one correspondence between D_k and G_k. It follows from this assumption that we can write the inverse of (11) as $p = p_k(x, y)$, $w = \Omega(x, y)$, where $p_k(x, y)$ is a function determined by a specific parameterization and depends on the region G_k. Let

$$\mathbf{e} = (1, 1), \qquad \mathbf{V}(\mathbf{r}) = \frac{1}{S(\Omega)} \begin{pmatrix} -\Omega_{xy} & \Omega_{xx} \\ \Omega_{yy} & -\Omega_{yx} \end{pmatrix},$$

$$\hat{\mathbf{r}}_k(p, w) = (\hat{x}_k(p, w), \hat{y}_k(p, w)), \qquad \nabla\delta(\mathbf{r}) = (\delta'(x)\delta(y), \delta(x)\delta'(y))^*.$$

The following statement (proved in the appendix) plays a key role in our further computations.

Lemma 1. *If we regard the vector function* $\hat{\mathbf{r}}(p,w)$ *of local coordinates on* $\Gamma_{k,w}$ *in* $G = G_k$ *as a functional of* $\Omega(\mathbf{r})$*, then its variational derivative is given by (we omit the subscripts* k *for brevity)*

$$\frac{\delta\hat{\mathbf{r}}(p,w)}{\delta\Omega(\mathbf{r})} = \frac{\hat{\mathbf{r}}_w(p,w)}{g(p,w)}\left(\delta(\Omega(x,y)-w)\delta(p(x,y)-p) - \mathbf{e}\frac{\partial}{\partial p}(\hat{\mathbf{r}}(p,w)\mathbf{V}(\mathbf{z})\,\nabla\delta(\mathbf{r}-\mathbf{z}))\right), \tag{12}$$

where $g(p,w) = \hat{x}_p\hat{y}_w - \hat{y}_p\hat{x}_w$*.*

We set

$$\zeta_k(p,w) = \int_w^{\omega_k} g_k(p,u)\,du \tag{13}$$

and let $F = F(\zeta_1,\dots,\zeta_N)$ be a smooth functional of the new variables. Obviously, it is also a functional of Ω, denoted by the same symbol F. We next introduce the range R_k of values of $\Omega(\mathbf{r})$, $\mathbf{r}\in G_k$, that is either the interval $(0,\omega_k)$ or $(\omega_k,0)$ depending on the sign of ω_k. Finally, we let $S(w) = \{k\in\{1,\dots,N\} \mid w\in R_k\}$ be the list of all regions containing a piece of $\Gamma_w = \{\mathbf{r}\mid\Omega(\mathbf{r}) = w\}$.

Proposition 4. *If* $\delta F/\delta\Omega(\mathbf{r})$ *is a smooth function of* $\mathbf{r}$*, then*

$$\frac{\delta F}{\delta\zeta_k(p,w)} = \left.\frac{\delta F}{\delta\Omega(\mathbf{r})}\right|_{\mathbf{r}=\mathbf{z}_k+\hat{\mathbf{r}}_k(p,w)}$$

for $w\neq\omega_k$*, and*

$$\begin{aligned}
&\{\zeta_k(p,w),F\} = L_k(F),\\
&L_k(F) = \frac{\partial}{\partial p}\left\{\sum_{j\in S(w)}\left(\left.\frac{\delta F}{\delta\zeta_j(q,w)}\right|_{q=p_j(\Delta\mathbf{z}_{kj}+\hat{\mathbf{r}}_k(p,w))}\right) - \nabla\left.\frac{\delta F}{\delta\Omega(\mathbf{r})}\right|_{\mathbf{r}=\mathbf{z}_k}\cdot\hat{\mathbf{r}}_k(p,w)\right\},
\end{aligned} \tag{14}$$

where $\Delta\mathbf{z}_{kj} = \mathbf{z}_k - \mathbf{z}_j$*.*

We note that the Poisson bracket $\{\zeta_k(p,w),\zeta_j(q,u)\}$ is undefined for $k=j$.

The main steps in deriving (14) are as follows. Using (13), the chain rule, and

$$\frac{\delta\zeta(p,w)}{\hat{\mathbf{r}}(q,u)} = \delta(p-q)\delta(w-u)\hat{\mathbf{r}}_p^{\perp}(q,u) - I_{(w,\omega_k)}(u)\delta'(p-q)\hat{\mathbf{r}}_u^{\perp}(q,u),$$

where $I_A(x)$ is the indicator of A and $(x,y)^{\perp} = (-y,x)$, we first obtain

$$\frac{\delta\zeta(p,w)}{\delta\Omega(\mathbf{r})} = \delta(\Omega(\mathbf{r})-w)\delta(p(\mathbf{r})-p) - \frac{\partial}{\partial p}\left(\hat{\mathbf{r}}(p,w)\mathbf{V}\nabla\delta(\mathbf{r}-\mathbf{z})\right),$$

which leads to the first equation in (14). We then substitute the obtained expression in (2) and split the integration over $\mathbb{R}^2$ into integrations over G_j, $j=1,\dots,n(w)$. We finally pass to the local coordinates $(\hat{x}_j,\hat{y}_j)$.

Setting $F=H$ in (14), we obtain Hamiltonian equations for the new variables

$$\frac{\partial\zeta_k(p,w)}{\partial t} = L_k(H). \tag{15}$$

To obtain a closed system in terms of the variables ζ_k, we again change the integration over the whole plane in (7) into integrations over the distinct G_k, $k=1,\dots,N$. The result is $H=\sum_{k,j=1}^N H_{kj}$, where

$$H_{kj} = \frac{1}{4\pi}\int_0^{\omega_k}\!\int_0^{\omega_j}\!\int_0^{L_k(w_1)}\!\int_0^{L_j(w_2)} w_1w_2\,\zeta_k(p_1,w_1)_{w_1}\zeta_j(p_2,w_2)_{w_2}\log D_{kj}(p_1,w_1,p_2,w_2)\,dp_1\,dp_2\,dw_1\,dw_2, \tag{16}$$

where

$$D_{kj} = \sqrt{\big(\Delta\xi_{kj} + \hat{x}_k(p_1, w_1) - \hat{x}_j(p_2, w_2)\big)^2 + \big(\Delta\eta_{kj} + \hat{y}_k(p_1, w_1) - \hat{y}_j(p_2, w_2)\big)^2},$$

$\Delta\xi_{kj} = \xi_k - \xi_j$, and $\Delta\eta_{kj} = \eta_k - \eta_j$. A cumbersome expression for the Hamiltonian is a trade-off for a diagonal Poisson bracket. Finally, $\hat{x}$ and $\hat{y}$ must be expressed in terms of ζ. This can be easily done in the case of a polar parameterization, which we consider below.

4. Polar parameterization

We now let $p = \varphi$ be a polar angle and introduce the local polar coordinates for each G_k

$$x = \xi_k + \rho_k(\varphi, w)\cos\varphi, \qquad y = \eta_k + \rho_k(\varphi, w)\sin\varphi,$$

$$(\varphi, w) \in D_k = [0, 2\pi] \times [0, \omega_k], \quad (x, y) \in G_k,$$

where $\rho_k(\varphi, w)$ is the distance from $\mathbf{z}_k$ to the point on the contour in the direction φ. In other words, the closed curve $\Gamma_{w,k}$ is covered by the equation $r = \rho_k(\varphi, w)$. It is easy to see that the new phase variable introduced in (13) now becomes $\zeta_k = \rho_k^2(\varphi, w)/2$ and (15) implies the following statement.

Proposition 5. *The function ρ_k satisfies the equation*

$$\frac{1}{4}\frac{\partial \rho_k^2(\varphi, w)}{\partial t} = \frac{\partial}{\partial\varphi}\left\{\frac{\delta H}{\delta\rho_k^2(w,\varphi)} + \sum_{j\neq k}\frac{\rho_j(\theta_{kj}, w)}{\rho_k(\varphi, w)}\frac{\delta H}{\delta\rho_j^2(\theta, w)}\bigg|_{\theta=\theta_{kj}(\varphi)} - \rho_k(\varphi, w)\left(D_\varphi\frac{\delta H}{\delta\rho_k^2}\right)_{w=\omega_k}\right\}, \tag{17}$$

where D_φ is the derivative in the direction given by φ and $\theta = \theta_{kj}(\varphi)$ is the solution of the equation

$$\frac{\Delta\eta_{kj} + \rho_j(\theta, w)\sin\theta}{\Delta\xi_{kj} + \rho_j(\theta, w)\cos\theta} = \tan\varphi.$$

As a result, expression (16) becomes

$$H_{kj} = \frac{1}{16\pi}\int_0^{\omega_k}\int_0^{\omega_k}\int_0^{2\pi}\int_0^{2\pi} w_1 w_2 (\rho_k^2)_{w_1}(\rho_j^2)_{w_2}\log D_{kj}(\varphi_1, w_1, \varphi_2, w_2)\, d\varphi_1\, d\varphi_2\, dw_1\, dw_2,$$

where

$$\rho_k = \rho_k(\varphi_1, w_1), \qquad \rho_j = \rho_j(\varphi_2, w_2),$$

$$D_{kj} = \sqrt{(\Delta\xi_{kj} + \rho_k\cos\varphi_1 - \rho_j\cos\varphi_2)^2 + (\Delta\eta_{kj} + \rho_k\sin\varphi_1 - \rho_j\sin\varphi_2)^2}.$$

To obtain a closed system for ρ_k, we must rewrite (17) in terms of the stream function

$$\psi_j(\theta, w) = \psi(\xi_j + \rho_j(\theta, w)\cos\theta, \eta_j + \rho_j(\theta, w)\sin\theta).$$

The result is

$$\frac{1}{2}\frac{\partial\rho_k(\varphi, w)}{\partial t} = -\frac{1}{\rho_k(\varphi, w)}\frac{\partial}{\partial\varphi}\left\{\psi_k(\varphi, w) + \sum_{j\neq k}\psi_j(\theta_{kj}, w) - \rho_k(\varphi, w)D_\varphi\psi(\varphi, \omega_k)\right\},$$

and we then substitute the expression

$$\psi_j(\theta, w) = -\frac{1}{2\pi} \sum_{\alpha=1}^{N} \int_0^{2\pi} \int_0^{\omega_\alpha} u\rho_\alpha(u, \theta) \frac{\partial \rho_\alpha(u, \theta)}{\partial u} \log D_{j\alpha}(\theta, w, \varphi, u)\, du\, d\varphi \tag{18}$$

derived from (6) for ψ.

We write the translation equations for the critical points in terms of the complex coordinates $z_k = \xi_k + i\eta_k$ ($z_k^\dagger = \xi_k - i\eta_k$) by substituting (18) in (9):

$$\frac{\partial z_k^\dagger}{\partial t} = \frac{1}{2\pi i} \int_0^{\omega_k} \int_0^{2\pi} \rho_k(\varphi, w) e^{-i\varphi}\, d\varphi\, dw + \sum_{j \neq k} \frac{1}{2\pi i} \int_0^{\omega_j} \int_0^{2\pi} \frac{w\rho_j(\varphi, w)\, \partial\rho_j(\varphi, w)/\partial w}{z_k - z_j - \rho_j(\varphi, w) e^{i\varphi}}\, dw\, d\varphi. \tag{19}$$

Finally, we express other well-known invariants of (1) in terms of the new variables. We first consider the Casimir functionals $K(\Omega) = \int K(\Omega(\mathbf{r}))\, d\mathbf{r}$, where $K(\,\cdot\,)$ is an arbitrary smooth function. Splitting the integral over $\mathbb{R}^2$ into the regions G_j, $j = 1, \ldots, N$, and passing to the new variables (ξ_j, η_j, ρ_j), we obtain

$$K = -\sum_j \frac{1}{2} \int_0^{\omega_j} K(w) \frac{\partial}{\partial w} \biggl(\int_0^{2\pi} \rho_j^2(\varphi, w)\, d\varphi \biggr) dw.$$

We note that the inner integral is simply the doubled area of the region bounded by $\Gamma_{w,j}$.

In the same manner, we can obtain the first moment

$$c = \int (x + iy)\Omega(\mathbf{r})\, d\mathbf{r}.$$

We have

$$c = \sum_j \int_0^{\omega_j} \int_0^{2\pi} \biggl(\frac{1}{2} z_j \rho_j^2(\varphi, w) + \frac{1}{3} \rho_j^3(\varphi, w) e^{i\varphi} \biggr) d\varphi\, dw.$$

5. Monopole and dipole

A monopole is defined by the conditions $N = 1$ and $\Omega(\mathbf{r}) > 0$, $\mathbf{r} \in \mathbb{R}^2$. We let $M = \omega_1 > 0$ denote the maximum vorticity value and introduce local polar coordinates with the pole at the maximum point $\mathbf{z} = (\xi, \eta)$. In other words, the closed curve Γ_w is covered by the equation $r = \rho(\varphi, w)$ in local polar coordinates. As already noted, $\rho(\varphi, w)$ does not remain a single-valued function in the evolution process except in trivial cases such as circular contours for all w. Therefore, if we interpret $\rho(\varphi, w)$ as a distance, then all the following equations hold only during a finite integration time (probably small). But if we treat $\rho(\varphi, w)$ as a generalized distance (a pseudo inverse of Ω with respect to the radial variable r), i.e.,

$$\rho(\varphi, w) = \int_0^\infty I_{(0,\infty)} \bigl(\Omega(\xi + r\cos\varphi, \eta + r\sin\varphi) - w \bigr) dr, \tag{20}$$

then the following evolution equations hold for all t because their derivation is based on the variational derivative of ρ with respect to Ω obtained from (20) rather than on the distance interpretation where it is assumed that the ray from $\mathbf{z}$ in the direction φ intersects Γ_w once. If it intersects the contour a few times, then ρ given by (20) is the sum of the distances to all the intersection points.

In the considered case, Eq. (17) becomes

$$\frac{1}{4} \frac{\partial \rho^2(\varphi, w)}{\partial t} = \frac{\partial}{\partial \varphi} \biggl\{ \frac{\delta H}{\delta \rho^2(\varphi, w)} - \rho(\varphi, w) \biggl(D_\varphi \frac{\delta H}{\delta \rho^2(\varphi, w)} \biggr)_{w=M} \biggr\}, \tag{21}$$

where $\rho = \rho(\varphi, w)$ and

$$H = \frac{1}{16\pi} \int_0^M \int_0^M \int_0^{2\pi} \int_0^{2\pi} w_1 w_2 (\rho_1^2)_{w_1} (\rho_2^2)_{w_2} \log D(\varphi_1, w_1, \varphi_2, w_2)\, d\varphi_1\, d\varphi_2\, dw_1\, dw_2,$$

$$\rho_1 = \rho(\varphi_1, w_1), \qquad \rho_2 = \rho(\varphi_2, w_2),$$

$$D = \sqrt{(\rho_1 \cos\varphi_1 - \rho_2 \cos\varphi_2)^2 + (\rho_1 \sin\varphi_1 - \rho_2 \sin\varphi_2)^2}.$$

Equation (21) first appeared in [10], where the singular term describing effects of the vortex motion on its shape was missing. Moreover, the expression for the Hamiltonian here is significantly simplified.

The translation equation in terms of the complex coordinate $z = \xi + i\eta$ becomes

$$\frac{\partial z^\dagger}{\partial t} = \frac{1}{2\pi i} \int_0^M \int_0^{2\pi} \rho(\varphi, w) e^{-i\varphi}\, d\varphi\, dw. \tag{22}$$

We note a similarity with the contour dynamics in [4], where a vorticity patch of value $\Omega(\mathbf{r}) = M$ bounded by a closed curve $r = \rho(\varphi)$ with zero vorticity outside was studied. It is easy to show that in this case, Eq. (1) again leads to a Hamiltonian system resulting in the evolution equation

$$\frac{1}{2} \frac{\partial}{\partial t} \rho^2(\varphi) = \frac{2}{M} \frac{\partial}{\partial \varphi} \frac{\delta H}{\delta \rho^2} = -\frac{\partial}{\partial \varphi} \psi(\varphi), \tag{23}$$

where $\psi(\varphi) = \psi(\varphi, r)|_{r=\rho(\varphi)}$ and $\psi(\varphi, r) = \psi(\xi + r\cos\varphi, \eta + r\sin\varphi)$. In this case the pole (ξ, η) is usually placed at the patch centroid. It is now easy to obtain a closed equation for $\rho(\varphi)$ from (23) using

$$\psi(\varphi, r) = -\frac{M}{4\pi} \int_0^{2\pi} \left[\rho^2(\theta) - r\big(\rho(\theta)\sin(\theta - \varphi)\big)_\theta\right] \log\big(r^2 + \rho^2(\theta) - 2r\rho(\theta)\cos(\theta - \varphi)\big) d\theta. \tag{24}$$

Strangely, in the literature, we could not find an absolutely correct closed equation obtained from (23) after differentiating in the right-hand side. For example, in both [4] and [11], expression (24) and Eq. (23) involving the stream function were correct, but a mistake was made when differentiating the stream function with respect to φ.

Returning to the case of smooth vorticity, we note that the stream function expression in the coordinates (φ, w)

$$\psi(\varphi, w) = -\frac{1}{4\pi} \int_0^{2\pi} \int_0^M u\rho(\theta, u)\rho_u(\theta, u) \log\big(\rho^2(\varphi, w) + \rho^2(\theta, u) - 2\rho(\theta, u)\rho(\varphi, w)\cos(\theta - \varphi)\big) du\, d\theta$$

is somewhat simpler than (24) because φ shows up only under the logarithm. Another advantage of a continuous monopole compared to a patch is that the maximum (the vortex head) moves along stream lines while the centroid of a patch certainly does not.

Summarizing, we can write a closed equation for $\rho(\varphi, w) = \rho(t, \varphi, w)$ in the form

$$\frac{\partial}{\partial t} \rho^2 = \frac{\partial}{\partial \varphi} N(\rho^2), \qquad \rho^2\big|_{t=0} = p(\varphi, w), \tag{25}$$

where $p(\,\cdot\,)$ is an initial condition and the nonlinear integro-differential operator is

$$N(\rho^2)(\varphi, w) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^M \big(u\rho^2(\theta, u)_u \log D + 4\rho(\theta, u)\rho(\varphi, w)\cos(\varphi - \theta)\big) du\, d\theta$$

with $D = \rho^2(\varphi, w) + \rho^2(\theta, u) - 2\rho(\theta, u)\rho(\varphi, w)\cos(\theta - \varphi)$.

Obviously, Eq. (25) is not simpler than the original Eq. (1) in the general case. But in one particular case, which we soon discuss, using (25) is more efficient than using (1). Namely, we suggest a natural asymptotic procedure bridging the contour dynamics and the smooth vorticity case. For this, we assume that the initial vorticity in Eq. (1) is represented as

$$\Omega(r, \varphi) = MS\left(\frac{r}{R(\varphi)}\right),$$

where the dimensionless function $S(x)$ defined on $[0, \infty)$ satisfies $S(0) = 1$, $S'(x) < 0$, and $S(\infty) = 0$ and $R(\varphi)$ is a certain spatial scale depending on the direction. We introduce the scaling

$$\Omega_\epsilon(r, \varphi) = MS\left(\left[\frac{r}{R(\varphi)}\right]^{1/\epsilon}\right), \tag{26}$$

which for small ϵ converts a continuously distributed vorticity into a patch,

$$\lim_{\epsilon \to 0} \Omega_\epsilon(r, \varphi) = \begin{cases} M, & r < R(\varphi), \\ 0, & r > R(\varphi). \end{cases}$$

We assume that the equation $r = R(\varphi)$ represents an ellipse, i.e., the solution of (25) with the initial elliptic patch Ω_0 is the well-known Kirchhoff vortex [11]. Any attempt to correct that solution for small ϵ fails because the derivative of $\Omega_\epsilon(r, \varphi)$ at $\epsilon = 0$ is infinite. Nevertheless, passing to

$$\rho_\epsilon(\varphi, w) = R(\varphi)\left[S^{-1}\left(\frac{w}{M}\right)\right]^\epsilon \tag{27}$$

obtained from (26), we obtain an analytic function of ϵ, which allows using a standard perturbation approach whose details are given in the appendix.

We show the results in Fig. 2, where it can be seen that the vorticity lines lose the elliptic shape but nevertheless preserve the central symmetry. The latter obviously follows from the original Eq. (1). Therefore, the integral in the right-hand side of (22) is zero, and the vortex center does not move.

The only purpose of the example was to show that the suggested perturbation procedure is well posed.

We now consider a dipole determined by the parameter values $N = 2$, $\omega_1 = M > 0$, and $\omega_2 = m < 0$. The next statement follows from (17) and general translation equation (19).

Proposition 6. *We have the equations*

$$\begin{aligned} \frac{1}{4}\frac{\partial \rho_1^2(\varphi, w)}{\partial t} &= \frac{\partial}{\partial \varphi}\left\{\frac{\delta H}{\delta \rho_1^2(\varphi, w)} - \rho_1(\varphi, w)\left(D_\varphi \frac{\delta H}{\delta \rho_1^2(\varphi, w)}\right)_{w=M}\right\}, \\ \frac{1}{4}\frac{\partial \rho_2^2(\varphi, w)}{\partial t} &= \frac{\partial}{\partial \varphi}\left\{\frac{\delta H}{\delta \rho_2^2(\varphi, w)} - \rho_2(\varphi, w)\left(D_\varphi \frac{\delta H}{\delta \rho_2^2(\varphi, w)}\right)_{w=m}\right\} \end{aligned} \tag{28}$$

and

$$\frac{\partial z_1^\dagger}{\partial t} = \frac{1}{2\pi i}\int_0^M \int_0^{2\pi} \rho_1(\varphi, w)e^{-i\varphi}\, d\varphi\, dw + \frac{1}{2\pi i}\int_0^m \int_0^{2\pi} \frac{w\rho_2(\varphi, w)\frac{\partial \rho_2(\varphi, w)}{\partial w}}{z_1 - z_2 - \rho_2(\varphi, w)e^{i\varphi}}\, dw\, d\varphi,$$

$$\frac{\partial z_2^\dagger}{\partial t} = \frac{1}{2\pi i}\int_0^m \int_0^{2\pi} \rho_2(\varphi, w)e^{-i\varphi}\, d\varphi\, dw + \frac{1}{2\pi i}\int_0^M \int_0^{2\pi} \frac{w\rho_1(\varphi, w)\frac{\partial \rho_1(\varphi, w)}{\partial w}}{z_2 - z_1 - \rho_1(\varphi, w)e^{i\varphi}}\, dw\, d\varphi,$$

where the expression for the Hamiltonian in terms of ρ_1, ρ_2, ξ_1, η_1, ξ_2, and η_2 can be obtained from Proposition 5.

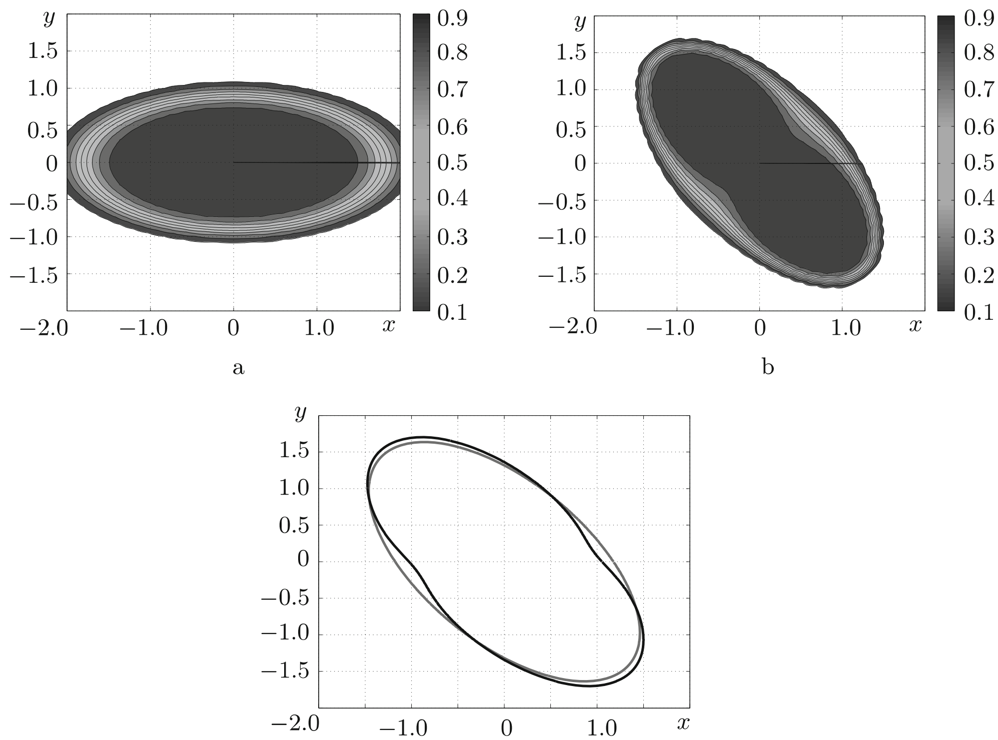

Fig. 2. Solution of (25) obtained perturbatively for $\epsilon = 0.05$ and $M = 1$: (a) the initial vortex, (b) the vortex at terminal instant ($T = 1$), and (c) the comparison vorticity line $w = 0.3$ for the Kirchhoff patch ($\epsilon = 0$) and the perturbed Kirchhoff patch corresponding to initial condition (26) with $\epsilon = 0.05$.

The evolution and translation equations in Proposition 6 were first announced in [12]. They are essentially simplified and corrected here.

To illustrate the use of (28), we consider a strong positive point vortex with a maximum M centered at the origin and a weak negative satellite with a minimum m, $|m| \ll M$, initially axisymmetric and centered at $(0, R)$. We assume that the stream function $\psi_1(r)$ of the positive vortex is not affected by the satellite. In addition we neglect the influence of the velocity field generated by the satellite on itself. In other words, we regard it as a passive scalar driven by the velocity field of the strong vortex. Hence, the first equation in (28) becomes $\rho_1(t, \varphi, w) = \rho_1(0, \varphi, w)$, and the second becomes a closed equation,

$$\frac{1}{2}\frac{\partial \rho_2^2(\varphi, w)}{\partial t} = -\frac{\partial}{\partial \varphi}\psi_1\big(\rho_2(\varphi, w)^2 + R^2 - 2R\rho_2(\varphi, w)\cos\varphi\big).$$

We note that w is included in the equation simply as a parameter.

Passing to the polar coordinates (r, θ) with a pole at the origin $(0, 0)$, i.e., setting $r^2 = \rho_2(\varphi, w)^2 + R^2 - 2R\rho_2(\varphi, w)\cos\varphi$ and $\sin\theta = (\rho_2/r)\sin\varphi$, we obtain

$$\frac{\partial r}{\partial t} + \frac{1}{r}\frac{\partial \psi_1(r)}{\partial r}\frac{\partial r}{\partial \theta} = 0.$$

This equation is integrable, but it makes a physical sense only if the background vortex is a point vortex, i.e., $\psi_1 = k \log r$, where k is its intensity. The solution is given by

$$r^2 - 2rR\cos\left(\theta - \frac{kt}{r^2}\right) = C(w), \tag{29}$$

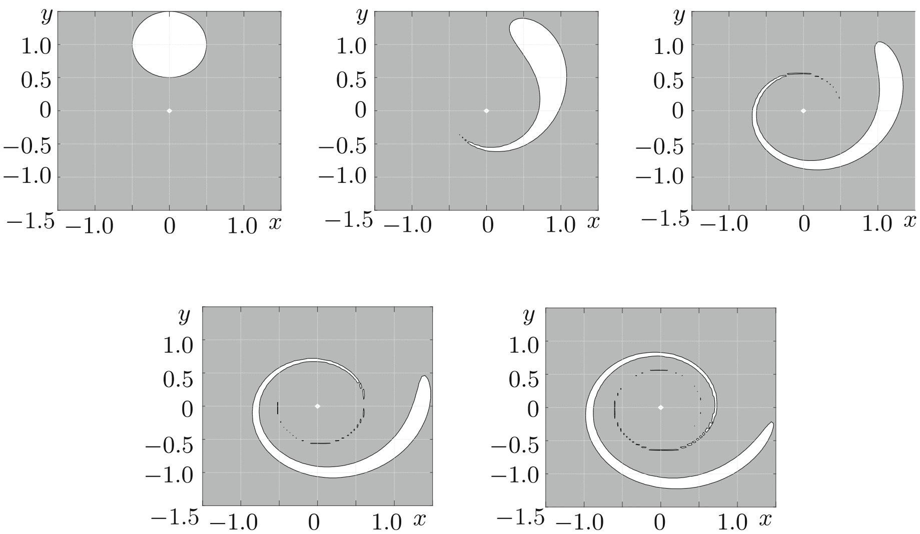

Fig. 3. Evolution of a weak satellite in a velocity field generated by a stationary point vortex at different instants.

where the constant $C(w)$ is defined by the vorticity level w and the shape of the initial satellite (see Fig. 3).

It can be shown that the vorticity line of level w spirals into the limit cycle $r = R - r_0(w)$, where $r_0(w)$ is the radius of the w-vorticity line for the initial satellite, and that the number of cycles in the spiral is $n \approx c(w)\omega_0 t$, where $\omega_0 = k/R^2$ is the angular velocity, t is time, and c is a constant depending on the vorticity level w. We note that integral (29) follows directly from the original Eq. (1) after linearization [13]. Moreover, physical aspects of the solution were discussed in [13], and the abovementioned assumptions were justified. In addition, we note that the case of a distributed intensive vortex was also considered in [13].

6. Discussion and conclusions

We have introduced a class of vorticities extending contour dynamics [4], [5] to the case of a continuously distributed vorticity and developed a Hamiltonian formalism for that class. Here, we also revealed the relation between our approach and contour dynamics. In this context, the proposed approach can be called "continuum contour dynamics."

In this section, we present explicit scalings transforming the suggested class into two well known models, the point vortex system [14] and the FAVOR model [6]. Regarding point vortices, we define the kth vortex with the function $\Omega_k(\mathbf{r}) = \Omega(\mathbf{r}) I_{G_k}(\mathbf{r}) = \widetilde{\Omega}_k(\mathbf{r} - \mathbf{z}_k)$ in the local coordinate system with the origin at $\mathbf{z}_k$. Then

$$\Omega_\epsilon(\mathbf{r}) = \frac{1}{\epsilon^2} \sum_k \widetilde{\Omega}_k\left(\frac{\mathbf{r} - \mathbf{z}_k}{\epsilon}\right) \to \sum_k \bar{\omega}_k \delta(\mathbf{r} - \mathbf{z}_k), \quad \epsilon \to 0,$$

where

$$\bar{\omega}_k = \frac{1}{2} \int_0^{2\pi} \int_0^{\omega_j} \rho_k^2(\varphi, w)\, dw\, d\varphi.$$

Regarding the FAVOR model, we define

$$R_k^2(\varphi) = -|\omega_k| \left.\frac{\partial \rho^2}{\partial w}\right|_{w=\omega_k}$$

as the characteristic spatial scale calculated in the vicinity of the kth vortex peak. The curve $r = R_k(\varphi)$ is obviously an ellipse, and the vorticity contour $\Omega_k = w$ with w close to ω_k is well approximated by

$r = c(w)R_k(\varphi)$ with a constant $c(w)$ depending on the vorticity level. We introduce the dimensionless distance $\tilde{r} = r/R_k(\varphi)$ and set $\widetilde{\Omega}_k(\varphi, \tilde{r}) = \Omega_k(\varphi, r)$, where (φ, r) are local polar coordinates. Let

$$\Omega_\epsilon(\mathbf{r}) = \sum_k \widetilde{\Omega}_k(\varphi, \tilde{r}^{1/\epsilon}).$$

Then

$$\Omega_\epsilon(\mathbf{r}) \to \sum_k \bar{\omega}_k I_{E_k}(\mathbf{r}), \quad \epsilon \to 0,$$

where $E_k = \{(\varphi, r) \mid r < R_k(\varphi\}$ is the Kirchhoff elliptic vortex [11].

The described limit procedures lead to the well-known Hamiltonian formulations of the point vertex system [2] and the FAVOR model [15], but the details of this approach based on Proposition 5 are beyond our scope here.

We further note that most of the above results can be extended to the corresponding class of vortices on an arbitrary two-dimensional Riemann manifold $\mathcal{M}$ with the metric $dS = s(x,y)\,dx\,dy$, $(x,y) \in D$, where D is a region in the plane (x,y) and $s = s(x,y)$ is the metric density. Such an extension is possible because the vorticity conservation equation on $\mathcal{M}$ similar to (1),

$$\frac{\partial \Omega}{\partial t} + s^{-1} J(\psi, \Omega) = 0,$$

can also be written in the Hamiltonian form

$$\frac{\partial q}{\partial t} = \{q, H\},$$

where $q = s\Omega$ is the phase variable and the noncanonical Poisson bracket is expressed in a form similar to (2),

$$\{F, G\} = \int_D \Omega(\mathbf{r}) J_{x,y}\left(\frac{\delta F}{\delta q(\mathbf{r})}, \frac{\delta G}{\delta q(\mathbf{r})}\right) d\mathbf{r}.$$

The Hamiltonian has the form $H = -(1/2)\int_D q\psi\, d\mathbf{r}$. The ideal hydrodynamics in the plane is given by $s \equiv 1$ and $D = \mathbb{R}^2$. For a sphere of unit radius, $x = \lambda$ is the longitude, $y = \theta$ is the latitude, and $D = [0, 2\pi] \times [0, \pi]$. Finally, for periodic boundary conditions on a rectangle $[0,a] \times [0,b]$, $\mathcal{M}$ is a torus, $s \equiv 1/ab$, and $D = [0, 2\pi] \times [0, 2\pi]$. Dipoles on a sphere were discussed in [12].

Finally, summarizing all the results, we conclude that from the application standpoint, there is not yet convincing evidence that the equations in terms of contours and the coordinates of vortex peaks have an advantage over traditional approaches. Nevertheless, certain similarities to contour dynamics, which has proved a useful theory, give hopes for a better future.

Theoretically, the suggested approach gives a useful insight into conservation laws concerning the vorticity topography. Moreover, we presented a solution of the traditionally interesting problem of diagonalizing the Poisson bracket for a certain class of vorticities. But we admit that the diagonalization was a goal in itself in this case, in contrast to the analogous problem for the Hasegawa–Mima equation [16], [17], which led to canonical variables and ultimately to advances in the theory of weak turbulence. We could formally introduce canonical variables for the Hamiltonian system considered here, but they would hardly have a clear physical meaning.

Appendix A: Proof of Lemma 1

The identity $p(\hat{x}(p,w), \hat{y}(p,w)) = p$ after differentiation with respect to p and w gives $p_x = \hat{y}_w/g$ and $p_y = -\hat{x}_w/g$. The same identity implies that

$$p_x \frac{\delta \hat{x}}{\delta \Omega(\mathbf{r})} + p_y \frac{\delta \hat{y}}{\delta \Omega(\mathbf{r})} = 0, \qquad \hat{y}_w \frac{\delta \hat{x}}{\delta \Omega(\mathbf{r})} - \hat{x}_w \frac{\delta \hat{y}}{\delta \Omega(\mathbf{r})} = 0. \tag{A.1}$$

We then find the variational derivatives of one more identity

$$\Omega(\xi + \hat{x}(p,w), \eta + \hat{y}(p,w)) = w$$

and obtain

$$\delta\Omega(\mathbf{r}) + \Omega_x(\delta\xi + \delta\hat{x}) + \Omega_y(\delta\eta + \delta\hat{y}) = 0. \tag{A.2}$$

It follows from $\Omega(x,y) = w$ that $\Omega_x = -\hat{y}_p/g$ and $\Omega_x = \hat{x}_p/g$. Substituting these expressions in (A.2) and solving (A.1) and (A.2) for $\delta\hat{x}/\delta\Omega(\mathbf{r})$ and $\delta\hat{y}/\delta\Omega(\mathbf{r})$, we obtain the result

$$\frac{\delta\hat{x}(p,w)}{\delta\Omega(\mathbf{r})} = \frac{\hat{x}_w(p,w)}{g(p,w)}\left(\delta(\Omega(x,y)-w)\delta(p(x,y)-p) - \hat{y}_p\frac{\delta\xi}{\delta\Omega(\mathbf{r})} - \hat{x}_p\frac{\delta\eta}{\delta\Omega(\mathbf{r})}\right),$$

$$\frac{\delta\hat{y}}{\delta\Omega(\mathbf{r})} = \frac{\delta\hat{x}}{\delta\Omega(\mathbf{r})}\frac{\hat{y}_w}{\hat{x}_w}.$$

Substituting expressions for the variational derivatives of ξ and η and converting them to the delta functions in x and y, we obtain (12).

Appendix B: Perturbation theory method for Eq. (25)

For the initial condition in (25), we assume that

$$p(\varphi,w) = p(\varphi,w;\epsilon) = p_0(\varphi,w) + \epsilon p_1(\varphi,w) + \dots,$$

and we similarly write the solution as

$$\rho^2(\varphi,w) = \rho_0^2(\varphi,w) + \epsilon\rho_1^2(\varphi,w) + \dots.$$

We obtain

$$\frac{\partial}{\partial t}\rho_n^2 = L_{n-1}(\rho_n^2), \qquad \rho_n^2\big|_{t=0} = p_n(\varphi,w), \tag{B.1}$$

where

$$L_{n-1}(\rho^2) = \frac{\delta N(\rho^2)}{\delta\rho^2}\bigg|_{\rho=\rho_{n-1}}$$

is the linearization of $N(\rho^2)$ at the previous correction.

We take the initial condition in form (27) and set $f(w) = \log S^{-1}(w/M)$. Hence, in the first order in ϵ,

$$p(\varphi,w) = R^2(\varphi) + 2\epsilon R^2(\varphi)f(w).$$

Because p_0 is independent of w, the zeroth approximation $\rho_0^2 = \rho_0^2(t,\varphi)$ is also independent of w and is in fact just the solution of the contour dynamics equation corresponding to the initial condition at $\epsilon = 0$. In addition, we assume that $R(\varphi)$ is symmetric about the pole and the singular term in (25) therefore vanishes ($\psi_\xi = \psi_\eta = 0$), i.e., the vortex center does not move. This implies a substantial simplification of linearized equation (B.1) for $n = 1$:

$$\frac{\partial}{\partial t}\rho_1^2(\varphi,w) = -\int_0^{2\pi} K(\varphi,\theta)\int_0^M \rho_1^2(\theta,u)\,du\,d\theta, \qquad \rho_1^2\big|_{t=0} = 2R^2(\varphi)f(w), \tag{B.2}$$

where the kernel is

$$K(\varphi,\theta) = \frac{1}{2\pi}\frac{\partial}{\partial\varphi}\log\bigl(\rho_0^2(\varphi) + \rho_0^2(\theta) - 2\rho_0(\theta)\rho_0(\varphi)\cos(\theta-\varphi)\bigr).$$

Integrating both sides of (B.2) over w, we obtain

$$\frac{\partial}{\partial t}z(\varphi) = -M\int_0^{2\pi} K(\varphi,\theta)z(\theta)\,d\theta, \qquad z\big|_{t=0} = 2\bar{f}R^2(\varphi),$$

where

$$z(\varphi) = \frac{1}{M}\int_0^M \rho_1^2(\varphi,w)dw, \qquad \bar{f} = \frac{1}{M}\int_0^M f(w)\,dw.$$

This equation is easy to solve numerically, and we can then recover $\rho_1(t,\varphi,w)$ itself using the initial condition given in (B.2). As a result, we obtain

$$\rho_1^2(t,\varphi,w) = z(t,\varphi) + 2R^2(\varphi)(t,\varphi)\big(f(w)-\bar{f}\,\big).$$

Inverting the function

$$\rho^2(t,\varphi,w) = \rho_0^2(t,\varphi) + \epsilon\rho_1^2(t,\varphi,w)$$

with respect to w, we obtain the first-order approximation for the vorticity itself:

$$\Omega_\epsilon(\xi + r\cos\varphi, \eta + r\sin\varphi) = MS\biggl(\exp\biggl\{\frac{r^2-\rho_0^2(t,\varphi)-\epsilon(z(t,\varphi)-z(0,\varphi))}{2\epsilon R^2(\varphi)}\biggr\}\biggr).$$

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Conflicts of interest. The author declares no conflicts of interest.

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