Theoretical and Mathematical Physics, **202**(3): 334–343 (2020)

PRIMITIVE SOLUTIONS OF THE KORTEWEG–DE VRIES EQUATION

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We survey recent results connected with constructing a new family of solutions of the Korteweg–de Vries equation, which we call primitive solutions. These solutions are constructed as limits of rapidly vanishing solutions of the Korteweg–de Vries equation as the number of solitons tends to infinity. A primitive solution is determined nonuniquely by a pair of positive functions on an interval on the imaginary axis and a function on the real axis determining the reflection coefficient. We show that elliptic one-gap solutions and, more generally, periodic finite-gap solutions are special cases of reflectionless primitive solutions.

Keywords: integrable system, Korteweg–de Vries equation, primitive solution

DOI: 10.1134/S0040577920030058

1. Introduction

The Korteweg–de Vries (KdV) equation

$$u_t(x,t) = 6u(x,t)u_x(x,t) - u_{xxx}(x,t) \tag{1}$$

plays a fundamental role in the modern theory of integrable systems and is the prototypical example of an infinite-dimensional integrable system. The KdV equation is the first equation of an infinite sequence of commuting equations called the KdV hierarchy. The auxiliary linear operator for the KdV hierarchy is the one-dimensional Schrödinger operator on the real axis

$$-\psi'' + u(x)\psi = E\psi, \quad -\infty < x < \infty. \tag{2}$$

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The research of S. A. Dyachenko and D. V. Zakharov was supported by the National Science Foundation (Grant No. DMS-1716822).

The research of V. E. Zakharov was supported by the National Science Foundation (Grant No. DMS-1715323). The results in Secs. 3–5 were obtained with support of a grant from the Russian Science Foundation (Project No. 19-72-30028).

Prepared from an English manuscript submitted by the authors; for the Russian version, see *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 202, No. 3, pp. 382–392, March, 2020. Received September 8, 2019. Revised September 8, 2019. Accepted October 8, 2019.

There are two important cases where the initial value problem for the KdV equation admits an analytic solution. If the initial condition $u(x, 0)$ tends to zero sufficiently rapidly as $x \to \pm\infty$, then the KdV equation can be solved using the inverse spectral transform (IST). In this case, the Schrödinger operator has a finite number of bound states and an absolutely continuous spectrum for positive energies. The corresponding solution of the KdV equation is a nonlinear superposition of a finite number of solitons, corresponding to the bound states, and a dissipating background. If the reflection coefficient is identically zero, then we obtain a family of multisoliton solutions of the KdV equation, which are given by an explicit algebraic formula.

We study the case of periodic initial data using algebro-geometric finite-gap solutions. Such a solution is determined by a hyperelliptic algebraic curve with real branch points and a divisor on it and can be explicitly given by the Matveev–Its formula in terms of the Riemann theta function of the spectral curve. Periodic finite-gap solutions are dense in the space of all periodic solutions and can hence be effectively approximated by them. It has long been assumed that periodic finite-gap solutions of the KdV equation can be obtained from N-soliton solutions in the limit $N \to \infty$, but a precise description of such a limit was unknown.

In [1]–[7], we constructed a new family of bounded solutions of the KdV equation, which we call *primitive solutions*, generalizing both rapidly vanishing and finite-gap periodic solutions. These solutions are obtained as the limit of rapidly vanishing solutions with N bound states as $N \to \infty$. A primitive solution is obtained by solving a contour problem determined on the complex plane by a pair of positive functions R_1 and R_2 on an interval on the imaginary axis and a function r on the real axis. The important case $r = 0$ corresponds to reflectionless primitive solutions. In [1]–[3], we studied reflectionless primitive solutions numerically and showed that they can exhibit quite complicated disordered behavior. In [5], [6], we considered reflectionless solutions with $R_1 = R_2$ and gave an algorithm for determining the corresponding solution $u(x, t)$ as a convergent Taylor series. In addition, we showed that finite-gap periodic solutions are primitive solutions.

2. Reformulation of IST as a $\bar{\partial}$-problem

We begin by recalling how to solve KdV equation (1) in the rapidly decreasing case using the IST (see [8]–[10]). We let $u(x, t)$ be a solution of the KdV equation and assume that $u(x, 0)$ tends to zero sufficiently rapidly as $x \to \pm\infty$. We regard $u(x, t)$ as a time-dependent potential of a Schrödinger operator $L(t)$ given by (2). Classical spectral theory indicates that $L(t)$ has an absolutely continuous spectrum $[0, \infty)$ and finitely many simple eigenstates with eigenvalues $-\kappa_1^2, \dots, -\kappa_N^2$. The spectral data for $L(t)$ satisfy the linear Gardner–Green–Kruskal–Miura (GGKM) equations, which can be explicitly solved, and the operator $L(t)$ can then be reconstructed from its spectral data.

Let $\psi_\pm(k, x, t)$ be the Jost solutions of the time-dependent Schrödinger equation:

$$L(t)\psi_\pm(k, x, t) = k^2 \psi_\pm(k, x, t).$$

The Jost solutions are analytic for $\operatorname{Im} k > 0$, are continuous for $\operatorname{Im} k \geq 0$, and have the asymptotic behavior as $k \to \infty$ with $\operatorname{Im} k > 0$

$$\psi_\pm(k, x, t) = e^{\pm ikx}\left(1 + Q_\pm(x, t)\frac{1}{2ik} + O\left(\frac{1}{k^2}\right)\right),$$

where

$$Q_+(x, t) = -\int_x^\infty u(y, t)\, dy, \qquad Q_-(x, t) = -\int_{-\infty}^x u(y, t)\, dy.$$

The Jost solutions satisfy the scattering relations

$$t(k)\psi_\mp(k,x,t) = \overline{\psi_\pm(k,x,t)} + r_\pm(k,t)\psi_\pm(k,x,t), \quad k \in \mathbb{R},$$

where $t(k)$ and $r_\pm(k,t)$ are the respective transmission and reflection coefficients. The *scattering data* for the Schrödinger operator $L(t)$ consists of the reflection coefficient $r(k,t) = r_+(k,t)$, the eigenvalues $\kappa_1, \ldots, \kappa_N$, and the phase coefficients $\gamma_1(t), \ldots, \gamma_N(t)$ defined by

$$\gamma_n(t) = \|\psi_+(i\kappa_n, x, t)\|_2^{-1}, \quad n = 1, \ldots, N.$$

If $u(x,t)$ satisfies KdV equation (1), then the κ_n are independent of t, while the time evolution of the quantities $r(k,t)$ and $\gamma_n(t)$ is given by the GGKM equations:

$$r(k,t) = r(k)e^{8ik^3t}, \quad r(k) = r(k,0), \qquad \gamma_n(t) = \gamma_n e^{4\kappa_n^3 t}, \quad \gamma_n = \gamma_n(0). \tag{3}$$

The constants κ_n and γ_n are positive, and the reflection coefficient $r(k)$ has the properties

$$r(-k) = \overline{r(k)}, \quad k \in \mathbb{R}, \qquad |r(k)| < 1 \quad \text{if } k \neq 0, \qquad r(0) = -1 \quad \text{if } |r(0)| = 1. \tag{4}$$

To reconstruct $u(x,t)$ from the spectral data, we consider the auxiliary function

$$\chi(k,x,t) = \begin{cases} t(k)\psi_-(k,x,t)e^{ikx}, & \operatorname{Im} k > 0, \\ \psi_+(-k,x,t)e^{ikx}, & \operatorname{Im} k < 0. \end{cases} \tag{5}$$

The function $\chi(k,x,t)$ has the following properties:

1. It is meromorphic on the complex-k plane away from the real axis and has the nontangential limits

$$\chi_\pm(k,x,t) = \lim_{\varepsilon \to 0} \chi(k \pm i\varepsilon, x, t), \quad k \in \mathbb{R}, \tag{6}$$

   on the real axis.

2. It has a jump on the real axis satisfying

$$\chi_+(k,x,t) - \chi_-(k,x,t) = r(k)e^{2ikx+8ik^3t}\chi_-(-k,x,t), \quad k \in \mathbb{R}. \tag{7}$$

3. It has simple poles at the points $i\kappa_1, \ldots, i\kappa_N$ and no other singularities. The residues at the poles satisfy the condition

$$\operatorname*{Res}_{i\kappa_n} \chi(k,x,t) = ic_n e^{-2\kappa_n x + 8\kappa_n^3 t}\chi(-i\kappa_n, x, t), \qquad c_n = \gamma_n^2. \tag{8}$$

4. It has the asymptotic behavior

$$\chi(k,x,t) = 1 + \frac{i}{2k}Q_+(x,t) + O\left(\frac{1}{k^2}\right), \quad |k| \to \infty, \quad \operatorname{Im} k \neq 0. \tag{9}$$

The solution $u(x,t)$ of the KdV equation is given in terms of χ by the formula

$$u(x,t) = \frac{d}{dx}Q_+(x,t). \tag{10}$$

An important class of solutions of the KdV equation, called multisoliton solutions, is obtained by choosing spectral data with $r(k) = 0$. In this case, the solution is given by the explicit formula

$$u(x,t) = -2\frac{d^2}{dx^2} \sum_{I \subset \{1,\ldots,N\}} \left[\prod_{\substack{\{i,j\} \subset I, \\ i<j}} \frac{(\kappa_i - \kappa_j)^2}{(\kappa_i + \kappa_j)^2} \prod_{i \in I} \frac{c_i}{2\kappa_i} e^{-2\kappa_i x} \right]. \tag{11}$$

3. Transplantation of poles and primitive solutions

Our initial papers [1]–[3] were motivated by the question of how we can pass to the limit $N \to \infty$ in formula (11). The resulting solutions of the KdV equation (and, more generally, the limits of generic rapidly decreasing solutions) are called primitive solutions, and they are constructed by the following three steps. First, following Manakov and Zakharov (see [11]), we reformulate boundary conditions (6)–(9) defining χ as a $\bar\partial$-problem. Second, we generalize this problem by allowing χ to have poles on the negative in addition to the positive imaginary axis. Finally, we pass to the limit as $N \to \infty$, and the poles of χ hence coalesce into jumps along two cuts on the imaginary axis.

Let $(r(k), \kappa_n, \gamma_n)$ be the scattering data for a Schrödinger operator and $c_n = \gamma_n^2$. We consider the distribution on the k plane, called the *dressing function*,

$$T(k) = \frac{i}{2}\delta(k_{\mathrm{I}})\theta(-k_{\mathrm{I}})r(k_{\mathrm{R}}) + \pi\delta(k_{\mathrm{R}})\sum_{n=1}^{N} c_n\delta(k_{\mathrm{I}} - \kappa_n). \tag{12}$$

Here, δ is the Dirac delta function, $k = k_{\mathrm{R}} + ik_{\mathrm{I}}$, θ is the Heaviside step function, and we use the conventions

$$\frac{\partial}{\partial\bar k}\frac{1}{k} = \pi\delta(k) = \pi\delta(k_{\mathrm{R}})\delta(k_{\mathrm{I}}), \qquad \int_{-\infty}^{\infty} f(x)\delta(x)\theta(\pm x)\,dx = \lim_{x\to 0^\pm} f(x).$$

A direct calculation shows that conditions (7) and (8) are equivalent to the $\bar\partial$-problem for the function χ (see [11])

$$\frac{\partial\chi}{\partial\bar k} = T(k)e^{2ikx+8ik^3t}\chi(-k,x,t), \quad \chi \to 1 \text{ as } k \to \infty. \tag{13}$$

The function χ solving this problem has a jump on the real axis determined by the reflection coefficient and has simple poles at the points $k = i\kappa_n$ on the positive imaginary axis. The reason for this lack of symmetry is that the IST is not symmetric under the spatial involution $x \mapsto -x$. We seek to pass to the limit as $N \to \infty$ and, in particular, to obtain finite-gap solutions as limits of multisoliton solutions. Because finite-gap solutions are periodic or quasiperiodic in x, we must first restore spatial symmetry, which we do by allowing χ to have poles on the negative in addition to the positive imaginary axis. This procedure was performed in [1]–[3] for reflectionless potentials (in the case $r(k) = 0$) and for arbitrary rapidly decreasing potentials in [7].

Let $(r(k), \kappa_1, \dots, \kappa_N, c_1, \dots, c_N)$ be the scattering data of a potential $u(x,t)$ rapidly decreasing at infinity and $\chi(k,x,t)$ be the function determined by $\bar\partial$-problem (13). We fix a subset $I \subset \{1, \dots, N\}$, and introduce the function

$$\tilde\chi(k,x,t) = \chi(k,x,t)\prod_{m\in I}\frac{k - i\kappa_m}{k + i\kappa_m}. \tag{14}$$

It has a jump on the real axis, tends to unity as $k \to \infty$, and has poles at $k = i\kappa_m$ for $m \notin I$ and at $k = -i\kappa_m$ for $m \in I$. These singularities can be encoded by requiring that $\tilde\chi$ solve the same $\bar\partial$-problem (13) as χ but with the dressing function

$$\widetilde T(k) = \frac{i}{2}\delta(k_{\mathrm{I}})\theta(-k_{\mathrm{I}})\tilde r(k_{\mathrm{R}}) + \pi\delta(k_{\mathrm{R}})\sum_{n=1}^{N}\tilde c_n\delta(k_{\mathrm{I}} - \tilde\kappa_n), \tag{15}$$

whose coefficients are equal to

$$\tilde{r}(k) = r(k)\prod_{m\in I}\left(\frac{k-i\kappa_m}{k+i\kappa_m}\right)^2, \qquad \tilde{\kappa}_n = \begin{cases} \kappa_n, & n\notin I, \\ -\kappa_n, & n\in I, \end{cases} \tag{16}$$

$$\tilde{c}_n = \begin{cases} c_n \prod_{m\in I}\left(\dfrac{\kappa_n-\kappa_m}{\kappa_n+\kappa_m}\right)^2, & n\notin I, \\ -\dfrac{4\kappa_n^2}{c_n}\prod_{m\in I\setminus\{n\}}\left(\dfrac{\kappa_n+\kappa_m}{\kappa_n-\kappa_m}\right)^2, & n\in I. \end{cases} \tag{17}$$

We note that

$$\tilde{r}(-k) = \overline{\tilde{r}(k)}, \qquad |\tilde{r}(k)| = |r(k)| \quad \text{for } k\in\mathbb{R}, \qquad \tilde{r}(0) = r(0),$$

and the function $\tilde{r}$ hence has the same properties (4) as r. We also note that $\tilde{c}_n$ is positive if $n\notin I$ and negative if $n\in I$, i.e., each coefficient $\tilde{c}_n$ has the same sign as $\tilde{\kappa}_n$.

The function $\tilde{\chi}$ has the asymptotic behavior as $|k|\to\infty$

$$\tilde{\chi}(k,x,t) = 1 + \frac{i}{2k}\widetilde{Q}_+(x,t) + O\left(\frac{1}{k^2}\right), \qquad \widetilde{Q}_+(x,t) = Q_+(x,t) - 4\sum_{m\in I}\kappa_m.$$

Therefore, $u(x,t)$ is obtained from $\tilde{\chi}(k,x,t)$ using the same formula (10).

We can conclude as follows. Let $u(x,t)$ be a rapidly vanishing solution of the KdV equation, $T(k)$ be distribution (12), and χ be the solution of $\bar{\partial}$-problem (13). If we choose any subset $I\subset\{1,\ldots,N\}$ and replace the distribution $T(k)$ with $\widetilde{T}(k)$ according to (15), then formula (10) with $\tilde{\chi}$ in place of χ gives the same solution $u(x,t)$ of the KdV equation. Hence, any rapidly vanishing solution of the KdV equation with N solitons can be obtained using the dressing method in 2^N different ways.

We now construct primitive potentials by passing to the limit as $N\to\infty$ in the distribution $\widetilde{T}$ (see [7]). We consider two positive, Hölder-continuous functions R_1 and R_2 on the interval $[k_1,k_2]$ and a function r on the real axis satisfying (4). We consider the dressing function

$$T(k) = \frac{i}{2}\delta(k_{\mathrm{I}})\theta(-k_{\mathrm{I}})r(k) + \pi\delta(k_{\mathrm{R}})\left[\int_{k_1}^{k_2} R_1(p)\delta(k_{\mathrm{I}}-p)\,dp - \int_{k_1}^{k_2} R_2(p)\delta(k_{\mathrm{I}}+p)\,dp\right]. \tag{18}$$

It is clear that by approximating the second and third integrals with a finite Riemann sum, we obtain a distribution of form (15), which, as seen above, describes a solution of the KdV equation rapidly decreasing at infinity.

Let χ be the solution of $\bar{\partial}$-problem (13) with dressing function (18). The function χ has a jump on the real axis and also on the intervals $[ik_1,ik_2]$ and $[-ik_2,-ik_1]$ on the imaginary axis and has the spectral representation

$$\chi(k,x,t) = 1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho(p,x,t)\,dp}{p-k} + \frac{i}{\pi}\int_{k_1}^{k_2}\frac{f(p,x,t)\,dp}{k-ip} + \frac{i}{\pi}\int_{k_1}^{k_2}\frac{g(p,x,t)\,dp}{k+ip}. \tag{19}$$

Substituting this representation in (13), we obtain the system of singular integral equations for ρ, f, and g

$$\begin{gathered}
\rho(k,x,t) = r(k,x,t)e^{-2ikx-8ik^3t} \times \\
\times \left[1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho(p,x,t)\,dp}{q+ik-\varepsilon} - \frac{i}{\pi}\int_{k_1}^{k_2}\frac{f(p,x,t)\,dp}{k+ip} + \frac{i}{\pi}\int_{k_1}^{k_2}\frac{g(p,x,t)\,dp}{-k+ip}\right], \quad k\in\mathbb{R}, \\
f(k,x,t) + \frac{R_1(k)}{\pi}e^{-2kx+8k^3t}\left[\int_{k_1}^{k_2}\frac{f(p,x,t)\,dp}{k+p} + \fint_{k_1}^{k_2}\frac{g(p,x,t)\,dp}{k-p}\right] = \\
= R_1(k)e^{-2kx+8k^3t}\left[1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho(p,x,t)\,dp}{p-ik}\right], \quad k\in[k_1,k_2], \\
g(k,x,t) + \frac{R_2(k)}{\pi}e^{2kx-8k^3t}\left[\fint_{k_1}^{k_2}\frac{f(p,x,t)\,dp}{k-p} + \int_{k_1}^{k_2}\frac{g(p,x,t)\,dp}{k+p}\right] = \\
= -R_2(k)e^{2kx-8k^3t}\left[1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho(p,x,t)\,dp}{p+ik}\right], \quad k\in[k_1,k_2].
\end{gathered} \tag{20}$$

The corresponding solution $u(x,t)$ of KdV equation (1), which we call a *primitive solution*, is given by the formula

$$u(x,t) = 2\frac{d}{dx}\left[-\frac{1}{2\pi}\int_{-\infty}^{\infty}\rho(p,x,t)\,dp + \frac{1}{\pi}\int_{k_1}^{k_2}[f(p,x,t)+g(p,x,t)]\,dp\right]. \tag{21}$$

For fixed instants of time, we obtain *primitive potentials* of Schrödinger operator (2).

We note that $\bar\partial$-problem (13), (18) defining a primitive solution has a certain gauge equivalence in the sense that a single primitive solution $u(x,t)$ of the KdV equation can be obtained from a family of different dressings (18). This is a consequence of our previous observation that a rapidly vanishing solution of the KdV equation with N bound states can be defined using 2^N different dressings of form (15).

If we set the reflection coefficient $r(k)$ to zero, then we obtain $\rho(k,x,t)=0$, and the resulting system of equations (here $k\in[k_1,k_2]$)

$$\begin{gathered}
f(k,x,t) + \frac{R_1(k)}{\pi}e^{-2kx+8k^3t}\left[\int_{k_1}^{k_2}\frac{f(p,x,t)\,dp}{k+p} + \fint_{k_1}^{k_2}\frac{g(p,x,t)\,dp}{k-p}\right] = R_1(k)e^{-2kx+8k^3t}, \\
g(k,x,t) + \frac{R_2(k)}{\pi}e^{2kx-8k^3t}\left[\fint_{k_1}^{k_2}\frac{f(p,x,t)}{k-p}dp + \int_{k_1}^{k_2}\frac{g(p,x,t)\,dp}{k+p}\right] = -R_2(k)e^{2kx-8k^3t}
\end{gathered} \tag{22}$$

describes reflectionless primitive potentials that we previously derived in [1]–[3]. The corresponding solution of the KdV equation is

$$u(x,t) = \frac{2}{\pi}\frac{d}{dx}\int_{k_1}^{k_2}\left[f(p,x,t)+g(p,x,t)\right]dp. \tag{23}$$

We do not know an analytic method for solving equations (20) in the general case. In [1]–[3], we studied these equations numerically (with $r(k)=0$). Discretizing the integrals using Riemann sums, we obtain a linear system that coincides with the system for multisoliton solutions of the KdV equation. In other words, rapidly vanishing solutions can approximate primitive solutions of the KdV equation, and multisoliton solutions can approximate reflectionless primitive solutions. Simulations with constant R_1 and R_2 show that a relatively ordered solution at $t=0$ quickly becomes disordered under the KdV flow.

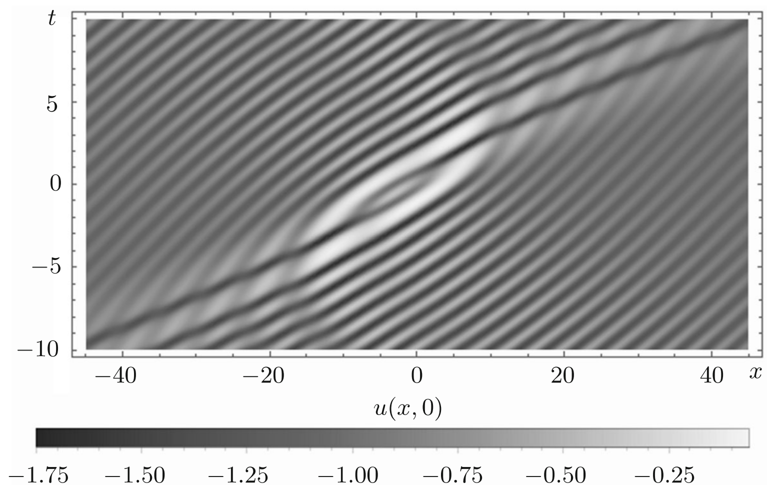

Fig. 1. A space–time plot of the primitive potential $u(x,t)$ determined by $k_1 = 1/4$, $k_2 = 1$, $R_1 = 10^2$, and $R_2 = 10^4$.

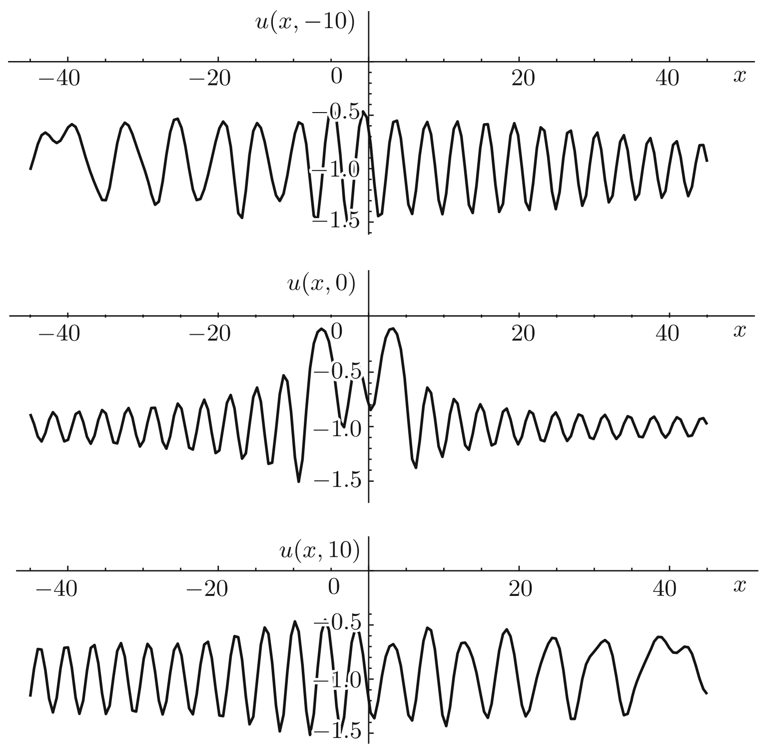

Fig. 2. Spatial plots of the primitive potential $u(x,t)$ shown in Fig. 1 at the instants $t = -10$, $t = 0$, and $t = 10$.

We show an example of a primitive potential with constant R_1 and R_2 in Figs. 1 and 2. Unfortunately, the condition number of the discretized system is exponential in x and requires the use of multiprecision arithmetic.

We can also consider solutions of the KdV equation obtained from (22) with $R_2 = 0$ (equivalently, with $R_1 = 0$). Such solutions were rigorously studied in [12]. These solutions are rapidly decreasing in one direction and tend to an elliptic one-gap potential in the other direction.

4. Algebro-geometric potentials as reflectionless primitive potentials

We now return to the question of constructing algebro-geometric finite-gap solutions of the KdV equation as limits of multisoliton solutions. In the simplest case, we want to construct the elliptic one-gap potential

$$u(x) = 2\wp(x + i\omega' - x_0) + e_3 \tag{24}$$

with the spectrum $[-k_2^2, -k_1^2] \cup [0, \infty)$, where

$$k_1^2 = e_2 - e_3, \qquad k_2^2 = e_1 - e_3, \qquad e_1 + e_2 + e_3 = 0. \tag{25}$$

In [2] (see Sec. 5), we showed that potential (24) is the reflectionless primitive potential corresponding to the dressing functions

$$R_1(k) = \sqrt{\frac{(k_2 - k)(k + k_1)}{(k - k_1)(k + k_2)}}, \qquad R_2(k) = \frac{1}{R_1(k)}. \tag{26}$$

At the same time, numerical experiments showed that the elliptic potential can also be constructed using the dressing functions $R_1 = R_2 = 1$. There is no contradiction here: as we noted above, a primitive potential can be given by a whole family of dressings of form (18). It is a curious fact that if we choose R_1 and R_2 to be distinct constants, then we (numerically) obtain solutions that do not appear to be finite-gap.

Finally, it was shown in [6] that any algebro-geometric finite-gap solution of the KdV equation including elliptic solutions is a reflectionless primitive solution. We formulate that theorem.

Theorem 1. *Let $0 < k_1 < k_2$, let $\kappa_1, \dots, \kappa_{2g}$ be an increasing sequence with*

$$0 < k_1 < \kappa_1 < \cdots < \kappa_{2g} < k_2, \tag{27}$$

and let $u(x,t)$ be a g-gap solution of the KdV equation with the spectrum

$$[-\kappa_{2g}^2, -\kappa_{2g-1}^2] \cup \cdots \cup [-\kappa_2^2, -\kappa_1^2] \cup [0, \infty). \tag{28}$$

Then there exist real constants $a_1, \dots, a_g$ such that $u(x,t)$ is the reflectionless primitive solution of the KdV equation determined by the dressing functions

$$\begin{aligned} R_1(k) &= \exp\biggl(\sum_{j=1}^{g} a_j k^{2j-1}\biggr) \sum_{l=1}^{g} \mathbb{1}_{[\kappa_{2l-1},\kappa_l]}(k), \\ R_2(k) &= \frac{1}{R_1(k)} = \exp\biggl(-\sum_{j=1}^{g} a_j k^{2j-1}\biggr) \sum_{l=1}^{g} \mathbb{1}_{[\kappa_{2l-1},\kappa_l]}(k), \end{aligned} \tag{29}$$

where $\mathbb{1}_{[\kappa_{2l-1},\kappa_l]}$ is the indicator function of $[\kappa_{2l-1}, \kappa_l]$. Conversely, any primitive solution $u(x,t)$ determined by dressing functions of form (29) *is an algebro-geometric finite-gap solution with spectrum* (28).

Because periodic finite-gap solutions of the KdV equation are dense in the space of all periodic solutions, it follows that the set of multisoliton solutions of KdV is dense in the space of periodic solutions of the KdV equation. Describing all pairs of dressing functions R_1 and R_2 such that the corresponding primitive solutions are algebro-geometric and determining what relation, if any, there is between generic primitive solutions and algebraic geometry remain open problems.

5. The symmetric case

In [5], we considered reflectionless primitive potentials determined by Eqs. (22) under the further assumption that $R_1(k) = R_2(k)$ for all $k \in [k_1, k_2]$. In this case, the jump coefficients f and g satisfy

$$g(p,x,t) = -f(p,-x,-t),$$

and the corresponding solution $u(x,t)$ of the KdV equation is symmetric in the sense that

$$u(-x,-t) = u(x,t).$$

The resulting integral equation for f can be solved recursively as a power series in $s = p^2$. For simplicity, we only give the equations for the coefficients of $f(p,x) = f(p,x,0)$:

$$f(p,x) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k} f_k(s) + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1} \sqrt{s}\, h_k(s), \qquad s = p^2. \tag{30}$$

We substitute this series in integral equations (22) and set $t = 0$. Collecting powers of x, we obtain the system of equations for $f_k(s)$ and $h_k(s)$, where k is a nonnegative integer and δ_{0k} is the Kronecker delta,

$$\begin{aligned}
\big(1 + R(\sqrt{s}\,)H\big) f_k(s) &= R(\sqrt{s}\,)\delta_{0k} - \sum_{i=0}^{k-1} \binom{2k}{2i} 2^{2k-2i} s^{k-i} f_i(s) - \sum_{j=0}^{k-1} \binom{2k}{2j+1} 2^{2k-2j-1} s^{k-j} h_j(s), \\
\big(1 - R(\sqrt{s}\,)H\big) h_k(s) &= -\sum_{i=0}^{k} \binom{2k+1}{2i} 2^{2k-2i+1} s^{k-i} f_i(s) - \sum_{j=0}^{k-1} \binom{2k+1}{2j+1} 2^{2k-2j} s^{k-j} h_j(s).
\end{aligned} \tag{31}$$

Here, H is the Hilbert transform on the interval $[k_1^2, k_2^2]$,

$$H[\psi(s)] = \frac{1}{\pi} \fint_{k_1^2}^{k_2^2} \frac{\psi(s')}{s'-s}\, ds'. \tag{32}$$

To solve Eqs. (31), we must invert the integral operators $1 \pm R(\sqrt{s}\,)H$ in the left-hand side. A calculation shows that the integral operator

$$L_\alpha[\psi(s)] = \psi(s) + \tan(\pi\alpha(s))H[\psi(s)],$$

where $\alpha(s)$ is a Hölder-continuous function on the interval $[k_1^2, k_2^2]$ such that $|\alpha(s)| < 1/2$ for all s, has a unique bounded inverse on $L^p(\mathbb{R})$ for $p > 1$ and $p \neq 2$ given by

$$L_\alpha^{-1}[\varphi(s)] = \cos^2(\pi\alpha(s))\varphi(s) - \sin(\pi\alpha(s))e^{-\pi H[\alpha(s)]} H[\cos(\pi\alpha(s))e^{\pi H[\alpha(s)]}\varphi(s)].$$

Using this formula, we can recursively solve system (31). The corresponding primitive potential is given by the formula

$$u(x) = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \int_{k_1^2}^{k_2^2} h_k(s')\, ds'. \tag{33}$$

It is easy to verify that this power series converges for all values of x.

Conflicts of interest. The authors declare no conflicts of interest.

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