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EFFECT OF $f(R)$-GRAVITY MODELS ON COMPACT STARS

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We study the possibility of forming anisotropic compact stars in the framework of $f(R)$-modified gravity in a static spherically symmetric space–time. We find the unknown coefficients involved in the metric using masses and radii of the compact stars 4U 1820-30, Cen X-3, EXO 1785-248, and LMC X-4. We obtain the hydrostatic equilibrium equation for different forces and use the generalized Tolman–Oppenheimer–Volkoff equation to analyze the behavior of stars. Moreover, we verify the regularity conditions, anisotropic behavior, energy conditions, and stability of the compact stars. We use the effective energy–momentum tensor in $f(R)$ gravity for the analysis. We show that in the framework of $f(R)$ gravity theory, these compact stars have physically acceptable patterns. Our results here also agree with those in general relativity, which is a special case of $f(R)$ gravity.

Keywords: anisotropic fluid, $f(R)$ gravity, compact star

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1. Introduction

Compact stars form in the final stage of stellar evolution when the outward radial pressure from nuclear fusion in the interior core can no longer oppose the existing gravitational forces. Consequently, a star undergoes the stellar death process and crashes to a denser state known as a compact star. Compact stars include white dwarfs, neutron stars, and black holes. Researchers have tried to understand the internal structure and physical properties of compact stars using the dynamics of Einstein's field equations. Schwarzschild [1] provided the first solution of the field equations, which describes the interior of compact object in hydrostatic equilibrium. When a cloud of gas contracts and condenses, it achieves an equilibrium state in which the thermal pressure together with normal liquid pressure balances the gravitational pressure and forms an astrophysical object like a compact star.

It was first assumed that the stellar core was a fluid with equal radial and tangential pressures [2]. But the chances of observing anisotropy in a compact star are much higher because there is a strong relativistic interaction between the particles and their behavior throughout the region becomes too irregular to form any uniform distribution [3]. In compact stars, one possible reason for the appearance of significant anisotropy is the relativistic nature of particles. We can therefore say that as a consequence of an anisotropic force, these objects become more compact compared with the case of an isotropic force, which allows a possible conversion of a neutron star into a strange star. The consequences of local anisotropy for models of relativistic spheres were discussed using the equation of state, and some useful results were obtained in [4]. In [5], Ruderman proposed that the nuclear regime at very high densities indicated that an anisotropic

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pressure could form in stars. In this case, the radial pressure differs from the tangential pressure inside the stellar core.

Because of the limitations of general relativity (GR) at large scales, modified theories of gravity have captured the attention of astrophysicists. Using these modified theories, they have tried to investigate theories of the collapse and stability of astronomical bodies. The modified theories are currently quite useful because they can help in describing the expansion of the Universe and related ideas. Such work has been done in the framework of modified theories of gravity, which are $f(R)$, $f(T)$, $f(R,T)$, $f(G)$, and $f(R,G)$ gravity [6]–[17]. In particular, $f(R)$ gravity seems attractive because it is a direct generalization of GR. Many researchers have discussed several forms of $f(R)$ theory and used them for different cosmological issues, for example, the late-time cosmic evolution [18]–[20], phantom fields [21]–[23], the inflationary epoch [24], and the history of the expansion of the Universe [25]–[27]. Applying $f(R)$ gravity becomes more appropriate if we assume basic principles such as the classification of singularities, the causal structure, and energy conditions [28]–[33]. The modified equations indicate how matter affects the space–time.

Studying compact stars has always been an interesting research topic. Anisotropic compact stars were discussed in GR with the cosmological constant [34]. In [35], the compact stellar structure was considered based on the energy density and radial pressure. A singularity-free solution for compact stellar objects was found in [36]. The homotopy perturbation method was used in [37] to obtain a stable relativistic structure, and three different models of perturbative $f(R)$ gravity were used in [38] to study a neutron star. In [39] based on several realistic equations of state, the existence of a relativistic stellar structure such as neutron stars with quark cores was discussed, and the obtained results were compared with the GR conclusions. It was found in that work that a more surprising celestial system can be obtained by applying a cubic $f(R)$ model. Cubic and quadratic $f(R)$ models were considered using a numerical approach [40], and the possibility of forming of anisotropic compact stars in $f(R)$ gravity was also studied [41]. In [42], neutron and quark stars were studied in the context of the $f(R){=}R{+}\alpha R^2$ model and the equivalent Brans–Dicke theory with a dilaton field in the framework of Einstein gravity. In [43], quark star models were presented with a realistic equation of state in nonperturbative $f(R)$ gravity.

Here, we mainly focus on analyzing some specific types of compact stars in $f(R)$ gravity with anisotropic configurations. In particular, we extend the work in [44] to modified $f(R)$ gravity and analyze the stability of the anisotropic compact stars 4U 1820-30, Cen X-3, EXO 1785-248 and LMC X-4. Using observational data, we discuss some structural properties of these four different compact stars: the distribution of density and pressure anisotropy, energy conditions, stability, the Tolman–Oppenheimer–Volkof (TOV) equation, and also the behavior of the equation of state parameter.

This paper has the following structure. In Sec. 2, we derive the field equations of $f(R)$ gravity. In Sec. 3, we discuss some physical properties of $f(R)$ models satisfying the regularity conditions and then analyze the stability of the solutions using the TOV equation and energy conditions. In Sec. 4, we present concluding remarks.

2. Modified field equations

The general action of $f(R)$ gravity is [12]

$$\mathcal{S}=\int\sqrt{-g}\biggl(\frac{1}{16\pi G}[f(R)]+\mathcal{L}_{\mathrm{m}}\biggr)d^4x, \tag{1}$$

where $f(R)$ is a function depending on the Ricci scalar and $\mathcal{L}_{\mathrm{m}}$ is the matter Lagrangian. It can be seen that this action is obtained by replacing R with $f(R)$ in the standard Einstein–Hilbert action. Varying the action with respect to the metric $g_{\mu\nu}$, we obtain the field equation

$$F(R)R_{\mu\nu}-\frac{1}{2}f(R)g_{\mu\nu}-\nabla_\mu\nabla_\nu F(R)+g_{\mu\nu}\,\Box F(R)=\kappa T_{\mu\nu}, \tag{2}$$

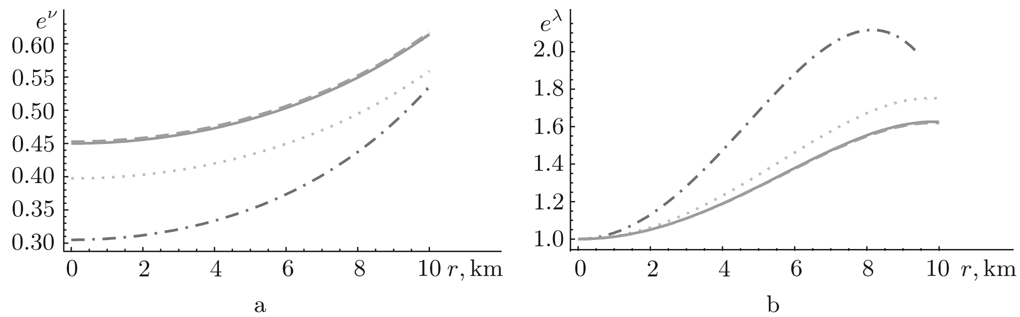

Fig. 1. Behavior of the metric potentials (a) e^{ν} and (b) e^{λ}: here and in all following plots, the dot-dashed, dotted, solid, and dashed curves correspond to the respective stars 4U 1820-30, Cen X-3, EXO 1785-248, and LMC X-4.

where $T_{\mu\nu}$ is the standard matter energy–momentum tensor, $F(R) = df(R)/dR$, $\Box = \nabla^{\mu}\nabla_{\mu}$, and ∇_{μ} as the covariant derivative. We can recover the GR field equation from (2) by replacing $f(R)$ with R. Equation (2) can also be alternatively written as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T^{\text{eff}}_{\mu\nu}, \tag{3}$$

where $G_{\mu\nu}$ is the usual Einstein tensor and $T^{\text{eff}}_{\mu\nu}$ is the effective energy–momentum tensor given by

$$T^{\text{eff}}_{\mu\nu} = \frac{1}{\kappa F(R)} \left(T_{\mu\nu} + \frac{f(R) - RF(R)}{2} g_{\mu\nu} + \nabla_{\mu}\nabla_{\nu} F(R) - g_{\mu\nu} \Box F(R) \right). \tag{4}$$

We consider the spherically symmetric space–time

$$ds^2 = e^{\nu}\, dt^2 - e^{\lambda}\, dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2 \tag{5}$$

with $\nu = Br^2 + 2\log C$ and $\lambda = \log(1 + ar^2 + br^4)$ known as the Tolman–Kuchowicz space–time [45]–[49]; here, a, b, B, and C are arbitrary constants. The behavior of the metric potentials in the framework of this model is shown in Fig. 1. For a physically justified model, the metric potential e^{ν} must be positive, must be free of singularities, and must be a monotonically increasing function of the radial coordinate. Our chosen metric potentials are consistent with these conditions. It is important to recall that in all plots, the dot-dashed, dotted, solid, and dashed curves correspond to the respective stars 4U 1820-30, Cen X-3, EXO 1785-248, and LMC X-4.

The energy–momentum tensor for an anisotropic fluid is defined as [50]

$$T^{\zeta}_{\eta} = (\rho + p_{\text{t}}) u^{\zeta} u_{\eta} - p_{\text{t}} g^{\zeta}_{\eta} + (p_{\text{r}} - p_{\text{t}}) v^{\zeta} v_{\nu}, \tag{6}$$

where u_{ζ} denotes the velocity four-vector with $u^{\zeta} u_{\eta} = -v^{\zeta} v_{\nu} = 1$. Here, ρ is the usual energy density, and p_{r} and p_{t} are the radial and transverse (tangential) fluid pressures.

Taking Eq. (6) together with Eqs. (4) and (5) into account, we can now write the effective radial and transverse pressures as

$$\begin{aligned} \rho^{\text{eff}} &= \frac{1}{F}\left(\rho + \left(\frac{f - RF}{2} \right) + e^{-\lambda} \left(F'' + F' \left(\frac{2}{r} - \frac{\lambda'}{2} \right) \right) \right), \\ p^{\text{eff}}_{\text{r}} &= \frac{1}{F}\left(p_{\text{r}} - \left(\frac{f + RF}{2} \right) - e^{-\lambda} F' \left(\frac{2}{r} + \frac{\nu'}{2} \right) \right), \\ p^{\text{eff}}_{\text{t}} &= \frac{1}{F}\left(p_{\text{t}} - \left(\frac{f + RF}{2} \right) - e^{-\lambda} \left(F'' + F' \left(\frac{1}{r} + \frac{\nu'}{2} - \frac{\lambda'}{2} \right) \right) \right). \end{aligned} \tag{7}$$

We analyze these field equations considering different $f(R)$ models.

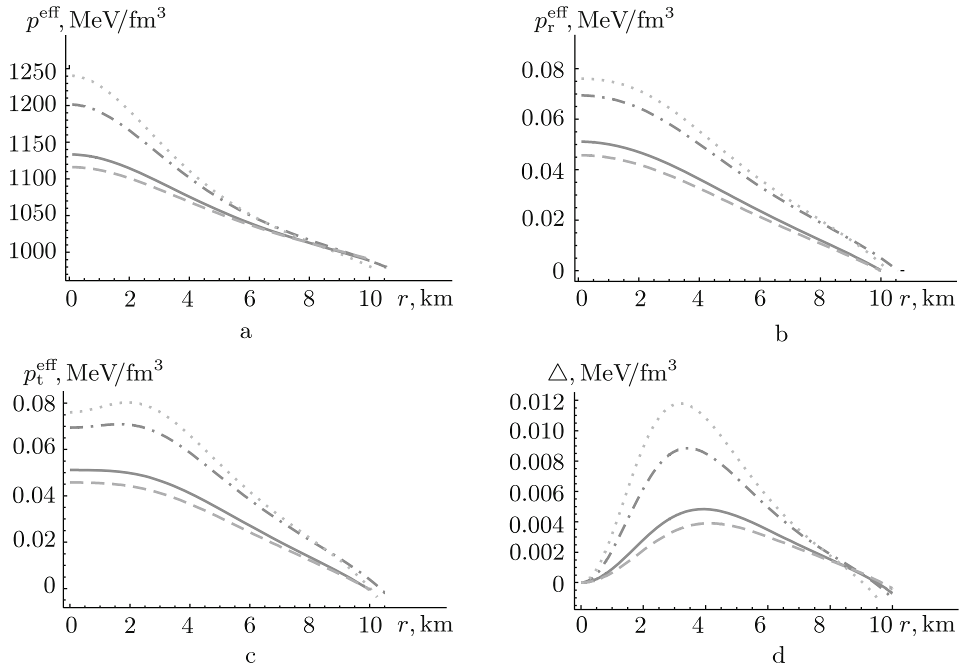

Fig. 2. Model 1: Behavior of (a) energy density, (b) radial pressure, (c) transverse pressure, and (d) anisotropic factor.

2.1. Model 1. We discuss the existence of compact stars for the model usually called a Starobinsky-like model [51]:

$$f(R) = R + mR^2, \tag{8}$$

where m is an arbitrary constant. The main feature of this model is the exponential expansion of the early Universe. In Fig. 2, we plot the functions $\rho^{\rm eff}$, $p_{\rm r}^{\rm eff}$, and $p_{\rm t}^{\rm eff}$ and the anisotropy factor $\triangle = p_{\rm t}^{\rm eff} - p_{\rm r}^{\rm eff}$ for this model. It is important to note here that we consider $\rho = 1000$, $p_{\rm r} \sim 0.01$, $p_{\rm t} \sim 0.01$, and $m = 1$ for model 1 throughout our further analysis.

2.2. Model 2. We also study the behavior of compact stars for the model [52]

$$f(R) = R + \alpha R(e^{-R/h} - 1), \tag{9}$$

where α and h are constants. In [52], this model was discussed in the context of an active program of investigating the acceleration of the late Universe complying with matter-dominated eras. In Fig. 3, we plot the functions $\rho^{\rm eff}$, $p_{\rm r}^{\rm eff}$, and $p_{\rm t}^{\rm eff}$ and the anisotropy factor $\triangle$ for this model. It is important to note here that we consider $\rho = 1000$, $p_{\rm r} \sim 0.01$, $p_{\rm t} \sim 0.01$, $h = 1$, and $\alpha = -0.1$ for model 2 throughout our further analysis to obtain physically acceptable results.

3. Physical analysis

Whether we consider an exterior or an interior space–time, the basic fundamental form for the boundary surface including the metric remains the same, and this ensures that the metric components in some coordinate systems are continuous in passing through the surface. According to Birkhoff's theorem in GR, the Schwarzschild solution is the unique spherically symmetric solution of the vacuum. But in modified theories of gravity, Birkhoff's theorem is not satisfied in most cases [53], [54], and it is therefore expected that in modified theories, the TOV equations for energy density and (possibly nonzero) pressure can admit

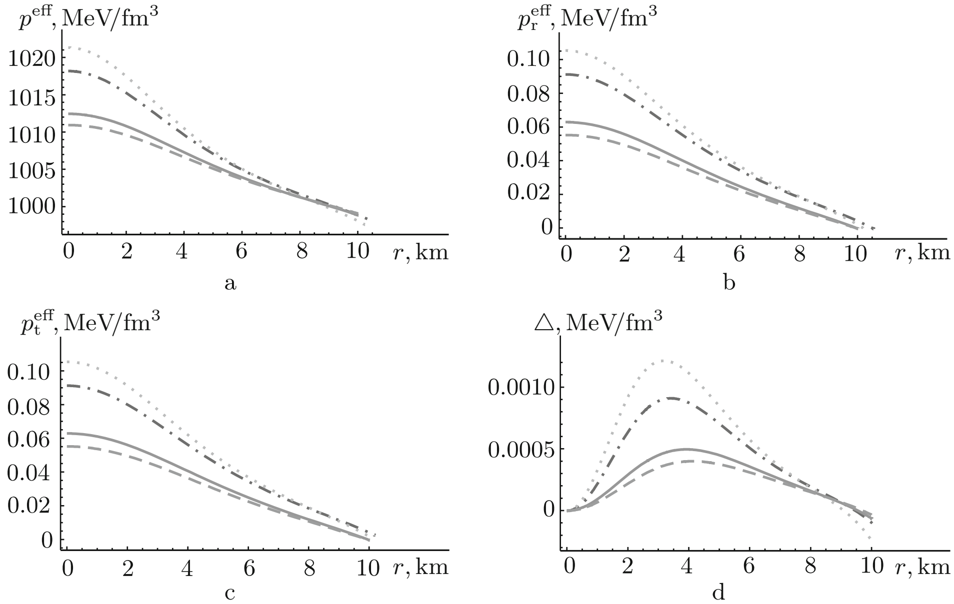

Fig. 3. Model 2: Behavior of (a) energy density, (b) radial pressure, (c) transverse pressure, and (d) anisotropic factor.

the Schwarzschild solution. We are interested in modeling realistic compact stars for which the exterior space–time must be static and asymptotically flat. To satisfy this condition, many authors have considered the Schwarzschild space–time

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\phi^2, \tag{10}$$

which has yielded some interesting results [38], [54]–[56]. At the boundary $r = R$, the continuity condition for the metric functions g_{tt}, g_{rr}, and $\delta g_{\text{tt}}/\delta r$ yield

$$g_{00}^- = g_{00}^+, \qquad g_{11}^- = g_{11}^+, \qquad \frac{\partial g_{\text{tt}}^-}{\partial r} = \frac{\partial g_{\text{tt}}^+}{\partial r}, \tag{11}$$

where $-$ and $+$ correspond to interior and exterior solutions. The conditions $e^\nu = e^{-\lambda}$ and $\lambda = (1-2M/r)^{-1}$ (briefly explained in [55], [57]) yield the constants a, b, B, and C:

$$\begin{gathered} a = \frac{1}{R(R-2M)} + \frac{M}{2R^3(R-2M)^4} - \frac{1}{R^2}, \qquad b = \frac{-M}{2R(R-2M)^4}, \\ B = \frac{M}{R^3}\left(1 - \frac{2M}{R}\right)^{-1}, \qquad C = \exp\left\{\frac{1}{2}\left(\log\left(1 - \frac{2M}{R}\right) - \frac{M}{R}\left(1 - \frac{2M}{R}\right)^{-1}\right)\right\}. \end{gathered} \tag{12}$$

For the specific values of M and R for given stars, the constants a, b, B, and C are given in Table 1.

Table 1

Star	Observed mass $M_\odot$	Predicted radius	a	b	B	C	$\frac{2M}{R}$
4U 1820-30	1.58 ± 0.06 [58]	10.56 ± 0.10	0.01753	−0.000093	0.003575	0.61099	0.443
Cen X-3	1.49 ± 0.08 [59]	10.40 ± 0.15	0.02012	−0.000116	0.003925	0.59987	0.454
LMC X-4	1.29 ± 0.05 [59]	10.00 ± 0.11	0.01157	−0.000057	0.002909	0.68747	0.367
EXO 1785-248	1.30 ± 0.20 [60]	10.02 ± 0.44	0.01291	−0.000066	0.003113	0.67084	0.384

Values of unknown constants for observed compact stars.

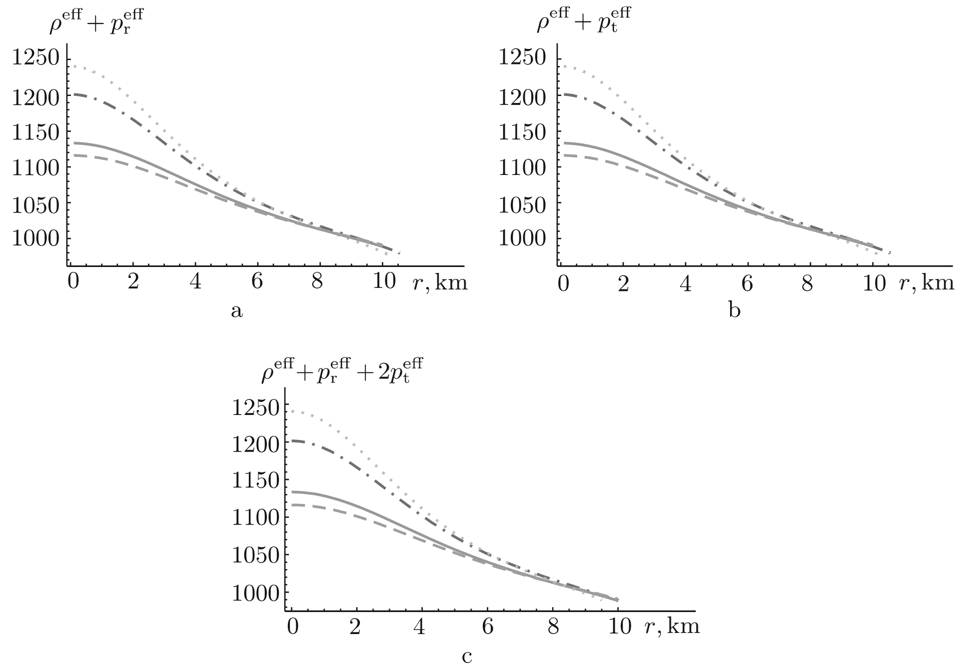

Fig. 4. Model 1: Behavior of energy constraints for compact stars: (a) NEC, (b) WEC, and (c) SEC.

3.1. Energy conditions. Energy conditions are considered very useful in analyzing some important cosmological questions. Some analytic models involving the energy conditions were recently discussed in [61]. We verify the viability of the energy conditions for modified gravity inside a sphere of anisotropic fluid: the null energy condition (NEC), the weak energy condition (WEC), and the strong energy condition (SEC) hold only if the corresponding inequalities

$$\begin{aligned} &\text{NEC:} \quad \rho^{\text{eff}} > 0, \quad \rho^{\text{eff}} + p_{\text{r}}^{\text{eff}} > 0, \\ &\text{WEC:} \quad \rho^{\text{eff}} > 0, \quad \rho^{\text{eff}} + p_{\text{r}}^{\text{eff}} > 0, \quad \rho^{\text{eff}} + p_{\text{t}}^{\text{eff}} > 0, \\ &\text{SEC:} \quad \rho^{\text{eff}} + p_{\text{r}}^{\text{eff}} > 0, \quad \rho^{\text{eff}} + p_{\text{r}}^{\text{eff}} + 2p_{\text{t}}^{\text{eff}} > 0 \end{aligned} \tag{13}$$

are satisfied simultaneously [62]. In Figs. 4 and 5, we show that all three conditions are satisfied for our chosen $f(R)$ models.

3.2. Equilibrium condition using the TOV equation. According to the TOV equation, the equilibrium condition between all forces (the gravitational force F_{g}, the anisotropic force F_{a}, and the hydrostatic force F_{h}) for a compact star has the form

$$\frac{\nu'}{2}(\rho^{\text{eff}} + p_{\text{r}}^{\text{eff}}) + \frac{dp_{\text{r}}^{\text{eff}}}{dr} + \frac{2}{r}(p_{\text{r}}^{\text{eff}} - p_{\text{t}}^{\text{eff}}) = 0. \tag{14}$$

This equation can also be written as $F_{\text{g}} + F_{\text{h}} + F_{\text{a}} = 0$, which shows the pattern of different forces satisfying the equilibrium condition for all forces. We plot F_{g}, F_{h}, and F_{a} for models 1 and 2 in Figs. 6 and 7.

3.3. Equation for the state parameter. The state parameter is a dimensionless parameter that represents the state of matter under some specific conditions. For anisotropic relativistic matter, the state parameter can be defined by

$$p_{\text{r}}^{\text{eff}} = w_{\text{r}}^{\text{eff}} \rho^{\text{eff}}, \qquad p_{\text{t}}^{\text{eff}} = w_{\text{t}}^{\text{eff}} \rho^{\text{eff}}. \tag{15}$$

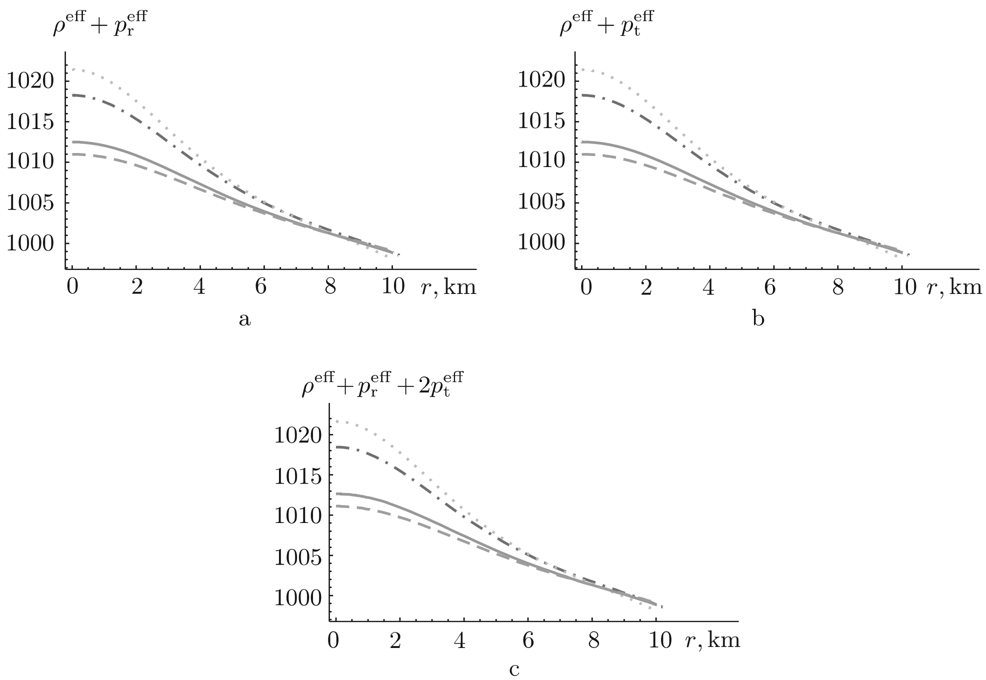

Fig. 5. Model 2: Behavior of energy constraints for compact stars: (a) NEC, (b) WEC, and (c) SEC.

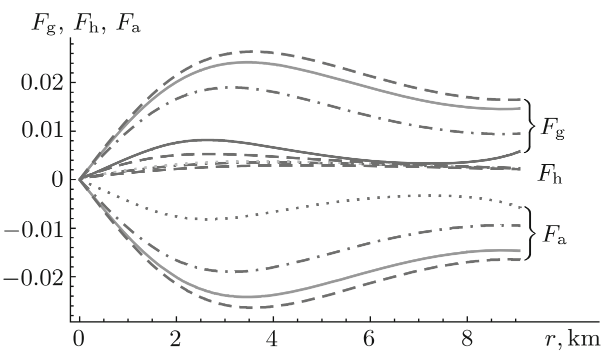

Fig. 6. Model 1: Behavior of $F_{\rm g}$, $F_{\rm h}$, and $F_{\rm a}$.

It is also important to note here that $w_{\rm r} = p_{\rm r}/\rho = 0.00001$ and $w_{\rm t} = p_{\rm t}/\rho = 0.00001$ (for both models), which means that the star consists of ordinary matter. The behaviors of $w_{\rm r}^{\rm eff}$ and $w_{\rm t}^{\rm eff}$ for the considered compact structures are plotted in Figs. 8 and 9. We note that $w_{\rm r}^{\rm eff}$ and $w_{\rm t}^{\rm eff}$ are between 0 and 1.

3.4. Energy density and pressure evolutions. It can be seen from Figs. 10 and 11 that the derivatives of $\rho^{\rm eff}$ and $p_{\rm r}^{\rm eff}$ with respect to the radial coordinate r are negative for all stars:

$$\frac{d\rho^{\rm eff}}{dr} < 0, \qquad \frac{dp_{\rm r}^{\rm eff}}{dr} < 0. \tag{16}$$

Figures 10 and 11 show that at $r = 0$ for our models,

$$\frac{d\rho^{\rm eff}}{dr} = \frac{dp_{\rm r}^{\rm eff}}{dr} = 0, \qquad \frac{d^2\rho^{\rm eff}}{dr^2} < 0, \qquad \frac{d^2 p_{\rm r}^{\rm eff}}{dr^2} < 0, \tag{17}$$

and we hence have maximum density and pressure at the center and can assume that the effect of $\rho^{\rm eff}$ decreases as r increases, which indicates high compaction at the stellar core. We propose that our chosen $f(R)$ models can ensure viable results in the outer region of the core.

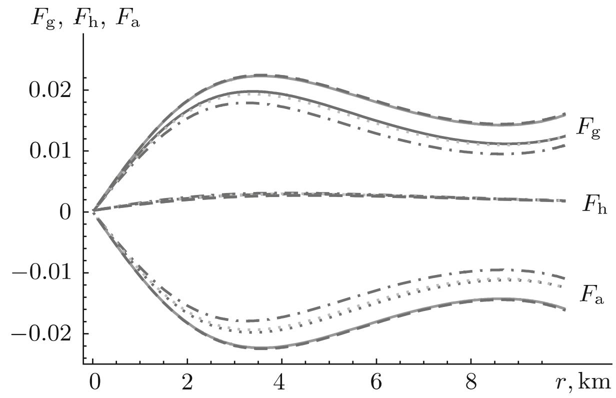

Fig. 7. Model 2: Behavior of F_g, F_h, and F_a.

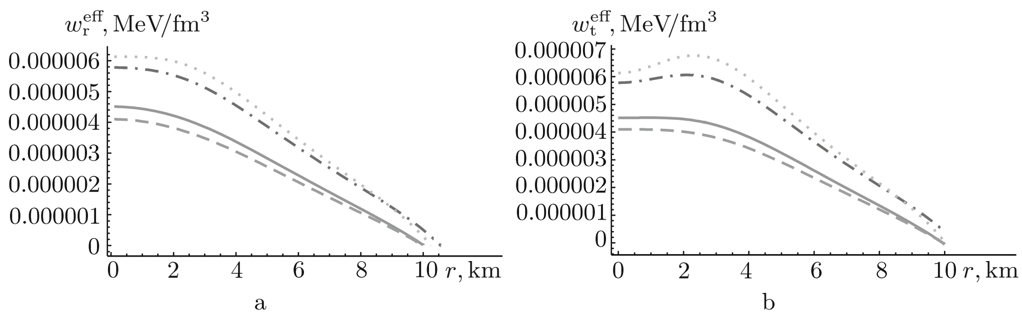

Fig. 8. Model 1: Evolution of the state parameter w_r^{eff} and w_t^{eff}.

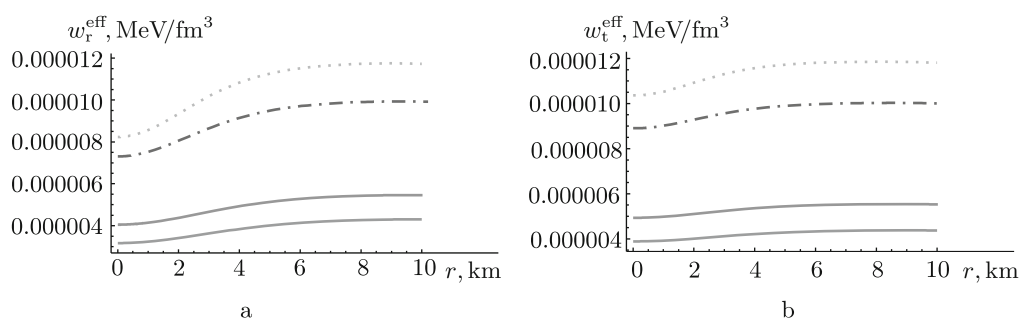

Fig. 9. Model 2: Evolution of the state parameter w_r^{eff} and w_t^{eff}.

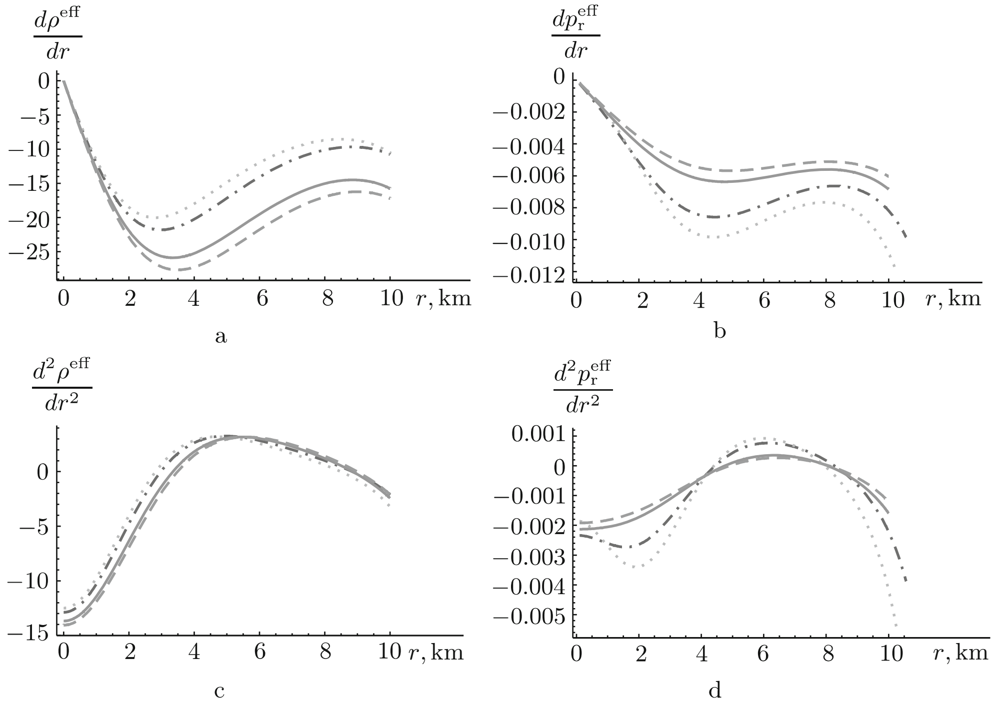

Fig. 10. Model 1: Behavior of $d\rho^{\text{eff}}/dr$ and $dp_{\text{r}}^{\text{eff}}/dr$.

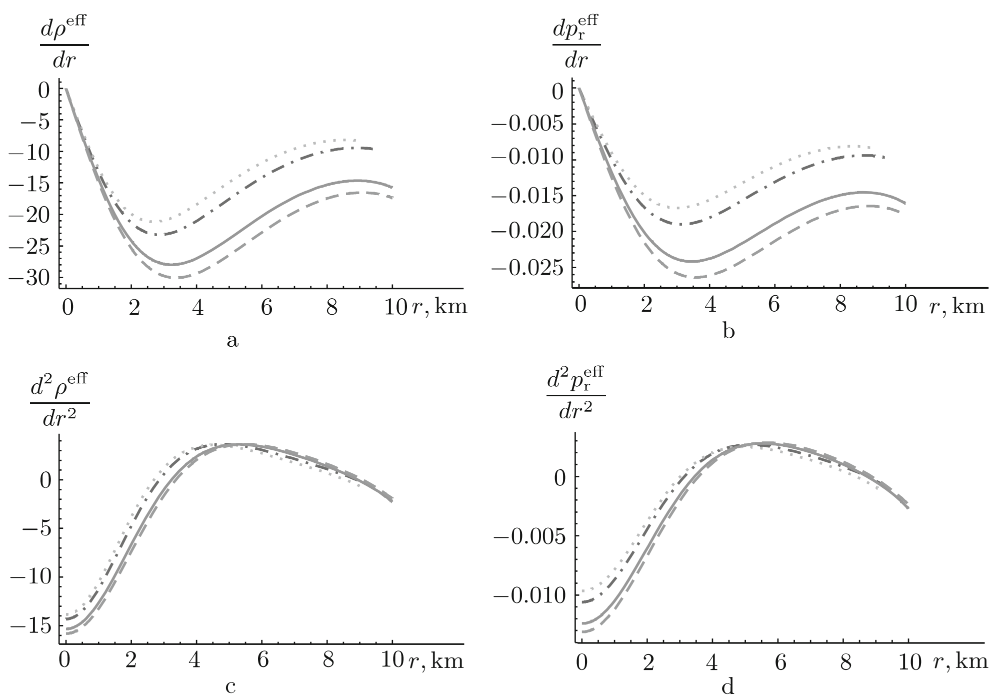

Fig. 11. Model 2: Behavior of $d\rho^{\text{eff}}/dr$ and $dp_{\text{r}}^{\text{eff}}/dr$.

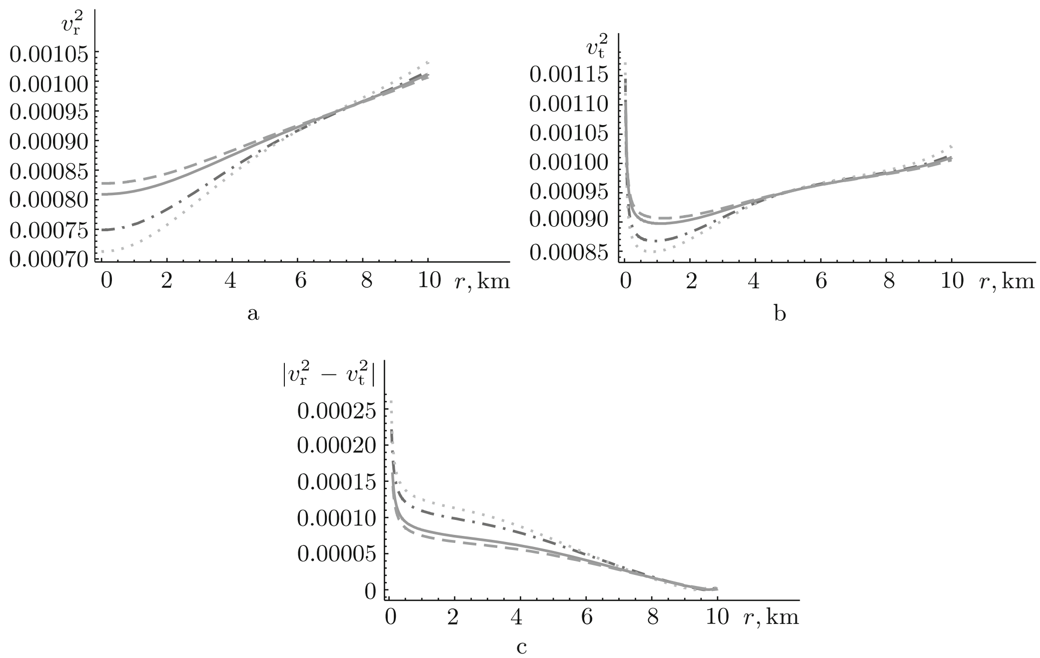

Fig. 12. Model 1: Plots for the radial and tangential speeds of sound $v_{\rm r}^2$ and $v_{\rm t}^2$ and the expression $|v_{\rm r}^2 - v_{\rm t}^2|$.

3.5. Herrera cracking concept. To verify the stability of our models, we use the cracking concept proposed by Herrera [63]. According to this concept, the radial and transverse speeds of sound should be positive and less than 1, which is also called the causality condition. For our proposed anisotropic models, we define the radial and transverse speeds of sound as

$$v_{\rm r}^2 = \frac{dp_{\rm r}^{\rm eff}}{d\rho^{\rm eff}}, \qquad v_{\rm t}^2 = \frac{dp_{\rm t}^{\rm eff}}{d\rho^{\rm eff}}. \tag{18}$$

In [64], another approach was applied to cracking inside a stagnant sphere, namely, the condition that $|v_{\rm r}^2 - v_{\rm t}^2| \le 1$ inside the stable region. Figures 12 and 13 show that our system agrees with both the causality condition and this last condition, which confirms the system stability.

3.6. Mass–radius relation. In the presence of an anisotropic matter distribution, the mass function is defined as

$$M(r) = 4\pi \int_0^R \rho r^2\, dr = \frac{R}{2}(1 - e^{-\lambda}). \tag{19}$$

The quantity M is the gravitational mass inside a sphere of radius r. It can be clearly seen from Fig. 14 that the stellar mass increases as r increases, and as the mass tends to zero, the radius r also tends to zero. Hence, the mass function is regular at the center of the star.

4. Concluding remarks

The $f(R)$ theory of gravity, which is a scientific and logical explanation of the problem of dark energy, has attracted much attention in modern cosmology. This theory has aroused a special interest because the $f(R)$ modifications of GR arises completely naturally in the low-energy effective actions of the quantum theory of gravity and in the quantization of the underlying fields in a curved space–time. This theory also simultaneously explains both the early-time inflation and the late-time acceleration of the Universe [11].

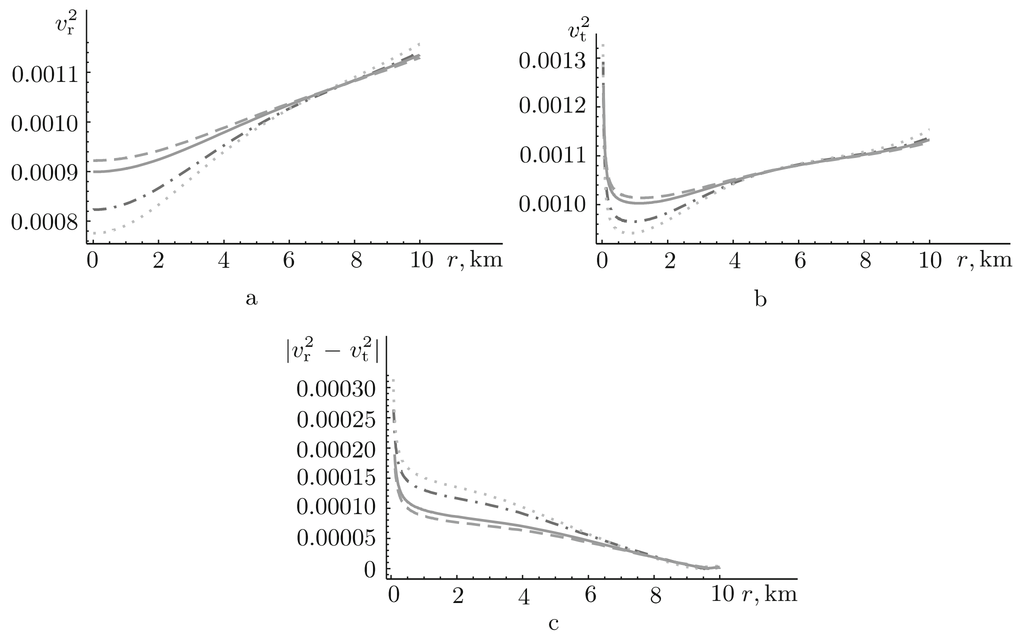

Fig. 13. Model 2: Plots for the radial and tangential speeds of sound v_r^2 and v_t^2 and the expression $|v_r^2 - v_t^2|$.

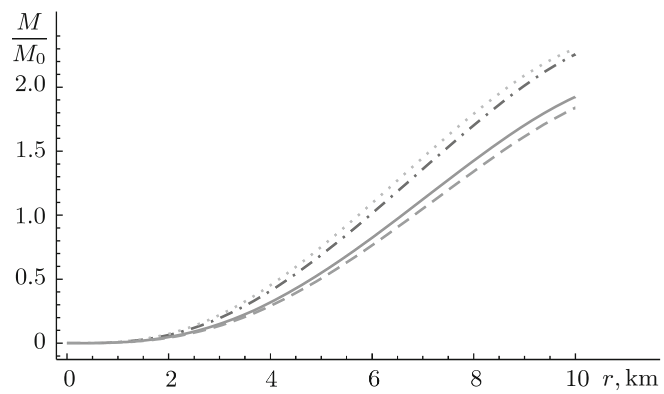

Fig. 14. Profile of the mass function.

Based on this theory, we have analyzed the behavior of some specific compact stars in the $f(R)$ theory of gravity. In particular, we considered 4U 1820-30, Cen X-3, EXO 1785-248, and LMC X-4 as representatives of compact stars and used their observed values of mass and radius to derive the values of the unknown metric coefficients. We considered two well-known $f(R)$ models: $f(R) = R + mR^2$ and $f(R) = R + \alpha R(e^{-R/h} - 1)$, where m, α, and h are arbitrary real constants. We further assumed that the stars fluid is anisotropic, $p_r \neq p_t$ in the stellar interiors. The physical interpretation of the results allowed drawing the following conclusions:

- Behavior of the metric potentials e^{λ} and e^{ν}: For a physically valid model, the metric potential e^{ν} must be free from singularities and must be a positive monotonically increasing function of the radial coordinate. The chosen metric potentials are consistent with all these conditions (see Fig. 1).

- Equation (17) confirms that the effective density and pressure attain their maximum values at the center for both models. Therefore, these models are free from a central singularity and are hence decreasing functions of the radial coordinate r for $0 \leq r \leq R$, where R is the radius of the stellar surface.

- For the proposed models, the anisotropy is nearly zero at the center (see Figs. 2 and 3), and $\Delta > 0$ as r increases, which represents a force due to anisotropy directed outward. This indicates the existence of more massive and compact configurations.

- It is well known that a stellar system is stable under fluctuations if it satisfies certain constraints on the radial and tangential speeds of sound. Figures 12 and 13 show that all five types of compact stars are stable, $0 \leq |v_{\mathrm{r}}^2 - v_{\mathrm{t}}^2| \leq 1$.

- The three energy conditions NEC, WEC and SEC are all satisfied for the considered stellar structures (see Figs. 4 and 5), which confirms the physical justification of the chosen $f(R)$ models. We also used the TOV equation to investigate the freedom from singularities and the stability of the stellar systems.

It is interesting that the effect of the $f(R)$ theory of gravity on the compact stellar structure demonstrate the same results as predicted in [44], where compact stars were considered in the GR framework.

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Conflicts of interest. The authors declare no conflicts of interest.

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