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COSET SPACE CONSTRUCTION FOR THE CONFORMAL GROUP: SPONTANEOUSLY BROKEN PHASE AND INVERSE HIGGS PHENOMENON

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We establish a mathematically rigorous way to construct effective theories resulting from the spontaneous breaking of conformal invariance. We show that the Namby–Goldstone field corresponding to spontaneously broken generators of special conformal transformations is always a nondynamical degree of freedom. We prove that the developed approach and the standard approach including application of the inverse Higgs mechanism are equivalent.

Keywords: conformal field theory, coset space technique, inverse Higgs phenomenon, spontaneous space–time symmetry breaking

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1. Introduction

The study of effective theories resulting from spontaneous symmetry breaking of the conformal invariance[1] has a long history. It was shown in one of the first works on this topic [1] that applying the standard coset space technique in such cases yields a Namby–Goldstone field (NGF) corresponding to spontaneously broken special conformal transformations (SCT) that is massive. It was suggested that this mechanism for forming a massive vector field as a result of spontaneous breaking of space–time symmetries is a new manifestation of the Higgs mechanism [2]–[4]. But it was later shown in [5]–[7] that the NGF for SCT never describes independent fluctuations of the vacuum and is hence a redundant field. The presence of similar redundant fields was then also found in several other cases of spontaneous breaking of space–time symmetries [8]–[10]. A way to construct effective theories without redundant fields was proposed in [8]. Namely, it was proposed to apply the so-called inverse Higgs constraint, which allows expressing redundant fields in terms of physical fields, i.e., fields describing independent vacuum fluctuations. The proposed prescription worked successfully in all known cases and was therefore accepted as the correct, standard way to eliminate unphysical degrees of freedom from an effective theory. In particular, in the case of spontaneous breaking of conformal invariance, this approach allows eliminating the NGF for SCT in favor of the dilaton field [6], [8], [11].

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[1]By definition, $O(2, d+1)$ is the conformal group in a $(d+1)$-dimensional space–time. It should be distinguished from the Weyl group, consisting of all possible local rescalings of the metric.

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Despite the wide application of the standard approach, it nevertheless lacks a rigorous mathematical justification. Indeed, imposing the inverse Higgs constraints is a mathematically consistent requirement that allows realizing a given scheme of spontaneous symmetry breaking by fewer fields than there are broken generators. Nevertheless, the necessity of using the inverse Higgs mechanism does not follow from the mathematical apparatus of the coset space technique, which makes this approach a successful prescription. Hence, the question of the mathematical justification of the standard approach remains open.

Until recently, there was a similar problem with constructing conformally invariant Lagrangians in the unbroken phase.[2] To ensure the applicability of the coset space technique in this case,[3] we must regard the SCT generators as spontaneously broken (more precisely, as nonlinearly realized generators) [1], [16], [17]. This raises the question of interpreting the coordinates associated with SCT in the used coset space. To illustrate this situation, we consider the scheme for nonlinear realization of the symmetries[4]

$$\mathrm{Conf}(d) \to SO(d), \tag{1}$$

where $\mathrm{Conf}(d)$ is the d-dimensional conformal group.[5] This corresponds to considering the coset space

$$g_{\mathrm{conf}} = e^{iP_\mu x^\mu} e^{iK_\nu y^\nu}, \tag{2}$$

where P_μ and K_ν are the respective generators of translations and SCT. The interpretation of the parameters x^μ is known: they are coordinates in the considered space. But it is unknown how y^ν should be interpreted: regard them as NGF [1] or as an additional set of coordinates corresponding to the so-called biconformal space [16], [17]? It was shown in [12] based on the method of induced representations [13]–[15] that y^ν plays a special role. Namely, y^ν should be regarded as field whose dependence on the coordinates is fixed by geometric considerations. But this means that y^ν can also play a special role in a spontaneously broken phase. If this is indeed so, then this observation can be the key for justifying the standard approach to the construction of effective theories in such cases.

Our aim here is to confirm the proposal indicated above. For this, we generalize the technique described in [12] to the case of spontaneously broken conformal invariance. It follows from the consideration that y^ν also depends on the coordinates fixed by geometric considerations, and it consequently decouples from the other fields. Analyzing the connection with the standard approach, we show that it can be regarded only as a convenient tool for constructing effective theories but not as a mathematically self-contained method.

This paper is structured as follows. In Sec, 2, we briefly review the standard technique and also generalize the results in [12] to the case of a spontaneously broken conformal invariance. In Sec. 3, we show that any effective Lagrangian obtained in the framework of one of the approaches can also be obtained in the framework of the other. This result allows establishing the status of the standard approach as a convenient tool for constructing effective theories. In Sec, 4, we summarize the results.

[2] We say that a theory is in the unbroken phase if all symmetry generators annihilate the vacuum. In [12], all generators were assumed to be unbroken, and the construction of conformally invariant theories was based on the method of induced representations [13]–[15]. The presence of the exponentials of SCT in the coset space used in [12] was necessitated by geometric considerations.

[3] The coset space technique is applicable only to a homogeneously reductive coset space G/H. To satisfy this requirement, it turns out to be necessary to include the exponentials of the SCT in the coset space. We recall that a coset space G/H is said to be homogeneously reductive if $[Z, V] \subset Z$ and $[V, V] \subset V$, where V are the basis generators of the algebra H and Z supplement them to the full basis in the algebra G.

[4] To apply the coset space technique, we must specify the nonlinearly realized generators. Because such generators are not necessarily broken, it is more appropriate to speak of a scheme of nonlinear realization rather than a scheme of spontaneous symmetry breaking. For example, regardless of whether translation generators are spontaneously broken, they must always be included in the coset space [18]. The coset space technique is also applicable for constructing gauge theories in the unbroken phase [19], [20].

[5] For simplicity (see footnote 6 below), we consider the Euclidean conformal group.

2. Spontaneously broken conformal invariance

2.1. Standard technique. We first consider the standard procedure for constructing effective Lagrangians resulting from spontaneous breaking of conformal invariance. We do not review the standard rules for applying the coset space technique, which can be found in [11], [18], [21] if needed.

We assume that the vacuum expectation value of some order parameter Φ spontaneously breaks the conformal invariance down to the Poincaré invariance. The scheme of nonlinear realization of symmetries hence has form (1), and the corresponding coset space is

$$g_{\rm br} = e^{iP_\mu x^\mu} e^{iK_\nu y^\nu(x)} e^{iD\pi(x)}, \tag{3}$$

where D is the generator of dilations. Following the standard rules for applying the coset space technique [18], we interpret $y^\nu(x)$ and $\pi(x)$ as fields and x^μ as coordinates of the Euclidean space.

To see that y^ν is a redundant degree of freedom, we consider the action of dilations and SCT on the order parameter,

$$\widehat{D}\Phi = \Delta_\Phi \Phi, \qquad \widehat{K}_\mu \Phi = 2x_\mu \Delta_\Phi \Phi, \tag{4}$$

where Δ_Φ is the scaling dimension of Φ and we take into account that Φ is independent of the coordinates. As can be seen from the presented formula, the action of SCT reduces to the coordinate-dependent action of dilations. Or, in other words, SCT do not have their "own" action on fields. This means that y^ν is not needed for describing all possible local fluctuations of the vacuum [7]–[10]. Hence, physical considerations show that y^ν is a redundant. To account for this fact in the framework of the coset space technique, it was proposed to use the so-called inverse Higgs mechanism, which is as follows.

The Maurer–Cartan forms for coset space (3),

$$g_H^{-1} dg_H = iP_\mu \omega_P^\mu + iK_\nu \omega_K^\nu + iD\omega_D + iL_{\mu\nu}\omega_L^{\mu\nu}, \tag{5}$$

where $L_{\mu\nu}$ are the generators of the Lorentz transformations, are given by

$$\begin{aligned} &\omega_K^\nu = e^{-\pi}(dy^\nu + 2y_\rho dx^\rho y^\nu - y^2 dx^\nu), && \omega_P^\mu = e^\pi dx^\mu, \\ &\omega_D = 2y_\rho dx^\rho + d\pi, && \omega_L^{\mu\nu} = y^\nu dx^\mu - y^\mu dx^\nu. \end{aligned} \tag{6}$$

All of these forms except $\omega_L^{\mu\nu}$ transform homogeneously under the action of all continuous elements of the conformal group. Therefore, the requirement that ω_D vanish,

$$\omega_D = 0 \quad \Rightarrow \quad y_\nu = -\frac{1}{2}\partial_\nu \pi, \tag{7}$$

is consistent with the action of all continuous elements of the conformal group. This prescription is known as the inverse Higgs constraint and allows expressing the field y^ν in terms of the dilaton field. With constraint (7) imposed, effective theories with the correct number of degrees of freedom can be constructed. Namely, applying the general rules of the formalism of the coset space technique [18], from Maurer–Cartan forms (6), we obtain the tetrads e_ν^μ, the covariant metric $g_{\mu\nu}$, and the covariant derivative of y^ν (which after requirement (7) is imposed plays the role of the covariant derivative of an arbitrary dilaton),

$$\begin{aligned} &\omega_P^\mu = e_\nu^\mu dx^\nu, \qquad g_{\mu\nu} \equiv e_\mu^\lambda \delta_{\lambda\rho} e_\nu^\rho = e^{2\pi}\eta_{\mu\nu}, \\ &D_\mu y^\nu\big|_{\rm iHc} = e^{-2\pi}\left(\frac{1}{2}\partial_\mu\pi\partial^\nu\pi - \frac{1}{2}\partial_\mu\partial^\nu\pi - \delta_\mu^\nu \partial_\lambda\pi\partial^\lambda\pi\right). \end{aligned} \tag{8}$$

For a matter field ψ, we can introduce the homogeneously transforming 1–form [18], [21]

$$D\psi = d\psi + \omega_L^{\mu\nu}\widehat{L}_{\mu\nu}\psi, \tag{9}$$

where $\widehat{L}_{\mu\nu}$ is a representation of $L_{\mu\nu}$ corresponding to ψ. With satisfaction of inverse Higgs constraint (7), this covariant derivative becomes

$$D_\mu\psi\big|_{\text{iHc}} = e^{-\pi}\biggl(\partial_\mu\psi + \frac{1}{2}(\delta_\mu^\rho\partial^\lambda\pi - \delta_\mu^\lambda\partial^\rho\pi)\widehat{L}_{\lambda\rho}\psi\biggr). \tag{10}$$

Any $SO(d)$-invariant Lagrangian constructed from (8) and (10) is also automatically conformally invariant. Similarly constructed Lagrangians also include only one NGF, as required by physical considerations.

The construction described above is the standard approach for constructing effective theories resulting from spontaneous breaking of conformal invariance. Despite its success in practical applications, requirement (7) remains its weak point. Indeed, on one hand, it is needed for eliminating redundant fields from the theory. On the other hand, it does not follow only from the formalism of the coset space technique that the field y^ν is always redundant. For example, based only on mathematical considerations, we can suppose that in some cases, the field $y^\nu(x)$ indeed describes a massive vector field or has another physical interpretation [5].

2.2. Two-orbit technique. Before presenting the generalization of the technique for constructing conformally invariant Lagrangians developed in [12] to the case of a spontaneously broken conformal invariance, we consider its application in the unbroken phase. Everywhere below, we call this technique the *two-orbit approach*.

The fields of d-dimensional conformal field theories are defined on a sphere S^d, which is equivalent to the Euclidean space supplemented by a point at infinity.[6] Indeed, SCT always map some point to infinity, and we must consequently also consider it. Hence, in accordance with the method of induced representations [13]–[15], to construct a conformally invariant theory, we must consider a coset space isomorphic to the sphere. This leads to the following difficulties. On one hand, the process of calculating Maurer–Cartan forms includes taking the logarithmic derivative. Therefore, the considered coset space must be parameterized by continuous elements of the group. But on the other hand, a sphere is not isomorphic to an orbit of any of its points under the action of continuous elements of a conformal group (only the inversion maps the north pole of the sphere to the south pole). To solve this problem, we can try to use the fact that a sphere can be obtained by gluing two Euclidean spaces together. A mathematically rigorous implementation of this idea leads to the following construction [12]: for constructing conformally invariant theories, we must consider coset space (2) where y^ν is a field with the fixed coordinate dependence

$$y^\nu(x) = \frac{x^\nu}{x^2}, \qquad \vec{x} \neq \vec{0}. \tag{11}$$

The field y^ν thus introduced describes the gluing map of coordinate charts around the north and south poles of the sphere and thus converts $2d$-dimensional coset space (2) into a d-dimensional sphere. We also note that condition (11) is admissible in the sense that it is invariant under the action of the conformal group. Indeed, expression (11) is a solution of the equation $\omega_K^\nu = 0$ [12]. Consequently, because ω_K^ν transforms homogeneously under the action of the conformal group, expression (11) is invariant.[7]

[6] In the case of a Minkowski space–time, we have not a point but a light cone at infinity. We therefore consider the Euclidean conformal group here.

[7] The action of the inversion exchanges the roles of x^μ and y^ν, and expression (11) is also invariant in this sense [12]. On the other hand, the second solution of the equation $\omega_K^\nu = 0$, namely, $y^\nu = 0$, is not invariant under the action of the inversion. We also note that in contrast to the approach adopted in [17], we regard $y^\nu(x)$ as a field, not a coordinate.

Condition (11) uniquely fixes the coordinate-dependence of y^ν, and the equation of motion for $y^\nu(x)$ must therefore admit (11) as a solution. This fact strongly constrains the admissible combinations of Maurer–Cartan forms. The reason for this is that the covariant derivative of an arbitrary (quasiprimary) field ψ has the form

$$D_\mu\psi = \partial_\mu\psi + 2y^\nu(\eta_{\nu\mu}\Delta + i\widehat{L}_{\mu\nu})\psi, \tag{12}$$

where Δ and $\widehat{L}_{\mu\nu}$ are representations of D and $L_{\mu\nu}$ corresponding to ψ. Then, on one hand, y^ν must have form (11). On the other hand, the (dynamical) field ψ couples to y^ν via the interaction term in covariant derivative (12). As a result of this, y^ν also has nontrivial dynamics, thus breaking condition (11). The only way to avoid this contradiction is to require that the interaction terms sum to a total derivative or totally disappear from the Lagrangian. This requirement substantially restricts the class of admissible Lagrangians and is a principally new constraint in the framework of the coset space technique. Remarkably, precisely it ensures a well-known property of conformal field theories: their virial current must be a total derivative of some function [22], [23].

In addition to the approach laid out above, we can approach the description of the procedure for constructing conformally invariant theories as follows. We regard y^ν in (2) as an auxiliary field introduced to ensure the applicability of the coset space technique. For only physical fields to remain in the theory, we must require that y^ν decouple from all matter fields. This is just a reformulation of the requirement established above. Although this reasoning is not rigorous, it indirectly supports the correctness of the two–orbit approach.

Generalizing the two–orbit approach to the case of a spontaneously broken conformal invariance is rather simple. Because conformal field theories are defined on a sphere, the interpretation of y^ν remains the same as before: it is a field whose equation of motion must admit (11) as a solution. This requirement follows from only the method of induced representations and is independent of whether the conformal invariance is spontaneously broken. Therefore, although the action of SCT do not annihilate the vacuum, the logic in [12] remains applicable. We can present one more argument supporting the proposed generalization . As previously mentioned, the virial current must be a total derivative as a consequence of requiring that y^ν decouple from the other fields of the theory. In a spontaneously broken phase, the virial current must still be a total derivative, which by reversing the logic leads to the previously established requirement.

We consider the application of the proposed technique. The conformal symmetry is the maximal space–time symmetry group that relativistic field theories can have [24], [25] (except supersymmetry, of course). Consequently, the most general scheme of spontaneous symmetry breaking including breaking conformal invariance has the form

$$\mathrm{Conf}(d) \times G_{\mathrm{int}} \to H, \tag{13}$$

where G_{int} is an internal symmetry group and H can include vector subgroups of space–time and internal symmetries. Everywhere below, we use standard terminology. In particular, we say that a generator is broken if it does not annihilate the vacuum. Applied to a conformal group, this leads to the following observation. The action of SCT on quasiprimary fields (elements of an irreducible representation) reduces to the coordinate-dependent action of a dilation, Lorentz transformation, and translation,

$$\widehat{K}_\mu\psi = (2x_\mu\widehat{D} - x^\nu\widehat{L}_{\mu\nu} + ix_\nu x^\nu\widehat{P}_\mu)\psi. \tag{14}$$

It follows from this formula that SCT are spontaneously broken if and only if at least one of the three generators D, P_μ, or $L_{\mu\nu}$ is broken. Indeed, if none of them are broken, then it follows from formula (14) that K_μ is also not broken, and conversely. Hence, if conformal invariance is spontaneously broken, then SCT are also necessarily broken. Consequently, the most general coset space corresponding to scheme (13) of nonlinear realization of symmetries can be written in the form

$$g_H = e^{iP_\mu x^\mu} e^{iK_\nu y^\nu} e^{iZ_a \xi^a}, \tag{15}$$

where Z_a are the broken generators different from K_ν, ξ^a are the corresponding NGF, a can denote both a space–time and an internal index, and we assume that translations are not spontaneously broken. Under the assumption that coset space (15) is homogeneously reductive, all Maurer–Cartan forms except ω_H^i,

$$g_H^{-1} dg_H = iP_\mu \omega_P^\mu + iK_\nu \omega_K^\nu + iZ_a \omega^a + iH_i \omega_H^i, \tag{16}$$

where H_i are the generators of H, transform homogeneously under the action of all continuous symmetries. For a matter field ψ, a homogeneously transforming 1-form is

$$\mathcal{D}\psi = d\psi + i\omega_H^i \widehat{H}_i \psi, \tag{17}$$

where $\widehat{H}_i$ is a representation of H_i corresponding to ψ. We then obtain G-invariant Lagrangians as H-invariant wedge products of ω_P^μ, ω_K^ν, ω_Z^a, ψ, and $\mathcal{D}\psi$ that admit (11) as a solution.

To understand which theories satisfy the indicated requirements, we note that in the general case, an arbitrary Lagrangian can be split into two parts,

$$\mathcal{L} = \mathcal{L}_{\text{kin}}(\omega_P^\mu, \omega_K^\nu) + \mathcal{L}_{\text{ph}}(\omega_P^\mu, \omega_K^\nu, \omega_Z^a, \psi, \mathcal{D}\psi), \tag{18}$$

where $\mathcal{L}_{\text{kin}}$ is a wedge product of only[8] ω_P^μ and ω_K^ν and $\mathcal{L}_{\text{ph}}$ contains all other terms. As is shown at the end of the next section, $\mathcal{L}_{\text{kin}}$ always admits (11) as a solution. Because such Lagrangians do not contain ξ^a and ψ, all Lagrangians with the same $\mathcal{L}_{\text{ph}}$ but different $\mathcal{L}_{\text{kin}}$ are physically equivalent. Hence, without loss of generality, we can set $\mathcal{L}_{\text{kin}}$ to zero, which we assume in what follows. We further note that because of the commutation relations of the conformal algebra, the fields y^ν appear in the Maurer–Cartan forms ω_K^ν and some appear in ω_Z^a and also in the 1-forms $\mathcal{D}\psi$. This results in the appearance of interaction terms between y^ν and other fields. Then, because ξ^a and ψ can have arbitrary dynamics, the equations of motion for y^ν admit (11) as a solution only if all interaction terms sum to a total derivative. Therefore, the only admissible Lagrangians are those in which y^ν decouples from the other fields, just as in the unbroken phase.

The described construction is the generalization of the technique described in [12] to the case of spontaneously broken conformal invariance. Its key finding is that the NGF for SCT does not represent perturbations of the vacuum but ensures the known property of conformal field theories: their virial current is a total derivative.

To illustrate the application of the developed approach, we consider the construction of effective Lagrangians in the case a spontaneous breaking of the conformal invariance to the Poincaré invariance,

$$\text{Conf}(d) \to SO(d). \tag{19}$$

From Eq. (6), we can obtain the covariant metric g_{mn} and the covariant derivatives of the fields π, y^ν, and ψ:

$$g_{mn} = e^{2\pi} \delta_{mn}, \qquad D_m \pi = e^{-\pi}(\partial_m \pi + 2y_m), \tag{20}$$

$$D_m y^\nu = e^{-2\pi}(\partial_m y^\nu + 2y_m y^\nu - \delta_m^\nu y^2), \qquad \mathcal{D}_m \psi = e^{-\pi}(\partial_m \psi + 2iy^n \widehat{L}_{mn} \psi), \tag{21}$$

[8] We can also write an analogous term $\mathcal{L}_{\text{kin}}$ in the unbroken phase. It corresponds to the kinetic term of y^ν and always admits (11) as a solution [12].

where $\widehat{L}_{mn}$ is a representation of L_{mn} corresponding to ψ and Latin letters denote indices that should be raised/lowered by the covariant metric. In accordance with the established requirements in the framework of the two-orbit approach, effective Lagrangians must (1) be constructed as $SO(d)$-invariant combinations of the covariant derivatives and (2) include y^ν only via a total derivative. For example, the simplest Lagrangian satisfying these requirements has the form

$$\mathcal{L} = \frac{1}{2} D_\mu \pi D^\mu \pi + D_\mu y^\mu = \frac{1}{2} e^{-2\pi} \partial_\mu \pi \partial^\mu \pi + \partial_\mu (e^{-2\pi} y^\mu). \tag{22}$$

Constructing more complicated effective theories in the framework of the two-orbit technique is problematic because of the need to satisfy the second requirement. In connection with this, the question arises of whether the proposed procedure for constructing effective theories can be simplified. As is shown in the next section, this is indeed possible, and such a simplification corresponds to applying the inverse Higgs mechanism.

3. Equivalence of the approaches

3.1. The case $d > 2$. It was established above that geometric considerations uniquely fix the coordinate-dependence of y^ν. At the same time, in the framework of the standard approach, y^ν and the dilaton field are connected by relation (7). Combining these statements, we can conclude that the dilation must also have a fixed coordinate-dependence. This statement seems meaningless because the dilaton must be a dynamical degree of freedom. In fact, such a contradiction arises because the standard approach is inconsistent with the inversion symmetry, and the presented argument demonstrates this explicitly. But a question then arises: Why does the standard approach allow reproducing all known conformally invariant Lagrangians although it is mathematically contradictory? The answer is that it indeed allows *formally* reproducing all possible effective Lagrangians. This section is devoted to proving this statement. Namely, we show that any effective Lagrangian obtained in the framework of the two-orbit technique can also be obtained using the standard technique, and vice versa. In this sense, the standard and the two-orbit approaches are equivalent.

To prove the equivalence of the approaches, we first consider the simplest scheme of spontaneous symmetry breaking: breaking the conformal group down to the Poincaré subgroup (see (19)). After consideration of this case, generalization to the case of an arbitrary scheme of nonlinear realization of symmetries is rather straightforward. In this case, the coset space and the Maurer–Cartan forms are respectively given by (3) and (6).

We first show that any effective Lagrangian obtained in the framework of the two-orbit approach can also be constructed using the standard technique. The two-orbit approach allows only Lagrangians such that y^ν is included via a total derivative. Consequently, there are in fact no equations restricting the dynamics of y^ν. It is therefore possible to set the field y^ν equal to any function compatible with the transformation properties of y^ν. In particular, as such a function, we can choose not (11) but expression (7) following from the inverse Higgs constraint. After it is substituted back in the Lagrangian, the quantity ω_D obviously vanishes. Because the Lagrangian obtained as a result of this procedure is conformally invariant, it must reduce to an $SO(d)$–invariant combination of the remaining Maurer–Cartan forms. We therefore have

$$\mathcal{L}_{\rm ph}\big|_{\rm iHc} = \widetilde{\mathcal{L}}(\omega_P^\mu, \omega_K^\nu, \psi, \mathcal{D}\psi)\big|_{\rm iHc}. \tag{23}$$

This expression shows that the Lagrangian $\mathcal{L}_{\rm ph}$ can be rewritten as an $SO(d)$-invariant combination of Maurer–Cartan forms (6) with application of the inverse Higgs constraint. This proves the first part of the statement.

To prove the converse, we note that the defined combination of covariant derivatives of fields has the same form as in the standard approach. Indeed, for Maurer–Cartan forms (6), the covariant metric and covariant derivatives of fields have the respective forms (20) and (21). For matter fields, the combination of covariant derivatives not including y^ν has the form

$$\widetilde{\mathcal{D}}_m\psi = \mathcal{D}_m\psi - (i\widehat{L}_{mn}\psi)D^n\pi. \tag{24}$$

The process of eliminating y^ν from $D_m y^\nu$ is slightly more complicated because $D_m y^\nu$ contains the derivative of y^ν. Because $D_m\pi$ transforms homogeneously under the action of all elements of the conformal group, we can find its covariant derivative under the assumption that it is an ordinary matter field [18], [21],

$$\mathcal{D}_m D_n\pi = e^{-\pi}(\partial_m D_n\pi + 2iy^\lambda\widehat{L}_{m\lambda}D_n\pi). \tag{25}$$

The modified covariant derivative of y^ν, which in fact does not include y^ν and plays the role of the covariant derivative of π, then has the form

$$\widetilde{D}_m y_n = D_m y_n - \frac{1}{2}\mathcal{D}_m D_n\pi - \frac{1}{4}g_{mn}D_k\pi D^k\pi. \tag{26}$$

Because covariant derivatives (24) and (26) do not include y^ν, any effective Lagrangian constructed as an $SO(d)$-invariant combination automatically satisfies all requirements of the two-orbit technique. The explicit form of covariant derivatives (24) and (26) coincides with the respective covariant derivatives (10) and (8) obtained in the framework of the standard approach. Consequently, any Lagrangian obtained in the framework of the standard approach can also be obtained in the framework of the two-orbit technique.

We note that the established fact of the correspondence of the modified covariant derivatives in the frameworks of the two-orbit technique and the standard approach can also be proved based on symmetry considerations. Namely, constraint (7) is the only relation between π and y^ν that is compatible with all continuous symmetries. Consequently, eliminating y^ν from the covariant derivatives must yield the same result as in the case where the inverse Higgs constraints are imposed. The presented argument is the fundamental reason that the two techniques must be equivalent.

To prove the equivalence of the approaches in the general case, we note that for $d > 2$, if the Lorentz invariance is broken, then the dilation symmetry is also broken. Indeed, the scaling dimension of operators $\mathcal{O}$ with a nonzero vacuum expectation value is bounded from below by the unitarity of the representation

$$\Delta_{\mathcal{O}} > 0. \tag{27}$$

Consequently, any nonzero operator $\mathcal{O}$ also leads to breaking the dilation invariance.

The obtained result allows generalizing the proof of the equivalence of the approaches to the general case described by nonlinear realization scheme (13). Because dilations are always spontaneously broken, the proof that any effective Lagrangian obtained using the two-orbit technique can also be obtained using the standard approach remains unchanged. To prove the converse statement, we note that the coset space corresponding to scheme (13) can be chosen in the form

$$g_H = e^{iP_\mu x^\mu}e^{iK_\nu y^\nu}e^{iD\pi}e^{iZ_a\xi^a}, \tag{28}$$

where Z_a denotes all broken generators except K_ν and D. Then, because dilations commute trivially with all generators except translations and SCT, the Maurer–Cartan 1-forms for dilations in the general case also have form (6). This allows eliminating y^ν from all covariant derivatives used in the two-orbit technique. By virtue of the symmetry considerations described above, the covariant derivatives thus obtained must coincide with their analogues in the standard approach. Therefore, the two techniques are equivalent for $d > 2$.

The proved equivalence of the approaches means that instead of applying the two-orbit technique, we can use the inverse Higgs mechanism. Although the standard approach is not mathematically rigorous (the inversion symmetry is broken), it nevertheless allows obtaining all possible effective Lagrangians. The inverse Higgs mechanism should hence be regarded as a convenient way to construct Lagrangians of effective theories.

We note that in the arguments above, it was assumed that the inverse Higgs constraints must imposed on the 1-form ω_D. But in the general case, part of the Lorentz group can also be spontaneously broken, and in this case as can be seen from explicit calculations, y^ν enters the corresponding Maurer–Cartan form linearly. If part of the Lorentz group is spontaneously broken (with indices α), then we can try to impose inverse Higgs constraints of the form

$$\omega_L^\alpha = 0, \tag{29}$$

which must hold for all α. But Eq. (29) as a system of equations for y^ν is overdetermined. Indeed, if the Lorentz group remains unbroken along n spatial directions, then in the coordinate form, Eq. (29) is a system of

$$d \times \frac{d(d-1) - n(n-1)}{2} > d \tag{30}$$

equations. Because (29) must hold off-shell and also because the NGF associated with the broken generators L_α are independent, all equations in system (29) are also independent. Consequently, these equations cannot be solved with $d > 2$, which makes application of this prescription impossible.

Concluding this section, we note that the Maurer–Cartan form for SCT for the most general scheme (13) of spontaneous symmetry breaking differs from the Maurer–Cartan form for the unbroken phase only by the common factor $e^{-\pi}$ [6], [12]. This observation allows using the arguments in [12] to prove that $\mathcal{L}_{\text{kin}}$ always admits (11) as a solution. Because the detailed proof of this statement would literally repeat the steps in [12], we do not present it.

3.2. Case $d = 2$. We now consider the special case $d = 2$. We note that in two-dimensional conformal field theories,[9] vacuum solutions breaking the Lorentz invariance but not the dilation invariance are possible.[10] In this case, imposing condition (29) allows expressing y^ν in terms of ω, the NGF for the broken $SO(2)$ symmetry. In the framework of the two-orbit approach, we can then use the covariant derivative of ω to eliminate y^ν from the other covariant derivatives. It follows from symmetry considerations that such covariant derivatives coincide with those obtained by imposing the inverse Higgs constraints. Further, repeating the arguments in Sec. 3.1, we can prove that the standard and two-orbit approaches are also equivalent in this case.

We now consider the case where both dilations and Lorentz transformations are spontaneously broken. We first note that in two dimensions, the conformal group (without inversion) is a direct product of two subgroups,

$$SO(1,3) = SU_+(2) \times SU_-(2). \tag{31}$$

Because the groups $SU(2)_+$ and $SU(2)_-$ are isomorphic, it suffices to consider only one of them: all arguments automatically transfer to the other. For definiteness, we consider $SU(2)_+$. The basis of the algebra of this group has the form

$$P_1 + iP_2, \qquad K_1 - iK_2, \qquad D - iL_{12}, \tag{32}$$

[9] Although the conformal group for $d = 2$ is larger than $O(1,3)$, we here understand the two-dimensional conformal group as $O(1,3)$.

[10] As an example, we consider a vector field with quadratic kinetic term. Such field has a zero scaling dimension. Hence, if its vacuum expectation value is nonzero, then only the Lorentz invariance is spontaneously broken.

where P_1, P_2 and K_1, K_2 are the respective generators of translations and SCT. The $SU(2)_+$ group acts on the holomorphic coordinate $x^+ \equiv x_1 + ix_2$ and after a spontaneous breaking generates two NGF, y^+ and π^+, corresponding to the broken generators $K_1 - iK_2$ and $D - iL_{12}$. In the framework of the standard approach, we can eliminate y^+ in favor of π^+,

$$y^+ = -\frac{1}{2}\partial_+\pi^+, \tag{33}$$

where ∂_+ is the operator of partial differentiation with respect to the coordinate x^+. In the framework of the two-orbit approach, y^+ must still be given by gluing map (11). Consequently, in all admissible Lagrangians in the framework of the two-orbit approach, y^+ must decouple from the other fields. This observation allows transferring the proof of the equivalence of the approaches in Sec. 3.1 to the considered case.

4. Discussion of results and conclusions

In the preceding section, we established the correspondence between the two-orbit and standard approaches. The first of them is mathematically self-contained because it follows directly from the method of induced representations. But its application in practice turns out to be difficult because of need to seek Lagrangians in which y^ν appears via a total derivative. On the other hand, it was shown that any effective Lagrangian obtained in the framework of the two-orbit approach can also be obtained in the framework of the standard technique. The latter does not have a rigorous mathematical justification but allows constructing effective Lagrangians in a simpler way. The standard approach can hence be regarded as a convenient tool for constructing effective theories, while the two-orbit approach is mathematically rigorous and therefore fundamental.

The obtained result that the NGF for SCT is always a redundant degree of freedom fully agrees with the results in [10], where it was shown that if the Noether currents associated with broken symmetries are functionally dependent, then certain NGFs are redundant. In the case of the conformal group, the action of the SCT reduces to the coordinate-dependent action of translations, dilations, and Lorentz transformations. Therefore, the Noether current for SCT are always functionally dependent, and breaking the SCT is always a consequence of breaking P_μ, D, or $L_{\mu\nu}$. Hence, the NGF for SCT are never independent perturbations of the vacuum and are always redundant fields.

Regarding the presented proof of the equivalence of two approaches, we note that they turned out to be equivalent as a result of requiring that y^ν enter Lagrangians only via a total derivative. Therefore, our results here do not allow elucidating the interpretation of the inverse Higgs mechanism in other cases [7]–[9], [19], [20], [26], [27].

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