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QUASIEXACT THEORY OF THREE-DIMENSIONAL OPTICAL SELF-FOCUSING

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We find a quasiexact three-dimensional analytic solution of the nonlinear Schrödinger equation describing the field of a stationary optical beam in an unbounded homogeneous nonlinear isotropic medium supporting a state of linear polarization.

Keywords: nonlinear homogeneous isotropic medium, nonlinear Schrödinger equation for stationary optical beams, quasiexact analytic solution

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The field of a stationary optical beam $E(x,y,z,t) = E_0(x,y,z)e^{i(\beta_0 z-\omega_0 t)}$ (where $\beta_0 = (\omega_0/c)n(\omega_0)$ is the central wave number and ω_0 is the carrier frequency) propagating in an unbounded nonlinear homogeneous isotropic medium can be described by the well-known nonlinear Schrödinger equation (NLSE) [1]

$$2i\beta_0\frac{\partial E_0}{\partial z} + \frac{\partial^2 E_0}{\partial x^2} + \frac{\partial^2 E_0}{\partial y^2} + 2\eta|E_0|^2 E_0 = 0. \qquad (1)$$

Here, E_0 is a slowly varying complex amplitude (its relative change over a distance of the order $1/\beta_0$ is small), and $\eta(\omega)$ is the nonlinearity coefficient. This equation holds in the paraxial approximation for a weakly nonlinear medium supporting a state of linear polarization, which allows using the scalar approach. For the one-dimensional Laplace operator, Eq. (1) was first solved by Zakharov and Shabat in their widely known paper [2].

The NLSE has been introduced in various branches of physics: it describes propagation of nonlinear Langmuir waves, deep-water waves, waves in electricity transmission lines, acoustic waves in bubbled fluid, and, first of all, propagation of optical radiation in nonlinear media. The last class of applications becomes especially important because of the development of laser technologies: the laser radiation intensity is so high that the nonlinear part of the medium susceptibility must be taken into account. There are dozens of monographs devoted to this equation, and the number of journal publications grows like a snow ball. Nevertheless, no single three-dimensional analytic solution of Eq. (1) in the form of a nonexpanding beam in an unbounded nonlinear medium has yet been found. Here, we obtain a simple analytic function that is an approximate solution of Eq. (1).

Because the considered transparent medium is homogeneous and isotropic, we are interested in axially symmetric beams. Passing to cylindrical coordinates in Eq.(1), we obtain the equation

$$2i\beta_0\frac{\partial E_0}{\partial z} + \frac{\partial^2 E_0}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_0}{\partial \rho} + 2\eta|E_0|^2 E_0 = 0. \qquad (2)$$

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We separate the variable z like time can be separated in quantum mechanics, i.e., we set

$$E_0(\rho, z) = R(\rho)e^{iqz}, \tag{3}$$

where $R(\rho)$ is a real function. We thus consider axially symmetric beams whose intensity is independent of the coordinate z. The wave numbers included in the Fourier representation of the field are distributed in a narrow interval of values near the wave number β_0, and the parameter q is the typical scale of this distribution. In other words, q defines a small correction to the wave number β_0. For the function R, we obtain the ordinary differential equation

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} = a^2R - 2\eta R^3, \tag{4}$$

where $\rho = \sqrt{x^2 + y^2}$ and $a = \sqrt{2q\beta_0}$.

For further study, we rescale Eq. (4) to the dimensionless form via

$$\xi = a\rho, \qquad R = \sqrt{\frac{a^2}{\eta}} f(\xi). \tag{5}$$

As a result, we obtain the equation in dimensionless variables

$$f''(\xi) + \frac{1}{\xi}f'(\xi) = f(\xi) - 2f^3(\xi). \tag{6}$$

We are only interested in solutions of this equation in a form of localized functions with an extremal value on the beam axis, i.e., at $\xi = 0$. We note that because Eq. (6) is symmetric under the transformation $f \leftrightarrow -f$, a solution with its sign changed is still a solution.

Let the maximum value $f(0) = A$. Then for the function $u(\xi)$ defined by the relation $f(\xi) = Au(\xi)$, we obtain the nonlinear spectral Cauchy–Dirichlet problem

$$u''(\xi) + \frac{1}{\xi}u'(\xi) = u(\xi) - \lambda u^3(\xi), \tag{7}$$

where $\lambda = 2A^2$ and the initial conditions are

$$u(0) = 1, \qquad u'(0) = 0. \tag{8}$$

The eigenvalues λ are defined by the condition that u and its derivative vanish at infinity.

A numerical study showed that the eigenvalue spectrum is a point spectrum. We managed to find only one eigenvalue,

$$\lambda = \pi \cosh\frac{8\pi}{25}, \tag{9}$$

and the corresponding localized eigenfunction, which is plotted in Fig. 1.

A similar numerical solution was first found in [3] and is now called the Townes mode. Other numerical solutions were presented in [4] and [5]. A summary of various numerical solutions was given in [6]. Some differences between solution shown in Fig. 1 and the Townes mode are due to different numerical coefficients in the considered equations.

Based on the graphical computer analysis, we conclude that such rapidly decaying functions are well approximated by the formula

$$u = \frac{1}{\sqrt{a_0 + a_1\cosh\xi + a_2\cosh 2\xi + \cdots + a_n\cosh n\xi}}$$

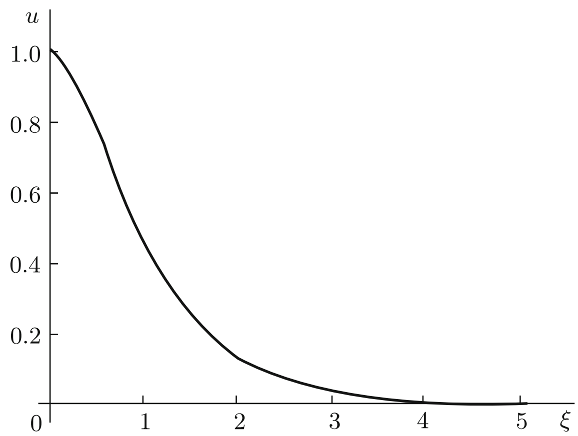

Fig. 1. Plot of the localized eigenfunction.

with varying coefficients a_i. This formula automatically satisfies the second initial condition in (8). For the first initial condition to also be satisfied, we set $a_0 = 1 - a_1 - a_2 - \cdots - a_n$, and the approximating function becomes

$$u = \frac{1}{\sqrt{1 + c_1 \sinh^2(\xi/2) + c_2 \sinh^2 \xi + \cdots + c_n \sinh^2(n\xi/2)}}. \tag{10}$$

Function (10) satisfies the initial conditions and is localized if the radicand is everywhere positive.

The number of varying coefficients $c_i = 2a_i$, and their numerical values are determined by the form of the approximating function and the required accuracy.

It turns out that only the two coefficients $c_1 = -13$ and $c_2 = 5$ suffice for the numerical solution plotted in Fig. 1. The graph of the function

$$u = \frac{1}{\sqrt{1 - 13 \sinh^2(\xi/2) + 5 \sinh^2 \xi}} \tag{11}$$

then merges with the graph of the numerical solution, which allows calling function (11) a quasiexact solution of problem (7), (8). To establish the quasiexactness criterion, we find the distance between the numerical solution and curve (11). In the metric generated by the scalar product, the distance between the functions is called the root-mean-square distance and can be calculated using the formula

$$\Delta = \left(\frac{1}{b - a} \int_a^b (\varphi - u)^2 \, d\xi \right)^{1/2}.$$

Integrating numerically from 0 to 5, we obtain $\Delta = 0.00337$. We note that the value of the numerical solution at the right end of the integration interval is 0.00462. We set the quasiexactness criterion: a smooth function given analytically is a quasiexact solution of the problem for which a numerical solution is found if the root-mean-square distance between them does not exceed 0.01. The same numerical value is accepted in optics for estimating the radiation quasimonochromaticity [7].

It is well known that the two-dimensional NLSE is a Hamiltonian-type equation [8] and the Hamiltonian vanishes on a soliton solution. It is interesting to find the value of the Hamiltonian deviation from zero on a quasiexact solution. In the context of the found solutions, we must calculate two integrals:

$$I_1 = \frac{1}{2} \int_0^\infty \left(\frac{df}{d\xi} \right)^2 2\pi\xi \, d\xi \quad \text{and} \quad I_2 = \frac{1}{2} \int_0^\infty 2\pi\xi f^4(\xi) \, d\xi,$$

where $f(\xi) = \sqrt{\lambda/2}\, u(\xi)$.

Taking formulas (9) and (11) into account, we obtain

$$I_1 \approx 2.973, \qquad I_2 \approx 2.986, \qquad H = I_1 - I_2 = -0.013.$$

The obtained result can be considered satisfactory because the Hamiltonian deviation from zero is 0.004 of the values I_1 and I_2.

Returning to the original notation, we have

$$E_0(x, y, z) = \pm \frac{\sqrt{\pi \cosh(8\pi/25)\beta_0 q/\eta}}{\sqrt{1 - 13\sinh^2(\xi/2) + 5\sinh^2 \xi}}\, e^{iqz}, \tag{12}$$

where

$$\xi = \sqrt{2q\beta_0(x^2 + y^2)}. \tag{13}$$

The signs $\pm$ in formula (12) take the above remark about the symmetry into account.

Obviously, the numerator of the fraction in (12) is equal to the peak value $E_{\max}$ of the field intensity on the beam axis. Expression (12) is then simplified,

$$E_0(x, y, z) = \pm \frac{E_{\max}}{\sqrt{1 - 13\sinh^2(\xi/2) + 5\sinh^2 \xi}}\, e^{iqz}, \tag{14}$$

where

$$q = \frac{\eta E_{\max}^2}{\beta_0 \pi \cosh(8\pi/25)}. \tag{15}$$

Formulas (13)–(15) are the basic result here. We note that there are other partial derivative equations with a cubic nonlinearity that can be reduced to spectral problem (7), (8) with the simple analytical solution (9)–(11).

Conflicts of interest. The author declares no conflicts of interest.

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