Theoretical and Mathematical Physics, **201**(2): 1545–1564 (2019)

BETHE VECTORS FOR ORTHOGONAL INTEGRABLE MODELS

A. N. Liashyk,[*] S. Z. Pakuliak,[†] E. Ragoucy,[‡] and N. A. Slavnov[†]

We consider quantum integrable models associated with the $\mathfrak{so}_3$ algebra and describe Bethe vectors of these models in terms of the current generators of the $\mathcal{D}Y(\mathfrak{so}_3)$ algebra. To implement this program, we use an isomorphism between the R-matrix and the Drinfeld current realizations of the Yangians and their doubles for classical type B-, C-, and D-series algebras. Using these results, we derive the actions of the monodromy matrix elements on off-shell Bethe vectors. We obtain recurrence relations for off-shell Bethe vectors and Bethe equations for on-shell Bethe vectors. The formulas for the action of the monodromy matrix elements can also be used to calculate scalar products in the models associated with the $\mathfrak{so}_3$ algebra.

Keywords: Yangian of a simple Lie algebra, Yangian double, algebraic Bethe ansatz

DOI: 10.1134/S0040577919110023

1. Introduction

The algebraic Bethe ansatz [1], [2] is a powerful method for investigating quantum integrable models. It has usually been applied to models associated with different deformations and generalizations of A-series algebras. Models corresponding to the B-, C-, and D-series algebras have been less investigated despite the pioneering papers [3], [4].

The nested algebraic Bethe ansatz [5]–[7] was mainly developed for quantum integrable models associated with A-series algebras of higher rank. A sufficiently powerful method for investigating the nested Bethe ansatz proposed in [8], [9] was recently reformulated using the language of the current realization of deformed infinite-dimensional algebras. In addition to the current representation, this approach also used the so-called RTT realization [10], in which the fundamental commutation relations of the monodromy matrices of the integrable models are determined by an R-matrix. Such an isomorphism between the current and RTT realizations was recently constructed in [11], [12] for the Yangians $Y(\mathfrak{so}_{2n+1})$, $Y(\mathfrak{sp}_{2n+2})$, and $Y(\mathfrak{so}_{2n+2})$, $n=1,2,\dots$, corresponding to the algebras of the classical B, C, and D series. This result immediately opens a possibility to apply the algebraic Bethe ansatz method to models with $\mathfrak{so}_{2n+1}$-, $\mathfrak{sp}_{2n+2}$, or $\mathfrak{so}_{2n+2}$ symmetries based on the current approach [13].

[*]Skolkovo Institute of Science and Technology, Moscow, Russia, e-mail: a.liashyk@gmail.com.

[†]Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia,
e-mail: stanislav.pakuliak@jinr.ru, nslavnov@mi-ras.ru.

[‡]Laboratoire de Physique Théorique LAPTh, CNRS and USMB, Annecy-le-Vieux, France,
e-mail: eric.ragoucy@lapth.cnrs.fr.

This research was performed at the Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, and is supported by a grant from the Russian Science Foundation (Project No. 19-11-00062).

Prepared from an English manuscript submitted by the authors; for the Russian version, see *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 201, No. 2, pp. 153–174, November, 2019. Received June 7, 2019. Revised June 7, 2019. Accepted June 13, 2019.

Here, we restrict ourself to the simplest case of $\mathfrak{so}_3$-invariant quantum integrable models. The corresponding R-matrix was found in the seminal paper by A. B. Zamolodchikov and Al. B. Zamolodchikov [14]. Our main task is to calculate the actions of monodromy matrix elements on off-shell Bethe vectors. The latter are defined in the framework of the approach presented in [15], [16]. The formulas giving the actions of the monodromy matrix elements turn out to be more important and more fundamental than the explicit formulas for the Bethe vectors in terms of the monodromy matrix elements. The action of the upper-triangular elements yields recurrence relations for the Bethe vectors, which can be used to obtain explicit expressions for the Bethe vectors. From the formulas for the action of the diagonal elements, we obtain Bethe equations as the condition for off-shell Bethe vectors to become on-shell vectors. The action of the lower-triangular monodromy matrix elements can be used to calculate scalar products of the Bethe vectors [17], [18].

This paper is organized as follows. In Sec. 2, we define a universal $\mathfrak{so}_{2n+1}$ quantum integrable model. In Sec. 3, we describe this model for $n = 1$ in the language of the double Yangian $\mathcal{D}Y(\mathfrak{so}_3)$ and also define projections onto intersections of the different type Borel subalgebras in this algebra and their properties. In Sec. 4, we define the universal off-shell Bethe vectors in terms of the current generators of the algebra $\mathcal{D}Y(\mathfrak{so}_3)$. This section also contains our main result: formulas describing the actions of the monodromy matrix elements on the Bethe vectors. Section 5 is devoted to the proof of these formulas.

2. Definition of the universal orthogonal integrable model

We define an $\mathfrak{so}_N$-invariant integrable model for $N = 2n + 1$. In this case, the $\mathfrak{so}_N$-invariant R-matrix $R(u, v)$ has the form [14]

$$R(u,v) = \mathbf{I}\otimes\mathbf{I} + \frac{c\,\mathbf{P}}{u-v} - \frac{c\,\mathbf{Q}}{u-v+c\kappa}. \tag{2.1}$$

Here, $\mathbf{I} = \sum_{i=1}^{N}\mathcal{E}_{ii}$ is the identity operator acting in the space $\mathbb{C}^N$, and $\mathcal{E}_{ij}$ are $N\times N$ matrices with the only nonzero entry equal to 1 at the intersection of the ith row and jth column. The operators $\mathbf{P}$ and $\mathbf{Q}$ act in $\mathbb{C}^N\otimes\mathbb{C}^N$ and are given by

$$\mathbf{P} = \sum_{i,j=1}^{N}\mathcal{E}_{ij}\otimes\mathcal{E}_{ji}, \qquad \mathbf{Q} = \sum_{i,j=1}^{N}\mathcal{E}_{ij}\otimes\mathcal{E}_{i'j'}, \tag{2.2}$$

where $i' = N + 1 - i$ and $j' = N + 1 - j$. Finally, c is a constant, u and v are arbitrary complex numbers called spectral parameters, and $\kappa = N/2 - 1$.

A universal orthogonal integrable model is defined by an $N\times N$ monodromy matrix $T(u)$ whose operator-valued entries $T_{i,j}(u)$ act in a Hilbert space $\mathcal{H}$ (the physical space of a quantum model). We do not specify the Hilbert space $\mathcal{H}$ nor any concrete representation of the operators $T_{i,j}(u)$. Such a monodromy matrix is then said to be universal. It satisfies an RTT algebra

$$R(u,v)(T(u)\otimes\mathbf{I})(\mathbf{I}\otimes T(v)) = (\mathbf{I}\otimes T(v))(T(u)\otimes\mathbf{I})R(u,v), \tag{2.3}$$

and this equation yields commutation relations of the monodromy matrix elements,

$$[T_{i,j}(u), T_{k,l}(v)] = \frac{c}{u-v}\big(T_{k,j}(v)T_{i,l}(u) - T_{k,j}(u)T_{i,l}(v)\big) + \\ + \frac{c}{u-v+c\kappa}\Big(\delta_{ki'}\sum_{p=1}^{N}T_{p,j}(u)T_{p',l}(v) - \delta_{lj'}\sum_{p=1}^{N}T_{k,p'}(v)T_{i,p}(u)\Big). \tag{2.4}$$

For any matrix X acting in $\mathbb{C}^N$, we let X^{t} denote the transposition

$$(X^{\mathrm{t}})_{i,j} = X_{j',i'} = X_{N+1-j,N+1-i}. \tag{2.5}$$

It is related to the "usual" transposition $(\,\cdot\,)^{\mathrm{T}}$ by a conjugation by the matrix $U = \sum_{i=1}^{N} \mathcal{E}_{ii'}$. We note that the R-matrix satisfies the relation

$$R(u,v)^{\mathrm{t}_1\mathrm{t}_2} = R(u,v), \tag{2.6}$$

where t_1 and t_2 denote transposition in the respective first and second spaces of $R(u,v)$. A direct consequence of commutation relations (2.4) is the equation [19]

$$T^{\mathrm{t}}(u-c\kappa)T(u) = T(u)T^{\mathrm{t}}(u-c\kappa) = z(u)\mathbf{I}, \tag{2.7}$$

where $z(u)$ is a scalar commuting with all the generators $T_{i,j}(u)$. In what follows, we set this central element equal to one: $z(u)=1$.

In our consideration, we restrict ourself to quantum integrable models such that the dependence of the universal monodromy matrix elements $T_{i,j}(u)$ on the parameter u is given by the series

$$T_{i,j}(u) = \delta_{ij}\mathbf{1} + \sum_{\ell\ge 0} T_{i,j}[\ell]u^{-\ell-1}, \tag{2.8}$$

where $\mathbf{1}$ and $T_{i,j}[\ell]$ are the respective identity and nontrivial operators acting in the Hilbert space $\mathcal{H}$. In this case, the universal monodromy matrix elements satisfying (2.4) and (2.7) can be identified with generating series of the generators of the Yangian $Y(\mathfrak{so}_N)$, and the Hilbert space $\mathcal{H}$ can be identified with the representation space of this infinite-dimensional algebra. In particular, a direct consequence of commutation relations (2.4) and expansion (2.8) are the commutation relations

$$\big[T_{i,j}(u), T_{k,l}[0]\big] = c\big(\delta_{il}T_{k,j}(u) - \delta_{kj}T_{i,l}(u) - \delta_{ik'}T_{l',j}(u) + \delta_{l'j}T_{i,k'}(u)\big), \tag{2.9}$$

which we use in what follows. We note that the Yangian $Y(\mathfrak{so}_N)$ is defined by relation (2.4) (with expansion (2.8)) and by the relation $z(u)=1$. If we do not impose this last relation and keep $z(u)$ arbitrary (but central), then we obtain a bigger algebra, conventionally denoted by $X(\mathfrak{o}_N)$ (see [20]).

It follows from (2.3) that $[\mathcal{T}(u),\mathcal{T}(v)]=0$, where $\mathcal{T}(u)=\sum_i T_{i,i}(u)$ is the universal transfer matrix. Hence, the transfer matrix is a generating function for the integrals of motion of the considered model.

The key problem of the algebraic Bethe ansatz is to find eigenvectors of the universal transfer matrix $\mathcal{T}(u)$ in the space $\mathcal{H}$. In this context, it is usually assumed that the physical space of the model has a special reference vector $|0\rangle \in \mathcal{H}$ such that

$$T_{i,j}(u)|0\rangle = 0 \quad \text{for } i>j, \qquad T_{i,i}(u)|0\rangle = \lambda_i(u)|0\rangle, \tag{2.10}$$

where $\lambda_i(u)$ are complex-valued functions. The eigenvectors of $\mathcal{T}(u)$ are then constructed as certain polynomials of the monodromy matrix entries $T_{i,j}(u)$ with $i<j$ acting on the reference vector $|0\rangle$. In the framework of the universal orthogonal model associated with R-matrix (2.1), the functions $\lambda_i(u)$ are free functional parameters modulo certain relations following from (2.7). We describe these relations below.

We let $\mathcal{B}_n$ denote the algebra of the monodromy matrix elements $T_{i,j}(u)$ satisfying (2.4) and (2.7) with $z(u)=1$ (we recall that $n=(N-1)/2$). The space $\mathcal{H}$ satisfying (2.10) then describes the class of $\mathcal{B}_n$ highest-weight representations.

3. Gauss coordinates of the universal monodromy

In the case of A-series algebras, an effective way to solve the eigenvalue problem for the transfer matrix is to use another set of generators associated with the Gauss coordinates of the monodromy matrix instead of the monodromy matrix elements [16], [13]. Moreover, using the recent result in [11], [12], we can verify that the Gauss coordinates of the monodromy matrix can be used effectively to resolve constraint (2.7) and to obtain a set of algebraically independent generators of the RTT algebras related to the classical B, C, and D series. On the other hand, the Gauss coordinates relate the RTT realization and the Drinfeld current representation [21] of the quantum affine algebras and the Yangian doubles. This allows constructing the off-shell Bethe vectors of the universal quantum integrable model in terms of the current generators of the corresponding infinite-dimensional algebra [22].

Hereafter, we restrict ourself to the $\mathfrak{so}_3$-invariant integrable models. Hence, we consider the $\mathcal{B}_1$ algebra with a 3×3 monodromy matrix and $\kappa = 1/2$ in Eq. (2.1).

The Gauss coordinates for the monodromy matrix $T(u)$ can be introduced in several different ways. We use the decomposition

$$T(u) = \mathbf{F}(u) \cdot \mathbf{D}(u) \cdot \mathbf{E}(u), \tag{3.1}$$

where $\mathbf{D}(u)$ is a diagonal matrix $\mathbf{D}(u) = \mathrm{diag}(k_1(u), k_2(u), k_3(u))$. The matrices $\mathbf{F}(u)$ and $\mathbf{E}(u)$ are upper-triangular and lower-triangular matrices:

$$\mathbf{F}(u) = \begin{pmatrix} 1 & \mathrm{F}_{2,1}(u) & \mathrm{F}_{3,1}(u) \\ 0 & 1 & \mathrm{F}_{3,2}(u) \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{E}(u) = \begin{pmatrix} 1 & 0 & 0 \\ \mathrm{E}_{1,2}(u) & 1 & 0 \\ \mathrm{E}_{1,3}(u) & \mathrm{E}_{2,3}(u) & 1 \end{pmatrix}. \tag{3.2}$$

Explicitly, the Gauss decomposition of the monodromy matrix associated with the $\mathcal{B}_1$ algebra is

$$T(u) = \begin{pmatrix} k_1 + \mathrm{F}_{2,1}k_2\mathrm{E}_{1,2} + \mathrm{F}_{3,1}k_3\mathrm{E}_{1,3} & \mathrm{F}_{2,1}k_2 + \mathrm{F}_{3,1}k_3\mathrm{E}_{2,3} & \mathrm{F}_{3,1}k_3 \\ k_2\mathrm{E}_{1,2} + \mathrm{F}_{3,2}k_3\mathrm{E}_{1,3} & k_2 + \mathrm{F}_{3,2}k_3\mathrm{E}_{2,3} & \mathrm{F}_{3,2}k_3 \\ k_3\mathrm{E}_{1,3} & k_3\mathrm{E}_{2,3} & k_3 \end{pmatrix}, \tag{3.3}$$

where we omit the dependence on the spectral parameter u for all the Gauss coordinates $\mathrm{E}_{i,j}(u)$, $\mathrm{F}_{j,i}(u)$, and $k_i(u)$ for brevity.

In terms of monodromy matrix elements, we can write formula (3.1) as

$$T_{i,j}(u) = \sum_{\max(i,j)\le \ell \le 3} \mathrm{F}_{\ell,i}(u) k_\ell(u) \mathrm{E}_{j,\ell}(u), \tag{3.4}$$

where we have $\mathrm{F}_{i,i}(u) = \mathrm{E}_{i,i}(u) = 1$ and $\mathrm{F}_{j,i}(u) = \mathrm{E}_{i,j}(u) = 0$ for $i > j$ according to (3.2). Conditions (2.10) are then ensured by the relations $E_{i,j}(u)|0\rangle = 0$ for $i < j$ and $k_i(u)|0\rangle = \lambda_i(u)|0\rangle$, which we hereafter assume.

3.1. Independent Gauss coordinates. It is easy to see from (3.4) that assuming the invertibility of $k_i(u)$, we can express all the Gauss coordinates in terms of the monodromy matrix elements. Because of relation (2.7), these Gauss coordinates are not independent. We find an independent set of generators for the $\mathcal{B}_1$ algebra and derive their commutation relations.

We say that an element of the $\mathcal{B}_1$ algebra is normal ordered if all the Gauss coordinates $\mathrm{F}_{j,i}(u)$ are on the left and all the Gauss coordinates $\mathrm{E}_{i,j}(u)$ are on the right of this element. It can be seen that Gauss decomposition (3.4) of the monodromy matrix is normal ordered by definition.

By condition (2.6), the transpose-inverse monodromy matrix $\widehat{T}(u)=(T^{-1}(u))^{\mathrm{t}}$ satisfies the same RTT commutation relations (2.3). To describe the matrix $\widehat{T}(u)$ in terms of the Gauss coordinates $\mathrm{F}_{j,i}(u)$, $\mathrm{E}_{i,j}(u)$, and $k_i(u)$, we must invert the matrices $\mathbf{F}(u)$, $\mathbf{E}(u)$, and $\mathbf{D}(u)$. They are given by the relations

$$\begin{aligned}&\mathbf{F}^{-1}(u)=\mathbf{I}+\sum_{i<j}\mathcal{E}_{ij}\widetilde{\mathrm{F}}_{j,i}(u),\qquad \mathbf{E}^{-1}(u)=\mathbf{I}+\sum_{i<j}\mathcal{E}_{ji}\widetilde{\mathrm{E}}_{i,j}(u),\\&\mathbf{D}^{-1}(u)=\operatorname{diag}(k_1^{-1}(u),k_2^{-1}(u),k_3^{-1}(u)),\end{aligned}\tag{3.5}$$

where $\widetilde{\mathrm{F}}_{i,i}(u)=\widetilde{\mathrm{E}}_{i,i}(u)=1$ and

$$\begin{aligned}&\widetilde{\mathrm{F}}_{i+1,i}(u)=-\mathrm{F}_{i+1,i}(u),\quad i=1,2,\qquad \widetilde{\mathrm{F}}_{3,1}(u)=-\mathrm{F}_{3,1}(u)+\mathrm{F}_{2,1}(u)\mathrm{F}_{3,2}(u),\\&\widetilde{\mathrm{E}}_{i,i+1}(u)=-\mathrm{E}_{i,i+1}(u),\quad i=1,2,\qquad \widetilde{\mathrm{E}}_{1,3}(u)=-\mathrm{E}_{1,3}(u)+\mathrm{E}_{2,3}(u)\mathrm{E}_{1,2}(u).\end{aligned}\tag{3.6}$$

The matrix elements of the transpose-inverse monodromy matrix can be easily expressed in terms of the original Gauss coordinates $\mathrm{F}_{j,i}(u)$, $\mathrm{E}_{i,j}(u)$, and $k_i(u)$:

$$\widehat{T}_{i,j}(u)=\sum_{1\le\ell\le\min(4-i,4-j)}\widetilde{\mathrm{E}}_{\ell,4-j}(u)k_\ell^{-1}(u)\widetilde{\mathrm{F}}_{4-i,\ell}(u)\tag{3.7}$$

or, explicitly,

$$\widehat{T}(u)=\begin{pmatrix}k_3^{-1}+\widetilde{\mathrm{E}}_{2,3}k_2^{-1}\widetilde{\mathrm{F}}_{3,2}+\widetilde{\mathrm{E}}_{1,3}k_1^{-1}\widetilde{\mathrm{F}}_{3,1} & k_2^{-1}\widetilde{\mathrm{F}}_{3,2}+\widetilde{\mathrm{E}}_{1,2}k_1^{-1}\widetilde{\mathrm{F}}_{3,1} & k_1^{-1}\widetilde{\mathrm{F}}_{3,1}\\ \widetilde{\mathrm{E}}_{2,3}k_2^{-1}+\widetilde{\mathrm{E}}_{1,3}k_1^{-1}\widetilde{\mathrm{F}}_{2,1} & k_2^{-1}+\widetilde{\mathrm{E}}_{1,2}k_1^{-1}\widetilde{\mathrm{F}}_{2,1} & k_1^{-1}\widetilde{\mathrm{F}}_{2,1}\\ \widetilde{\mathrm{E}}_{1,3}k_1^{-1} & \widetilde{\mathrm{E}}_{1,2}k_1^{-1} & k_1^{-1}\end{pmatrix},\tag{3.8}$$

where, as in (3.4), we omit the spectral parameter dependence in the Gauss coordinates $\widetilde{\mathrm{F}}_{j,i}(u)$, $\widetilde{\mathrm{E}}_{i,j}(u)$, and $k_i(u)$. To fix the set of algebraically independent generators of the $\mathcal{B}_1$ algebra, we consider the relations

$$T_{i,j}(u-c/2)=\widehat{T}_{i,j}(u),\quad 2\le i,j\le 3.\tag{3.9}$$

For $i,j=3$, we have

$$k_1(u)=k_3^{-1}(u-c/2).\tag{3.10}$$

To proceed further, we must normally order the Gauss coordinates in the monodromy matrix elements in (3.7) using

$$\begin{aligned}&k_3^{-1}(u)\mathrm{F}_{3,2}(u)k_3(u)=\mathrm{F}_{3,2}(u+c),\qquad k_3(u)\mathrm{E}_{2,3}(u)k_3^{-1}(u)=\mathrm{E}_{2,3}(u+c),\\&[\mathrm{E}_{2,3}(u),\mathrm{F}_{3,2}(u-c)]=k_2(u-c)k_3^{-1}(u-c)-k_2(u)k_3^{-1}(u).\end{aligned}\tag{3.11}$$

These equations are particular cases of the commutation relations

$$\begin{aligned}&k_3(u)\mathrm{F}_{3,2}(v)k_3^{-1}(u)=f(u,v)\mathrm{F}_{3,2}(v)-g(u,v)\mathrm{F}_{3,2}(u),\\&k_3^{-1}(u)\mathrm{E}_{2,3}(v)k_3(u)=f(u,v)\mathrm{E}_{2,3}(v)-g(u,v)\mathrm{E}_{2,3}(u),\end{aligned}\tag{3.12}$$

and

$$[\mathrm{E}_{2,3}(u),\mathrm{F}_{3,2}(v)]=g(u,v)\big(k_2(v)k_3^{-1}(v)-k_2(u)k_3^{-1}(u)\big),\tag{3.13}$$

where

$$g(u,v)=\frac{c}{u-v},\qquad f(u,v)=1+g(u,v)=\frac{u-v+c}{u-v}. \tag{3.14}$$

Formulas (3.12) and (3.13) can be obtained from (2.4) by correspondingly setting the subscripts $\{i,j,k,l\}$ equal to $\{3,3,2,3\}$, $\{3,3,3,2\}$, and $\{2,3,3,2\}$. We note that these commutation relations are of the $\mathfrak{gl}$ type because the second line in (2.4) does not contribute. Finally, using Eqs. (3.9) for $\{i,j\}=\{2,3\}$, $\{i,j\}=\{3,2\}$, and $\{i,j\}=\{2,2\}$, we obtain

$$\mathrm{F}_{2,1}(u)=-\mathrm{F}_{3,2}(u+c/2),\qquad \mathrm{E}_{1,2}(u)=-\mathrm{E}_{2,3}(u+c/2) \tag{3.15}$$

and the constraint

$$k_2(u)=k_3(u+c/2)k_3^{-1}(u-c/2)k_2^{-1}(u+c/2). \tag{3.16}$$

Therefore, taking (3.10) and (3.15) into account, we can restrict ourself to the Gauss coordinates $k_2(u)$, $k_3(u)$, $\mathrm{F}_{3,2}(u)$, and $\mathrm{E}_{2,3}(u)$ and constraint (3.16).[1] The latter can also be interpreted as condition (2.7) for the central element $z(u)$ expressed in terms of the diagonal Gauss coordinates as

$$z(u)=k_1(u)k_3(u-c/2)=k_2(u)k_2(u+c/2)k_3(u-c/2)k_3^{-1}(u+c/2)=1. \tag{3.17}$$

Alternatively, we can choose $k_1(u)$, $k_2(u)$, $\mathrm{F}_{2,1}(u)$, and $\mathrm{E}_{1,2}(u)$ as a set of generators of the $\mathcal{B}_1$ algebra with the constraint

$$k_2(u)=k_1^{-1}(u+c/2)k_1(u-c/2)k_2^{-1}(u-c/2). \tag{3.18}$$

In addition to commutation relations (3.12), we also need the commutation relations of the Gauss coordinates $\mathrm{F}_{3,2}(v)$ and $\mathrm{E}_{2,3}(v)$ with the diagonal coordinate $k_2(u)$ and between themselves. These commutation relations follow from (2.4):

$$k_2(u)\mathrm{F}_{3,2}(v)k_2^{-1}(u)=\frac{f(u,v)}{f(u,v+c/2)}\mathrm{F}_{3,2}(v)+g(u,v)\mathrm{F}_{3,2}(u)+g(v,u+c/2)\mathrm{F}_{3,2}(u+c/2), \tag{3.19}$$

$$k_2^{-1}(u)\mathrm{E}_{2,3}(v)k_2(u)=\frac{f(u,v)}{f(u,v+c/2)}\mathrm{E}_{2,3}(v)+g(u,v)\mathrm{E}_{2,3}(u)+g(v,u+c/2)\mathrm{E}_{2,3}(u+c/2) \tag{3.20}$$

and

$$(u-v+c/2)\mathrm{F}_{3,2}(u)\mathrm{F}_{3,2}(v)-(u-v-c/2)\mathrm{F}_{3,2}(v)\mathrm{F}_{3,2}(u)=\frac{c}{2}\big(\mathrm{F}_{3,2}^2(u)+\mathrm{F}_{3,2}^2(v)\big), \tag{3.21}$$

$$(u-v-c/2)\mathrm{E}_{2,3}(u)\mathrm{E}_{2,3}(v)-(u-v+c/2)\mathrm{E}_{2,3}(v)\mathrm{E}_{2,3}(u)=-\frac{c}{2}\big(\mathrm{E}_{2,3}^2(u)+\mathrm{E}_{2,3}^2(v)\big). \tag{3.22}$$

Deriving (3.21) and (3.22) from RTT commutation relations (2.4), we also obtain the relations

$$\mathrm{F}_{3,1}(v)=-\frac{1}{2}\mathrm{F}_{3,2}^2(v),\qquad \mathrm{E}_{1,3}(v)=-\frac{1}{2}\mathrm{E}_{2,3}^2(v). \tag{3.23}$$

We note that because of expansion (2.8), the following zero modes of the monodromy matrix elements vanish: $T_{i,i'}[0]=0$ for $i=1,2,3$.

We also note that constraint (3.17) implies a relation between the eigenvalues $\lambda_i(u)$ given by (2.10):

$$\lambda_1(u)\lambda_3(u-c/2)=\lambda_2(u)\lambda_2(u+c/2)\lambda_3(u-c/2)\lambda_3^{-1}(u+c/2)=1. \tag{3.24}$$

Therefore, $\lambda_i(u)$ are free functional parameters satisfying condition (3.24).

[1]Using the results in [23], we can similarly determine a set of generators of the $\mathcal{B}_n$ algebra corresponding to the classical algebra $\mathfrak{so}_{2n+1}$.

3.2. Yangian double and its current realization. Here, we describe the construction of the Yangian double and define projections on intersections of the different Borel subalgebras of this algebra. This is necessary for the current realization of the off-shell Bethe vectors.

Summarizing the results in the preceding subsection, we conclude that the $\mathcal{B}_1$ algebra of the elements of monodromy matrix (3.1) with

$$\mathbf{F}(u) = \begin{pmatrix} 1 & -\mathrm{F}_{3,2}(u+c/2) & -\mathrm{F}^2_{3,2}(u)/2 \\ 0 & 1 & \mathrm{F}_{3,2}(u) \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{E}(u) = \begin{pmatrix} 1 & 0 & 0 \\ -\mathrm{E}_{2,3}(u+c/2) & 1 & 0 \\ -\mathrm{E}^2_{2,3}(u)/2 & \mathrm{E}_{2,3}(u) & 1 \end{pmatrix}, \tag{3.25}$$

$$\mathbf{D}(u) = \mathrm{diag}(k_3^{-1}(u-c/2), k_2(u), k_3(u)) \tag{3.26}$$

together with constraint (3.16) and series expansion (2.8) is isomorphic to the Yangian $Y(\mathfrak{so}_3)$ [24]. According to the quantum double construction [24], the Yangian double [25] $\mathcal{D}Y(\mathfrak{so}_3)$ associated with the $\mathcal{B}_1$ algebra is a Hopf algebra for a pair of matrices $T^{\pm}(u)$ satisfying the commutation relations with R-matrix (2.1)

$$R(u,v)(T^{\mu}(u)\otimes\mathbf{I})(\mathbf{I}\otimes T^{\nu}(v)) = (\mathbf{I}\otimes T^{\nu}(v))(T^{\mu}(u)\otimes\mathbf{I})R(u,v), \tag{3.27}$$

where μ and ν independently take the values $+$ and $-$. Both matrices $T^{\pm}(u)$ have Gauss decomposition (3.1) with matrices (3.25) and (3.26) and constraint (3.16). To distinguish them, we equip the Gauss coordinates with the superscripts $\pm$.

The difference between the matrices $T^{+}(u)$ and $T^{-}(u)$ is in the different series expansions in the spectral parameter u. The matrix $T^{+}(u)$ is expanded over negative powers of u as in (2.8). It is therefore identified with the universal monodromy matrix $T(u)$ given by (3.1). In contrast, the monodromy matrix $T^{-}(u)$ is given by the series

$$T^{-}_{i,j}(u) = \delta_{ij}\mathbf{1} + \sum_{\ell<0} T_{i,j}[\ell]u^{-\ell-1} \tag{3.28}$$

in nonnegative powers of u. We let $\mathcal{DB}_1$ denote the algebra generated by the matrices $T^{\pm}(u)$ satisfying commutation relations (3.27).

According to [12], we can write the commutation relations in the double $\mathcal{DB}_1$ in terms of the formal generating series

$$\begin{aligned} F(u) &= \mathrm{F}^{+}_{3,2}(u) - \mathrm{F}^{-}_{3,2}(u) = \sum_{\ell\in\mathbb{Z}} F[\ell]u^{-\ell-1}, \\ E(u) &= \mathrm{E}^{+}_{2,3}(u) - \mathrm{E}^{-}_{2,3}(u) = \sum_{\ell\in\mathbb{Z}} E[\ell]u^{-\ell-1}, \\ k^{\pm}_j(u) &= 1 + \sum_{\substack{\ell\geq 0\\ \ell<0}} k_j[\ell]u^{-\ell-1}, \end{aligned} \tag{3.29}$$

as

$$k^{\pm}_2(u)F(v)(k^{\pm}_2(u))^{-1} = f(u,v)f(v,u+c/2)F(v), \tag{3.30}$$

$$k^{\pm}_3(u)F(v)(k^{\pm}_3(u))^{-1} = f(u,v)F(v), \tag{3.31}$$

$$(k^{\pm}_2(u))^{-1}E(v)k^{\pm}_2(u) = f(u,v)f(v,u+c/2)E(v), \tag{3.32}$$

$$(k^{\pm}_3(u))^{-1}E(v)k^{\pm}_3(u) = f(u,v)E(v), \tag{3.33}$$

$$(u-v+c/2)F(u)F(v) = (u-v-c/2)F(v)F(u), \tag{3.34}$$

$$(u-v-c/2)\ E(u)E(v) = (u-v+c/2)E(v)E(u), \tag{3.35}$$

$$[E(u),F(v)] = c\delta(u,v)\big(k_2^+(u)(k_3^+(u))^{-1} - k_2^-(v)(k_3^-(v))^{-1}\big). \tag{3.36}$$

Here, $\delta(u,v)$ in (3.36) is the additive δ-function given by the formal series

$$\delta(u,v) = \frac{1}{u}\sum_{\ell\in\mathbb{Z}} \frac{v^\ell}{u^\ell}. \tag{3.37}$$

The rational functions in the right-hand sides of Eqs. (3.30)–(3.33) should be understood as power series in v/u for $k_2^+(u)$ and $k_3^+(u)$ and in u/v for $k_2^-(u)$ and $k_3^-(u)$. We call the generating series $F(u)$ and $E(u)$ *currents* and the diagonal Gauss coordinates $k_2^\pm(u)$ and $k_3^\pm(u)$ *Cartan currents*.

A coproduct in $\mathcal{DB}_1$ is given by the standard formula

$$\Delta(T_{i,j}^\pm(u)) = \sum_{k=1}^{3} T_{k,j}^\pm(u)\otimes T_{i,k}^\pm(u), \tag{3.38}$$

where the monodromy matrix elements $T_{i,j}^+(u)$ and $T_{i,j}^-(u)$ form two Borel subalgebras, each isomorphic to $\mathcal{B}_1$. Each of these $\mathcal{B}_1$ algebras is a natural Hopf subalgebra of $\mathcal{DB}_1$. We let $U^\pm$ denote these standard Borel subalgebras.

It is well known [22] that another decomposition of the whole algebra into two dual subalgebras can be associated with the current realization of the double $\mathcal{DB}_1$. One *current* subalgebra U_F is formed by the current $F(u)$ and the Cartan currents $k_3^+(u)$ and $k_2^+(u)$, and the other current subalgebra U_E is formed by the current $E(u)$ and the "negative" Cartan currents $k_2^-(u)$ and $k_3^-(u)$. It follows from (3.38) that these new current Borel subalgebras are not Hopf subalgebras with respect to coproduct (3.38). For the subalgebras U_F and U_E to become Hopf subalgebras in $\mathcal{DB}_1$, we introduce a new, so-called Drinfeld coproduct $\Delta^{(\mathrm{D})}$. It is related to the original coproduct (3.38) by the twisting procedure (see [22] and the references therein).

For the generating series of the $\mathcal{DB}_1$ algebra, the Drinfeld coproduct in the current Borel subalgebra U_F $(j=2,3)$ is given by

$$\Delta^{(\mathrm{D})}k_j^+(u) = k_j^+(u)\otimes k_j^+(u), \qquad \Delta^{(\mathrm{D})}F(u) = \mathbf{1}\otimes F(u) + F(u)\otimes k_2^+(u)(k_3^+(u))^{-1}. \tag{3.39}$$

In the dual current Borel subalgebra U_E, it acts on the generators $k_j^-(u)$ and $E(u)$ as

$$\Delta^{(\mathrm{D})}k_j^-(u) = k_j^-(u)\otimes k_j^-(u), \qquad \Delta^{(\mathrm{D})}E(u) = E(u)\otimes\mathbf{1} + k_2^-(u)(k_3^-(u))^{-1}\otimes E(u). \tag{3.40}$$

It is obvious that there are nonempty intersections of the Borel subalgebras of different types,

$$\begin{aligned} U_F^- &= U_F\cap U^-, \qquad & U_F^+ &= U_F\cap U^+,\\ U_E^- &= U_E\cap U^-, \qquad & U_E^+ &= U_E\cap U^+. \end{aligned} \tag{3.41}$$

and these intersections are subalgebras in $\mathcal{DB}_1$ [22]. Furthermore, they are coideals with respect to Drinfeld coproduct (3.39) and (3.40)

$$\begin{aligned} \Delta^{(\mathrm{D})}(U_F^+) &= U_F\otimes U_F^+, \qquad & \Delta^{(\mathrm{D})}(U_F^-) &= U_F^-\otimes U_F,\\ \Delta^{(\mathrm{D})}(U_E^+) &= U_E\otimes U_E^+, \qquad & \Delta^{(\mathrm{D})}(U_E^-) &= U_E^-\otimes U_E. \end{aligned} \tag{3.42}$$

According to the general theory of the Cartan–Weyl construction, we can impose a global ordering of the generators in $\mathcal{DB}_1$. There are two different choices for such an ordering. We let the symbol $\lessdot$ denote the ordering relation and introduce the cycling ordering between elements of the subalgebras $U_F^{\pm}$ and $U_E^{\pm}$ as

$$\ldots \lessdot U_F^- \lessdot U_F^+ \lessdot U_E^+ \lessdot U_E^- \lessdot U_F^- \lessdot \ldots. \tag{3.43}$$

Using this ordering rule, we can say that arbitrary elements $\mathcal{F} \in U_F$ and $\mathcal{E} \in U_E$ are ordered if they are represented in the form

$$\mathcal{F} = \mathcal{F}_- \cdot \mathcal{F}_+, \qquad \mathcal{E} = \mathcal{E}_+ \cdot \mathcal{E}_-, \quad \mathcal{F}_\pm \in U_F^\pm, \quad \mathcal{E}_\pm \in U_E^\pm. \tag{3.44}$$

According to the general theory [22], we can define the projections of any ordered elements from the subalgebras U_F and U_E onto subalgebras (3.41) using the formulas

$$\begin{aligned} &P_f^+(\mathcal{F}_- \cdot \mathcal{F}_+) = \varepsilon(\mathcal{F}_-)\mathcal{F}_+, \qquad P_f^-(\mathcal{F}_- \cdot \mathcal{F}_+) = \mathcal{F}_-\varepsilon(\mathcal{F}_+), \quad \mathcal{F}_\pm \in U_F^\pm, \\ &P_e^+(\mathcal{E}_+ \cdot \mathcal{E}_-) = \mathcal{E}_+\varepsilon(\mathcal{E}_-), \qquad P_e^-(\mathcal{E}_+ \cdot \mathcal{E}_-) = \varepsilon(\mathcal{E}_+)\mathcal{E}_-, \quad \mathcal{E}_\pm \in U_E^\pm, \end{aligned} \tag{3.45}$$

where the counit map $\varepsilon\colon \mathcal{DB}_1 \to \mathbb{C}$ is defined by the rules

$$\varepsilon(F[\ell]) = \varepsilon(E[\ell]) = 0, \qquad \varepsilon(k_j[\ell]) = 0. \tag{3.46}$$

Let $\overline{U}_F$ be the extension of the algebra U_F formed by the infinite sums of the ordered products $\mathcal{A}_{i_1}[\ell_1]\cdots\mathcal{A}_{i_a}[\ell_a]$ with $\ell_1 \le \cdots \le \ell_a$, where $\mathcal{A}_{i_l}[\ell_l]$ is either $F[\ell_l]$ or $k_{i_l}[\ell_l]$. We similarly define $\overline{U}_E$ as the extension of U_E by infinite sums of ordered products $\mathcal{B}_{i_1}[\ell_1]\cdots\mathcal{B}_{i_b}[\ell_b]$ with $\ell_1 \ge \cdots \ge \ell_b$, where $\mathcal{B}_{i_l}[\ell_l]$ is either $E[\ell_l]$ or $k_{i_l}[\ell_l]$. The following can be proved [22]:

1. The action of projections (3.45) extends to the respective algebras $\overline{U}_F$ and $\overline{U}_E$.
2. For any $\mathcal{F} \in \overline{U}_F$ with $\Delta^{(\mathrm{D})}(\mathcal{F}) = \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)}$, we have
$$\mathcal{F} = P_f^-(\mathcal{F}^{(2)}) \cdot P_f^+(\mathcal{F}^{(1)}). \tag{3.47}$$
3. For any $\mathcal{E} \in \overline{U}_E$ with $\Delta^{(\mathrm{D})}(\mathcal{E}) = \mathcal{E}^{(1)} \otimes \mathcal{E}^{(2)}$, we have
$$\mathcal{E} = P_e^+(\mathcal{E}^{(1)}) \cdot P_e^-(\mathcal{E}^{(2)}). \tag{3.48}$$

The formal definitions of projections (3.45) are useful for proving fundamental properties of projections (3.47) and (3.48) onto intersections of the different types of Borel subalgebras. In practical calculations, we often use a more "physical" method. For example, to calculate the projection P_f^+ of the product of the currents $F(u_i)$, we replace each current with the difference of the Gauss coordinates $F(u_i) = \mathrm{F}_{3,2}^+(u_i) - \mathrm{F}_{3,2}^-(u_i)$ and then use the commutation relation

$$\left(u - v + \frac{c}{2}\right)\mathrm{F}_{3,2}^+(u)\mathrm{F}_{3,2}^-(v) - \left(u - v - \frac{c}{2}\right)\mathrm{F}_{3,2}^-(v)\mathrm{F}_{3,2}^+(u) = \frac{c}{2}\big((\mathrm{F}_{3,2}^+(u))^2 + (\mathrm{F}_{3,2}^-(v))^2\big) \tag{3.49}$$

to move all negative Gauss coordinates $\mathrm{F}_{3,2}^-(u_i)$ to the left. Eventually, after such a normal ordering of all the terms in the product of currents, the action of the projection P_f^+ means cancellation of all summands having at least one "negative" Gauss coordinate $\mathrm{F}_{3,2}^-(u_i)$ on the left. The actions of the projections P_f^-, P_e^+, and P_e^- can be defined similarly.

4. Universal Bethe vectors for $\mathcal{B}_1$ algebra

A direct application of the theory of projections is to the construction of the universal off-shell Bethe vectors by calculating the projections of the products of currents. For this, we identify the monodromy matrix of some model with the generating series $T^+_{i,j}(u)$ satisfying the RTT relation with the corresponding R-matrix. We then define a universal off-shell Bethe vector of this model as the projection P^+_f applied to the product of currents corresponding to the simple roots of the underlying finite-dimensional algebra. Because the universal monodromy matrix elements $T^+_{i,j}(u)$ are expressed in terms of the Gauss coordinates, which are themselves related to the currents according to formulas (3.29), we can compute the action of the monodromy matrix elements on these Bethe vectors. This leads to recurrence relations for the latter. On the other hand, we can compute the projection of the product of currents to obtain the structure of the universal Bethe vector. In all these calculations, the main technical tool is the possibility to represent the product of currents in a normal-ordered form using Eqs. (3.47) or (3.48). In this section, we implement this program in the case of the Yangian double $\mathcal{DB}_1$.

4.1. Off-shell Bethe vectors and projections. We introduce rational functions

$$\mathfrak{g}(u,v)=\frac{c/2}{u-v},\qquad \mathfrak{f}(u,v)=\frac{u-v+c/2}{u-v},\qquad \mathfrak{h}(u,v)=\frac{\mathfrak{f}(u,v)}{\mathfrak{g}(u,v)}=\frac{u-v+c/2}{c/2}. \tag{4.1}$$

They correspond to a rescaling $c\to c/2$ in functions (3.14). For a set of complex parameters $\bar u=\{u_1,\dots,u_r\}$ of cardinality r, we also introduce a product

$$\gamma(\bar u)=\prod_{i<j}^{r}\mathfrak{f}(u_j,u_i) \tag{4.2}$$

and a normalized ordered product of the currents

$$\mathbb{F}_r(\bar u)=\gamma(\bar u)\mathcal{F}(\bar u)=\gamma(\bar u)F(u_r)F(u_{r-1})\cdots F(u_1). \tag{4.3}$$

We note that according to commutation relations (3.34), this normalized product is symmetric under any permutation of the parameters u_j.

In what follows, we consider the projection of $\mathbb{F}(\bar u)$ (called the *pre-Bethe vector*)

$$\widehat{\mathbb{B}}_r(\bar u)=P^+_f(\mathbb{F}(\bar u))=\gamma(\bar u)P^+_f(F(u_r)F(u_{r-1})\cdots F(u_1)) \tag{4.4}$$

and the universal off-shell Bethe vector

$$\mathbb{B}_r(\bar u)=\widehat{\mathbb{B}}_r(\bar u)|0\rangle=\gamma(\bar u)P^+_f(F(u_r)F(u_{r-1})\cdots F(u_1))|0\rangle. \tag{4.5}$$

We call the complex variables $\bar u$ in (4.4) and (4.5) the *Bethe parameters*.

The pre-Bethe vector $\widehat{\mathbb{B}}_r(\bar u)$ and Bethe vector itself $\mathbb{B}_r(\bar u)$ are symmetric under permutations of the Bethe parameters. The term “off-shell” means that the parameters u_i are generic complex numbers. If they satisfy a set of equations called Bethe equations, then the Bethe vectors become eigenvectors of the universal transfer matrix $\mathcal{T}(u)$ and are called on-shell Bethe vectors.

In this section, we calculate projection (4.5) and obtain an expression for the Bethe vector in terms of the Gauss coordinates $\mathrm{F}^+_{3,2}(u_i)$. This allows calculating the action of the monodromy matrix elements $T^+_{i,j}(z)$ given by (3.4) on Bethe vectors (4.5). The action formulas for $T^+_{i,j}(z)$ with $i<j$ yield recurrence relations for the Bethe vectors in terms of the upper-triangular elements of the monodromy matrix. The action of the

diagonal elements $T^+_{i,i}(z)$ lead to the Bethe equations. Finally, the action of the lower-triangular elements $T^+_{i,j}(z)$ for $i > j$ can be used to calculate the scalar products of Bethe vectors, which are a necessary tool for studying correlation functions of the quantum integrable model in the algebraic Bethe ansatz framework.

First, we calculate the projection of the product of currents in (4.5). For this, we use an approach first implemented in [15]. We rewrite commutation relation (3.21) between $\mathrm{F}^{\pm}_{3,2}(u)$ and $\mathrm{F}^{-}_{3,2}(v)$ in the form

$$F(u)\mathrm{F}^{-}_{3,2}(v) = \frac{\mathfrak{f}(v,u)}{\mathfrak{f}(u,v)}\mathrm{F}^{-}_{3,2}(v)F(u) + \mathfrak{h}^{-1}(u,v)X(u), \tag{4.6}$$

where we let $X(u)$ denote the combination of the Gauss coordinates $X(u) = (\mathrm{F}^{+}_{3,2}(u))^2 - (\mathrm{F}^{-}_{3,2}(u))^2$. Using this commutation relation, we can write

$$P^+_f(F(u_r)\cdots F(u_2)\mathrm{F}^{-}_{3,2}(u_1)) = \sum_{j=2}^{r}\mathfrak{h}^{-1}(u_j,u_1)\prod_{s=2}^{j-1}\frac{\mathfrak{f}(u_1,u_s)}{\mathfrak{f}(u_s,u_1)}\,Y_j, \tag{4.7}$$

where the element $Y_j \in U^+_F$ is given by

$$Y_j = P^+_f(F(u_r)\cdots F(u_{j+1})X(u_j)F(u_{j-1})\cdots F(u_2))$$

and does not explicitly depend on the spectral parameter u_1.

Substituting $u_1 = u_m$, $m = 2,\ldots,r$, in (4.7) and replacing $\mathrm{F}^{-}_{3,2}(u_1)$ with the difference $\mathrm{F}^{+}_{3,2}(u_1) - F(u_1)$ in the left-hand side of this equation, we obtain

$$P^+_f(F(u_r)\cdots F(u_2))\mathrm{F}^{+}_{3,2}(u_m) = \sum_{j=2}^{r}\mathfrak{h}^{-1}(u_j,u_m)\prod_{s=2}^{j-1}\frac{\mathfrak{f}(u_m,u_s)}{\mathfrak{f}(u_s,u_m)}\,Y_j, \tag{4.8}$$

where we use the properties of projections (3.45) and the fact that the square of the total current $F^2(u) = 0$ vanishes by virtue of commutation relations (3.34).

We can regard (4.7) as a system of linear equations for the unknown elements $Y_j \in U^+_F$, which can be found as linear combinations of the elements $P^+_f(F(u_r)\cdots F(u_2))\mathrm{F}^{+}_{3,2}(u_m)$. Solving Eqs. (4.8) for Y_j and substituting them in (4.7), we obtain

$$P^+_f(F(u_r)\cdots F(u_1)) = P^+_f(F(u_r)\cdots F(u_2))\mathrm{F}^{+}_{3,2}(u_1;u_2,\ldots,u_r), \tag{4.9}$$

where

$$\mathrm{F}^{+}_{3,2}(u_1;u_2,\ldots,u_r) = \mathrm{F}^{+}_{3,2}(u_1) - \sum_{j=2}^{r}\mathfrak{h}^{-1}(u_j,u_1)\prod_{\substack{s=2,\\ s\neq j}}^{r}\frac{\mathfrak{f}(u_s,u_j)}{\mathfrak{f}(u_s,u_1)}\mathrm{F}^{+}_{3,2}(u_j). \tag{4.10}$$

An off-shell Bethe vector $\mathbb{B}_r(\bar u)$ can thus be expressed as the ordered product of linear combinations of the Gauss coordinates acting on the reference vector

$$\mathbb{B}_r(\bar u) = \gamma(\bar u)\prod_{1\le j\le r}^{\longleftarrow}\mathrm{F}^{+}_{3,2}(u_j;u_{j+1},\ldots,u_r)|0\rangle, \tag{4.11}$$

where the ordered product $\overleftarrow{\prod_j} A_j$ of the noncommuting components A_j means $A_rA_{r-1}\cdots A_1$.

4.2. Action of monodromy matrix elements on Bethe vectors. We hereafter use a special notation for products of rational functions (4.1) and the eigenvalues $\lambda_i(u)$ given by (2.10). If any of these functions depends on a set of variables (or two sets of variables), then we take the product over this set. In particular,

$$\lambda_k(\bar{u}) = \prod_{u_i\in\bar{u}} \lambda_k(u_i), \qquad \mathfrak{h}(u,\bar{v}) = \prod_{v_j\in\bar{v}} \mathfrak{h}(u,v_j), \qquad \mathfrak{f}(\bar{u},\bar{v}) = \prod_{u_i\in\bar{u}}\prod_{v_j\in\bar{v}} \mathfrak{f}(u_i,v_j), \tag{4.12}$$

and so on. We also introduce subsets $\bar{u}_i = \bar{u}\setminus\{u_i\}$ and $\bar{u}_{i,j} = \bar{u}\setminus\{u_i,u_j\}$ and extend this convention to products over these subsets, for example,

$$\mathfrak{g}(v_i,\bar{v}_i) = \prod_{\substack{v_j\in\bar{v},\\ v_j\neq v_i}} \mathfrak{g}(v_i,v_j), \qquad \mathfrak{f}(\bar{u}_{i,j},\{u_i,u_j\}) = \prod_{\substack{u_k\in\bar{u},\\ u_k\notin\{u_i,u_j\}}} \mathfrak{f}(u_k,u_i)\mathfrak{f}(u_k,u_j). \tag{4.13}$$

By definition, any product over the empty set is equal to 1. A double product is equal to 1 if at least one of the sets is empty.

Theorem 4.1. *The action of the monodromy matrix element $T_{i,j}(z)$ on an off-shell Bethe vector $\mathbb{B}_r(\bar{u})$ given by* (4.5) *yields a linear combination of off-shell Bethe vectors*

$$T_{i,j}(z)\mathbb{B}_r(\bar{u}) = s(i,j)\lambda_3(z) \sum_{\{\bar{\eta}_{\rm I},\bar{\eta}_{\rm II},\bar{\eta}_{\rm III}\}\vdash\bar{\eta}} \frac{\lambda_2(\bar{\eta}_{\rm III})}{\lambda_3(\bar{\eta}_{\rm III})} \frac{\mathfrak{f}(\bar{\eta}_{\rm I},\bar{\eta}_{\rm II})\mathfrak{f}(\bar{\eta}_{\rm I},\bar{\eta}_{\rm III})\mathfrak{f}(\bar{\eta}_{\rm II},\bar{\eta}_{\rm III})}{\mathfrak{h}(\bar{\eta}_{\rm I},z)\,\mathfrak{h}(z+c/2,\bar{\eta}_{\rm III})} \mathbb{B}_{r-i+j}(\bar{\eta}_{\rm II}), \tag{4.14}$$

where $s(i,j) = 2^{i-j+1}(-1)^{\delta_{i1}+\delta_{j1}}$. The sum is taken over partitions of the set $\bar{\eta} = \{\bar{u},z,z+c/2\}$ into several disjoint subsets $\{\bar{\eta}_{\rm I},\bar{\eta}_{\rm II},\bar{\eta}_{\rm III}\}\vdash\bar{\eta}$ with cardinalities $\#\bar{\eta}_{\rm I} = i-1$ and $\#\bar{\eta}_{\rm III} = 3-j$.

Proof. This theorem is proved in the next section.

4.3. Actions of upper-triangular monodromy matrix elements. The action of upper-triangular monodromy matrix elements on the Bethe vector $\mathbb{B}_r(\bar{u})$ are the simplest. In particular, it follows from the restrictions on the cardinalities of the subsets $\#\bar{\eta}_{\rm I}$ and $\#\bar{\eta}_{\rm III}$ that $\bar{\eta}_{\rm I} = \bar{\eta}_{\rm III} = \varnothing$ for the action of $T_{1,3}(z)$. The sum over partitions disappears, and we immediately obtain

$$T_{1,3}(z)\mathbb{B}_r(\bar{u}) = -\frac{\lambda_3(z)}{2}\mathbb{B}_{r+2}(\bar{u},z,z+c/2). \tag{4.15}$$

For the action of $T_{1,2}(z)$, we have $\#\bar{\eta}_{\rm I} = 0$ and $\#\bar{\eta}_{\rm III} = 1$. The sum over partitions in (4.14) becomes

$$T_{1,2}(z)\mathbb{B}_r(\bar{u}) = -\lambda_3(z) \sum_{\{\bar{\eta}_{\rm II},\bar{\eta}_{\rm III}\}\vdash\bar{\eta}} \frac{\lambda_2(\bar{\eta}_{\rm III})}{\lambda_3(\bar{\eta}_{\rm III})} \frac{\mathfrak{f}(\bar{\eta}_{\rm II},\bar{\eta}_{\rm III})}{\mathfrak{h}(z+c/2,\bar{\eta}_{\rm III})} \mathbb{B}_{r+1}(\bar{\eta}_{\rm II}), \tag{4.16}$$

where either $\bar{\eta}_{\rm III} = z$ or $\bar{\eta}_{\rm III} = z+c/2$ or $\bar{\eta}_{\rm III} = u_i$ with $i=1,\dots,r$. It is easy to see that the case $\bar{\eta}_{\rm III} = z+c/2$ does not contribute, because $\mathfrak{f}(z,z+c/2) = 0$. Hence, we obtain

$$\begin{aligned} T_{1,2}(z)\mathbb{B}_r(\bar{u}) = {}& -\lambda_2(z)\mathfrak{f}(\bar{u},z)\mathbb{B}_{r+1}(\bar{u},z+c/2) - \\ & -\lambda_3(z)\sum_{i=1}^{r} \mathfrak{g}(z,u_i)\mathfrak{f}(\bar{u}_i,u_i)\frac{\lambda_2(u_i)}{\lambda_3(u_i)}\mathbb{B}_{r+1}(\bar{u}_i,z,z+c/2). \end{aligned} \tag{4.17}$$

We similarly derive the action of $T_{2,3}(z)$:

$$T_{2,3}(z)\mathbb{B}_r(\bar{u}) = \lambda_3(z)\mathfrak{f}(z+c/2,\bar{u})\mathbb{B}_{r+1}(\bar{u},z) - \\ - \lambda_3(z)\sum_{i=1}^{r}\frac{\mathfrak{f}(u_i,\bar{u}_i)}{\mathfrak{h}(z,u_i)}\mathbb{B}_{r+1}(\bar{u}_i,z,z+c/2). \tag{4.18}$$

Formulas (4.15)–(4.18) yield recurrence relations for the Bethe vectors:

$$\mathbb{B}_{r+1}(\bar{u},z+c/2) = -\frac{T_{1,2}(z)\mathbb{B}_r(\bar{u})}{\lambda_2(z)\,\mathfrak{f}(\bar{u},z)} - \frac{2T_{1,3}(z)}{\lambda_2(z)\,\mathfrak{f}(\bar{u},z)}\sum_{i=1}^{r}\frac{\mathfrak{f}(\bar{u}_i,u_i)}{\mathfrak{h}(u_i,z+c/2)}\frac{\lambda_2(u_i)}{\lambda_3(u_i)}\mathbb{B}_{r-1}(\bar{u}_i) \tag{4.19}$$

and

$$\mathbb{B}_{r+1}(\bar{u},z) = \frac{T_{2,3}(z)\mathbb{B}_r(\bar{u})}{\lambda_3(z)\,\mathfrak{f}(z+c/2,\bar{u})} - \frac{2T_{1,3}(z)}{\lambda_3(z)\,\mathfrak{f}(z+c/2,\bar{u})}\sum_{i=1}^{r}\frac{\mathfrak{f}(u_i,\bar{u}_i)}{\mathfrak{h}(z,u_i)}\mathbb{B}_{r-1}(\bar{u}_i). \tag{4.20}$$

These recursions allow constructing Bethe vectors successively in terms of polynomials in $T_{i,j}$ with $i<j$ applied to $|0\rangle$ starting from the initial condition $\mathbb{B}_0(\varnothing) = |0\rangle$ and $\mathbb{B}_1(u) = T_{2,3}(u)|0\rangle/\lambda_3(u)$. We failed to solve these recurrence relations and obtain explicit compact expressions for generic Bethe vectors in terms of such polynomials, but such explicit formulas are not required in most problems involving the algebraic Bethe ansatz.

4.4. Actions of diagonal monodromy matrix elements. The actions of the diagonal operators can be derived in the same manner. We omit the details of this derivation and show how action formula (4.14) implies the Bethe equations in the case of the action of diagonal elements $T_{i,i}(z)$.

Setting $i=j=1$ in (4.14), we obtain

$$T_{1,1}(z)\mathbb{B}_r(\bar{u}) = \lambda_1(z)\mathfrak{f}(\bar{u},z)\mathfrak{f}(\bar{u},z+c/2)\mathbb{B}_r(\bar{u}) + \\ + 2\lambda_2(z)\sum_{i=1}^{r}\mathfrak{g}(z+c/2,u_i)\frac{\lambda_2(u_i)}{\lambda_3(u_i)}\mathfrak{f}(\bar{u}_i,z)\mathfrak{f}(\bar{u}_i,u_i)\mathbb{B}_r(\bar{u}_i,z+c/2) + \\ + 2\lambda_3(z)\sum_{i<j}\mathfrak{g}(z,\{u_i,u_j\})\frac{\lambda_2(u_i)}{\lambda_3(u_i)}\frac{\lambda_2(u_j)}{\lambda_3(u_j)}\mathfrak{f}(\bar{u}_{i,j},\{u_i,u_j\})\mathbb{B}_r(\bar{u}_{i,j},z,z+c/2). \tag{4.21}$$

Here, the first term in right-hand side corresponds to $\bar{\eta}_{\rm III} = \{z,z+c/2\}$, the second term corresponds to $\bar{\eta}_{\rm III} = \{u_i,z+c/2\}$, and the third term corresponds to $\bar{\eta}_{\rm III} = \{u_i,u_j\}$, $i<j$. The term corresponding to the subset $\bar{\eta}_{\rm III} = \{u_i,z\}$ vanishes because $\mathfrak{f}(z,z+c/2)=0$. To obtain the first term in the right-hand side of (4.21), we also use (3.17).

The action of $T_{2,2}(z)$ on the Bethe vector is

$$T_{2,2}(z)\mathbb{B}_r(\bar{u}) = \lambda_2(z)\mathfrak{f}(\bar{u},z)\mathfrak{f}(z+c/2,\bar{u})\mathbb{B}_r(\bar{u}) + \\ + 2\lambda_3(z)\sum_{i=1}^{r}\mathfrak{g}(z,u_i)\frac{\lambda_2(u_i)}{\lambda_3(u_i)}\mathfrak{f}(z+c/2,\bar{u}_i)\mathfrak{f}(\bar{u}_i,u_i)\mathbb{B}_r(\bar{u}_i,z) - \\ - 2\lambda_2(z)\sum_{i=1}^{r}\frac{\mathfrak{f}(\bar{u}_i,z)}{\mathfrak{h}(z,u_i)}\mathfrak{f}(u_i,\bar{u}_i)\mathbb{B}_r(\bar{u}_i,z+c/2) + \\ + 2\lambda_3(z)\sum_{i\neq j}^{r}\frac{\lambda_2(u_j)}{\lambda_3(u_j)}\frac{\mathfrak{f}(u_i,\bar{u}_{i,j})\mathfrak{f}(\bar{u}_j,u_j)}{\mathfrak{h}(z,u_i)\mathfrak{h}(u_j,z+c/2)}\mathbb{B}_r(\bar{u}_{i,j},z,z+c/2). \tag{4.22}$$

Here, the terms in the right-hand side correspond to the partitions such that $\{\bar\eta_{\rm I},\bar\eta_{\rm III}\}$ are either $\{u_i,u_j\}$ or $\{u_i,z\}$ or $\{z+c/2,u_i\}$ or $\{z+c/2,z\}$. Contributions of other partitions vanish because $\mathfrak{f}(z,z+c/2)=0$.

The action of $T_{3,3}(z)$ is given by

$$\begin{aligned}T_{3,3}(z)\mathbb{B}_r(\bar u) = \lambda_3(z)\mathfrak{f}(z,\bar u)\mathfrak{f}(z+c/2,\bar u)\mathbb{B}_r(\bar u) + \\ + 2\lambda_3(z)\sum_{i=1}^{r}\mathfrak{g}(u_i,z)\mathfrak{f}(z+c/2,\bar u_i)\mathfrak{f}(u_i,\bar u_i)\mathbb{B}_r(\bar u_i,z) + \\ + 2\lambda_3(z)\sum_{i<j}^{r}\frac{\mathfrak{f}(\{u_i,u_j\},\bar u_{i,j})}{\mathfrak{h}(z,\{u_i,u_j\})}\,\mathbb{B}_r(\bar u_{i,j},z,z+c/2).\end{aligned} \tag{4.23}$$

Summing all three Eqs. (4.21)–(4.23) and gathering the coefficients of the Bethe vectors $\mathbb{B}_r(\bar u)$, $\mathbb{B}_r(\bar u_i,z)$, $\mathbb{B}_r(\bar u_i,z+c/2)$, and $\mathbb{B}_r(\bar u_{i,j},z,z+c/2)$, we see that if the Bethe parameters satisfy the system of Bethe equations

$$\frac{\lambda_2(u_i)}{\lambda_3(u_i)} = \frac{\mathfrak{f}(u_i,\bar u_i)}{\mathfrak{f}(\bar u_i,u_i)}, \tag{4.24}$$

then the Bethe vector $\mathbb{B}_r(\bar u)$ becomes the eigenvector of the transfer matrix $\mathcal{T}(z)$,

$$\mathcal{T}(z)\mathbb{B}_r(\bar u) = \tau(z|\bar u)\mathbb{B}_r(\bar u), \tag{4.25}$$

with the eigenvalue

$$\tau(z|\bar u) = \lambda_1(z)\mathfrak{f}(\bar u,z)\mathfrak{f}(\bar u,z+c/2) + \lambda_2(z)\mathfrak{f}(\bar u,z)\mathfrak{f}(z+c/2,\bar u) + \lambda_3(z)\mathfrak{f}(z,\bar u)\mathfrak{f}(z+c/2,\bar u). \tag{4.26}$$

We note that taking constraint (3.24) into account, we can also write the Bethe equations in the form

$$\frac{\lambda_1(u_i-c/2)}{\lambda_2(u_i-c/2)} = \frac{\mathfrak{f}(u_i,\bar u_i)}{\mathfrak{f}(\bar u_i,u_i)}. \tag{4.27}$$

It is easy to see that systems of equations (4.24) and (4.27) are equivalent to the condition that eigenvalue (4.26) has no poles at $z=u_i$ and $z=u_i-c/2$.

We do not give explicit formulas for the action of lower-triangular elements of $T(u)$: they are quite cumbersome. We only note that these formulas can be used to compute scalar products of Bethe vectors. We thus obtain

$$S_r^{\mathfrak{so}_3}(\bar u|\bar v) = 2^{-r}S_r^{\mathfrak{gl}_2}(\bar u|\bar v)\big|_{c\to c/2}, \tag{4.28}$$

where $S_r^{\mathfrak{so}_3}(\bar u|\bar v)$ denotes the scalar product of $\mathbb{B}_r(\bar v)$ with the dual vector of $\mathbb{B}_r(\bar u)$ and $S_r^{\mathfrak{gl}_2}(\bar u|\bar v)$ is the same object for the generalized model based on $Y(\mathfrak{gl}_2)$. Equation (4.28) is unsurprising because the Yangians $Y(\mathfrak{so}_3)$ and $Y(\mathfrak{gl}_2)$ are isomorphic. In fact, the equality can already be seen at the level of Bethe vectors:

$$\mathbb{B}_r^{\mathfrak{so}_3}(\bar u) = 2^{-r/2}\,\mathbb{B}_r^{\mathfrak{gl}_2}(\bar u)\big|_{c\to c/2}. \tag{4.29}$$

We note that the isomorphism is rather explicit and simple in the current representation but is more involved in terms of the monodromy matrix elements (see [20] for an explicit construction in the RTT representation).

5. Proofs

To prove Theorem 4.1, we need a special representation for pre-Bethe vectors (4.4) in terms of the normalized product of currents (4.3) and the "negative" Gauss coordinates $\mathrm{F}^-_{3,2}(u_i)$. This representation is a direct consequence of projection properties (3.47). To formulate it, we introduce several additional notions.

For any formal series $G(\bar u)$ depending on a set of parameters $\bar u = \{u_1, \dots, u_r\}$, we define a deformed symmetrization by the sum

$$\overline{\mathrm{Sym}}_{\bar u}\, G(\bar u) = \sum_{\sigma\in S_r} \prod_{\substack{\ell<\ell',\\ \sigma(\ell)>\sigma(\ell')}} \frac{\mathfrak{f}(u_{\sigma(\ell')}, u_{\sigma(\ell)})}{\mathfrak{f}(u_{\sigma(\ell)}, u_{\sigma(\ell')})} G({}^{\sigma}\bar u), \tag{5.1}$$

where S_r is a permutation group of the set $\bar u$ and ${}^{\sigma}\bar u = \{u_{\sigma(1)}, \dots, u_{\sigma(r)}\}$. If $c = 0$, then the deformed symmetrization coincides with the usual symmetrization Sym_u.

Let $\mathcal{F}(\bar u)$ be the ordered product of the currents $F(u_i)$ defined by (4.3). Then $\mathcal{F}(\bar u) \in \overline{U}_F$ is a formal series of the generators of the $\mathcal{DB}_1$ algebra. It can be represented in the normal ordered form via (3.47), coproduct properties (3.39), and commutation relations (3.30):

$$\mathcal{F}(\bar u) = \overline{\mathrm{Sym}}_{\bar u}\biggl(\sum_{s=0}^{r} \frac{1}{s!(r-s)!} P^-_f\bigl(F_2(u_r)\cdots F_2(u_{s+1})\bigr)\cdot P^+_f\bigl(F_2(u_s)\cdots F_2(u_1)\bigr)\biggr). \tag{5.2}$$

Multiplying both sides by the factor $\gamma(\bar u)$ given by (4.2) and using the fact that

$$\gamma(\bar u)\,\overline{\mathrm{Sym}}_{\bar u}(G(\bar u)) = \mathrm{Sym}_{\bar u}(\gamma(\bar u) G(\bar u)) \tag{5.3}$$

for any formal series $G(\bar u)$, we obtain the ordering rule for the normalized symmetric product of the currents

$$\mathbb{F}_r(\bar u) = \mathrm{Sym}_{\bar u}\biggl(\sum_{s=0}^{r} \frac{\prod_{j=s+1}^{r}\prod_{i=1}^{s}\mathfrak{f}(u_j, u_i)}{s!(r-s)!} P^-_f\bigl(\mathbb{F}_{r-s}(u_r, \dots, u_{s+1})\bigr)\cdot P^+_f\bigl(\mathbb{F}_s(u_s, \dots, u_1)\bigr)\biggr).$$

Because both $P^-_f(\mathbb{F}_{r-s}(u_r, \dots, u_{s+1}))$ and $P^+_f(\mathbb{F}_s(u_s, \dots, u_1))$ are symmetric over their arguments, the sum over permutations within the subsets $\{u_r, \dots, u_{s+1}\}$ and $\{u_s, \dots, u_1\}$ leads to cancellation of the combinatorial factor $s!(r-s)!$. The sum over permutations of the whole set $\bar u$ thus becomes a sum over partitions of this set into two nonintersecting subsets $\bar u_{\mathrm{I}}$ and $\bar u_{\mathrm{II}}$ with the cardinalities $\#\,\bar u_{\mathrm{I}} = s$ and $\#\,\bar u_{\mathrm{II}} = r-s$ for any s:

$$\mathbb{F}_r(\bar u) = \sum_{\{\bar u_{\mathrm{I}}, \bar u_{\mathrm{II}}\}\vdash \bar u} \mathfrak{f}(\bar u_{\mathrm{II}}, \bar u_{\mathrm{I}}) P^-_f\bigl(\mathbb{F}_{r-s}(\bar u_{\mathrm{II}})\bigr)\cdot P^+_f\bigl(\mathbb{F}_s(\bar u_{\mathrm{I}})\bigr). \tag{5.4}$$

Using the relation $P^-_f(F_2(u)) = -\mathrm{F}^-_{3,2}(u)$ and calculating the projection via (3.49)

$$P^-_f\bigl(\mathbb{F}_2(\{u_j, u_i\})\bigr) = \bigl(\mathfrak{f}(u_j, u_i)\mathrm{F}^-_{3,2}(u_j) - \mathfrak{g}(u_j, u_i)\mathrm{F}^-_{3,2}(u_i)\bigr)\mathrm{F}^-_{3,2}(u_i),$$

we can write the pre-Bethe vector $P^+_f(\mathbb{F}_r(\bar u))$ in the form

$$\begin{aligned} P^+_f(\mathbb{F}_r(\bar u)) = \mathbb{F}_r(\bar u) + \sum_{i=1}^{r} \mathfrak{f}(u_i, \bar u_i)\mathrm{F}^-_{3,2}(u_i) P^+_f\bigl(\mathbb{F}_{r-1}(\bar u_i)\bigr) -{}& \\ - \sum_{i<j}^{r} \mathfrak{f}(\{u_j, u_i\}, \bar u_{i,j})\bigl(\mathfrak{f}(u_j, u_i)\mathrm{F}^-_{3,2}(u_j) - \mathfrak{g}(u_j, u_i)\mathrm{F}^-_{3,2}(u_i)\bigr) \times{}& \\ \times \mathrm{F}^-_{3,2}(u_i) P^+_f\bigl(\mathbb{F}_{r-2}(\bar u_{i,j})\bigr) + \mathbb{W}.& \end{aligned} \tag{5.5}$$

Here, $\mathbb{W}$ denotes all the terms that contain at least three "negative" Gauss coordinates $\mathrm{F}^-_{3,2}(u_i)$ on the left of the product.

5.1. Action of the elements $T_{i,3}(z)$. Formula (4.9) with $r=2$, $u_2=u_1+c/2$, and $u_1=z$ is

$$P_f^+(F(z+c/2)F(z)) = \mathrm{F}^+_{3,2}(z+c/2)\Big(\mathrm{F}^+_{3,2}(z) - \frac{1}{2}\mathrm{F}^+_{3,2}(z+c/2)\Big). \tag{5.6}$$

According to commutation relation (3.21) at $u=z+c/2$ and $v=z$, this projection is equal to

$$P_f^+(F(z+c/2)F(z)) = \frac{1}{2}(\mathrm{F}^+_{3,2}(z))^2. \tag{5.7}$$

This means that the monodromy matrix element $T^+_{1,3}(z)$ can be expressed in terms of the current generators as

$$T^+_{1,3}(z) = -P_f^+(F(z+c/2)F(z)k_3^+(z)). \tag{5.8}$$

On the other hand, the properties of the projection onto U_F^+ imply that for any $\mathcal{F}_1,\mathcal{F}_2\in\overline{U}_F$,

$$P_f^+(\mathcal{F}_1\cdot P_f^+(\mathcal{F}_2)) = P_f^+(\mathcal{F}_1)\cdot P_f^+(\mathcal{F}_2). \tag{5.9}$$

Commutation relations (2.4) in the $\mathcal{DB}_1$ algebra,

$$f(u,z)T^+_{1,3}(z)T^-_{2,3}(u) = T^-_{2,3}(u)T^+_{1,3}(z) + g(u,z)T^-_{1,3}(u)T^+_{2,3}(z), \tag{5.10}$$

and property (5.9) yield

$$P_f^+(T^+_{1,3}(z)\mathrm{F}^-_{3,2}(u)) = 0 \tag{5.11}$$

or, equivalently,

$$P_f^+(F(z+c/2)F(z)k_3^+(z)\mathrm{F}^-_{3,2}(u)) = 0. \tag{5.12}$$

The action of $T^+_{1,3}(z)$ on Bethe vector is then

$$\begin{aligned} T^+_{1,3}(z)\cdot\mathbb{B}_r(\bar u) &= -P_f^+(F(z+c/2)F(z)k_3^+(z))\cdot P_f^+(\mathbb{F}_r(\bar u))|0\rangle = \\ &= -P_f^+(F(z+c/2)F(z)k_3^+(z)\cdot P_f^+(\mathbb{F}_r(\bar u)))|0\rangle = \\ &= -\lambda_3(z)f(z,\bar u)P_f^+(F(z+c/2)F(z)\mathbb{F}_r(\bar u))|0\rangle = \\ &= -\frac{\lambda_3(z)}{2}\mathbb{B}_{r+2}(z+c/2,z,\bar u). \end{aligned} \tag{5.13}$$

Here, we use (5.5) and (5.12) to replace $P_f^+(\mathbb{F}(\bar u))$ with $\mathbb{F}(\bar u)$ in the second line of (5.13) and the obvious relations $\mathfrak{f}(z+c/2,z)=2$ and $\mathfrak{f}(z+c/2,u)\mathfrak{f}(z,u)=f(z,u)$. We have thus proved action formula (4.15).

Combining formulas (2.4) for the values of the indices $\{i,j,k,l\}$ to $\{2,3,2,3\}$ and $\{3,3,1,3\}$ in the double Yangian $\mathcal{DB}_1$, we obtain the commutation relations

$$\begin{aligned} T^+_{2,3}(z)T^-_{2,3}(u) &= \frac{\mathfrak{f}(z+c/2,u)}{\mathfrak{f}(u+c/2,z)}T^-_{2,3}(u)T^+_{2,3}(z) + \\ &+ \frac{2}{\mathfrak{h}(z,u)}\Big(T^+_{1,3}(z)T^-_{3,3}(u) - \frac{\mathfrak{f}(z+c/2,u)\mathfrak{f}(z,u)}{\mathfrak{f}(u+c/2,z)\mathfrak{f}(u,z)}T^-_{1,3}(u)T^+_{3,3}(z)\Big). \end{aligned} \tag{5.14}$$

Using explicit expressions for the monodromy matrix elements in terms of Gauss coordinates, from (5.14), we conclude that

$$P_f^+(T^+_{2,3}(z)\mathrm{F}^-_{3,2}(u)) = \frac{2}{\mathfrak{h}(z,u)}T^+_{1,3}(z). \tag{5.15}$$

Combining this formula with Eq. (5.11), we obtain

$$P_f^+(T_{2,3}^+(z)\mathrm{F}_{3,2}^-(u)\mathrm{F}_{3,2}^-(v)) = 0. \tag{5.16}$$

Equations (5.15) and (5.16) together with representation (5.5) now yield

$$\begin{aligned} T_{2,3}^+(z)\cdot\mathbb{B}_r(\bar u) &= P_f^+(F(z)k_3^+(z))\cdot P_f^+(\mathbb{F}_r(\bar u))|0\rangle = \\ &= \lambda_3(z)f(z,\bar u)P_f^+(F(z)\mathbb{F}_r(\bar u))|0\rangle + 2\sum_{i=1}^r \frac{\mathfrak{f}(u_i,\bar u_i)}{\mathfrak{h}(z,u_i)}P_f^+(T_{1,3}^+(z)P_f^+(\mathbb{F}_{r-1}(\bar u_i)))|0\rangle = \\ &= \lambda_3(z)\biggl(\mathfrak{f}(z+c/2,\bar u)\mathbb{B}_{r+1}(\{z,\bar u\}) - \sum_{i=1}^r \frac{\mathfrak{f}(u_i,\bar u_i)}{\mathfrak{h}(z,u_i)}\mathbb{B}_{r+1}(\{z+c/2,z,\bar u\})\biggr). \end{aligned}$$

We have thus proved (4.18).

Taking the equality $T_{3,3}^+(z) = k_3^+(z)$ into account, from commutation relations (2.4) in the double Yangian $\mathcal{DB}_1$, we obtain the formula

$$T_{3,3}^+(z)\mathrm{F}_{3,2}^-(u) = f(z,u)\mathrm{F}_{3,2}^-(u)T_{3,3}^+(z) + g(u,z)T_{2,3}^+(z), \tag{5.17}$$

which together with (5.11), (5.15), and (5.16) yields the equalities

$$\begin{aligned} &P_f^+(T_{3,3}^+(z)\mathrm{F}_{3,2}^-(u)) = g(u,z)T_{2,3}^+(z), \\ &P_f^+(T_{3,3}^+(z)\mathrm{F}_{3,2}^-(u)\mathrm{F}_{3,2}^-(v)) = \frac{2g(u,z)}{\mathfrak{h}(z,v)}T_{1,3}^+(z), \\ &P_f^+(T_{3,3}^+(z)\mathrm{F}_{3,2}^-(u)\mathrm{F}_{3,2}^-(v)\mathrm{F}_{3,2}^-(w)) = 0. \end{aligned} \tag{5.18}$$

Using these equations and the first three terms in the right-hand side of (5.5), we prove (4.23).

All the other action formulas in Theorem 4.1 can be proved similarly. But it is clear that to calculate the action of the lower-triangular elements of the monodromy matrix, we should compute more terms in the right-hand side of representation (5.5) contained in the term $\mathbb{W}$. This makes the calculations rather cumbersome. Instead, there is a more elegant way to calculate these actions using the zero modes of the monodromy matrix and commutation relations (2.9). We explain this approach in the next subsection.

5.2. Action of zero modes. In what follows, we use the zero modes of the lower-triangular elements of the monodromy matrix,

$$T_{2,1}^+[0] = -T_{3,2}^+[0] = -\mathrm{E}_{2,3}^+[0], \tag{5.19}$$

and the action of the zero mode $\mathrm{E}_{2,3}^+[0]$ on the Bethe vector $\mathbb{B}_r(\bar u)$. To calculate this action, we use again representation (5.5) and the commutation relations following from (3.13) and (3.30)

$$\mathrm{E}_{2,3}^+[0]\mathrm{F}_{3,2}^-(u) = \mathrm{F}_{3,2}^-(u)\mathrm{E}_{2,3}^+[0] + c\bigl(k_2^-(u)(k_3^-(u))^{-1} - 1\bigr), \tag{5.20}$$

$$\mathrm{E}_{2,3}^+[0]F(u) = F(u)\mathrm{E}_{2,3}^+[0] + c\bigl(k_2^+(u)(k_3^+(u))^{-1} - k_2^-(u)(k_3^-(u))^{-1}\bigr), \tag{5.21}$$

$$k_2^+(u)(k_3^+(u))^{-1}F(v) = \frac{\mathfrak{f}(v,u)}{\mathfrak{f}(u,v)}F(v)k_2^+(u)(k_3^+(u))^{-1}. \tag{5.22}$$

Equation (5.20) means that only two terms in the right-hand side of (5.5) contribute to the action of the zero mode $\mathrm{E}^+_{2,3}[0]$ on the Bethe vector. Because $\mathrm{E}^+_{2,3}[0]|0\rangle = 0$, this action follows from the chain of equations

$$\begin{aligned}
\mathrm{E}^+_{2,3}[0]\mathbb{B}_r(\bar u) &= [\mathrm{E}^+_{2,3}[0], P^+_f(\mathbb{F}_r(\bar u))]|0\rangle = \\
&= P^+_f\left(\left[\mathrm{E}^+_{2,3}[0], \left(\mathbb{F}_r(\bar u) - \sum_{i=1}^{r} \mathfrak{f}(u_i,\bar u_i)\mathrm{F}^-_{3,2}(u_i)\mathbb{F}_{r-1}(\bar u_i)\right)\right]\right)|0\rangle = \\
&= cP^+_f\left(\sum_{i=1}^{r} \mathfrak{f}(u_i,\bar u_i)\big(k^+_2(u_i)(k^+_3(u_i))^{-1} - 1\big)\mathbb{F}_{r-1}(\bar u_i)\right)|0\rangle = \\
&= c\sum_{i=1}^{r} \mathfrak{f}(u_i,\bar u_i)\left(\frac{\mathfrak{f}(\bar u_i,u_i)}{\mathfrak{f}(u_i,\bar u_i)}\frac{\lambda_2(u_i)}{\lambda_3(u_i)} - 1\right)\mathbb{B}_{r-1}(\bar u_i).
\end{aligned} \tag{5.23}$$

We note that if the set $\bar u$ satisfies Bethe equations (4.24), then the zero mode $\mathrm{E}^+_{2,3}[0]$ annihilates the on-shell Bethe vectors, which are highest-weight vectors of the finite-dimensional $\mathfrak{so}_3$ algebra generated by the zero modes. This a typical property of on-shell Bethe vectors in the models associated with Yangians. Indeed, these models are invariant under the action of the finite-dimensional algebra generated by the zero modes, and the highest-weight property is related to the completeness of the Bethe ansatz. This was previously noted for models based on the $\mathfrak{gl}_N$ symmetry [26].

Formula (2.9) at $\{i,j,k,l\} = \{1,3,3,2\}$ yields

$$T^+_{1,2}(z) = -T^+_{2,3}(z) + c^{-1}[\mathrm{E}^+_{2,3}[0], T^+_{1,3}(z)], \tag{5.24}$$

which can be used to obtain the action of the monodromy matrix element $T^+_{1,2}(z)$ from the already known actions of $T^+_{2,3}(z)$, $T^+_{1,3}(z)$, and $\mathrm{E}^+_{2,3}[0]$. To prove the rest of Theorem 4.1, we use the relations

$$\begin{aligned}
&T^+_{2,2}(z) = -T^+_{3,3}(z) + c^{-1}[\mathrm{E}^+_{2,3}[0], T^+_{2,3}(z)], \qquad T^+_{1,1}(z) = T^+_{2,2}(z) - c^{-1}[\mathrm{E}^+_{2,3}[0], T^+_{1,2}(z)],\\
&T^+_{3,2}(z) = c^{-1}[\mathrm{E}^+_{2,3}[0], T^+_{3,3}(z)], \qquad T^+_{2,1}(z) = c^{-1}[\mathrm{E}^+_{2,3}[0], T^+_{1,1}(z)],\\
&T^+_{3,1}(z) = -c^{-1}[\mathrm{E}^+_{2,3}[0], T^+_{3,2}(z)],
\end{aligned}$$

which imply the action formulas for $T^+_{2,2}(z)$, $T^+_{1,1}(z)$, $T^+_{3,2}(z)$, $T^+_{2,1}(z)$, and $T^+_{3,1}(z)$.

6. Conclusion

We have begun a program of investigating $\mathfrak{so}_N$-invariant quantum integrable models using the current formalism of deformed Kac–Moody algebras and method of projections onto intersections of different Borel subalgebras in these algebras. This method was formulated in [22], [16] and developed in [13]. In the framework of this method, the off-shell Bethe vectors are defined in terms of the generators of the corresponding infinite-dimensional algebra. The RTT formulation of this algebra uses the same R-matrix as the intertwining relations of the monodromy matrix of $\mathfrak{so}_N$-invariant quantum integrable model. Our main goal here was to find formulas for the action of the monodromy matrix elements on Bethe vectors using the current approach and the projection method. We showed that our approach allows obtaining such formulas and expressing the result of these actions as linear combinations of off-shell Bethe vectors.

We note that we do not use explicit representations for the Bethe vectors in terms of the monodromy matrix elements acting on the reference vector. Such explicit representations are still lacking, although the recurrence relations derived here allow finding them, at least for the vectors with a small number of

Bethe parameters. We showed that the projection method allows completely abandoning the use of such representations.

In this paper, we restricted ourself to the simplest case of the $\mathfrak{so}_3$-invariant quantum integrable models. As noted (see the end of Sec. 4), they are equivalent to the models built on the $\mathfrak{gl}_2$ algebra. Nevertheless, the calculations leading to the Bethe vectors and the action formulas differ quite noticeably. For models with $\mathfrak{gl}_2$ symmetries, they reflect the general $\mathfrak{gl}_N$ scheme, and for models with $\mathfrak{so}_3$ symmetries, they are closer to the $\mathfrak{so}_N$ approach. In this sense, although the presented calculations do not give new results on the scalar products of Bethe vectors, they shed some light on the case of integrable models based on orthogonal and symplectic Yangians. Indeed, it is clear that the method introduced here can be generalized to $\mathfrak{so}_N$- and $\mathfrak{sp}_{2n}$-invariant models. The corresponding results will be published later.

Conflicts of interest. The authors declare no conflicts of interest.

REFERENCES

1. E. K. Sklyanin, L. A. Takhtadzhyan, and L. D. Faddeev, "Quantum inverse problem method: I," *Theor. Math. Phys.*, **40**, 688–706 (1979).
2. L. A. Takhtadzhyan and L. D. Faddeev, "The quantum method of the inverse problem and the Heisenberg XYZ model," *Russian Math. Surveys*, **34**, 11–68 (1979).
3. N. Yu. Reshetikhin, "A method of functional equations in the theory of exactly solvable quantum systems," *Lett. Math. Phys.*, **7**, 205–213 (1983).
4. N. Yu. Reshetikhin, "O(N) invariant quantum field theoretical models: Exact solution," *Nucl. Phys. B*, **251**, 565–580 (1985).
5. P. P. Kulish and N. Yu. Reshetikhin, *Soviet Phys. JETP*, **53**, 108–114 (1981).
6. P. P. Kulish and N. Yu. Reshetikhin, *J. Soviet Math.*, **34**, 1948–1971 (1989).
7. P. P. Kulish and N. Yu. Reshetikhin, "Diagonalization of $GL(N)$ invariant transfer matrices and quantum N-wave system (Lee model)," *J. Phys. A: Math. Gen.*, **16**, L591–L596 (1983).
8. A. N. Varchenko and V. O. Tarasov, "Jackson integral representations for solutions of the Knizhnik–Zamolodchikov quantum equation," *St. Petersburg Math. J.*, **6**, 275–313 (1995); arXiv:hep-th/9311040v3 (1993).
9. V. Tarasov and A. Varchenko, "Combinatorial formulae for nested Bethe vector," *SIGMA*, **9**, 048 (2013); arXiv:math/0702277v3 (2007).
10. N. Y. Reshetikhin and M. A. Semenov-Tian-Shansky, "Central extensions of quantum current groups," *Lett. Math. Phys.*, **19**, 133–142 (1990).
11. N. Jing, M. Liu, and A. Molev, "Isomorphism between the R-matrix and Drinfeld presentations of Yangian in types B, C and D," *Commun. Math. Phys.*, **361**, 827–872 (2018).
12. N. Jing, M. Liu, and F. Yang, "Double Yangians of the classical types and their vertex representations," arXiv:1810.06484v2 [math.QA] (2018).
13. A. A. Hutsalyuk, A. Liashyk, S. Z. Pakulyak, E. Ragoucy, and N. A. Slavnov, "Current presentation for the super-Yangian double $DY(\mathfrak{gl}(m|n))$ and Bethe vectors," *Russian Math. Surveys*, **72**, 33–99 (2017); arXiv:1611.09020v2 [cs.CL] (2016).
14. A. B. Zamolodchikov and Al. B. Zamolodchikov, "Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models," *Ann. Phys.*, **120**, 253–291 (1979).
15. S. Z. Pakuliak and S. M. Khoroshkin, "Weight function for the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_3)$," *Theor. Math. Phys.*, **145**, 1373–1399 (2005); arXiv:math/0610433v1 (2006).
16. S. Khoroshkin and S. Pakuliak, "A computation of an universal weight function for the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$," *J. Math. Kyoto Univ.*, **48**, 277–321 (2008); arXiv:0711.2819v2 [math.QA] (2007).
17. S. Belliard, S. Pakuliak, E. Ragoucy, and N. A. Slavnov, "Highest coefficient of scalar products in $SU(3)$-invariant models," *J. Stat. Mech.*, **9**, P09003 (2012); arXiv:1206.4931v1 [math-ph] (2012).

18. A. Hutsalyuk, A. Liashyk, S. Z. Pakuliak, E. Ragoucy, and N. A. Slavnov, "Scalar products of Bethe vectors in the models with $\mathfrak{gl}(m|n)$) symmetry," *Nucl. Phys. B*, **923**, 277–311 (2017); arXiv:1704.08173v3 [math-ph] (2017).
19. A. Molev, *Yangian and Classical Lie Algebras* (Math. Surv. Monogr., Vol. 143), Amer. Math. Soc., Providence, R. I. (2007).
20. D. Arnaudon, A. Molev, and E. Ragoucy, "On the R-matrix realization of Yangians and their representations," *Ann. Henri Poincaré*, **7**, 1269–1325 (2006); arXiv:math.QA/0511481v1 (2005).
21. V. G. Drinfel'd, "A new realization of Yangians and of quantum affine algebras," *Soviet Dokl. Math.*, **36**, 212–216 (1988).
22. B. Enriquez, S. Khoroshkin, and S. Pakuliak, "Weight functions and Drinfeld currents," *Commun. Math. Phys.*, **276**, 691–725 (2007).
23. A. Liashyk, S. Z. Pakuliak, E. Ragoucy, and N. A. Slavnov, "New symmetries of $\mathfrak{gl}(N)$-invariant Bethe vectors," *J. Stat. Mech.*, **2019**, 044001 (2019); arXiv:1810.00364v2 [math-ph] (2018).
24. V. G. Drinfeld, "Quantum groups," *J. Soviet Math.*, **41**, 898–915 (1988).
25. S. M. Khoroshkin and V. N. Tolstoy, "Yangian double," *Lett. Math. Phys.*, **36**, 373–402 (1996).
26. E. Mukhin, V. Tarasov, and A. Varchenko, "Bethe eigenvectors of higher transfer matrices," *J. Stat. Mech.*, **2006**, P08002 (2006); arXiv:math/0605015v2 (2006).