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TRANSLATION INVARIANCE OF THE PERIODIC GIBBS MEASURES FOR THE POTTS MODEL ON THE CAYLEY TREE

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We study the Potts model with a zero external field on the Cayley tree. For the antiferromagnetic Potts model with q states on a second-order Cayley tree and for the ferromagnetic Potts model with q states on a kth-order Cayley tree, we show that all periodic Gibbs measures are translation-invariant for all parameter values.

Keywords: Cayley tree, configuration, Potts model, Gibbs measure, periodic measure, translation-invariant measure

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1. Introduction

Solutions of problems arising in studying the thermodynamic properties of physical and biological systems primarily lead to problems in the theory of Gibbs measures. It is known that each limit Gibbs measure is associated with one phase of the physical system. The Potts model is a generalization of the Ising model and is well studied on the lattice $\mathbb{Z}^d$ and on the Cayley tree. The concept of the Gibbs measure for the Potts model on the Cayley tree is introduced in the conventional manner (see [1]–[4]).

The ferromagnetic Potts model with three states on a second-order Cayley tree was studied in [5], where it was shown that a critical temperature $T_{\rm c}$ exists such that three translation-invariant and uncountable number of non-translation-invariant Gibbs measures exist for $T < T_{\rm c}$. The results in [5] were generalized to the Potts model with a finite number of states on a Cayley tree of an arbitrary (finite) order in [6].

On a Cayley tree of an arbitrary order, the translation-invariant Gibbs measure of the antiferrimagnetic Potts model with q states and with an external field was shown to be unique (see [4]). The Potts model with a countable number of states and with a nonzero external field was studied in [7], where it was proved that the model has a unique translation-invariant Gibbs measure.

Periodic Gibbs measures were studied in [8], where it was proved that under certain conditions, all periodic Gibbs measures are translation-invariant. In particular, under certain conditions, for the ferromagnetic Potts model with three states on a Cayley tree of an arbitrary order and for the antiferromagnetic Potts model with three states on a second-order Cayley tree, all periodic Gibbs measures are translation-invariant. Moreover, the conditions were found under which the Potts model with a nonzero external field has periodic Gibbs measures. The results in [8] were followed up in [9], where the existence of at least three periodic Gibbs measures with the period two on a third- or fourth-order Cayley tree for the Potts model with three states and a zero external field was proved. In [10], the Potts model with q states on a Cayley tree of order

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$k \geq 3$ was studied, and on certain invariants, the existence of periodic (not translation-invariant) Gibbs measures was shown under certain conditions on the model parameters. Moreover, the lower bound of the number of the existing periodic Gibbs measures was shown. In [11], translation-invariant Gibbs measures for the ferromagnetic Potts model with q states were described in detail, and the number of them was shown not to exceed $2^q - 1$, and the problem of extremes of these measures was studied in [12]. In [13], the results reported in [10] were improved, and explicit equations were derived for the translation-invariant Gibbs measures for the Potts model with three states on a Cayley tree of order $k = 3$.

Here, we generalize certain results in [8]. For the antiferromagnetic Potts model with q states on a second-order Cayley tree, we show that all periodic Gibbs measures are translation-invariant for any parameter value. Moreover, for the ferromegnatic Potts model with q states on a kth-order Cayley tree, we show that all the periodic Gibbs measures are translation-invariant.

2. Definitions and known facts

The Cayley tree. We assume that $\Im^k = (V, L)$ is a Cayley tree of order $k \geq 1$, i.e., an infinite graph without cycles and with exactly $k+1$ edges at each vertex of the graph. Here, V is the set of all vertices $\Im^k$, and L is the set of all edges. Two vertices x and y are called *nearest neighbors* if there exists an edge $l \in L$ that connects them. We then write $l = \langle x, y \rangle$.

For a fixed point $x^0 \in V$, we assume that $W_n = \{x \in V \mid d(x, x^0) = n\}$, $V_n = \bigcup_{m=0}^{n} W_m$, and $L_n = \{\langle x, y \rangle \in L \mid x, y \in V_n\}$, where $d(x, y)$ is the distance between the vertices x and y on the Cayley tree, i.e., the number of edges of the minimum path connecting the vertices x and y. We write $x \prec y$ if the path from x^0 to y passes through x. A vertex y is called the direct descendant of x if $y \succ x$ and x and y are nearest neighbors. The set of the direct descendants of x is denoted by $S(x)$, i.e., if $x \in W_n$, then $S(x) = \{y_i \in W_{n+1} \mid d(x, y_i) = 1,\ i = 1, 2, \ldots, k\}$.

The Potts model. We assume that $\Phi = \{1, 2, \ldots, q\}$, $q \geq 2$, and $\sigma \in \Omega = \Phi^V$ is a configuration, i.e., $\sigma = \{\sigma(x) \in \Phi \colon x \in V\}$. For the subset $A \subset V$, we define Ω_A as the set of all configurations defined on A and taking values in Φ.

We consider the Hamiltonian of the Potts model

$$H(\sigma) = -J \sum_{\langle x,y \rangle \in L} \delta_{\sigma(x)\sigma(y)} - \alpha \sum_{x \in V} \delta_{1\sigma(x)}, \tag{1}$$

where $J \in R$, $\alpha \in R$ is the external field, $\langle x, y \rangle$ are nearest neighbors, and δ_{ij} is the Kronecker symbol.

The Gibbs measure. For each n, the measure μ_n on Ω_{V_n} is defined as

$$\mu_n(\sigma_n) = Z_n^{-1} \exp\biggl\{-\beta H_n(\sigma_n) + \sum_{x \in W_n} \tilde{h}_{\sigma(x),x}\biggr\}, \tag{2}$$

where $\{\tilde{h}_x = (\tilde{h}_{1,x}, \ldots, \tilde{h}_{q,x}) \in \mathbb{R}^q,\ x \in V\}$ is the set of vectors, $\beta = 1/T$ (T is the temperature, $T > 0$), $\sigma_n = \{\sigma(x),\ x \in V_n\} \in \Omega_{V_n}$, Z_n^{-1} is the normalizing factor, and $H_n(\sigma_n) = -J \sum_{\langle x,y \rangle \in L_n} \delta_{\sigma(x)\sigma(y)} - \alpha \sum_{x \in V_n} \delta_{1\sigma(x)}$.

The consistency condition for the measures $\mu_n(\sigma_n)$, $n \geq 1$, and $\sigma_{n-1} \in \Phi^{V_{n-1}}$ is

$$\sum_{\omega_n \in \Phi^{W_n}} \mu_n(\sigma_{n-1} \vee \omega_n) = \mu_{n-1}(\sigma_{n-1}). \tag{2'}$$

Here, $\sigma_{n-1} \vee \omega_n$ is the union of configurations, i.e., $\sigma_{n-1} \vee \omega_n \in \Phi^{V_n}$ such that $(\sigma_{n-1} \vee \omega_n)\big|_{V_{n-1}} = \sigma_{n-1}$ and $(\sigma_{n-1} \vee \omega_n)\big|_{W_n} = \omega_n$.

We assume that μ_n, $n \geq 1$, is the sequence of the measures on Ω_{V_n} satisfying consistency condition (2′). In this case, there exists a unique measure μ on Φ^V such that $\mu(\{\sigma|_{V_n} = \sigma_n\}) = \mu_n(\sigma_n)$ for all n and $\sigma_n \in \Phi^{V_n}$. This measure is called the splitting Gibbs measure corresponding to Hamiltonian (1) and the vector-valued function $\tilde{h}_x$, $x \in V$.

According to [14], measures (2) satisfy consistency condition (2′) if and only if the equation

$$h_x = \sum_{y \in S(x)} F(h_y, \theta, \alpha) \tag{3}$$

is satisfied for any $x \in V$, where $F\colon h = (h_1, \dots, h_{q-1}) \in \mathbb{R}^{q-1} \to F(h, \theta, \alpha) = (F_1, \dots, F_{q-1}) \in \mathbb{R}^{q-1}$ is defined as

$$F_i = \alpha\beta\delta_{1i} + \log \frac{(\theta-1)e^{h_i} + \sum_{j=1}^{q-1} e^{h_j} + 1}{\theta + \sum_{j=1}^{q-1} e^{h_j}},$$

$\theta = e^{J\beta}$, $S(x)$ is the set of direct descendants of x, and $h_x = (h_{1,x}, \dots, h_{q-1,x})$ under the condition that $h_{i,x} = \tilde{h}_{i,x} - \tilde{h}_{q,x}$, $i = 1, \dots, q-1$.

It is known that there exists a one-to-one correspondence between the set of V vertices of the Cayley tree of order $k \geq 1$ and the group G_k that is the free product of $k+1$ cyclic groups of second order with the corresponding generators $a_1, a_2, \dots, a_{k+1}$.

We assume that $\widehat{G}_k$ is the normal divisor of the finite index of the group G_k.

Definition 1. A set of vectors $h = \{h_x,\ x \in G_k\}$ is said to be $\widehat{G}_k$-periodic if $h_{yx} = h_x$ for any $x \in G_k$ and $y \in \widehat{G}_k$.

A $\widehat{G}_k$-periodic set is said to be translation-invariant.

Definition 2. A measure μ is said to be $\widehat{G}_k$-periodic if it corresponds to a $\widehat{G}_k$-periodic set of vectors h.

The following theorem was proved in [8].

Theorem 1 [8]**.** *Let H be the normal divisor of the finite index in G_k. Then for the Potts model, all H-periodic Gibbs measures are either $G_k^{(2)}$-periodic or translation-invariant, where $G_k^{(2)}$ is the subgroup comprising words of even length.*

3. Antiferromagnetic case

We consider the case $q \geq 3$, $\alpha = 0$, i.e., $\sigma\colon V \to \Phi = \{1, 2, \dots, q\}$. By Theorem 1, the only $G_k^{(2)}$-periodic Gibbs measures correspond to the set of vectors $h = \{h_x \in \mathbb{R}^{q-1} \colon x \in G_k\}$ of the form

$$h_x = \begin{cases} h & \text{for even } |x|, \\ l & \text{for odd } |x|. \end{cases}$$

Here, $h = (h_1, h_2, \dots, h_{q-1})$ and $l = (l_1, l_2, \dots, l_{q-1})$. By virtue of Eq. (3), we then have

$$h_i = k \log \frac{(\theta-1)e^{l_i} + \sum_{j=1}^{q-1} e^{l_j} + 1}{\sum_{j=1}^{q-1} e^{l_j} + \theta}, \qquad l_i = k \log \frac{(\theta-1)e^{h_i} + \sum_{j=1}^{q-1} e^{h_j} + 1}{\sum_{j=1}^{q-1} e^{h_j} + \theta}, \quad i = \overline{1, q-1}.$$

We introduce the notation $e^{h_i} = x_i$ and $e^{l_i} = y_i$. We can then rewrite the last system of equations for $i = \overline{1, q-1}$ as

$$x_i = \left(\frac{(\theta-1)y_i + \sum_{j=1}^{q-1} y_j + 1}{\sum_{j=1}^{q-1} y_j + \theta} \right)^k, \qquad y_i = \left(\frac{(\theta-1)x_i + \sum_{j=1}^{q-1} x_j + 1}{\sum_{j=1}^{q-1} x_j + \theta} \right)^k. \tag{4}$$

We consider the map $W\colon \mathbb{R}^{q-1}\times\mathbb{R}^{q-1}\to\mathbb{R}^{q-1}\times\mathbb{R}^{q-1}$ defined by the equations

$$x_i'=\left(\frac{(\theta-1)y_i+\sum_{j=1}^{q-1}y_j+1}{\sum_{j=1}^{q-1}y_j+\theta}\right)^k,\qquad y_i'=\left(\frac{(\theta-1)x_i+\sum_{j=1}^{q-1}x_j+1}{\sum_{j=1}^{q-1}x_j+\theta}\right)^k. \tag{5}$$

We note that system (4) is the equation $z=W(z)$. To solve system of equations (4), we must find the fixed points of (5): $z'=W(z)$, where $z=(x_1,\ldots,x_{q-1},y_1,\ldots,y_{q-1})$.

Lemma 1. *The following sets are invariant with respect to the map W:*

$$I_1=\{z\in\mathbb{R}^{2q-2}\colon x_1=x_2=\cdots=x_{q-1}=y_1=y_2=\cdots=y_{q-1}\},$$

$$I_2=\{z\in\mathbb{R}^{2q-2}\colon x_1=x_2=\cdots=x_{q-1},\ y_1=y_2=\cdots=y_{q-1}\},$$

$$I_3=\{z\in\mathbb{R}^{2q-2}\colon x_i=y_i,\ i=1,2,\ldots,q-1\},$$

$$I_4=\{z\in\mathbb{R}^{2q-2}\colon x_i=y_{q-i},\ i=1,2,\ldots,q-1\},$$

$$I_5=\{z\in\mathbb{R}^{2q-2}\colon x_1=y_1=1\},\qquad I_6=\{z\in\mathbb{R}^{2q-2}\colon x_{q-1}=y_{q-1}=1\}.$$

The lemma is proved similarly to Lemma 2 in [8].

Remark 1. The map W can have invariant sets that differ from the sets $I_1,\ldots,I_6$, i.e., the sets $I_1,\ldots,I_6$ do not fully describe all invariant sets of W.

Lemma 2. *The Gibbs measures for the Potts model on the invariant sets I_1 and I_3 are translation-invariant.*

The proof is obvious because we have $h_x=\text{const}$ on the invariant sets I_1 and I_3.

Remark 2. 1. For $q=2$, the Potts model coincides with the Ising model. The Ising case was first rigorously analyzed in [15].

2. In the case $k=2$, $q=3$, $J<0$, and $\alpha=0$, it was proved that all $G_k^{(2)}$-periodic Gibbs measures on the invariant sets $I_1,\ldots,I_6$ are translation-invariant (see [8]).

We have the following theorem.

Theorem 2. *Let $k=2$, $q\ge 2$, $J<0$, and $\alpha=0$. Then the $G_k^{(2)}$-periodic Gibbs measure for the Potts model is unique. Moreover, this measure coincides with the unique translation-invariant Gibbs measure.*

Proof. We note that $x_i=y_i=1$, $i=1,2,\ldots,q-1$, is a solution of system of equations (4), which is composed of $2q-2$ equations. We show that system (4) has no other solutions. For this, we substitute the expressions for y_i in the right-hand sides of the first $q-1$ equations in (4). We then obtain the equalities

$$\begin{aligned}
&\sqrt{x_1}=\frac{\theta\tilde{x}_1+\tilde{x}_2+\cdots+\tilde{x}_{q-1}+\gamma}{\tilde{x}_1+\tilde{x}_2+\cdots+\tilde{x}_{q-1}+\theta\gamma},\qquad \sqrt{x_2}=\frac{\theta\tilde{x}_2+\tilde{x}_1+\cdots+\tilde{x}_{q-1}+\gamma}{\tilde{x}_1+\tilde{x}_2+\cdots+\tilde{x}_{q-1}+\theta\gamma},\\
&\sqrt{x_3}=\frac{\theta\tilde{x}_3+\tilde{x}_1+\tilde{x}_2+\cdots+\tilde{x}_{q-1}+\gamma}{\tilde{x}_1+\tilde{x}_2+\cdots+\tilde{x}_{q-1}+\theta\gamma},\qquad \ldots,\\
&\sqrt{x_{q-1}}=\frac{\theta\tilde{x}_{q-1}+\tilde{x}_1+\tilde{x}_2+\cdots+\tilde{x}_{q-2}+\gamma}{\tilde{x}_1+\tilde{x}_2+\cdots+\tilde{x}_{q-1}+\theta\gamma},
\end{aligned} \tag{6}$$

where

$$\tilde{x}_1 = (\theta x_1 + x_2 + \cdots + x_{q-1} + 1)^2, \qquad \tilde{x}_2 = (\theta x_2 + x_1 + \cdots + x_{q-1} + 1)^2,$$

$$\tilde{x}_3 = (\theta x_3 + x_1 + x_2 + \cdots + x_{q-1} + 1)^2, \qquad \ldots,$$

$$\tilde{x}_{q-1} = (\theta x_{q-1} + x_1 + x_2 + \cdots + x_{q-2} + 1)^2,$$

$$\gamma = (x_1 + x_2 + \cdots + x_{q-1} + \theta)^2.$$

We subtract unity from both sides of all equalities in (6):

$$\sqrt{x_1} - 1 = \frac{(\theta-1)(\tilde{x}_1 - \gamma)}{\tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_{q-1} + \theta\gamma},$$

$$\sqrt{x_2} - 1 = \frac{(\theta-1)(\tilde{x}_2 - \gamma)}{\tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_{q-1} + \theta\gamma},$$

$$\sqrt{x_3} - 1 = \frac{(\theta-1)(\tilde{x}_3 - \gamma)}{\tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_{q-1} + \theta\gamma}, \qquad \ldots,$$

$$\sqrt{x_{q-1}} - 1 = \frac{(\theta-1)(\tilde{x}_{q-1} - \gamma)}{\tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_{q-1} + \theta\gamma}.$$

We introduce the notation $L = (\theta-1)/(\tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_{q-1} + \theta\gamma)$ and rewrite the last system of equations as

$$\sqrt{x_i} - 1 = L(\tilde{x}_i - \gamma), \quad i = 1, 2, \ldots, q-1. \tag{7}$$

We calculate the differences $\tilde{x}_i - \gamma$, $i = 1, 2, \ldots, q-1$,

$$\tilde{x}_i - \gamma = (\theta-1)(x_i - 1)\left[(\theta+1)(x_i+1) + 2\left(\sum_{i=1}^{q-1} x_i - x_i\right)\right],$$

and substitute them in (7). After some transformation, we then have

$$(\sqrt{x_i} - 1)\left[1 - L(\theta-1)(\sqrt{x_i}+1)\left((\theta+1)(x_i+1) + 2\left(\sum_{i=1}^{q-1} x_i - x_i\right)\right)\right] = 0.$$

Hence, for $i = 1, 2, \ldots, q-1$, we have $x_i = 1$ or

$$1 - L(\theta-1)(\sqrt{x_i}+1)\left[(\theta+1)(x_i+1) + 2\left(\sum_{i=1}^{q-1} x_i - x_i\right)\right] = 0. \tag{8}$$

We note that the solution $x_i = 1$, $i = 1, 2, \ldots, q-1$, corresponds to a translation-invariant Gibbs measure. Therefore, we consider the case where $x_i \neq 1$. We rewrite system of equations (8) as

$$\frac{1}{L} = (\theta-1)(\sqrt{x_i}+1)\left[(\theta+1)(x_i+1) + 2\left(\sum_{i=1}^{q-1} x_i - x_i\right)\right], \quad i = 1, 2, \ldots, q-1.$$

For $i \neq j$, $i, j = 1, 2, \ldots, q-1$, we then have

$$(\theta+1)(\sqrt{x_i}+1)(x_i+1) + 2(\sqrt{x_i}+1)\left(\sum_{i=1}^{q-1} x_i - x_i\right) =$$

$$= (\theta+1)(\sqrt{x_j}+1)(x_j+1) + 2(\sqrt{x_j}+1)\left(\sum_{i=1}^{q-1} x_i - x_j\right).$$

After some transformation, we have the equality

$$(\sqrt{x_i}-\sqrt{x_j}\,)\left[(\theta+1)(x_i+x_j+1)+(\theta-1)(\sqrt{x_i}+\sqrt{x_j}+\sqrt{x_ix_j}\,)+2\left(\sum_{i=1}^{q-1}x_i-x_i-x_j\right)\right]=0.$$

Hence, $x_i=x_j$ or

$$(\theta+1)(x_i+x_j+1)+(\theta-1)(\sqrt{x_i}+\sqrt{x_j}+\sqrt{x_ix_j}\,)+2\left(\sum_{i=1}^{q-1}x_i-x_i-x_j\right)=0. \tag{9}$$

In the case $x_i=x_j$, we have the solution $x_i=x_j=1$, which corresponds to the translation-invariant Gibbs measure. Let $x_i\neq x_j$. Then it follows from (9) that

$$\theta(x_i+x_j+1+\sqrt{x_i}+\sqrt{x_j}+\sqrt{x_ix_j}\,)+2(x_1+x_2+\cdots+x_{q-1}-x_i-x_j)+$$
$$+x_i+x_j+1-(\sqrt{x_i}+\sqrt{x_j}+\sqrt{x_ix_j}\,)=0. \tag{10}$$

We prove that Eq. (10) has no solutions. For this, it suffices to prove that the inequality $x_i+x_j+1>\sqrt{x_i}+\sqrt{x_j}+\sqrt{x_ix_j}$ holds. Introducing notation $\sqrt{x_i}=s$ and $\sqrt{x_j}=t$, we obtain a quadratic inequality in s, $s^2-(t+1)s+t^2-t+1>0$, and its discriminant $D=-3(t-1)^2<0$ is negative for $t\neq 1$. Therefore, system of equations (4) has solutions only of a form $x_i=x_j$, i.e., $z=(x_1,\ldots,x_{q-1},y_1,\ldots,y_{q-1})\in I_1$. Therefore, all $G_k^{(2)}$-periodic Gibbs measures are translation-invariant, and uniqueness of the $G_k^{(2)}$-periodic Gibbs measure follows from the uniqueness of the translation-invariant Gibbs measure for the antiferromagnetic Potts model (see [4]). The theorem is proved.

Remark 3. For the antiferromagnetic Potts model, the unique translation-invariant Gibbs measure is associated with a solution of system of equations (4) like $x_i=y_i=1$, $i=1,2,\ldots,q-1$.

4. Ferromagnetic case

The following theorem was proved in [8].

Theorem 3 [8]**.** *For the Potts model with a zero external field for $k\geq 1$, $q=3$, and $J>0$, all $G_k^{(2)}$-periodic Gibbs measures are translation-invariant.*

The following theorem generalizes the statement of Theorem 3.

Theorem 4. *Let $k\geq 2$, $q\geq 3$, $J>0$, and $\alpha=0$. Then all $G_k^{(2)}$-periodic Gibbs measures for the Potts model are translation-invariant.*

Proof. We consider the differences x_i-y_i, $i=1,2,\ldots,q-1$, in system of equations (4):

$$x_i-y_i=\left[\frac{(\theta-1)y_i+\sum_{j=1}^{q-1}y_j+1}{\sum_{j=1}^{q-1}y_j+\theta}-\frac{(\theta-1)x_i+\sum_{j=1}^{q-1}x_j+1}{\sum_{j=1}^{q-1}x_j+\theta}\right]A_i=$$
$$=\frac{A_i}{XY}\left[(\theta^2-\theta)(y_i-x_i)+(\theta-1)\left(y_i\sum_{j=1}^{q-1}x_j-x_i\sum_{j=1}^{q-1}y_j\right)+(\theta-1)\left(\sum_{j=1}^{q-1}y_j-\sum_{j=1}^{q-1}x_j\right)\right],$$

where

$$A_i = \left(\frac{(\theta-1)y_i + \sum_{j=1}^{q-1} y_j + 1}{\sum_{j=1}^{q-1} y_j + \theta}\right)^{k-1} + \cdots + \left(\frac{(\theta-1)x_i + \sum_{j=1}^{q-1} x_j + 1}{\sum_{j=1}^{q-1} x_j + \theta}\right)^{k-1},$$

$$X = \sum_{j=1}^{q-1} x_j + \theta, \qquad Y = \sum_{j=1}^{q-1} y_j + \theta, \quad i = \overline{1, q-1}.$$

After some transformations, we obtain the expression

$$x_i - y_i = \frac{A_i(\theta-1)}{XY}\left(\theta(y_i - x_i) + y_i \sum_{j=1}^{q-1}(x_j - y_j) + (y_i - x_i)\sum_{j=1}^{q-1} y_j + \sum_{j=1}^{q-1} y_j - \sum_{j=1}^{q-1} x_j\right).$$

As a result, we have the system of equations

$$\begin{aligned}
&a_{11}(x_1 - y_1) + a_{12}(x_2 - y_2) + \cdots + a_{1q-1}(x_{q-1} - y_{q-1}) = 0,\\
&a_{21}(x_1 - y_1) + a_{22}(x_2 - y_2) + \cdots + a_{2q-1}(x_{q-1} - y_{q-1}) = 0,\\
&\qquad\vdots\\
&a_{q-11}(x_1 - y_1) + a_{q-12}(x_2 - y_2) + \cdots + a_{q-1q-1}(x_{q-1} - y_{q-1}) = 0,
\end{aligned} \tag{11}$$

where

$$a_{ii} = 1 + \frac{A_i(\theta-1)(\theta + 1 + \sum_{j=1}^{q-1} y_j - y_i)}{XY}, \qquad a_{il} = \frac{(\theta-1)(1-y_i)A_i}{XY}, \quad i \neq l, \quad i, l = \overline{1, q-1}.$$

This system is known to have a zero solution if the determinant

$$\det \mathbf{A} = \begin{vmatrix}
1 + \dfrac{A_1(\theta-1)(1+Y-y_1)}{XY} & \dfrac{(\theta-1)(1-y_1)A_1}{XY} & \cdots & \dfrac{(\theta-1)(1-y_1)A_1}{XY}\\
\dfrac{(\theta-1)(1-y_2)A_2}{XY} & 1 + \dfrac{A_2(\theta-1)(1+Y-y_2)}{XY} & \ddots & \vdots\\
\vdots & \ddots & \ddots & \dfrac{(\theta-1)(1-y_{q-1})A_{q-1}}{XY}\\
\dfrac{(\theta-1)(1-y_{q-1})A_{q-1}}{XY} & \dfrac{(\theta-1)(1-y_{q-1})A_{q-1}}{XY} & \cdots & 1 + \dfrac{A_{q-1}(\theta-1)(1+Y-y_{q-1})}{XY}
\end{vmatrix}$$

is nonzero where $\mathbf{A}$ is the matrix of the given system. We rewrite the determinant of $\mathbf{A}$:

$$\det \mathbf{A} = C\begin{vmatrix}
1 + B_1 & 1 & \cdots & 1\\
1 & 1 + B_2 & \ddots & \vdots\\
\vdots & \ddots & \ddots & 1\\
1 & \cdots & 1 & 1 + B_{q-1}
\end{vmatrix},$$

where

$$C = \frac{A_1(\theta-1)(1-y_1)}{XY}\,\frac{A_2(\theta-1)(1-y_2)}{XY}\cdots\frac{A_{q-1}(\theta-1)(1-y_{q-1})}{XY},$$

$$B_i = \frac{XY + A_i(\theta-1)Y}{A_i(\theta-1)(1-y_i)}.$$

We show that $\det \mathbf{A} \neq 0$. For this, we use the following lemma.

Lemma 3 [16]**.** *The determinant*

$$\begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 1+a_n \end{vmatrix} = a_1 a_2 \cdots a_n \left(1 + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

Using Lemma 3, we calculate the determinant of A:

$$\det \mathbf{A} = \frac{1}{X^{q-1}Y^{q-1}} (XY + A_1(\theta-1)Y)(XY + A_2(\theta-1)Y) \times \cdots \times$$
$$\times (XY + A_{q-1}(\theta-1)Y)\left(1 + \frac{A_1(\theta-1)(1-y_1)}{XY + A_1(\theta-1)Y} + \right.$$
$$\left. + \frac{A_2(\theta-1)(1-y_2)}{XY + A_2(\theta-1)Y} + \cdots + \frac{A_{q-1}(\theta-1)(1-y_{q-1})}{XY + A_{q-1}(\theta-1)Y}\right).$$

Expanding, we obtain the equality $\det \mathbf{A} = P/X^{q-1}Y^{q-1}$, where

$$P = (XY + A_1(\theta-1)Y)(XY + A_2(\theta-1)Y) \cdots (XY + A_{q-1}(\theta-1)Y) +$$
$$+ (XY + A_2(\theta-1)Y)(XY + A_3(\theta-1)Y) \cdots (XY + A_{q-1}(\theta-1)Y) \times$$
$$\times A_1(\theta-1)(1-y_1) + (XY + A_1(\theta-1)Y)(XY + A_3(\theta-1)Y) \times \cdots \times$$
$$\times (XY + A_{q-1}(\theta-1)Y)A_2(\theta-1)(1-y_2) + \cdots + (XY + A_1(\theta-1)Y) \times$$
$$\times (XY + A_2(\theta-1)Y) \cdots (XY + A_{q-2}(\theta-1)Y)A_{q-1}(\theta-1)(1-y_{q-1}).$$

We then group the expression for P into powers of (XY):

$$P = (XY)^{q-1} + (XY)^{q-2}(\theta-1) \times$$
$$\times \left[Y \sum_{j=1}^{q-1} A_j + (1-y_1)A_1 + (1-y_2)A_2 + \cdots + (1-y_n)A_n\right] + (XY)^{q-3}(\theta-1)^2 Y \times$$
$$\times \left[Y(A_1A_2 + \cdots + A_{q-2}A_{q-1}) + \sum_{i=1}^{q-1}\left(\sum_{j=1}^{q-1} A_j - A_i\right)A_i(1-y_i)\right] + \cdots +$$
$$+ (XY)^{q-1-i}(\theta-1)^i Y^{i-1}[Y(A_1A_2 \cdots A_i + \cdots + A_{q-i}A_{q-i+1} \ldots A_{q-1}) +$$
$$+ (A_2A_3 \ldots A_i + \cdots + A_{q-i+1} \cdots A_{q-1})A_1(1-y_1) +$$
$$+ (A_1A_3 \ldots A_i + \cdots + A_{q-i+1} \cdots A_{q-1})A_2(1-y_2) + \cdots +$$
$$+ (A_1A_2 \cdots A_{i-1} + \cdots + A_{q-i}A_{q-i+1} \cdots A_{q-2})A_{q-1}(1-y_{q-1})] + \cdots +$$
$$+ (\theta-1)^{q-1}Y^{q-2}[Y(A_1A_2 \cdots A_{q-1}) + A_1A_2 \cdots A_{q-1}(1-y_1) + \cdots +$$
$$+ A_1A_2 \cdots A_{q-1}(1-y_{q-1})].$$

We show that $P > 0$. With the relations $\theta > 1$, $X > 0$, $Y > 0$, and $A_i > 0$ taken into account, it suffices to show that the expressions in square brackets are positive. Indeed, after some transformations, the expression in the first square brackets is

$$Y\sum_{j=1}^{q-1} A_j + (1-y_1)A_1 + (1-y_2)A_2 + \cdots + (1-y_n)A_n = (\theta+1)\sum_{j=1}^{q-1} A_j + \sum_{j\neq k} y_j A_k > 0,$$

the expression in the second square brackets is

$$Y(A_1A_2 + \cdots + A_{q-2}A_{q-1}) + \biggl(\sum_{j=1}^{q-1} A_j - A_1\biggr)A_1(1-y_1) + \cdots + \biggl(\sum_{j=1}^{q-1} A_j - A_{q-1}\biggr)A_{q-1}(1-y_{q-1}) =$$

$$= (\theta+1)(A_1A_2 + \cdots + A_{q-2}A_{q-1}) + y_1 \sum_{j\neq k\neq 1} A_jA_k + \cdots + y_{q-1} \sum_{j\neq k\neq q-1} A_jA_k > 0,$$

and the expression in the ith square bracket is

$$\begin{aligned} &Y(A_1A_2\cdots A_i + \cdots + A_{q-i}A_{q-i+1}\cdots A_{q-1}) + \\ &\quad + (A_2A_3\cdots A_i + \cdots + A_{q-i+1}\cdots A_{q-1})A_1(1-y_1) + \\ &\quad + (A_1A_3\cdots A_i + \cdots + A_{q-i+1}\cdots A_{q-1})A_2(1-y_2) + \cdots + \\ &\quad + (A_1A_2\cdots A_{i-1} + \cdots + A_{q-i}A_{q-i+1}\cdots A_{q-2})A_{q-1}(1-y_{q-1})) = \\ &= (\theta+1)(A_1\cdots A_i + \cdots + A_{q-i}\cdots A_{q-1}) + y_1(A_2\cdots A_{i+1} + \cdots + A_{q-i}\cdots A_{q-1}) + \cdots + \\ &\qquad + y_{q-1}(A_1\cdots A_i + \cdots + A_{q-i-1}\cdots A_{q-2}) > 0. \end{aligned}$$

Finally, the expression in the last square brackets is

$$YA_1A_2\cdots A_{q-1} + A_1A_2\cdots A_{q-1}(1-y_1) + \cdots + A_1A_2\cdots A_{q-1}(1-y_{q-1}) =$$

$$= A_1A_2\cdots A_{q-1}(\theta+1) > 0.$$

Hence, $P > 0$. Therefore, system of equations (11) has solutions only of the form $x_i = y_i$, i.e., $z = (x_1, \ldots, x_{q-1}, y_1, \ldots, y_{q-1}) \in I_3$. According to Lemma 2, we have thus obtained the sought result. The theorem is proved.

Remark 4. For the ferromagnetic Potts model, not more than $2^q - 1$ translation-invariant Gibbs measures exist (see [11]).

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