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ALGEBRO-GEOMETRIC INTEGRATION OF THE MODIFIED BELOV–CHALTIKIAN LATTICE HIERARCHY

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Using the Lenard recurrence relations and the zero-curvature equation, we derive the modified Belov–Chaltikian lattice hierarchy associated with a discrete 3×3 matrix spectral problem. Using the characteristic polynomial of the Lax matrix for the hierarchy, we introduce a trigonal curve $\mathcal{K}_{m-2}$ of arithmetic genus $m-2$. We study the asymptotic properties of the Baker–Akhiezer function and the algebraic function carrying the data of the divisor near P_{∞_1}, P_{∞_2}, P_{∞_3}, and P_0 on $\mathcal{K}_{m-2}$. Based on the theory of trigonal curves, we obtain the explicit theta-function representations of the algebraic function, the Baker–Akhiezer function, and, in particular, solutions of the entire modified Belov–Chaltikian lattice hierarchy.

Keywords: modified Belov–Chaltikian lattice hierarchy, trigonal curve, quasiperiodic solution

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1. Introduction

Differential–difference equations have been at the center of considerable research activity, to a great extent because of their burgeoning relevance in various applications. For instance, the Toda lattice models the motions of a sequence of identical particles that interact with their nearest neighbors through exponential forces [1], [2], the Kac–van Moerbeke lattice describes the population dynamics of competing species in biology, a model in plasma physics is encountered when studying the collapse of Langmuir waves [3], [4], the nonlinear self-dual network equation describes the propagation of electrical signals in a cascade of four-terminal nonlinear LC self-dual circuits, and so forth [5]–[7]. On the other hand, this class of differential–difference equations has various beautiful algebraic and geometric properties, such as Lax pairs, the Painlevé property, N-soliton solutions, bi-Hamiltonian structures, infinite conservation laws, general symmetries, and a prolongation structure, to name a few [8].

The importance of seeking quasiperiodic solutions of discrete soliton equations is well known. In addition to being interesting in themselves, quasiperiodic solutions are also reducible to multisoliton and other solutions. Over the past decades, a fairly satisfactory understanding has been obtained for quasiperiodic solutions of soliton equations associated with 2×2 matrix spectral problems, including the continuous and discrete cases [9]–[22]. But we encounter great difficulty in trying to study quasiperiodic solutions of soliton equations related to 3×3 matrix spectral problems because it involves the theory of trigonal curves [23]–[32], not the hyperelliptic curves for the 2×2 problems. Nevertheless, certain quasiperiodic solutions of

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the Boussinesq equation related to a third-order differential operator were found as special solutions of the Kadomtsev–Peviashvili equation or by the reduction theory of the Riemann theta function [23]–[28]. A unified framework proposed in [33], [34] yields all quasiperiodic solutions of the entire Boussinesq hierarchy associated with a third-order differential operator. In [35], a general method was developed for introducing a trigonal curve using the characteristic polynomial of the Lax matrix. Based on that, we successfully obtained quasiperiodic solutions for the modified Boussinesq, Kaup–Kupershmidt, and coupled mKdV hierarchies associated with continuous 3×3 matrix spectral problems [35]–[37].

Our main aim here is to show that the above approaches apply equally well to the discrete case. For most continuous cases, it suffices to consider only the infinite points on the corresponding three-sheeted Riemann surface, but when the research focus shifts from partial differential equations to differential–difference equations, a major problem in this direction is to analyze not only the infinite points but also the zero points on the associated three-sheeted Riemann surface. They are equally important in the discrete case. We must give detailed asymptotic expansions of the meromorphic function and the Baker–Akhiezer function near the infinite and zero points. On the other hand, the Riemann theta-function representation of the Baker–Akhiezer function depends on the second Abel differentials and also the Abel differentials of the third kind because of the discrete variable.

Starting from a discrete 3×3 matrix spectral problem, we here derive quasiperiodic solutions of the entire modified Belov–Chaltikian lattice hierarchy based on the theory of trigonal curves. The first nontrivial member in the hierarchy is

$$u_{n,t} = \frac{1}{v_{n+1}} - \frac{1}{v_{n-2}}, \qquad v_{n,t} = \frac{u_{n+1}}{v_{n+1}} - \frac{u_n}{v_{n-1}}, \tag{1.1}$$

where $u_n = u(n,t)$ and $v_n = v(n,t)$ are functions of the discrete variable $n \in \mathbb{Z}$ and the continuous variable $t \in \mathbb{R}$.

This paper has the following structure. In Sec. 2, we give details about the construction of the modified Belov–Chaltikian lattice hierarchy related to a discrete 3×3 matrix spectral problem with two potentials. In Sec. 3, we introduce the Baker–Akhiezer function, an algebraic function carrying the data of the divisor, and the trigonal curve $\mathcal{K}_{m-2}$ of arithmetic genus $m-2$ using the characteristic polynomial of the Lax matrix for the hierarchy. We then derive Dubrovin-type equations. In Sec. 4, we obtain explicit theta-function representations of the Baker–Akhiezer function, the algebraic function, and, in particular, solutions of the entire modified Belov–Chaltikian lattice hierarchy. Using essential properties of the algebraic function ϕ carrying the data of the divisor and the Baker–Akhiezer function ψ_3 near the three infinite points P_{∞_s}, $s = 1, 2, 3$ and the triple point P_0, we derive their Riemann theta-function representations. The Riemann–Jacobi inverse problem is obtained by comparing the asymptotic expansion of the meromorphic function and its Riemann theta-function representation, whence we obtain quasiperiodic solutions of the entire modified Belov–Chaltikian lattice hierarchy by virtue of the Riemann theta functions.

2. The modified Belov–Chaltikian lattice hierarchy

Throughout this paper, we assume the following hypothesis.

Hypothesis. *The functions u and v satisfy $u(\,\cdot\,,t), v(\,\cdot\,,t) \in \mathbb{C}^{\mathbb{Z}}$, $t \in \mathbb{R}$, $u(n,\,\cdot\,), v(n,\,\cdot\,) \in C^1(\mathbb{R})$, $n \in \mathbb{Z}$, $v(n,t) \neq 0$, and $(n,t) \in \mathbb{Z} \times \mathbb{R}$, where $\mathbb{C}^{\mathbb{Z}}$ denotes the set of all complex-valued sequences indexed by $\mathbb{Z}$.*

The shift operators and the difference operator are defined by

$$Ef(n) = f(n+1), \qquad E^{-1}f(n) = f(n-1), \qquad \Delta f(n) = (E-1)f(n), \quad n \in \mathbb{Z}.$$

For convenience, we usually write $f(n)=f$, $E^{\pm}f=f^{\pm}$, and $f(n+k)=E^k f$, $n,k\in\mathbb{Z}$. We consider the discrete 3×3 matrix spectral problem

$$E\psi = U\psi, \qquad \psi=\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}, \qquad U=\begin{pmatrix}0 & \lambda & 0\\ u & \lambda v & 1\\ 1 & 0 & 0\end{pmatrix}, \tag{2.1}$$

where $u=u(n,t)$ and $v=v(n,t)$ are two potentials and λ is a constant spectral parameter. For this, we introduce two sets of Lenard recurrence relations:

$$K\hat{g}_j = J\hat{g}_{j+1}, \qquad \hat{g}_j=(\hat{a}_j,\hat{b}_j,\hat{c}_j)^{\mathrm{T}}, \quad j\geq 0, \tag{2.2}$$

$$K\check{g}_j = J\check{g}_{j+1}, \qquad \check{g}_j=(\check{a}_j,\check{b}_j,\check{c}_j)^{\mathrm{T}}, \quad j\geq 0, \tag{2.3}$$

with the starting points

$$\hat{g}_0=\begin{pmatrix}0\\ 1/v^-\\ 0\end{pmatrix}, \qquad \check{g}_0=\begin{pmatrix}\check{a}_0\\ -\check{c}_0/v^-\\ \check{c}_0\end{pmatrix}, \tag{2.4}$$

where $\check{a}_0$ and $\check{c}_0$ satisfy $v^-\check{a}_0\check{a}_0^+=\check{c}_0^2-1$ and $\check{c}_0^+=u\check{a}_0^+-\check{c}_0$ with $\check{a}_0\neq 0$ and the two difference operators K and J are defined as

$$\begin{aligned} K&=\begin{pmatrix}0 & E^2-E^{-1} & 0\\ E^3-1 & EuE-u & 0\\ E^{-1}uE-uE & EvE-vE & E^2-E^{-1}\end{pmatrix},\\ J&=\begin{pmatrix}vE^2-E^{-1}vE & u(vE-EvE) & u(1-E^2)\\ 0 & v(vE-EvE) & v(E-E^2)\\ E^{-1}uE-uE & EvE-vE & E^2-E^{-1}\end{pmatrix}. \end{aligned} \tag{2.5}$$

The respective recurrence relations (2.2) and (2.3) then determine $\hat{g}_j$ and $\check{g}_j$ uniquely up to a term in $\operatorname{Ker} J$, which we always assume to be zero. For example,

$$\hat{g}_1=\left(\frac{1}{v^-v^{--}}, -\frac{u}{v(v^-)^2}-\frac{u^-}{(v^-)^2v^{--}}, \frac{u}{vv^-}\right)^{\mathrm{T}}.$$

To generate a hierarchy of nonlinear evolution equations associated with spectral problem (2.1), we solve the stationary zero-curvature equation

$$(EV)U-UV=0, \qquad V=\begin{pmatrix}\lambda V_{11} & \lambda V_{12} & \lambda V_{13}\\ V_{21} & \lambda V_{22} & V_{23}\\ V_{31} & \lambda V_{32} & \lambda V_{33}\end{pmatrix}, \tag{2.6}$$

which is equivalent to

$$
\begin{aligned}
&uV_{12}^{+}+V_{13}^{+}-V_{21}=0,\\
&V_{11}^{+}+vV_{12}^{+}-V_{22}=0,\\
&V_{12}^{+}-V_{23}=0,\\
&\lambda uV_{22}^{+}+V_{23}^{+}-\lambda uV_{11}-\lambda vV_{21}-V_{31}=0,\\
&V_{21}^{+}+\lambda vV_{22}^{+}-uV_{12}-\lambda vV_{22}-V_{32}=0,\\
&V_{22}^{+}-uV_{13}-vV_{23}-V_{33}=0,\\
&uV_{32}^{+}+V_{33}^{+}-V_{11}=0,\\
&V_{31}^{+}+\lambda vV_{32}^{+}-V_{12}=0,\\
&V_{32}^{+}-V_{13}=0.
\end{aligned}
\tag{2.7}
$$

We now define each element V_{ij} of the 3×3 matrix V as

$$
\begin{aligned}
&V_{11}=c, && V_{12}=b, && V_{13}=a^{+},\\
&V_{21}=a^{++}+ub^{+}, && V_{22}=vb^{+}+c^{+}, && V_{23}=b^{+},\\
&V_{31}=b^{-}-\lambda v^{-}a, && V_{32}=a, && V_{33}=c^{-}-u^{-}a.
\end{aligned}
\tag{2.8}
$$

Substituting (2.8) in (2.7) yields the Lenard equation

$$KG=\lambda JG,\qquad G=(a,b,c)^{\mathrm{T}}. \tag{2.9}$$

Expanding a, b, and c into Laurent polynomials in λ, we obtain

$$a=\sum_{j\geq0}a_j\lambda^{-j},\qquad b=\sum_{j\geq0}b_j\lambda^{-j},\qquad c=\sum_{j\geq0}c_j\lambda^{-j}. \tag{2.10}$$

Equation (2.9) is equivalent to the recurrence relation

$$KG_j=JG_{j+1},\qquad JG_0=0,\quad j\geq0, \tag{2.11}$$

where $G_j=(a_j,b_j,c_j)^{\mathrm{T}}$. Because the equation $JG_0=0$ has a solution

$$G_0=\alpha_0\hat{g}_0+\beta_0\check{g}_0, \tag{2.12}$$

we can express G_j as

$$G_j=\alpha_0\hat{g}_j+\beta_0\check{g}_j+\cdots+\alpha_j\hat{g}_0+\beta_j\check{g}_0,\quad j\geq0, \tag{2.13}$$

where α_j and β_j are arbitrary constants.

Let ψ satisfy discrete spectral problem (2.1) and the auxiliary problem

$$\psi_{t_r}=\widetilde{V}^{(r)}\psi,\qquad \widetilde{V}^{(r)}=\begin{pmatrix}\lambda\widetilde{V}_{11}^{(r)} & \lambda\widetilde{V}_{12}^{(r)} & \lambda\widetilde{V}_{13}^{(r)}\\ \widetilde{V}_{21}^{(r)} & \lambda\widetilde{V}_{22}^{(r)} & \widetilde{V}_{23}^{(r)}\\ \widetilde{V}_{31}^{(r)} & \lambda\widetilde{V}_{32}^{(r)} & \lambda\widetilde{V}_{33}^{(r)}\end{pmatrix}, \tag{2.14}$$

where each $\widetilde{V}_{ij}^{(r)} = V_{ij}(\tilde{a}^{(r)}, \tilde{b}^{(r)}, \tilde{c}^{(r)})$,

$$\tilde{a}^{(r)} = \sum_{j=0}^{r} \tilde{a}_j \lambda^{r-j}, \qquad \tilde{b}^{(r)} = \sum_{j=0}^{r} \tilde{b}_j \lambda^{r-j}, \qquad \tilde{c}^{(r)} = \sum_{j=0}^{r} \tilde{c}_j \lambda^{r-j}, \tag{2.15}$$

and $\widetilde{G}_j = (\tilde{a}_j, \tilde{b}_j, \tilde{c}_j)^{\mathrm{T}}$ is determined by

$$\widetilde{G}_j = \tilde{\alpha}_0 \hat{g}_j + \cdots + \tilde{\alpha}_j \hat{g}_0, \quad j \geq 0, \tag{2.16}$$

where $\{\tilde{\alpha}_j\}$ are constants chosen independent of $\{\alpha_j\}$. The compatibility condition for (2.1) and (2.14) then yields the zero-curvature equation $U_{t_r} - (E\widetilde{V}^{(r)})U + U\widetilde{V}^{(r)} = 0$, which is equivalent to a hierarchy of differential–difference equations,

$$(u_{t_r}, v_{t_r})^{\mathrm{T}} = X_r, \quad r \geq 0, \tag{2.17}$$

where the vector fields

$$X_j = X(u, v; \underline{\tilde{\alpha}}^{(j)}) = \mathcal{P}(K\widetilde{G}_j) = \mathcal{P}(J\widetilde{G}_{j+1}), \quad j \geq 0,$$

$\underline{\tilde{\alpha}}^{(j)} = (\tilde{\alpha}_0, \ldots, \tilde{\alpha}_j)$, and $\mathcal{P}$ is the projective map $(\gamma_1, \gamma_2, \gamma_3)^{\mathrm{T}} \to (\gamma_1, \gamma_2)^{\mathrm{T}}$. The first two nontrivial members in lattice hierarchy (2.17) are

$$u_{t_0} = \tilde{\alpha}_0 \left(\frac{1}{v^+} - \frac{1}{v^{--}} \right), \qquad v_{t_0} = \tilde{\alpha}_0 \left(\frac{u^+}{v^+} - \frac{u}{v^-} \right) \tag{2.18}$$

and

$$\begin{aligned} u_{t_1} &= \tilde{\alpha}_0 \left[-\frac{u^{++}}{v^{++}(v^+)^2} - \frac{u^+}{(v^+)^2 v} + \frac{u^-}{v^-(v^{--})^2} + \frac{u^{--}}{(v^{--})^2 v^{---}} \right] + \tilde{\alpha}_1 \left(\frac{1}{v^+} - \frac{1}{v^{--}} \right), \\ v_{t_1} &= \tilde{\alpha}_0 \left[\frac{1}{v^{++}v^+} - \frac{1}{v^- v^{--}} - \frac{u^{++}u^+}{v^{++}(v^+)^2} - \frac{(u^+)^2}{(v^+)^2 v} + \frac{u^2}{v(v^-)^2} + \frac{uu^-}{(v^-)^2 v^{--}} \right] + \tilde{\alpha}_1 \left(\frac{u^+}{v^+} - \frac{u}{v^-} \right). \end{aligned} \tag{2.19}$$

Especially for $\tilde{\alpha}_0 = 1$ and $t_0 = t$, Eq. (2.18) reduces to (1.1).

3. The Baker–Akhiezer function

In this section, we introduce the associated Baker–Akhiezer function. We then define a trigonal curve $\mathcal{K}_{m-2}$ of degree m using the characteristic polynomial of the Lax matrix, whence we derive the algebraic function carrying the data of the divisor. We decompose the modified Belov–Chaltikian lattice hierarchy into a system of solvable ordinary differential equations.

We now introduce the Baker–Akhiezer function $\psi(P, n, n_0, t_r, t_{0,r})$ by

$$\begin{aligned} &E\psi(P, n, n_0, t_r, t_{0,r}) = U(u(n, t_r), v(n, t_r); \lambda(P))\psi(P, n, n_0, t_r, t_{0,r}), \\ &\psi_{t_r}(P, n, n_0, t_r, t_{0,r}) = \widetilde{V}^{(r)}(u(n, t_r), v(n, t_r); \lambda(P))\psi(P, n, n_0, t_r, t_{0,r}), \\ &V^{(p)}(u(n, t_r), v(n, t_r); \lambda(P))\psi(P, n, n_0, t_r, t_{0,r}) = y(P)\psi(P, n, n_0, t_r, t_{0,r}), \\ &\psi_3(P, n_0, n_0, t_{0,r}, t_{0,r}) = 1, \quad (n, t_r) \in \mathbb{Z} \times \mathbb{R}, \end{aligned} \tag{3.1}$$

where $V^{(p)} = (\lambda^p V)_+$, $V_{ij}^{(p)} = V_{ij}(a^{(p)}, b^{(p)}, c^{(p)})$, and

$$a^{(p)} = \sum_{j=0}^{p} a_j \lambda^{p-j}, \qquad b^{(p)} = \sum_{j=0}^{p} b_j \lambda^{p-j}, \qquad c^{(p)} = \sum_{j=0}^{p} c_j \lambda^{p-j}. \tag{3.2}$$

The compatibility conditions for the first three expressions in (3.1) yield

$$U_{t_r} - (E\widetilde{V}^{(r)})U + U\widetilde{V}^{(r)} = 0, \tag{3.3}$$

$$-(EV^{(p)})U + UV^{(p)} = 0, \tag{3.4}$$

$$-V^{(p)}_{t_r} + [\widetilde{V}^{(r)}, V^{(p)}] = 0. \tag{3.5}$$

A direct calculation shows that $yI - V^{(p)}$ also satisfies Lax equations (3.4) and (3.5), which implies that the characteristic polynomial $\mathcal{F}_{m-2}(\lambda, y) = \det(yI - V^{(p)})$ of the Lax matrix $V^{(p)}$ is a constant independent of n and t_r, with the expansion [38]

$$\det(yI - V^{(p)}) = y^3 - y^2 R_m(\lambda) + yS_m(\lambda) - T_m(\lambda), \tag{3.6}$$

where $R_m(\lambda)$, $S_m(\lambda)$, and $T_m(\lambda)$ are polynomials with constant coefficients λ

$$\begin{aligned}
R_m(\lambda) &= \lambda V^{(p)}_{11} + \lambda V^{(p)}_{22} + \lambda V^{(p)}_{33} = \lambda(\alpha_0\lambda^p + \dots),\\
S_m(\lambda) &= \begin{vmatrix} \lambda V^{(p)}_{11} & \lambda V^{(p)}_{12} \\ V^{(p)}_{21} & \lambda V^{(p)}_{22} \end{vmatrix} + \begin{vmatrix} \lambda V^{(p)}_{11} & \lambda V^{(p)}_{13} \\ V^{(p)}_{31} & \lambda V^{(p)}_{33} \end{vmatrix} + \begin{vmatrix} \lambda V^{(p)}_{22} & V^{(p)}_{23} \\ \lambda V^{(p)}_{32} & \lambda V^{(p)}_{33} \end{vmatrix} = \lambda(-\beta_0^2\lambda^{2p+1} + \dots),\\
T_m(\lambda) &= \begin{vmatrix} \lambda V^{(p)}_{11} & \lambda V^{(p)}_{12} & \lambda V^{(p)}_{13} \\ V^{(p)}_{21} & \lambda V^{(p)}_{22} & V^{(p)}_{23} \\ V^{(p)}_{31} & \lambda V^{(p)}_{32} & \lambda V^{(p)}_{33} \end{vmatrix} = \lambda(-\alpha_0\beta_0^2\lambda^{3p+2} + \dots).
\end{aligned} \tag{3.7}$$

We then naturally obtain the trigonal curve $\mathcal{K}_{m-2}$ of degree m by

$$\mathcal{K}_{m-2}\colon \mathcal{F}_{m-2}(\lambda, y) = y^3 - y^2 R_m(\lambda) + yS_m(\lambda) - T_m(\lambda) = 0, \tag{3.8}$$

where $m = 3p + 3$ for $\alpha_0\beta_0 \neq 0$. We assume that $\beta_0(\alpha_0 + \beta_0)(\alpha_0 - \beta_0) \neq 0$. Then the trigonal curve $\mathcal{K}_{m-2}$ has three different infinite points P_{∞_1}, P_{∞_2}, and P_{∞_3}, which are not branch points. For convenience, we also let the same symbol $\mathcal{K}_{m-2}$ denote the compactification of $\mathcal{K}_{m-2}$. Hence, $\mathcal{K}_{m-2}$ becomes a three-sheeted Riemann surface of arithmetic genus $m - 2$ if it is nonsingular and irreducible. A point P on $\mathcal{K}_{m-2}$ is represented as $P = (\lambda, y)$ satisfying (3.8) together with P_{∞_s}, $s = 1, 2, 3$. The complex structure on $\mathcal{K}_{m-2}$ is standardly defined by introducing local coordinates $\eta\colon P \to (\lambda - \lambda')$ near points $P' \in \mathcal{K}_{m-2}$ that are neither branch nor infinite points of $\mathcal{K}_{m-2}$, $\zeta\colon P \to \lambda^{-1}$ near the infinite points $P_{\infty_s} \in \mathcal{K}_{m-2}$, $s = 1, 2, 3$, $\xi\colon P \to \lambda^{1/3}$ near the point $P_0 = (0, 0)$, which is the triple point of $\mathcal{K}_{m-2}$, and similarly at other branch points of $\mathcal{K}_{m-2}$.

An algebraic function carrying the data of the divisor is closely related to $\psi(P, n, n_0, t_r, t_{0,r})$: the meromorphic function $\phi(P, n, t_r)$ on $\mathcal{K}_{m-2}$ defined by

$$\phi(P, n, t_r) = \frac{\psi_1(P, n, n_0, t_r, t_{0,r})}{\psi_3(P, n, n_0, t_r, t_{0,r})}, \quad P \in \mathcal{K}_{m-2}, \quad (n, t_r) \in \mathbb{Z} \times \mathbb{R}, \tag{3.9}$$

which with (3.1) implies that

$$\phi = \frac{yV_{12}^{(p)} + C_m}{yV_{32}^{(p)} + A_m} = \frac{\lambda F_{m-2}}{y^2 V_{12}^{(p)} - y(C_m + V_{12}^{(p)} R_m) + D_m} = \\ = \frac{y^2 V_{32}^{(p)} - y(A_m + V_{32}^{(p)} R_m) + B_m}{E_{m-2}}, \tag{3.10}$$

where

$$\begin{aligned} A_m &= V_{12}^{(p)} V_{31}^{(p)} - \lambda V_{11}^{(p)} V_{32}^{(p)}, \\ B_m &= \lambda V_{32}^{(p)} (\lambda V_{22}^{(p)} V_{33}^{(p)} - V_{23}^{(p)} V_{32}^{(p)}) + \lambda V_{31}^{(p)} (V_{12}^{(p)} V_{33}^{(p)} - V_{13}^{(p)} V_{32}^{(p)}), \\ C_m &= \lambda V_{13}^{(p)} V_{32}^{(p)} - \lambda V_{12}^{(p)} V_{33}^{(p)}, \\ D_m &= \lambda V_{12}^{(p)} (\lambda V_{11}^{(p)} V_{22}^{(p)} - V_{12}^{(p)} V_{21}^{(p)}) + \lambda V_{13}^{(p)} (\lambda V_{11}^{(p)} V_{32}^{(p)} - V_{12}^{(p)} V_{31}^{(p)}), \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} E_{m-2} &= \lambda V_{32}^{(p)} (V_{21}^{(p)} V_{32}^{(p)} - V_{22}^{(p)} V_{31}^{(p)}) + V_{31}^{(p)} (\lambda V_{11}^{(p)} V_{32}^{(p)} - V_{12}^{(p)} V_{31}^{(p)}), \\ F_{m-2} &= V_{12}^{(p)} (V_{12}^{(p)} V_{23}^{(p)} - \lambda V_{13}^{(p)} V_{22}^{(p)}) + \lambda V_{13}^{(p)} (V_{12}^{(p)} V_{33}^{(p)} - V_{13}^{(p)} V_{32}^{(p)}). \end{aligned} \tag{3.12}$$

For later use, we also introduce

$$\mathcal{A}_m = \lambda V_{21}^{(p)} V_{32}^{(p)} - \lambda V_{22}^{(p)} V_{31}^{(p)}. \tag{3.13}$$

We can easily show that there exist various interrelations between the polynomials A_m, B_m, C_m, D_m, E_{m-2}, F_{m-2}, R_m, S_m, T_m, and $\mathcal{A}_m$, some of which we list:

$$\begin{aligned} V_{32}^{(p)} \lambda F_{m-2} &= V_{12}^{(p)} D_m - (V_{12}^{(p)})^2 S_m - C_m^2 - V_{12}^{(p)} C_m R_m, \\ A_m \lambda F_{m-2} &= (V_{12}^{(p)})^2 T_m + C_m D_m, \\ V_{12}^{(p)} E_{m-2} &= V_{32}^{(p)} B_m - (V_{32}^{(p)})^2 S_m - A_m^2 - V_{32}^{(p)} A_m R_m, \\ C_m E_{m-2} &= (V_{32}^{(p)})^2 T_m + A_m B_m, \end{aligned} \tag{3.14}$$

$$\begin{aligned} E_{m-2} &= V_{32}^{(p)} \mathcal{A}_m - V_{31}^{(p)} A_m, \\ F_{m-2} &= -E_{m-2}^{+}. \end{aligned} \tag{3.15}$$

By inspection of (2.13) and (3.12), we infer that $E_{m-2}(\lambda, n, t_r)$ and $F_{m-2}(\lambda, n, t_r)$ are polynomials in λ of degree $m-2$. Hence, we can write them in the form of two finite products as

$$\begin{aligned} E_{m-2}(\lambda, n, t_r) &= \beta_0(\alpha_0^2 - \beta_0^2) \frac{1}{v^{--}(n, t_r)} \check{a}_0(n, t_r) \prod_{j=1}^{m-2} (\lambda - \mu_j(n, t_r)), \\ F_{m-2}(\lambda, n, t_r) &= \beta_0(\beta_0^2 - \alpha_0^2) \frac{1}{v^{-}(n, t_r)} \check{a}_0^{+}(n, t_r) \prod_{j=1}^{m-2} (\lambda - \mu_j^{+}(n, t_r)). \end{aligned} \tag{3.16}$$

Defining $\{\hat{\mu}_j(n,t_r)\}_{j=1,\dots,m-2}\subset\mathcal{K}_{m-2}$ and $\{\hat{\mu}_j^+(n,t_r)\}_{j=1,\dots,m-2}\subset\mathcal{K}_{m-2}$ by

$$\hat{\mu}_j(n,t_r)=(\mu_j(n,t_r),y(\hat{\mu}_j(n,t_r)))=\left(\mu_j(n,t_r),-\frac{A_m(\mu_j(n,t_r),n,t_r)}{V_{32}^{(p)}(\mu_j(n,t_r),n,t_r)}\right)=$$
$$=\left(\mu_j(n,t_r),-\frac{\mathcal{A}_m(\mu_j(n,t_r),n,t_r)}{V_{31}^{(p)}(\mu_j(n,t_r),n,t_r)}\right), \tag{3.17}$$
$$\hat{\mu}_j^+(n,t_r)=(\mu_j^+(n,t_r),y(\hat{\mu}_j^+(n,t_r)))=\left(\mu_j^+(n,t_r),-\frac{C_m(\mu_j^+(n,t_r),n,t_r)}{V_{12}^{(p)}(\mu_j^+(n,t_r),n,t_r)}\right), \tag{3.18}$$

we can easily see that the two representations of $\hat{\mu}_j(n,t_r)$ are equivalent. In fact, from (3.15) and (3.16), we deduce that

$$E_{m-2}|_{\lambda=\mu_j(n,t_r)}=(V_{32}^{(p)}\mathcal{A}_m-V_{31}^{(p)}A_m)|_{\lambda=\mu_j(n,t_r)}=0, \tag{3.19}$$

which means that

$$\frac{A_m(\mu_j(n,t_r),n,t_r)}{V_{32}^{(p)}(\mu_j(n,t_r),n,t_r)}=\frac{\mathcal{A}_m(\mu_j(n,t_r),n,t_r)}{V_{31}^{(p)}(\mu_j(n,t_r),n,t_r)}. \tag{3.20}$$

The dynamics of the zeros $\{\mu_j(n,t_r)\}_{j=1,\dots,m-2}$ of $E_{m-2}(\lambda,n,t_r)$ are then described in terms of Dubrovin-type equations as follows.

Lemma 1. *Let the zeros $\{\mu_j(n,t_r)\}_{j=1,\dots,m-2}$ of $E_{m-2}(\lambda,n,t_r)$ remain distinct for $(n,t_r)\in\mathbb{Z}\times\mathbb{R}$. Then $\{\mu_j(n,t_r)\}_{j=1,\dots,m-2}$ satisfy the system of differential equations*

$$\begin{aligned}\mu_{j,t_r}(n,t_r)=&[\widetilde{V}_{32}^{(r)}(\mu_j(n,t_r),n,t_r)V_{31}^{(p)}(\mu_j(n,t_r),n,t_r)-\\&-\widetilde{V}_{31}^{(r)}(\mu_j(n,t_r),n,t_r)V_{32}^{(p)}(\mu_j(n,t_r),n,t_r)]\times\\&\times\frac{3y^2(\hat{\mu}_j(n,t_r))-2y(\hat{\mu}_j(n,t_r))R_m(\mu_j(n,t_r))+S_m(\mu_j(n,t_r))}{\beta_0(\alpha_0^2-\beta_0^2)\frac{\check{a}_0(n,t_r)}{v^{--}(n,t_r)}\prod_{\substack{k=1\\k\neq j}}^{m-2}(\mu_j(n,t_r)-\mu_k(n,t_r))},\quad 1\le j\le m-2.\end{aligned} \tag{3.21}$$

Proof. From (3.5) and (3.11)–(3.15), we obtain

$$\begin{aligned}E_{m-2,t_r}=&(\lambda V_{21}^{(p)}(V_{32}^{(p)})^2+\lambda V_{31}^{(p)}V_{32}^{(p)}(V_{11}^{(p)}-V_{22}^{(p)})-V_{12}^{(p)}(V_{31}^{(p)})^2)_{t_r}=\\=&(3\lambda\widetilde{V}_{33}^{(r)}-\widetilde{R}_m)E_{m-2}+\widetilde{V}_{31}^{(r)}(3V_{12}^{(p)}\mathcal{A}_m-3\lambda V_{11}^{(p)}A_m+2A_mR_m+V_{32}^{(p)}S_m)-\\&-\widetilde{V}_{32}^{(r)}(3\lambda V_{21}^{(p)}A_m-3\lambda V_{22}^{(p)}\mathcal{A}_m+2\mathcal{A}_mR_m+V_{31}^{(p)}S_m)=\\=&(3\lambda\widetilde{V}_{33}^{(r)}-\widetilde{R}_m)E_{m-2}+3\widetilde{V}_{31}^{(r)}(V_{12}^{(p)}\mathcal{A}_m-\lambda V_{11}^{(p)}A_m)-\\&-3\widetilde{V}_{32}^{(r)}(\lambda V_{21}^{(p)}A_m-\lambda V_{22}^{(p)}\mathcal{A}_m)+\\&+2(\widetilde{V}_{31}^{(r)}A_m-\widetilde{V}_{32}^{(r)}\mathcal{A}_m)R_m+(\widetilde{V}_{31}^{(r)}V_{32}^{(p)}-\widetilde{V}_{32}^{(r)}V_{31}^{(p)})S_m,\end{aligned} \tag{3.22}$$

where $\widetilde{R}_m=\lambda(\widetilde{V}_{11}^{(r)}+\widetilde{V}_{22}^{(r)}+\widetilde{V}_{33}^{(r)})$. Taking (3.17) into account, we obtain

$$\left.\frac{A_m}{V_{32}^{(p)}}\right|_{\lambda=\mu_j(n,t_r)}=\left.\frac{\mathcal{A}_m}{V_{31}^{(p)}}\right|_{\lambda=\mu_j(n,t_r)}=-y(\hat{\mu}_j(n,t_r)). \tag{3.23}$$

Substituting (3.23) in (3.22) naturally yields

$$\begin{aligned}
E_{m-2,t_r}|_{\lambda=\mu_j(n,t_r)} = & -3y(\hat{\mu}_j(n,t_r))(\widetilde{V}_{31}^{(r)}A_m - \widetilde{V}_{32}^{(r)}\mathcal{A}_m)|_{\lambda=\mu_j(n,t_r)} + \\
& + (-2y(\hat{\mu}_j(n,t_r))R_m(\mu_j(n,t_r)) + \\
& + S_m(\mu_j(n,t_r)))(\widetilde{V}_{31}^{(r)}V_{32}^{(p)} - \widetilde{V}_{32}^{(r)}V_{31}^{(p)})|_{\lambda=\mu_j(n,t_r)} = \\
= & [3y^2(\hat{\mu}_j(n,t_r)) - 2y(\hat{\mu}_j(n,t_r))R_m(\mu_j(n,t_r)) + S_m(\mu_j(n,t_r))] \times \\
& \times (\widetilde{V}_{31}^{(r)}V_{32}^{(p)} - \widetilde{V}_{32}^{(r)}V_{31}^{(p)})|_{\lambda=\mu_j(n,t_r)}.
\end{aligned} \tag{3.24}$$

On the other hand, (3.16) implies that

$$E_{m-2,t_r}|_{\lambda=\mu_j(n,t_r)} = -\beta_0(\alpha_0^2 - \beta_0^2)\frac{\check{a}_0(n,t_r)}{v^{--}(n,t_r)}\mu_{j,t_r}(n,t_r)\prod_{\substack{k=1\\k\neq j}}^{m-2}(\mu_j(n,t_r) - \mu_k(n,t_r)). \tag{3.25}$$

Comparing (3.24) and (3.25), we can write the expression for $\mu_{j,t_r}(n,t_r)$. ■

4. Quasiperiodic solutions

In this section, we derive explicit Riemann theta-function representations for the meromorphic function $\phi(P,n,t_r)$, the Baker–Akhiezer function $\psi_3(P,n,n_0,t_r,t_{0,r})$, and in particular for the potentials $u(n,t_r)$ and $v(n,t_r)$ for the entire hierarchy of differential–difference equations.

Lemma 2. *Let* $P \in \mathcal{K}_{m-1} \setminus \{P_{\infty_1}, P_{\infty_2}, P_{\infty_3}, P_0\}$, $(n,t_r) \in \mathbb{Z}\times\mathbb{R}$. *Then*

$$\phi(P,n,t_r) \underset{\zeta\to 0}{=} \begin{cases} v^{--}(n,t_r)\zeta^{-1} + \dfrac{u^{--}(n,t_r)}{v^{---}(n,t_r)} + O(\zeta), & P \to P_{\infty_1}, \\ \dfrac{1-\check{c}_0(n,t_r)}{v^{-}(n,t_r)\check{a}_0(n,t_r)} + O(\zeta), & P \to P_{\infty_2}, \qquad \zeta = \lambda^{-1}, \\ -\dfrac{1+\check{c}_0(n,t_r)}{v^{-}(n,t_r)\check{a}_0(n,t_r)} + O(\zeta), & P \to P_{\infty_3}, \end{cases} \tag{4.1}$$

$$\phi(P,n,t_r) \underset{\xi\to 0}{=} \xi + O(\xi^2), \qquad P \to P_0, \qquad \xi = \lambda^{1/3}. \tag{4.2}$$

Proof. A direct calculation shows that $\phi(P,n,t_r)$ satisfies the Riccati-type equation

$$\begin{aligned}
&\phi^{++}(P,n,t_r)\phi^{+}(P,n,t_r)\phi(P,n,t_r) = \\
&\qquad = \lambda[(u(n,t_r) + v(n,t_r)\phi^{+}(P,n,t_r))\phi(P,n,t_r) + 1].
\end{aligned} \tag{4.3}$$

In terms of the local coordinate $\zeta = \lambda^{-1}$ near P_{∞_s}, $s = 1,2,3$, we substitute three sets of ansatzes,

$$s = 1\colon \quad \phi(P,n,t_r) \underset{\zeta\to 0}{=} \kappa_{1,-1}(n,t_r)\zeta^{-1} + \kappa_{1,0}(n,t_r) + O(\zeta),$$

$$s = 2\colon \quad \phi(P,n,t_r) \underset{\zeta\to 0}{=} \kappa_{2,0}(n,t_r) + O(\zeta),$$

$$s = 3\colon \quad \phi(P,n,t_r) \underset{\zeta\to 0}{=} \kappa_{3,0}(n,t_r) + O(\zeta),$$

in (4.3). Comparing like powers of ζ yields (4.1). Similarly, choosing the local coordinate $\xi = \lambda^{1/3}$ near P_0 and substituting

$$\phi(P, n, t_r) \underset{\xi \to 0}{=} \kappa_{0,1}(n, t_r)\xi + O(\xi^2)$$

in (4.3), we obtain (4.2). ∎

Examining expression (3.10), we find that $\hat{\mu}_1^+(n, t_r), \ldots, \hat{\mu}_{m-2}^+(n, t_r)$ are $m-2$ zeros and $\hat{\mu}_1(n, t_r), \ldots, \hat{\mu}_{m-2}(n, t_r)$ are $m-2$ poles of the meromorphic function $\phi(P, n, t_r)$. Combining the asymptotic expansions near P_0 and P_{∞_1} in Lemma 2, we finally obtain the divisor of $\phi(P, n, t_r)$:

$$(\phi(P, n, t_r)) = \mathcal{D}_{P_0, \hat{\mu}_1^+(n,t_r), \ldots, \hat{\mu}_{m-2}^+(n,t_r)}(P) - \mathcal{D}_{P_{\infty_1}, \hat{\mu}_1(n,t_r), \ldots, \hat{\mu}_{m-2}(n,t_r)}(P). \tag{4.4}$$

We equip the Riemann surface $\mathcal{K}_{m-2}$ with the homology basis, $\{a_j, b_j\}_{j=1}^{m-2}$, where the cycles are independent and have the intersection numbers

$$a_j \circ b_k = \delta_{jk}, \qquad a_j \circ a_k = 0, \qquad b_j \circ b_k = 0, \quad j, k = 1, \ldots, m-2.$$

For now, as our basis, we choose the set

$$\varpi_l = \frac{1}{3y^2 - 2yR_m + S_m} \begin{cases} \lambda^{l-1}\, d\lambda, & 1 \le l \le 2p+1, \\ \left(y - \dfrac{R_m}{3}\right)\lambda^{l-2p-2}\, d\lambda, & 2p+2 \le l \le m-2, \end{cases} \tag{4.5}$$

which are $m-2$ linearly independent holomorphic differentials on $\mathcal{K}_{m-2}$. Using the homology basis $\{a_j\}_{j=1}^{m-2}$ and $\{b_j\}_{j=1}^{m-2}$, we can construct the period matrices $A = (A_{jk})$ and $B = (B_{jk})$ as

$$A_{jk} = \int_{a_k} \varpi_j, \qquad B_{jk} = \int_{b_k} \varpi_j. \tag{4.6}$$

We can show that A and B are invertible. We now define the matrices C and τ by $C = A^{-1}$, $\tau = A^{-1}B$. We can show that τ is symmetric ($\tau_{jk} = \tau_{kj}$) and has a positive-definite imaginary part ($\operatorname{Im}\tau > 0$) [39], [40]. If we normalize ϖ_l into a new basis $\underline{\omega} = (\omega_1, \ldots, \omega_{m-2})$,

$$\omega_j = \sum_{l=1}^{m-2} C_{jl}\varpi_l, \tag{4.7}$$

then we have

$$\int_{a_k} \omega_j = \delta_{jk}, \qquad \int_{b_k} \omega_j = \tau_{jk}, \quad j, k = 1, \ldots, m-2.$$

Let $\mathcal{T}_{m-2}$ be the period lattice $\{\underline{z} \in \mathbb{C}^{m-2} | \underline{z} = \underline{N} + \underline{M}\tau, \underline{N}, \underline{M} \in \mathbb{Z}^{m-2}\}$. The complex torus $\mathcal{J}_{m-2} = \mathbb{C}^{m-2}/\mathcal{T}_{m-2}$ is called the Jacobian variety of $\mathcal{K}_{m-2}$. An Abel map $\underline{\mathcal{A}}\colon \mathcal{K}_{m-2} \to \mathcal{J}_{m-2}$ is defined by

$$\underline{\mathcal{A}}(P) = (\mathcal{A}_1(P), \ldots, \mathcal{A}_{m-2}(P)) = \left(\int_{Q_0}^{P} \omega_1, \ldots, \int_{Q_0}^{P} \omega_{m-2}\right) \pmod{\mathcal{T}_{m-2}}, \tag{4.8}$$

with the natural linear extension to the quotient group $\operatorname{Div}(\mathcal{K}_{m-2})$,

$$\underline{\mathcal{A}}\left(\sum n_k P_k\right) = \sum n_k \underline{\mathcal{A}}(P_k), \tag{4.9}$$

where the same path is chosen from Q_0 to P for all $j = 1, \ldots, m-2$. We define

$$\underline{\rho}(n, t_r) = \underline{\mathcal{A}}\left(\sum_{k=1}^{m-2} \hat{\mu}_k(n, t_r)\right) = \left(\sum_{k=1}^{m-2} \int_{Q_0}^{\hat{\mu}_k(n,t_r)} \underline{\omega}\right) \pmod{\mathcal{T}_{m-2}}, \tag{4.10}$$

where $\underline{\rho}(n, t_r)$ is linearized on $\mathcal{J}_{m-2}$ in what follows.

Let $\theta(\underline{z})$ denote the Riemann theta function associated with $\mathcal{K}_{m-2}$ equipped with an appropriately fixed homology basis:

$$\theta(\underline{z}) = \sum_{\underline{N} \in \mathbb{Z}^{m-2}} \exp\{\pi i \langle \underline{N}\tau, \underline{N}\rangle + 2\pi i \langle \underline{N}, \underline{z}\rangle\}. \tag{4.11}$$

Here, $\underline{z} = (z_1, \ldots, z_{m-2}) \in \mathbb{C}^{m-2}$ is a complex vector. The angle brackets denote the Euclidean scalar product:

$$\langle \underline{N}, \underline{z}\rangle = \sum_{i=1}^{m-2} N_i z_i, \qquad \langle \underline{N}\tau, \underline{N}\rangle = \sum_{i,j=1}^{m-2} \tau_{ij} N_i N_j. \tag{4.12}$$

Expression (4.11) implies that

$$\theta(\underline{z} + \underline{N} + \underline{M}\tau) = \exp\{-\pi i \langle \underline{M}\tau, \underline{M}\rangle - 2\pi i \langle \underline{M}, \underline{z}\rangle\}\theta(\underline{z}). \tag{4.13}$$

For brevity, we define the function $\underline{z}\colon \mathcal{K}_{m-2} \times \sigma^{m-2}\mathcal{K}_{m-2} \to \mathbb{C}^{m-2}$ as

$$\underline{z}(P, \underline{\hat{\mu}}(n, t_r)) = \underline{\Lambda} - \underline{\mathcal{A}}(P) + \underline{\rho}(n, t_r), \quad P \in \mathcal{K}_{m-2}, \tag{4.14}$$

where $\sigma^{m-2}\mathcal{K}_{m-2}$ denotes the $(m-2)$th symmetric power of $\mathcal{K}_{m-2}$ and $\underline{\Lambda} = (\Lambda_1, \ldots, \Lambda_{m-2})$ is the vector of Riemann constants depending on the base point Q_0 according to

$$\Lambda_j = \frac{1}{2}(1 + \tau_{jj}) - \sum_{\substack{l=1 \\ pl \neq j}}^{m-2} \int_{a_l} \omega_l(P) \int_{Q_0}^{P} \omega_j, \quad j = 1, \ldots, m-2. \tag{4.15}$$

The normalized Abelian differential of the third kind $\omega^{(3)}_{P_0, P_{\infty_1}}(P)$ is holomorphic on $\mathcal{K}_{m-2} \setminus \{P_0, P_{\infty_1}\}$ with simple poles at P_0 and P_{∞_1} with the respective residues $+1$ and -1, i.e.,

$$\begin{aligned} \omega^{(3)}_{P_0, P_{\infty_1}}(P) &\underset{\xi \to 0}{=} (\xi^{-1} + O(1))\, d\xi, && P \to P_0, \quad \xi = \lambda^{1/3}, \\ \omega^{(3)}_{P_0, P_{\infty_1}}(P) &\underset{\zeta \to 0}{=} (-\zeta^{-1} + \omega_0^{\infty_1} + O(\zeta))\, d\zeta, && P \to P_{\infty_1}, \quad \zeta = \lambda^{-1}. \end{aligned} \tag{4.16}$$

We then have

$$\begin{aligned} \int_{Q_0}^{P} \omega^{(3)}_{P_0, P_{\infty_1}} &\underset{\xi \to 0}{=} \log \xi + e_0^{(3)}(Q_0) + O(\xi), && P \to P_0, \quad \xi = \lambda^{1/3}, \\ \int_{Q_0}^{P} \omega^{(3)}_{P_0, P_{\infty_1}} &\underset{\zeta \to 0}{=} -\log \zeta + e_1^{(3)}(Q_0) + \omega_0^{\infty_1}\zeta + O(\zeta^2), && P \to P_{\infty_1}, \quad \zeta = \lambda^{-1}, \end{aligned} \tag{4.17}$$

where $e_0^{(3)}(Q_0)$ and $e_1^{(3)}(Q_0)$ are integration constants and Q_0 is an appropriately chosen base point on $\mathcal{K}_{m-1} \setminus \{P_{\infty_1}, P_{\infty_2}, P_{\infty_3}, P_0\}$. The b-periods of the differential $\omega^{(3)}_{P_0, P_{\infty_1}}(P)$ are denoted by

$$\underline{U}^{(3)} = (U_1^{(3)}, \ldots, U_{m-2}^{(3)}), \qquad U_j^{(3)} = \frac{1}{2\pi i} \int_{b_j} \omega^{(3)}_{P_0, P_{\infty_1}}, \quad j = 1, \ldots, m-2. \tag{4.18}$$

The Riemann theta-function representation of the meromorphic function $\phi(P, n, t_r)$ is then given by the following theorem.

Theorem 1. *If the curve $\mathcal{K}_{m-2}$ is nonsingular and irreducible, $P=(\lambda,y)\in\mathcal{K}_{m-2}\setminus\{P_{\infty_1},P_{\infty_2},P_{\infty_3},P_0\}$, $(n,t_r),(n_0,t_{0,r})\in\mathbb{Z}\times\mathbb{R}$, and $\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}$ is also nonspecial for each $(n,t_r)\in\mathbb{Z}\times\mathbb{R}$, then*

$$\phi(P,n,t_r)=\frac{\theta(\underline{z}(P_0,\underline{\hat{\mu}}(n,t_r)))}{\theta(\underline{z}(P_0,\underline{\hat{\mu}}^+(n,t_r)))}\frac{\theta(\underline{z}(P,\underline{\hat{\mu}}^+(n,t_r)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n,t_r)))}\exp\biggl(\int_{Q_0}^{P}\omega^{(3)}_{P_0,P_{\infty_1}}-e^{(3)}_0(Q_0)\biggr). \tag{4.19}$$

Proof. Let Φ be defined by the right-hand side of (4.19) with the aim to prove that $\phi=\Phi$. It follows from (4.17) that

$$\begin{aligned}
&\exp\biggl(\int_{Q_0}^{P}\omega^{(3)}_{P_0,P_{\infty_1}}(P)-e^{(3)}_0(Q_0)\biggr)\underset{\xi\to0}{=}\xi+O(\xi^2),\\
&P\to P_0,\quad \xi=\lambda^{1/3},\\
&\exp\biggl(\int_{Q_0}^{P}\omega^{(3)}_{P_0,P_{\infty_1}}(P)-e^{(3)}_0(Q_0)\biggr)\underset{\zeta\to0}{=}\zeta^{-1}e^{e^{(3)}_1(Q_0)-e^{(3)}_0(Q_0)}+O(1),\\
&P\to P_{\infty_1},\quad \zeta=\lambda^{-1}.
\end{aligned} \tag{4.20}$$

Using (4.4), we immediately see that ϕ has simple zeros at $\underline{\hat{\mu}}^+(n,t_r)$ and P_0, and simple poles at $\underline{\hat{\mu}}(n,t_r)$ and P_{∞_1}. By (4.20) and a special case of Riemann's vanishing theorem, we see that Φ has the same properties. Using the Riemann–Roch theorem, we conclude that the holomorphic function $\Phi/\phi=\gamma$, where γ is a constant. Using (4.20) and Lemma 2, we obtain

$$\frac{\Phi}{\phi}\underset{\xi\to0}{=}\frac{(\xi+O(\xi^2))(1+O(\xi))}{\xi+O(\xi^2)}\underset{\xi\to0}{=}1+O(\xi),\quad P\to P_0,\quad \xi=\lambda^{1/3}. \tag{4.21}$$

We then conclude that $\gamma=1$, which completes the proof of (4.19). ■

To study the properties of $\psi_3(P,n,n_0,t_r,t_{0,r})$, we first do some preparation. We obtain the expression for $\psi_3(P,n,n_0,t_r,t_{0,r})$ from (3.1) and (3.9):

$$\begin{aligned}
\psi_3(P,n,n_0,t_r,t_{0,r})=\ &\exp\biggl(\int_{t_{0,r}}^{t_r}\Bigl[\widetilde{V}^{(r)}_{31}(\lambda,n_0,t')\phi(P,n_0,t')+\widetilde{V}^{(r)}_{32}(\lambda,n_0,t')\times\\
&\times\phi^+(P,n_0,t')\phi(P,n_0,t')+\lambda\widetilde{V}^{(r)}_{33}(\lambda,n_0,t')\Bigr]\,dt'\biggr)\times\\
&\times\begin{cases}\displaystyle\prod_{n'=n_0}^{n-1}\phi(P,n',t_r), & n\ge n_0+1,\\ 1, & n=n_0,\\ \displaystyle\prod_{n'=n}^{n_0-1}\phi(P,n',t_r)^{-1}, & n\le n_0-1.\end{cases}
\end{aligned} \tag{4.22}$$

By inspection, we verify that

$$\psi_3(P,n,n_0,t_r,t_{0,r})=\psi_3(P,n_0,n_0,t_r,t_{0,r})\psi_3(P,n,n_0,t_r,t_r). \tag{4.23}$$

Motivated by the integrand in (4.22) and because $\widetilde{V}^{(r)}_{31}=\tilde{b}^{(r)-}-\lambda v^-\tilde{a}^{(r)}$, $\widetilde{V}^{(r)}_{32}=\tilde{a}^{(r)}$, and $\widetilde{V}^{(r)}_{33}=\tilde{c}^{(r)^-}-u^-\tilde{a}^{(r)}$, we define the function

$$I_r(P,n,t_r)=(\tilde{b}^{(r)-}-\lambda v^-\tilde{a}^{(r)})\phi+\tilde{a}^{(r)}\phi^+\phi+\lambda(\tilde{c}^{(r)^-}-u^-\tilde{a}^{(r)}), \tag{4.24}$$

whose homogeneous case is denoted by

$$\widehat{I}_r(P,n,t_r) = (\hat{b}^{(r)-} - \lambda v^- \hat{a}^{(r)})\phi + \hat{a}^{(r)}\phi^+\phi + \lambda(\hat{c}^{(r)^-} - u^-\hat{a}^{(r)}), \tag{4.25}$$

where $\hat{g}^{(r)} = (\hat{a}^{(r)}, \hat{b}^{(r)}, \hat{c}^{(r)})^{\mathrm{T}}$ denotes the corresponding homogeneous case of $\tilde{g}^{(r)} = (\tilde{a}^{(r)}, \tilde{b}^{(r)}, \tilde{c}^{(r)})^{\mathrm{T}}$, i.e.,

$$\hat{g}^{(r)} = \tilde{g}^{(r)}|_{\tilde{\alpha}_0=1,\, \tilde{\alpha}_1=\cdots=\tilde{\alpha}_r=0} = \sum_{l=0}^{r} \hat{g}_l \lambda^{r-l}, \tag{4.26}$$

where $\hat{g}_l = (\hat{a}_l, \hat{b}_l, \hat{c}_l)^{\mathrm{T}}$ is defined in (2.2). Hence,

$$I_r(P,n,t_r) = \sum_{l=0}^{r} \tilde{\alpha}_{r-l} \widehat{I}_l(P,n,t_r). \tag{4.27}$$

Lemma 3. *Let* $(n,t_r) \in \mathbb{Z} \times \mathbb{R}$. *Then*

$$\widehat{I}_r(P,n,t_r) \underset{\zeta\to 0}{=} \begin{cases} \zeta^{-r-1} - v^{--}\hat{b}^-_{r+1} - \hat{c}^-_{r+1} + O(\zeta), & P \to P_{\infty_1}, \\ (u^- + v^-\kappa_{2,0})\hat{a}_{r+1} - \hat{c}^-_{r+1} + O(\zeta), & P \to P_{\infty_2}, \\ (u^- + v^-\kappa_{3,0})\hat{a}_{r+1} - \hat{c}^-_{r+1} + O(\zeta), & P \to P_{\infty_3}, \end{cases} \qquad \zeta = \lambda^{-1}, \tag{4.28}$$

$$\widehat{I}_r(P,n,t_r) \underset{\xi\to 0}{=} O(\xi), \quad P \to P_0, \quad \xi = \lambda^{1/3}. \tag{4.29}$$

Proof. By induction, we prove that the first expression in (4.28) in this lemma holds. The other three expressions can be proved similarly. For $r = 0$, $\hat{a}^{(r)} = \hat{c}^{(r)} = 0$ and $\hat{b}^{(r)} = 1/v^-$. It is then easy to see that

$$\begin{aligned} \widehat{I}_0(P,n,t_r) &= \frac{1}{v^{--}}\phi = \zeta^{-1} + \frac{u^{--}}{v^{--}v^{---}} + O(\zeta) = \\ &= \zeta^{-1} - v^{--}\hat{b}^-_1 - \hat{c}^-_1 + O(\zeta), \quad P \to P_{\infty_1}. \end{aligned} \tag{4.30}$$

We suppose that $\widehat{I}_r(P,n,t_r)$ has the expansion

$$\widehat{I}_r(P,n,t_r) \underset{\zeta\to 0}{=} \zeta^{-r-1} + \sum_{j=0}^{\infty} \sigma_j(n,t_r)\zeta^j, \quad P \to P_{\infty_1}, \tag{4.31}$$

for some coefficients $\{\sigma_j(n,t_r)\}$, $j \geq 0$, to be determined. Differentiating (3.9) with respect to t_r and using (3.1) yield

$$\phi_{t_r} = \left(\frac{\psi_3^+}{\psi_3}\right)_{t_r} = \frac{\psi_3^+}{\psi_3}\left(\frac{\psi^+_{3,t_r}}{\psi_3^+} - \frac{\psi_{3,t_r}}{\psi_3}\right) = \phi\Delta(\widetilde{V}^{(r)}_{31}\phi + \widetilde{V}^{(r)}_{32}\phi^+\phi + \lambda\widetilde{V}^{(r)}_{33}) = \phi\Delta I_r. \tag{4.32}$$

Furthermore, we obtain

$$\phi(P,n,t_r)_{t_r} = \phi(P,n,t_r)\Delta\widehat{I}_r(P,n,t_r). \tag{4.33}$$

Using (2.17) and Lemma 2 and comparing like powers of ζ in (4.33), we obtain

$$\begin{aligned} \Delta\sigma_0 &= \frac{(v^{--})_{t_r}}{v^{--}} = v^{--}\hat{b}^-_{r+1} - v^-\hat{b}_{r+1} + \hat{c}^-_{r+1} - \hat{c}_{r+1} = \Delta(-v^{--}\hat{b}^-_{r+1} - \hat{c}^-_{r+1}), \\ \Delta\sigma_1 &= \frac{1}{v^{--}}\left(\left(\frac{u^{--}}{v^{---}}\right)_{t_r} - \frac{u^{--}}{v^{---}}\Delta\sigma_0\right) = \\ &= \frac{\hat{a}_{r+1}}{v^{---}} - \frac{\hat{a}^{--}_{r+1}}{v^{--}} + \frac{u^{--}\hat{b}^-_{r+1}}{v^{---}} - \frac{u^{--}\hat{b}^{--}_{r+1}}{v^{--}} = \\ &= \Delta\left(-\frac{\hat{a}_{r+1} + u^{--}\hat{b}^-_{r+1}}{v^{---}} - v^{--}\hat{b}^-_{r+2} - \hat{c}^-_{r+2}\right), \end{aligned} \tag{4.34}$$

whence we can infer that

$$\begin{aligned}\sigma_0(n,t_r) &= -v^{--}\hat{b}^-_{r+1} - \hat{c}^-_{r+1},\\ \sigma_1(n,t_r) &= -\frac{\hat{a}_{r+1} + u^{--}\hat{b}^-_{r+1}}{v^{---}} - v^{--}\hat{b}^-_{r+2} - \hat{c}^-_{r+2},\end{aligned} \tag{4.35}$$

where the summation constants are taken as zero because there are no arbitrary constants in the expansion of $\phi(P,n,t_r)$ near P_{∞_1} nor in the coefficients $\hat{a}_r$, $\hat{b}_r$, and $\hat{c}_r$ with the condition $\Delta\Delta^{-1} = \Delta^{-1}\Delta = 1$. It follows that

$$\begin{aligned}\widehat{I}_{r+1}(P,n,t_r) &= \zeta^{-1}\widehat{I}_r + (\hat{b}^-_{r+1} - \zeta^{-1}v^-\hat{a}_{r+1})\phi + \hat{a}_{r+1}\phi^+\phi + \zeta^{-1}(\hat{c}^-_{r+1} - u^-\hat{a}_{r+1}) =\\ &= \zeta^{-r-2} - v^{--}\hat{b}^-_{r+2} - \hat{c}^-_{r+2} + O(\zeta).\end{aligned} \tag{4.36}$$

We have thus proved that $\widehat{I}_r(P,n,t_r)$ has the expansion in (4.28) near P_{∞_1}. ■

From Lemma 3 and (4.27), we obtain

$$I_r(P,n,t_r) \underset{\zeta\to 0}{=} \begin{cases} \sum_{l=0}^{r} \tilde{\alpha}_{r-l}\zeta^{-l-1} + \tilde{\alpha}_{r+1} - v^{--}\tilde{b}^-_{r+1} - \tilde{c}^-_{r+1} + O(\zeta), & P \to P_{\infty_1},\\ (u^- + v^-\kappa_{2,0})\tilde{a}_{r+1} - \tilde{c}^-_{r+1} + O(\zeta), & P \to P_{\infty_2},\\ (u^- + v^-\kappa_{3,0})\tilde{a}_{r+1} - \tilde{c}^-_{r+1} + O(\zeta), & P \to P_{\infty_3}, \end{cases} \qquad \zeta = \lambda^{-1}, \tag{4.37}$$

$$I_r(P,n,t_r) \underset{\xi\to 0}{=} O(\xi), \quad P \to P_0, \quad \xi = \lambda^{1/3}. \tag{4.38}$$

Let $\omega^{(2)}_{P_{\infty_1},j}$ be the normalized second-kind differential holomorphic on $\mathcal{K}_{m-2} \setminus \{P_{\infty_1}\}$ with a pole of order $j \geq 2$ at P_{∞_1} satisfying

$$\begin{aligned}&\int_{a_k} \omega^{(2)}_{P_{\infty_1},j} = 0, \quad k = 1,\ldots,m-2,\\ &\omega^{(2)}_{P_{\infty_1},j}(P) \underset{\zeta\to 0}{=} (\zeta^{-j} + O(1))\,d\zeta, \quad P \to P_{\infty_1}, \quad \zeta = \lambda^{-1}.\end{aligned} \tag{4.39}$$

Moreover, using (4.37) and (4.38), we introduce the Abelian differential

$$\widetilde{\Omega}^{(2)}_r(P) = \sum_{l=0}^{r} \tilde{\alpha}_{r-l}(l+1)\omega^{(2)}_{P_{\infty_1},l+2}(P). \tag{4.40}$$

Integrating (4.40) yields

$$\int_{Q_0}^{P} \widetilde{\Omega}^{(2)}_r \underset{\zeta\to 0}{=} \begin{cases} -\sum_{l=0}^{r} \tilde{\alpha}_{r-l}\zeta^{-l-1} + \tilde{e}^{(2)}_1(Q_0) + O(\zeta), & P \to P_{\infty_1},\\ \tilde{e}^{(2)}_2(Q_0) + O(\zeta), & P \to P_{\infty_2},\\ \tilde{e}^{(2)}_3(Q_0) + O(\zeta), & P \to P_{\infty_3}, \end{cases} \qquad \zeta = \lambda^{-1}, \tag{4.41}$$

$$\int_{Q_0}^{P} \widetilde{\Omega}^{(2)}_r \underset{\xi\to 0}{=} \tilde{e}^{(2)}_0(Q_0) + O(\xi), \quad P \to P_0, \quad \xi = \lambda^{1/3}, \tag{4.42}$$

where $\tilde{e}_1^{(2)}(Q_0)$, $\tilde{e}_2^{(2)}(Q_0)$, $\tilde{e}_3^{(2)}(Q_0)$, and $\tilde{e}_0^{(2)}(Q_0)$ are integration constants. The b-periods of the differential $\widetilde{\Omega}_r^{(2)}(P)$ are denoted by

$$\underline{\widetilde{U}}_r^{(2)} = (\widetilde{U}_{r,1}^{(2)}, \ldots, \widetilde{U}_{r,m-2}^{(2)}), \qquad \widetilde{U}_{r,j}^{(2)} = \frac{1}{2\pi i}\int_{b_j} \widetilde{\Omega}_r^{(2)}, \quad j = 1, \ldots, m-2. \tag{4.43}$$

After this preparation, we can give the theta-function representation of the Baker–Akhiezer function $\psi_3(P, n, n_0, t_r, t_{0,r})$ in the following theorem.

Theorem 2. *Let the curve $\mathcal{K}_{m-2}$ be nonsingular and irreducible. Let $P = (\lambda, y) \in \mathcal{K}_{m-2} \setminus \{P_{\infty_1}, P_{\infty_2}, P_{\infty_3}, P_0\}$ and $(n, n_0, t_r, t_{0,r}) \in \mathbb{Z}^2 \times \mathbb{R}^2$. If $\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}$ is nonspecial for each $(n, t_r) \in \mathbb{Z} \times \mathbb{R}$, then*

$$\begin{aligned}\psi_3(P, n, n_0, t_r, t_{0,r}) = {} & \frac{\theta(\underline{z}(P_0, \underline{\hat{\mu}}(n_0, t_{0,r})))}{\theta(\underline{z}(P_0, \underline{\hat{\mu}}(n, t_r)))}\frac{\theta(\underline{z}(P, \underline{\hat{\mu}}(n, t_r)))}{\theta(\underline{z}(P, \underline{\hat{\mu}}(n_0, t_{0,r})))} \times \\ & \times \exp\bigg((n - n_0)\bigg(\int_{Q_0}^{P} \omega_{P_0, P_{\infty_1}}^{(3)} - e_0^{(3)}(Q_0)\bigg) + \\ & + (t_r - t_{0,r})\bigg(\tilde{e}_0^{(2)}(Q_0) - \int_{Q_0}^{P} \widetilde{\Omega}_r^{(2)}\bigg)\bigg).\end{aligned} \tag{4.44}$$

Proof. Letting $\Psi_3(P, n, n_0, t_r, t_{0,r})$ denote the right-hand side of (4.44), our goal is to prove that $\Psi_3(P, n, n_0, t_r, t_{0,r}) = \psi_3(P, n, n_0, t_r, t_{0,r})$. In fact,

$$\Psi_3(P, n, n_0, t_r, t_{0,r}) = \Psi_3(P, n, n_0, t_r, t_r)\Psi_3(P, n_0, n_0, t_r, t_{0,r}). \tag{4.45}$$

It is easy to see from (4.19) and (4.22) that

$$\psi_3(P, n, n_0, t_r, t_r) = \begin{cases} \prod\limits_{n'=n_0}^{n-1} \phi(P, n', t_r), & n \geq n_0 + 1, \\ 1, & n = n_0, \\ \prod\limits_{n'=n}^{n_0-1} \phi(P, n', t_r)^{-1}, & n \leq n_0 - 1, \end{cases} = \Psi_3(P, n, n_0, t_r, t_r). \tag{4.46}$$

By (4.23) and (4.45), it remains to identify

$$\psi_3(P, n_0, n_0, t_r, t_{0,r}) = \Psi_3(P, n_0, n_0, t_r, t_{0,r}). \tag{4.47}$$

In what follows, we inspect the zeros and poles of $\psi_3(P, n_0, n_0, t_r, t_{0,r})$ on $\mathcal{K}_{m-2} \setminus \{P_{\infty_1}, P_{\infty_2}, P_{\infty_3}, P_0\}$. Using (3.10), (3.12)–(3.16), and Lemma 1, we can compute

$$\begin{aligned}I_r(P, n, t_r) &= \widetilde{V}_{31}^{(r)}\phi + \widetilde{V}_{32}^{(r)}\phi^+\phi + \lambda\widetilde{V}_{33}^{(r)} = \widetilde{V}_{31}^{(r)}\phi + \widetilde{V}_{32}^{(r)}\frac{y - V_{31}^{(p)}\phi - \lambda V_{33}^{(p)}}{V_{32}^{(p)}} + \lambda\widetilde{V}_{33}^{(r)} = \\ &= \bigg(\widetilde{V}_{31}^{(r)} - \widetilde{V}_{32}^{(r)}\frac{V_{31}^{(p)}}{V_{32}^{(p)}}\bigg)\phi + y\frac{\widetilde{V}_{32}^{(r)}}{V_{32}^{(p)}} - \widetilde{V}_{32}^{(r)}\frac{\lambda V_{33}^{(p)}}{V_{32}^{(p)}} + \lambda\widetilde{V}_{33}^{(r)} = \bigg(\widetilde{V}_{31}^{(r)} - \widetilde{V}_{32}^{(r)}\frac{V_{31}^{(p)}}{V_{32}^{(p)}}\bigg) \times \\ &\quad \times \frac{y^2 V_{32}^{(p)} - y(A_m + V_{32}^{(p)}R_m) + V_{12}^{(p)}\mathcal{A}_m + V_{32}^{(p)}S_m + A_m R_m - \lambda V_{11}^{(p)}A_m}{E_{m-2}} +\end{aligned}$$

$$+\, y\frac{\widetilde{V}_{32}^{(r)}}{V_{32}^{(p)}} - \widetilde{V}_{32}^{(r)}\frac{\lambda V_{33}^{(p)}}{V_{32}^{(p)}} + \lambda\widetilde{V}_{33}^{(r)} = \frac{1}{E_{m-2}}\bigg[\frac{1}{3}E_{m-2,t_r} + \frac{1}{3}\widetilde{R}_m E_{m-2} +$$

$$+\, (\widetilde{V}_{31}^{(r)}V_{32}^{(p)} - \widetilde{V}_{32}^{(r)}V_{31}^{(p)})\bigg(y^2 - yR_m + \frac{2}{3}S_m\bigg) +$$

$$+\, (\widetilde{V}_{32}^{(r)}\mathcal{A}_m - \widetilde{V}_{31}^{(r)}A_m)\bigg(y - \frac{1}{3}R_m\bigg)\bigg] =$$

$$\underset{\lambda\to\mu_j(n,t_r)}{=} -\frac{\mu_{j,t_r}(n,t_r)}{\lambda - \mu_j(n,t_r)} + O(1) \underset{\lambda\to\mu_j(n,t_r)}{=} \partial_{t_r}\log(\lambda - \mu_j(n,t_r)) + O(1).$$

Then

$$\psi_3(P,n_0,n_0,t_r,t_{0,r}) = \exp\bigg(\int_{t_{0,r}}^{t_r} I_r(P,n_0,t')\,dt'\bigg) = \frac{\lambda - \mu_j(n_0,t_r)}{\lambda - \mu_j(n_0,t_{0,r})}O(1) =$$

$$= \begin{cases} (\lambda - \mu_j(n_0,t_r))O(1) & \text{for } P \text{ near } \hat{\mu}_j(n_0,t_r) \neq \hat{\mu}_j(n_0,t_{0,r}), \\ O(1) & \text{for } P \text{ near } \hat{\mu}_j(n_0,t_r) = \hat{\mu}_j(n_0,t_{0,r}), \\ (\lambda - \mu_j(n_0,t_{0,r}))^{-1}O(1) & \text{for } P \text{ near } \hat{\mu}_j(n_0,t_{0,r}) \neq \hat{\mu}_j(n_0,t_r), \end{cases} \tag{4.48}$$

where $O(1) \neq 0$. Hence, all zeros and poles of $\psi_3(P,n_0,n_0,t_r,t_{0,r})$ and $\Psi_3(P,n_0,n_0,t_r,t_{0,r})$ on $\mathcal{K}_{m-2} \setminus \{P_{\infty_1}, P_{\infty_2}, P_{\infty_3}, P_0\}$ are simple and coincide. It can be easily seen from (4.37), (4.38), (4.41), and (4.42) that the singularities of $\psi_3(P,n_0,n_0,t_r,t_{0,r})$ and $\Psi_3(P,n_0,n_0,t_r,t_{0,r})$ at P_{∞_1}, P_{∞_2}, P_{∞_3}, and P_0 coincide. It follows from the Riemann–Roch uniqueness that $\Psi_3(P,n_0,n_0,t_r,t_{0,r})/\psi_3(P,n_0,n_0,t_r,t_{0,r}) = \gamma$, where γ is a constant. Using (4.38) and (4.42), we obtain

$$\frac{\Psi_3(P,n_0,n_0,t_r,t_{0,r})}{\psi_3(P,n_0,n_0,t_r,t_{0,r})} \underset{\xi\to 0}{=} \frac{(1+O(\xi))(1+O(\xi))}{1+O(\xi)} \underset{\xi\to 0}{=} 1 + O(\xi), \quad P \to P_0. \tag{4.49}$$

We then conclude that $\gamma = 1$. ∎

Straightening the flows by the Abel map is described in our next result.

Theorem 3. *Let $(n,t_r), (n_0,t_{0,r}) \in \mathbb{Z}\times\mathbb{R}$. Then*

$$\underline{\rho}(n,t_r) = \underline{\rho}(n_0,t_{0,r}) - \underline{U}^{(3)}(n-n_0) + \underline{\widetilde{U}}_r^{(2)}(t_r - t_{0,r}) \pmod{\mathcal{T}_{m-2}}. \tag{4.50}$$

Proof. We introduce the meromorphic differential on $\mathcal{K}_{m-2}$

$$\Omega(n,n_0,t_r,t_{0,r}) = \frac{\partial}{\partial\lambda}\log(\psi_3(P,n,n_0,t_r,t_{0,r}))\,d\lambda. \tag{4.51}$$

From representation (4.44), we obtain

$$\Omega(n,n_0,t_r,t_{0,r}) = (n-n_0)\omega_{P_0,P_{\infty_1}}^{(3)} - (t_r - t_{0,r})\widetilde{\Omega}_r^{(2)} + \sum_{j=1}^{m-2}\omega_{\hat{\mu}_j(n,t_r),\hat{\mu}_j(n_0,t_{0,r})}^{(3)} + \widetilde{\omega}, \tag{4.52}$$

where $\widetilde{\omega}$ denotes a holomorphic differential on $\mathcal{K}_{m-2}$, i.e., $\widetilde{\omega} = \sum_{j=1}^{m-2} e_j\omega_j$ for some $e_j \in \mathbb{C}$, $j = 1,\ldots,m-2$. Because $\psi_3(P,n,n_0,t_r,t_{0,r})$ is single-valued on $\mathcal{K}_{m-2}$, all a- and b-periods of Ω are integer multiples of $2\pi i$ and hence

$$2\pi i M_k = \int_{a_k}\Omega(n,n_0,t_r,t_{0,r}) = \int_{a_k}\widetilde{\omega} = e_k, \quad k = 1,\ldots,m-2, \tag{4.53}$$

for some $M_k \in \mathbb{Z}$. Similarly, for some $N_k \in \mathbb{Z}$,

$$2\pi i N_k = \int_{b_k} \Omega(n, n_0, t_r, t_{0,r}) = (n - n_0) \int_{b_k} \omega^{(3)}_{P_0, P_{\infty_1}} - (t_r - t_{0,r}) \int_{b_k} \widetilde{\Omega}^{(2)}_r +$$

$$+ \sum_{j=1}^{m-2} \int_{b_k} \omega^{(3)}_{\hat{\mu}_j(n,t_r), \hat{\mu}_j(n_0,t_{0,r})} + \int_{b_k} \widetilde{\omega} = 2\pi i (n - n_0) U^{(3)}_k - 2\pi i (t_r - t_{0,r}) \widetilde{U}^{(2)}_{r,k} +$$

$$+ 2\pi i \sum_{j=1}^{m-2} \int_{\hat{\mu}_j(n_0,t_{0,r})}^{\hat{\mu}_j(n,t_r)} \omega_k + 2\pi i \sum_{j=1}^{m-2} M_j \int_{b_k} \omega_j =$$

$$= 2\pi i (n - n_0) U^{(3)}_k - 2\pi i (t_r - t_{0,r}) \widetilde{U}^{(2)}_{r,k} +$$

$$+ 2\pi i \sum_{j=1}^{m-2} \left[\int_{Q_0}^{\hat{\mu}_j(n,t_r)} \omega_k - \int_{Q_0}^{\hat{\mu}_j(n_0,t_{0,r})} \omega_k \right] + 2\pi i \sum_{j=1}^{m-2} M_j \tau_{jk}. \tag{4.54}$$

Hence, we have

$$\underline{N} = (n - n_0) \underline{U}^{(3)} - (t_r - t_{0,r}) \underline{\widetilde{U}}^{(2)}_r + \sum_{j=1}^{m-2} \int_{Q_0}^{\hat{\mu}_j(n,t_r)} \underline{\omega} - \sum_{j=1}^{m-2} \int_{Q_0}^{\hat{\mu}_j(n_0,t_{0,r})} \underline{\omega} + \underline{M} \tau, \tag{4.55}$$

where $\underline{N} = (N_1, \ldots, N_{m-2})$ and $\underline{M} = (M_1, \ldots, M_{m-2}) \in \mathbb{Z}^{m-2}$. By symmetry of τ, this is equivalent to (4.50). ■

Our principal result, the Riemann theta-function representations for solutions of the entire hierarchy now quickly follow from the prepared material.

Theorem 4. *Let the curve $\mathcal{K}_{m-2}$ be nonsingular and irreducible and $(n, t_r) \in \mathbb{Z} \times \mathbb{R}$. If $\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}$ is nonspecial, then $u(n, t_r)$ and $v(n, t_r)$ have the forms*

$$u(n, t_r) = \frac{\theta(\underline{K}^{(1)} - \underline{U}^{(3)} n + \underline{\widetilde{U}}^{(2)}_r t_r) \theta(\underline{K}^{(2)} - \underline{U}^{(3)} n + \underline{\widetilde{U}}^{(2)}_r t_r)}{\theta(\underline{K}^{(3)} - \underline{U}^{(3)} n + \underline{\widetilde{U}}^{(2)}_r t_r) \theta(\underline{K}^{(0)} - \underline{U}^{(3)} n + \underline{\widetilde{U}}^{(2)}_r t_r)} e^{2(e^{(3)}_1(Q_0) - e^{(3)}_0(Q_0))} \times$$

$$\times \left(\omega_0^{\infty_1} - \sum_{j=0}^{m-2} d^{(\infty_1)}_{j,0} \partial_{z_j} \log \frac{\theta(\underline{K}^{(2)} - \underline{U}^{(3)} n + \underline{\widetilde{U}}^{(2)}_r t_r)}{\theta(\underline{K}^{(1)} - \underline{U}^{(3)} n + \underline{\widetilde{U}}^{(2)}_r t_r)} \right), \tag{4.56}$$

$$v(n, t_r) = \frac{(\theta(\underline{K}^{(2)} - \underline{U}^{(3)} n + \underline{\widetilde{U}}^{(2)}_r t_r))^2}{\theta(\underline{K}^{(3)} - \underline{U}^{(3)} n + \underline{\widetilde{U}}^{(2)}_r t_r) \theta(\underline{K}^{(1)} - \underline{U}^{(3)} n + \underline{\widetilde{U}}^{(2)}_r t_r)} e^{e^{(3)}_1(Q_0) - e^{(3)}_0(Q_0)},$$

where

$$\underline{K}^{(s)} = \underline{\Lambda} - \underline{\mathcal{A}}(P_0) + \underline{\rho}(n_0, t_{0,r}) + \underline{U}^{(3)} n_0 - \underline{U}^{(3)} s - \underline{\widetilde{U}}^{(2)}_r t_{0,r}, \quad s = 0, 1, 2, 3,$$

$$d^{(\infty_1)}_{j,0} = -\frac{1}{\alpha_0^2 - \beta_0^2} C_{j,2p+1} - \frac{2\alpha_0}{3(\alpha_0^2 - \beta_0^2)} C_{j,m-2}.$$

Proof. Choosing the local coordinate $\zeta = \lambda^{-1}$ near P_{∞_1}, from (3.1), we obtain

$$y = V^{(p)}_{31} \phi + V^{(p)}_{32} \phi^+ \phi + \lambda V^{(p)}_{33} \underset{\zeta \to 0}{=} \zeta^{-p-1} (\alpha_0 + O(\zeta)), \quad P \to P_{\infty_1}. \tag{4.57}$$

From (3.7) and (4.5), we obtain

$$\omega_j = \sum_{l=1}^{m-2} C_{jl}\varpi_l = \sum_{l=1}^{2p+1} C_{jl}\frac{\lambda^{l-1}\,d\lambda}{3y^2-2yR_m+S_m} + \sum_{l=2p+2}^{m-2} C_{jl}\frac{(y-R_m/3)\lambda^{l-2p-2}\,d\lambda}{3y^2-2yR_m+S_m} =$$

$$\underset{\zeta\to 0}{=} (d_{j,0}^{(\infty_1)} + O(\zeta))\,d\zeta, \quad P\to P_{\infty_1}, \quad j=1,\dots,m-2. \tag{4.58}$$

Combining (4.10) and (4.14), we obtain the asymptotic expansion

$$\frac{\theta(\underline{z}(P,\underline{\hat{\mu}}^+(n,t_r)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n,t_r)))} = \frac{\theta(\underline{\Lambda}-\underline{\mathcal{A}}(P)+\underline{\rho}^+(n,t_r))}{\theta(\underline{\Lambda}-\underline{\mathcal{A}}(P)+\underline{\rho}(n,t_r))} =$$

$$= \frac{\theta\big(\underline{\Lambda}-\underline{\mathcal{A}}(P_{\infty_1})+\underline{\rho}^+(n,t_r)+\int_P^{P_{\infty_1}}\underline{\omega}\big)}{\theta\big(\underline{\Lambda}-\underline{\mathcal{A}}(P_{\infty_1})+\underline{\rho}(n,t_r)+\int_P^{P_{\infty_1}}\underline{\omega}\big)} \underset{\zeta\to 0}{=}$$

$$\underset{\zeta\to 0}{=} \frac{\theta(\dots,\Lambda_j-\mathcal{A}_j(P_{\infty_1})+\rho_j^+(n,t_r)-d_{j,0}^{(\infty_1)}\zeta+O(\zeta^2),\dots)}{\theta(\dots,\Lambda_j-\mathcal{A}_j(P_{\infty_1})+\rho_j(n,t_r)-d_{j,0}^{(\infty_1)}\zeta+O(\zeta^2),\dots)} \underset{\zeta\to 0}{=}$$

$$\underset{\zeta\to 0}{=} \frac{\theta_1^+ - \sum_{j=1}^{m-2} d_{j,0}^{(\infty_1)}\,\partial_{z_j}\theta_1^+\zeta + O(\zeta^2)}{\theta_1 - \sum_{j=1}^{m-2} d_{j,0}^{(\infty_1)}\,\partial_{z_j}\theta_1\zeta + O(\zeta^2)} \underset{\zeta\to 0}{=}$$

$$\underset{\zeta\to 0}{=} \frac{\theta_1^+}{\theta_1}\left(1 - \sum_{j=1}^{m-2} d_{j,0}^{(\infty_1)}\,\partial_{z_j}\log\frac{\theta_1^+}{\theta_1}\zeta + O(\zeta^2)\right), \quad P\to P_{\infty_1}, \tag{4.59}$$

where $\theta_1 = \theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}(n,t_r)))$ and $\theta_1^+ = \theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}^+(n,t_r)))$. Expanding ϕ given by (4.19) near P_{∞_1}, we obtain

$$\phi \underset{\zeta\to 0}{=} \zeta^{-1}\frac{\theta_0\theta_1^+}{\theta_0^+\theta_1}e^{e_1^{(3)}(Q_0)-e_0^{(3)}(Q_0)}\left[1 + \left(\omega_0^{\infty_1} - \sum_{j=1}^{m-2} d_{j,0}^{(\infty_1)}\,\partial_{z_j}\log\frac{\theta_1^+}{\theta_1}\right)\zeta + O(\zeta^2)\right], \tag{4.60}$$

where $\theta_0 = \theta(\underline{z}(P_0,\underline{\hat{\mu}}(n,t_r)))$ and $\theta_0^+ = \theta(\underline{z}(P_0,\underline{\hat{\mu}}^+(n,t_r)))$. Comparing with the asymptotic expansion near P_{∞_1} in Lemma 2, we find

$$u(n,t_r) = \frac{\theta(\underline{z}(P_0,\underline{\hat{\mu}}^+(n,t_r)))\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}^{+++}(n,t_r)))}{\theta(\underline{z}(P_0,\underline{\hat{\mu}}^{+++}(n,t_r)))\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}^+(n,t_r)))}e^{2(e_1^{(3)}(Q_0)-e_0^{(3)}(Q_0))} \times$$

$$\times\left(\omega_0^{\infty_1} - \sum_{j=0}^{m-2} d_{j,0}^{(\infty_1)}\,\partial_{z_j}\log\frac{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}^{+++}(n,t_r)))}{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}^{++}(n,t_r)))}\right), \tag{4.61}$$

$$v(n,t_r) = \frac{\theta(\underline{z}(P_0,\underline{\hat{\mu}}^{++}(n,t_r)))\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}^{+++}(n,t_r)))}{\theta(\underline{z}(P_0,\underline{\hat{\mu}}^{+++}(n,t_r)))\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}^{++}(n,t_r)))}e^{e_1^{(3)}(Q_0)-e_0^{(3)}(Q_0)}.$$

Applying Abel's theorem to (4.4) yields

$$\underline{\rho}^+(n,t_r) + \underline{\mathcal{A}}(P_0) = \underline{\rho}(n,t_r) + \underline{\mathcal{A}}(P_{\infty_1}), \tag{4.62}$$

whence we infer that

$$\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}^+(n,t_r))) = \theta(\underline{z}(P_0,\underline{\hat{\mu}}(n,t_r))). \tag{4.63}$$

Combining (4.61), (4.63), and Theorem 3, we can write $u(n,t_r)$ and $v(n,t_r)$ briefly in (4.56). ■

5. Conclusions

We have constructed quasiperiodic solutions of the Belov–Chaltikian lattice hierarchy. Starting from a discrete 3×3 matrix spectral problem, we derived a hierarchy of the modified Belov–Chaltikian lattice equation. We then defined a trigonal curve with three different infinite points and a zero point using the characteristic polynomial of the Lax matrix for the hierarchy, whence we introduced the associated meromorphic function and Baker–Akhiezer function. In view of their asymptotic expansions near the three infinite points and the zero point and the Abel differentials, we obtained Riemann theta-function representations of the meromorphic function, the Baker–Akhiezer function, and, in particular, solutions of the entire modified Belov–Chaltikian lattice hierarchy.

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