Theoretical and Mathematical Physics, **198**(3): 392–411 (2019)

TIME EVOLUTION OF QUADRATIC QUANTUM SYSTEMS: EVOLUTION OPERATORS, PROPAGATORS, AND INVARIANTS

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We use the evolution operator method to describe time-dependent quadratic quantum systems in the framework of nonrelativistic quantum mechanics. For simplicity, we consider a free particle with a variable mass $M(t)$, a particle with a variable mass $M(t)$ in an alternating homogeneous field, and a harmonic oscillator with a variable mass $M(t)$ and frequency $\omega(t)$ subject to a variable force $F(t)$. To construct the evolution operators for these systems in an explicit disentangled form, we use a simple technique to find the general solution of a certain class of differential and finite-difference nonstationary Schrödinger-type equations of motion and also the operator identities of the Baker–Campbell–Hausdorff type. With known evolution operators, we can easily find the most general form of the propagators, invariants of any order, and wave functions and establish a unitary relation between systems. Results known in the literature follow from the obtained general results as particular cases.

Keywords: nonstationary quadratic system, evolution operator, propagator, invariant, unitary relation

DOI: 10.1134/S004057791903005X

1. Introduction

A principal problem in quantum theory is studying the time evolution of physical systems. The evolution of nonrelativistic quantum systems from a given initial state $\psi(t_0)$ to a subsequent state $\psi(t)$ is defined by the nonstationary Schrödinger equation [1]

$$\widehat{S}(t)\psi(t) = 0, \qquad \widehat{S}(t) = i\hbar\partial_t - H(t) \tag{1.1}$$

or the unitary evolution operator $U(t,t_0)$,

$$\psi(t) = U(t,t_0)\psi(t_0), \tag{1.2}$$

where $\widehat{S}(t)$ is the Schrödinger operator, $H(t)$ is the system Hamiltonian, and $t > t_0$. If we substitute (1.2) in (1.1), then we find that the operator U satisfies the Schrödinger equation $\widehat{S}(t)U(t,t_0) = 0$ with the initial

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This research is supported by the Science Development Foundation under the President of the Republic of Azerbaijan (Research Grants Nos. EIF-KETPL-2-2015-1(2015)-1(25)-56/02/1 and EIF/MQM/Elm-Tehsil-1-2016-1(26)-71/11/1).

The research of A. I. Ahmadov is supported by Baku State University (Research Grants "50 + 50," 2018–2019).

Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 198, No. 3, pp. 451–472, March, 2019. Received December 26, 2017. Revised June 8, 2018. Accepted June 8, 2018.

state $U(t_0, t_0) = 1$. According to the principles of quantum theory, information about the dynamics of a quantum system is encoded in the matrix elements of the evolution operator

$$U(t, t_0) = \mathrm{T}\exp\left\{-\frac{i}{\hbar}\int_{t_0}^{t} H(t')\,dt'\right\}, \tag{1.3}$$

where T is the chronological ordering operator.

The kernel of operator (1.3) is the Green's function (or the Feynman propagator) in the appropriate coordinate, momentum, mixed, or other representation. It follows from definition (1.3) that the evolution operator has the group property: for $t > t_1 > t_0$,

$$\mathrm{T}\exp\left\{-\frac{i}{\hbar}\int_{t_0}^{t} H(t')\,dt'\right\} = \mathrm{T}\exp\left\{-\frac{i}{\hbar}\int_{t_1}^{t} H(t')\,dt'\right\}\ \mathrm{T}\exp\left\{-\frac{i}{\hbar}\int_{t_0}^{t_1} H(t')\,dt'\right\}. \tag{1.4}$$

Exactly solvable problems play a special role in nonstationary problems in quantum mechanics. But constructing exact analytic solutions of equations of motion is a difficult problem and is not always solvable. Schrödinger equation (1.1) can be exactly solved only in some cases (see, e.g., [2]–[21]). Nonstationary quantum mechanical problems can usually be solved using approximate methods, for example, time-dependent perturbation theory, the adiabatic approximation, the sudden perturbation method, and numerical methods among others. Constructing exact analytic solutions of the nonstationary Schrödinger equation describing the time-evolution of quantum systems is of great interest because exact solutions allow following variations of physical quantities characterizing the considered system most closely, i.e., obtaining more physical information about a system.

One more reason for focusing much attention on describing the behavior of physical systems analytically is as follows. Exact solutions can serve as models of real physical processes and allow, first, analyzing both the mathematical side of the problem and the physical features of the considered process deeply and comprehensively and, second, justifying the approximate methods used to solve a problem.

In particular, the nonstationary quadratic quantum systems from a free particle to a time-dependent harmonic oscillator are exactly solvable nonrelativistic systems. These systems have long attracted the attention of physicists because they are important for applications in many branches of quantum theory: statistical mechanics, superconductivity theory, atomic physics, molecular spectroscopy, nuclear physics, quantum field theory, etc. (see the references in [9], [10]). Various methods are used to study nonstationary quantum systems such as the method of invariants (integrals of motion) [4], [7], the path integral method [2], the space–time transformation method [17], the generating function method [3], [21], and the evolution operator method [22], [23]. We note that although the evolution operator method (S-matrix theory) plays a central role in quantum field theory [22], it has rarely been used to solve problems in quantum mechanics [18], [23]–[25].

Here, our purpose is to study physical properties of time-dependent quadratic nonrelativistic quantum systems using the evolution operator method. We consider a free particle with a variable mass $M(t)$, a particle with a variable mass $M(t)$ in an alternating homogeneous field, and a harmonic oscillator with a variable mass $M(t)$ and frequency $\omega(t)$ subject to a variable force $F(t)$. To obtain the evolution operators of the considered quantum systems in an explicit disentangled form, we use a simple technique to find the general solution of a certain class of differential and finite-difference nonstationary Schrödinger-type equations of motion (1.1). We show that the evolution operator method allows obtaining solutions of the nonstationary Schrödinger equation for quadratic systems directly and relatively easily. For these models, we find the general form of the evolution operators, which we use to construct the propagator, invariants, and wave functions. From the obtained general expressions, we obtain the known results as particular cases.

This paper is organized as follows. In Sec. 2, we find the general form of the solution of a certain class of differential and finite-difference nonstationary Schrödinger-type equations of motion; as a result, we find operator identities of the Baker–Campbell–Hausdorff type. We consider particular cases of these operator identities. We devote Sec. 3 to the evolution operators. In Sec. 4, we obtain propagators and their particular cases. We construct the invariants in Sec. 5 and the wave functions in Sec. 6.

2. Operator identities of the Baker–Campbell–Hausdorff type

In this section, we use a simple tool to derive some new operator identities of the Baker–Campbell–Hausdorff type, which we use in the following sections. For this, we consider the nonstationary Schrödinger-type equation of motion in the momentum y-representation [26]:

$$i\hbar\,\partial_t u(y,t) = [H_0(y,t) - i\hbar F(t) g(y)\partial_y] u(y,t). \tag{2.1}$$

Here, $H_0(y,t)$ is the time-dependent "free Hamiltonian," $F(t)$ is the time-dependent force, $g(y)$ is some function included in the "potential energy," and the dependence of the functions H_0, F, and g on their arguments is arbitrary. The Hamiltonian in (2.1) is a Hermitian operator (but it can also be non-Hermitian).

We present particular cases of Eq. (2.1):

1. If $H_0(y,t) = y^2/2M(t)$ and $g(y)=1$, where $y=p$, then it coincides with the Schrödinger equation describing a nonrelativistic particle with a variable mass $M(t)$ in an alternating homogeneous field [1].
2. If $H_0(y,t) = mc^2\cosh y$ and $g(y) = 1/mc$, where $y=\chi=\log[(p+p_0)/mc]$ is the rapidity, and $p_0=\sqrt{p^2+m^2c^2}$, then it describes the motion of a relativistic quantum particle in an alternating homogeneous field [18].
3. With $H_0(y,t) = 3iy^2/2$ and $-\hbar F(t) g(y) = y^3$, it was used in [27] to analyze differences between nonlinear classical and quantum dynamics by calculating the time evolution of the Wigner function for simple polynomial Hamiltonians.

We find a solution of Eq. (2.1) by the evolution operator method with the initial condition $u(y,t_0) = \varphi(y)$, where $\varphi(y)$ is the initial "wave" function. We pass to a new variable η in Eq. (2.1) [19]:

$$i\hbar\,\partial_t u[G^{-1}(\eta),t] = \{H_0[G^{-1}(\eta),t] - i\hbar F(t)\partial_\eta\} u[G^{-1}(\eta),t], \tag{2.2}$$

where

$$\eta = \int_{y_0}^{y} \frac{dy'}{g(y')} = G(y) - G(y_0) \tag{2.3}$$

and y_0 is an arbitrary constant. The function $g(y)$ must be such that integral (2.3) exists. For simplicity, we hereafter assume that the lower integration limit y_0 is a root of the equation $G(y_0)=0$. Then $\eta = G(y)$ and $y = G^{-1}(\eta)$.

It is easy to verify that the solution of Eq. (2.2) can be written as [18]

$$u(y,t) = \exp\biggl\{-\frac{i}{\hbar}\int_{t_0}^{t} H_0[G^{-1}(G(y)-\delta(t)+\delta(t')),t']\,dt'\biggr\}\varphi[G^{-1}(G(y)-\delta(t))], \tag{2.4}$$

where $\delta(t) = \int_{t_0}^{t} F(t')\,dt'$ is a force pulse. We introduce the notation

$$A = -\int_{t_0}^{t} h_0(y,t')\,dt' - \delta(t) g(y)\partial_y, \qquad A_0 = -\delta(t) g(y)\partial_y,$$

$$A_1 = -\int_{t_0}^{t} h_0[G^{-1}(G(y)-\delta(t)+\delta(t')),t']\,dt', \qquad A_2 = -\int_{t_0}^{t} h_0[G^{-1}(G(y)+\delta(t'))t']\,dt',$$

where $h_0(y,t) = (i/\hbar)H_0(y,t)$. It follows from (2.4) that, first, the evolution operator for Eq. (2.1) can be written in a disentangled form

$$U_g(t,t_0) \equiv e^{A} = e^{\lambda A_1} e^{A_0} e^{\lambda_1 A_2}, \tag{2.5}$$

where $\lambda = \lambda(t)$ is an arbitrary function of time and $\lambda_1 = 1 - \lambda$, and, second, the action of the operator e^{A_0} on an arbitrary function $\varphi(y)$ is defined by the formula

$$e^{-\delta g(y)\partial_y}\varphi(y) = \varphi[G^{-1}(G(y) - \delta)]. \tag{2.6}$$

We consider particular cases of operator identity (2.5) and formula (2.6). It is clear that choosing different functions $h_0(y,t)$ and $g(y)$ in (2.5), we can obtain different Baker–Campbell–Hausdorff-type identities, which we can use either to solve differential and finite-difference equations by the operator method or to disentangle exponential expressions that include noncommuting operators.

1. Let $g(y) = 1$. Then $\eta = G(y) = y$, $y = G^{-1}(\eta) = \eta$. Setting $h_0(y,t) = \varepsilon_1 y^n$ at $n \neq -1$ and $F(t) = F_0 = \text{const}$, we obtain the operator identity

$$\begin{aligned} e^{\alpha y^n + \beta\partial_y} = \exp\Big\{\lambda \frac{\alpha}{\beta(n+1)}[(y+\beta)^{n+1} - y^{n+1}]\Big\} \times \\ \times e^{\beta\partial_y} \exp\Big\{\lambda_1 \frac{\alpha}{\beta(n+1)}[y^{n+1} - (y-\beta)^{n+1}]\Big\}. \end{aligned} \tag{2.7}$$

If $n = -1$, then instead of (2.7), we obtain

$$e^{\alpha/y + \beta\partial_y} = \exp\Big\{\lambda\frac{\alpha}{\beta}\log\Big|\frac{y+\beta}{y}\Big|\Big\} e^{\beta\partial_y} \exp\Big\{\lambda_1\frac{\alpha}{\beta}\log\Big|\frac{y}{y-\beta}\Big|\Big\}. \tag{2.8}$$

Formula (2.7) can be written for any two operators $\widehat{A}$ and $\widehat{B}$ with the commutation relation $[\widehat{A},\widehat{B}] = -c$, where c is a number:

$$\begin{aligned} e^{\alpha\widehat{A}^n + \beta\widehat{B}} = \exp\Big\{\lambda \frac{\alpha}{\beta c(n+1)}[(\widehat{A}+\beta c)^{n+1} - \widehat{A}^{n+1}]\Big\} \times \\ \times e^{\beta\widehat{B}} \exp\Big\{\lambda_1 \frac{\alpha}{\beta(n+1)c}[\widehat{A}^{n+1} - (\widehat{A}-\beta c)^{n+1}]\Big\}. \end{aligned} \tag{2.9}$$

Identity (2.8) can be also generalized to the case of two operators $\widehat{A}$ and $\widehat{B}$ satisfying the commutation relation $[\widehat{A},\widehat{B}] = c\widehat{A}^2$:

$$e^{\alpha\widehat{A} + \beta\widehat{B}} = \exp\Big\{\frac{\lambda\alpha}{\beta c}\log|1 + \beta c\widehat{A}\,|\Big\} e^{\beta\widehat{B}} \exp\Big\{-\frac{\lambda_1\alpha}{\beta c}\log|1 - \beta c\widehat{A}\,|\Big\}. \tag{2.10}$$

2. Let $g(y) = y$. Then $\eta = G(y) = \log y$, $y = G^{-1}(\eta) = e^{\eta}$. If we choose the function h_0 in the form $h_0 = \varepsilon_1 y^{\nu}$, $\nu \in \mathbb{R}$, and assume a constant force $F(t) = F_0 = \text{const}$, then we obtain the identity

$$e^{\alpha y^{\nu} + \beta y\partial_y} = \exp\Big\{\frac{\lambda\alpha}{\nu\beta}(e^{\nu\beta} - 1)y^{\nu}\Big\} e^{\beta y\partial_y} \exp\Big\{\frac{\lambda_1\alpha}{\nu\beta}(1 - e^{-\nu\beta})y^{\nu}\Big\} \tag{2.11}$$

from (2.5). This identity can be generalized to the case of any two operators $\widehat{A}$ and $\widehat{B}$ with the commutation relation $[\widehat{A},\widehat{B}] = -c\widehat{A}$:

$$e^{\alpha\widehat{A}^n + \beta\widehat{B}} = \exp\Big\{\frac{\lambda\alpha}{n\beta c}(e^{n\beta c} - 1)\widehat{A}^n\Big\} e^{\beta\widehat{B}} \exp\Big\{\frac{\lambda_1\alpha}{n\beta c}(1 - e^{-n\beta c})\widehat{A}^n\Big\}, \quad n \in \mathbb{N}. \tag{2.12}$$

3. Now let $g(y) = 1/y$. Then $\eta = G(y) = y^2/2$ and $y = G^{-1}(\eta) = \sqrt{2\eta}$. Setting $h_0(y,t) = \varepsilon_1 y^4$ and $F(t) = F_0 = \text{const}$, we obtain the identity

$$e^{\alpha\widehat{A}+\beta\widehat{B}} = \exp\left\{\lambda\alpha\left[\widehat{A} + \frac{1}{2}\beta c\widehat{A}^{1/2} + \frac{1}{12}\beta^2c^2\right]\right\}e^{\beta\widehat{B}}\exp\left\{\lambda_1\alpha\left[\widehat{A} - \frac{1}{2}\beta c\widehat{A}^{1/2} + \frac{1}{12}\beta^2c^2\right]\right\}, \tag{2.13}$$

which holds for any two operators $\widehat{A}$ and $\widehat{B}$ satisfying the commutation relation $[\widehat{A},\widehat{B}] = -c\widehat{A}^{1/2}$.

We give one more important particular case of formula (2.6), where $g(y) = -y^\nu$, $\nu \in \mathbb{R}$, and $\nu \neq 1$:

$$e^{\delta y^\nu \partial_y}\varphi(y) = \varphi[(y^{\nu_1} + \nu_1\delta)^{1/\nu_1}], \tag{2.14}$$

where $\nu_1 = 1 - \nu$. For $\nu = 1$, the right-hand side of Eq. (2.1) is equal to

$$\lim_{\nu_1\to 0}\varphi[(y^{\nu_1} + \nu_1\delta)^{1/\nu_1}] = \varphi[\lim_{\nu_1\to 0}(y^{\nu_1} + \nu_1\delta)^{1/\nu_1}] = \varphi(e^\delta y). \tag{2.15}$$

We now write operator identity (2.5) in the configurational x-representation:

$$e^B = e^{\lambda B_1}e^{B_0}e^{\lambda_1 B_2}, \tag{2.16}$$

where we introduce the notation

$$\begin{aligned} &B = -\int_{t_0}^t h_0(-i\hbar\partial_x, t')\,dt' + \frac{i}{\hbar}\delta(t)g(-i\hbar\partial_x)x, \qquad B_0 = \frac{i}{\hbar}\delta(t)g(-i\hbar\partial_x)x, \\ &B_1 = -\int_{t_0}^t h_0[G^{-1}(G(-i\hbar\partial_x) - \delta(t) + \delta(t')), t']\,dt', \\ &B_2 = -\int_{t_0}^t h_0[G^{-1}(G(-i\hbar\partial_x) + \delta(t')), t']\,dt'. \end{aligned} \tag{2.17}$$

From the identity in general form (2.16), we can also obtain various operator identities of the Baker–Campbell–Hausdorff type. We here consider only two of them, which we use to disentangle the evolution operators of quadratic systems. We set $g(-i\hbar\partial_x) = 1$ in the first case and $g(-i\hbar\partial_x) = -i\hbar\partial_x$ in the second case and then obtain the respective identities

$$\begin{aligned} \exp\left\{-\int_{t_0}^t h_0(-i\hbar\partial_x, t')\,dt' + \frac{i}{\hbar}\delta(t)x\right\} &= \exp\left\{-\lambda\int_{t_0}^t h_0[-i\hbar\partial_x - \delta(t) + \delta(t'), t']dt'\right\}\times \\ &\times e^{i\hbar^{-1}\delta(t)x}\exp\left\{-\lambda_1\int_{t_0}^t h_0[-i\hbar\partial_x + \delta(t'), t']\,dt'\right\}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} \exp\left\{-\int_{t_0}^t h_0(-i\hbar\partial_x, t')\,dt + \delta(t)\partial_x x\right\} &= \exp\left\{-\lambda\int_{t_0}^t h_0(-i\hbar\partial_x e^{-\delta(t)+\delta(t')}, t')\,dt'\right\}\times \\ &\times e^{\delta(t)\partial_x x}\exp\left\{-\lambda_1\int_{t_0}^t h_0(-i\hbar\partial_x e^{\delta(t')}, t')\,dt'\right\}. \end{aligned} \tag{2.19}$$

If we now set $h_0(-i\hbar\partial_x, t) = -a(t)\partial_x^n$, $n \in \mathbb{N}$, in these identities, then we obtain two necessary identities:

$$\exp\biggl\{\int_{t_0}^t a(t')\,dt'\partial_x^n + \frac{i}{\hbar}\delta(t)x\biggr\} = \exp\biggl\{\lambda\Bigl(\frac{i}{\hbar}\Bigr)^n \int_{t_0}^t a(t')[-i\hbar\partial_x - \delta(t) + \delta(t')]^n\,dt'\biggr\} \times \\ \times e^{i\hbar^{-1}\delta(t)x} \exp\biggl\{\lambda_1\Bigl(\frac{i}{\hbar}\Bigr)^n \int_{t_0}^t a(t')[-i\hbar\partial_x + \delta(t')]^n\,dt'\biggr\}, \tag{2.20}$$

$$\exp\biggl\{\int_{t_0}^t a(t')\,dt'\partial_x^n + \delta(t)\partial_x x\biggr\} = \exp\biggl\{\lambda e^{-n\delta(t)} \int_{t_0}^t a(t') e^{n\delta(t')}\,dt'\,\partial_x^n\biggr\} \times \\ \times e^{\delta(t)\partial_x x} \exp\biggl\{\lambda_1 \int_{t_0}^t a(t') e^{n\delta(t')}\,dt'\,\partial_x^n\biggr\}. \tag{2.21}$$

3. Evolution operators

In what follows, the evolution operator of a quantum system, which (as stated above) contains complete information about the system, plays a very important role. In this section, we construct the evolution operators for the considered systems. It is well known that they can be described by Hermitian Hamiltonians.

3.1. Free particle with a variable mass. We consider the time-dependent Schrödinger equation in $\mathbb{R}$ for a free particle with a variable mass $M(t)$:

$$\widehat{S}_{\mathrm{F}}(x,t)\psi_{\mathrm{F}}(x,t) = 0, \qquad \widehat{S}_{\mathrm{F}}(x,t) = i\hbar\partial_t + \frac{\hbar^2}{2M(t)}\partial_x^2 + V_0(t), \tag{3.1}$$

where $V_0(t)$ is a potential well (barrier) whose depth (height) varies with time. The evolution operator for Eq. (3.1) has the simplest form:

$$U_{\mathrm{F}}(x,t) = e^{i\Lambda_0(t) + i\hbar S_2(t)\partial_x^2}, \tag{3.2}$$

where

$$\Lambda_0(t) = \frac{1}{\hbar}\int_{t_0}^t V_0(t')\,dt', \qquad S_2(t) = \int_{t_0}^t \frac{dt'}{2M(t')}.$$

3.2. Particle with a variable mass in an alternating homogeneous field. A particle with a variable mass in an alternating homogeneous field can be described by the Schrödinger equation

$$\widehat{S}_{\mathrm{L}}(x,t)\psi_{\mathrm{L}}(x,t) = 0, \qquad \widehat{S}_{\mathrm{L}}(x,t) = i\hbar\partial_t + \frac{\hbar^2}{2M(t)}\partial_x^2 + F(t)x + V_0(t). \tag{3.3}$$

In this case, the evolution operator in a disentangled form is given by the formula [18], [23]

$$U_{\mathrm{L}}(x,t) = V_{\mathrm{L}}(x,t)U_{\mathrm{F}}(x,t), \tag{3.4}$$

where we introduce the notation

$$V_{\mathrm{L}}(x,t) = e^{i\varphi_0(x,t)} e^{-S_1(t)\partial_x} = e^{-S_1(t)\partial_x} e^{i\hbar^{-1}[M(t)\dot{S}_1(t)x + \sigma_{\mathrm{L}}(t)]}, \tag{3.5}$$

$$\varphi_0(x,t) = \frac{1}{\hbar}[x\delta(t) - S_0(t)], \tag{3.6}$$

$$S_0(t) = \int_{t_0}^t \frac{\delta^2(t')}{2M(t')}\,dt', \qquad S_1(t) = \int_{t_0}^t \frac{\delta(t')}{M(t')}\,dt'. \tag{3.7}$$

Here, σ_{L} is the classical action for a particle in an alternating homogeneous field,

$$\sigma_{\text{L}}(t) = \delta(t)S_1(t) - S_0(t) = \int_{t_0}^{t}\left[\frac{1}{2}M(t')\dot{S}_1^2(t') + F(t')S_1(t')\right]dt',$$

and the function $S_1(t)$ satisfies the equation

$$\frac{d}{dt}[M(t)\dot{S}_1(t)] = F(t) \tag{3.8}$$

with the initial conditions $S_1(t_0) = 0$ and $\dot{S}_1(t_0) = 0$. For $M(t) = m = \text{const}$, we have

$$S_0(t) = \frac{\delta_2(t)}{2m}, \qquad S_1(t) = \frac{\delta_1(t)}{m}, \qquad S_2(t) = \frac{\tau}{2m},$$

where $\delta_1(t) = \int_{t_0}^{t}\delta(t')\,dt'$, $\delta_2(t) = \int_{t_0}^{t}\delta^2(t')\,dt'$, and $\tau = t - t_0$.

Formulas (3.2) and (3.4) establish a unitary relation between a free quantum particle with a variable mass and a quantum particle with a variable mass in an alternating homogeneous field [23]:

$$\begin{aligned} &\psi_{\text{L}}(x,t) = V_{\text{L}}(x,t)\psi_{\text{F}}(x,t), \qquad &&\psi_{\text{F}}(x,t) = V_{\text{L}}^{-1}(x,t)\psi_{\text{L}}(x,t),\\ &\widehat{S}_{\text{L}} = V_{\text{L}}\widehat{S}_{\text{F}}V_{\text{L}}^{-1}, &&\widehat{S}_{\text{F}} = V_{\text{L}}^{-1}\widehat{S}_{\text{L}}V_{\text{L}}. \end{aligned}$$

3.3. Harmonic oscillator with a variable mass and frequency subject to a variable force. A harmonic oscillator with a variable mass and frequency subject to a variable force can be described by the Schrödinger equation

$$\begin{aligned} &\widehat{S}_{\text{H}}(x,t)\psi_{\text{H}}(x,t) = 0,\\ &\widehat{S}_{\text{H}}(x,t) = i\hbar\partial_t + \frac{\hbar^2}{2M(t)}\partial_x^2 - \frac{1}{2}M(t)\omega^2(t)x^2 + F(t)x + V_0(t). \end{aligned} \tag{3.9}$$

To find the evolution operator, we first reduce problem (3.9) to the case where the oscillator is not subject to a force. For this, we perform the unitary transformation of the wave function ψ_{H} (cf. [3], [21]):

$$\psi_{\text{H}}(x,t) = U_1(x,t)\psi_{\text{H}}^{(0)}(x,t), \tag{3.10}$$

where the unitary operator U_1 is

$$U_1(x,t) = e^{-\xi(t)\partial_x}e^{i\hbar^{-1}[M(t)\dot{\xi}(t)x + \sigma_{\text{H}}(t)]}, \tag{3.11}$$

the function $\xi(t)$ satisfies the classical equation of motion

$$\frac{d}{dt}[M(t)\dot{\xi}(t)] + M(t)\omega^2(t)\xi(t) = F(t), \tag{3.12}$$

and $\sigma_{\text{H}}(t)$ is the classical action of a harmonic oscillator with $F(t) \neq 0$,

$$\sigma_{\text{H}}(t) = \int_{t_0}^{t}\left[\frac{1}{2}M(t')\dot{\xi}^2(t') - \frac{1}{2}M(t')\omega^2(t')\xi^2(t') + F(t')\xi(t')\right]dt'. \tag{3.13}$$

Without loss of generality, we can take a real solution $\xi(t)$ of Eq. (3.12), which together with its first derivative $\dot{\xi}(t)$ vanishes at the initial instant: $\xi(t_0) = 0$ and $\dot{\xi}(t_0) = 0$. We hence obtain the initial condition for operator (3.11) in the form $U_1(x, t_0) = 1$. In addition, it is clear that if $F(t) \equiv 0$, then necessarily $\xi(t) \equiv 0$.

As a result, (3.9) becomes the Schrödinger equation for a harmonic oscillator with a variable mass and frequency:

$$\widehat{S}_{\mathrm{H}}^{(0)}(x,t)\psi_{\mathrm{H}}^{(0)}(x,t) = 0, \qquad \widehat{S}_{\mathrm{H}}^{(0)}(x,t) = i\hbar\partial_t + \frac{\hbar^2}{2M(t)}\partial_x^2 - \frac{1}{2}M(t)\omega^2(t)x^2 + V_0(t). \tag{3.14}$$

We next choose the wave function $\psi_{\mathrm{H}}^{(0)}$ in the form

$$\psi_{\mathrm{H}}^{(0)}(x,t) = U_2(x,t)\psi_{\mathrm{H}_2}(x,t), \qquad U_2(x,t) = e^{i\alpha(t)x^2}, \tag{3.15}$$

where the real function $\alpha(t)$ satisfying the initial condition $\alpha(t_0) \equiv 0$ is defined from the nonlinear first-order differential equation (Riccati equation)

$$\dot{\alpha}(t) + \frac{2\hbar}{M(t)}\alpha^2(t) = -\frac{1}{2\hbar}M(t)\omega^2(t). \tag{3.16}$$

It is well known that introducing a new function $\eta(t)$ instead of $\alpha(t)$ via the formula

$$\alpha(t) = \frac{M(t)\dot{\eta}(t)}{2\hbar\eta(t)}, \tag{3.17}$$

we can reduce Riccati equation (3.16) to a linear homogeneous second-order differential equation

$$\frac{d}{dt}[M(t)\dot{\eta}(t)] + M(t)\omega^2(t)\eta(t) = 0. \tag{3.18}$$

It follows from the initial condition $\alpha(t_0) = 0$ that $\eta(t_0) \neq 0$ and $\dot{\eta}(t_0) = 0$. Substituting (3.15) in (3.14) yields the equation for $\psi_{\mathrm{H}_2}(x,t)$:

$$\begin{aligned} &\widehat{S}_{\mathrm{H}_2}(x,t)\psi_{\mathrm{H}_2}(x,t) = 0, \\ &\widehat{S}_{\mathrm{H}_2}(x,t) = i\hbar\partial_t + \frac{\hbar^2}{2M(t)}\partial_x^2 + \frac{i\hbar^2}{M(t)}\alpha(t)(\partial_x x + x\partial_x) + V_0(t). \end{aligned} \tag{3.19}$$

Taking formula (2.21) with $n = 2$ into account, we can now represent the evolution operator $U_3(x,t)$ for Eq. (3.19) in a disentangled form as

$$U_3(x,t) = e^{i\Lambda_0(t) + \frac{b(t)}{2}(\partial_x x + x\partial_x)} e^{iS(t)\partial_x^2}, \tag{3.20}$$

where

$$b(t) = -2\hbar\int_{t_0}^{t}\frac{\alpha(t')}{M(t')}\,dt', \qquad S(t) = \hbar\int_{t_0}^{t}\frac{e^{2b(t')}}{2M(t')}\,dt' \tag{3.21}$$

or, in terms of the function $\eta(t)$ in (3.17),

$$b(t) = \log\frac{\eta(t_0)}{\eta(t)}, \qquad S(t) = \hbar\eta^2(t_0)\int_{t_0}^{t}\frac{dt'}{2M(t')\eta^2(t')}. \tag{3.22}$$

It is now clear that evolution operator (3.9) for harmonic oscillator with a variable mass and frequency subject to a variable force is equal to the product of the unitary operators $U_1(x,t)$ in (3.11), $U_2(x,t)$ in (3.15), and $U_3(x,t)$ in (3.20), i.e., $U_{\rm H} = U_1 U_2 U_3$ or, explicitly,

$$U_{\rm H}(x,t) = e^{-\frac{b(t)}{2}+i\Lambda_0(t)+\frac{i}{\hbar}\sigma_{\rm H}(t)} e^{-\xi(t)\partial_x} e^{\frac{i}{\hbar}M(t)\dot{\xi}(t)x} e^{i\alpha(t)x^2} e^{b(t)\partial_x x} e^{iS(t)\partial_x^2}. \tag{3.23}$$

We represent it as

$$U_{\rm H}(x,t) = U_1(x,t)U_{\rm H}^{(0)}(x,t), \tag{3.24}$$

where

$$U_{\rm H}^{(0)}(x,t) = e^{\frac{b(t)}{2}+i\Lambda_0(t)} e^{i\alpha(t)x^2} e^{b(t)x\partial_x} e^{iS(t)\partial_x^2} \tag{3.25}$$

is the evolution operator for a harmonic oscillator with a variable mass and frequency. It follows from (3.24) that the harmonic oscillator problem is unitarily equivalent to the problem of an oscillator subject to a force. By virtue of the operator relation $e^{b\partial_x x}e^{iS\partial_x^2} = e^{iSe^{-2b}\partial_x^2}e^{b\partial_x x}$ (see identities (2.21)), we rewrite $U_{\rm H}^{(0)}(x,t)$ given by (3.25) as

$$U_{\rm H}^{(0)}(x,t) = e^{\frac{b(t)}{2}+i\Lambda_0(t)} e^{i\alpha(t)x^2} e^{iSe^{-2b}\partial_x^2} e^{bx\partial_x}. \tag{3.26}$$

We note that there is a close analogy between the operators $V_{\rm L}$ given by (3.5) and U_1 given by (3.11). First, they are operators "generating" the action of the force: $V_{\rm L}$ generates the action of $F(t)$ on a free particle, and U_1 generates the action of $F(t)$ on an oscillator; second, both operators are expressed in terms of a classical action: $V_{\rm L}$ is expressed in terms of the classical action of a particle in a homogeneous field, and U_1 is expressed in terms of the classical action of an oscillator in a homogeneous field. In addition, Eq. (3.12) for the "coordinate" $\xi(t)$ of a classical oscillator in a homogeneous field with $\omega(t) = 0$ transforms into Eq. (3.8) for the "coordinate" $S_1(t)$ of a classical particle in a homogeneous field. Therefore, in the limit as $\omega \to 0$, we have the relations

$$\lim_{\omega\to 0}\xi(t) = S_1(t), \qquad \lim_{\omega\to 0}S(t) = \hbar S_2(t). \tag{3.27}$$

The first relation is obvious, and we prove the second. As $\omega \to 0$, the limits

$$\lim_{\omega\to 0}\alpha(t) = 0, \qquad \lim_{\omega\to 0}\dot{\eta}(t) = 0, \tag{3.28}$$

must exist from physical considerations, i.e., $\eta(t) \to$ const or $\eta(t) \to 0$. Consequently, as $\omega \to 0$, equalities (3.27) follow from Eq. (3.21). It is clear from limit relations (3.27) and (3.28) that

$$\lim_{\omega\to 0}U_{\rm H}^{(0)}(x,t) = U_{\rm F}(x,t), \qquad \lim_{\omega\to 0}U_{\rm H}(x,t) = U_{\rm L}(x,t), \tag{3.29}$$

i.e., the evolution operator for an oscillator in the limit $\omega \to 0$ coincides with the evolution operator for a free particle, and the evolution operator for an oscillator subject to a force coincides with the evolution operator for a particle in a homogeneous field.

We introduce the notation

$$V_{\rm FH}^{(0)}(x,t) = e^{b(t)/2}e^{i\alpha(t)x^2}e^{b(t)x\partial_x}, \qquad V_{\rm FH}(x,t) = U_1(x,t)V_{\rm FH}^{(0)}(x,t) \tag{3.30}$$

and the notions of a renormalized mass $M_{\rm Ren}(t) = M(t)e^{-2b(t)}$, the renormalized free Schrödinger equation

$$\widehat{S}_{\rm F}^{\rm Ren}(x,t)\psi_{\rm F}^{\rm Ren}(x,t) = 0, \qquad \widehat{S}_{\rm F}^{\rm Ren}(x,t) = i\hbar\partial_t + \frac{\hbar^2}{2M_{\rm Ren}(t)}\partial_x^2 + V_0(t), \tag{3.31}$$

and the renormalized evolution operator for a free particle

$$U_{\text{F}}^{\text{Ren}}(x,t) = U_{\text{F}}(x,t)\big|_{M\to M_{\text{Ren}}} = e^{i\Lambda_0(t)}e^{iS(t)\partial_x^2}. \tag{3.32}$$

Here, the difference from formulas (3.1) and (3.2) is that the renormalized mass appears in expressions (3.31) and (3.32).

Comparing $U_{\text{H}}^{(0)}$ given by (3.25), U_{H} given by (3.24), and $U_{\text{F}}^{\text{Ren}}$ given by (3.32), we conclude that the harmonic oscillator problem (subject to a force or not) is unitarily equivalent to the problem of a free particle with a renormalized mass, i.e., we have the relations

$$U_{\text{H}}^{(0)}(x,t) = V_{\text{FH}}^{(0)}(x,t)U_{\text{F}}^{\text{Ren}}(x,t), \qquad U_{\text{H}}(x,t) = V_{\text{FH}}(x,t)U_{\text{F}}^{\text{Ren}}(x,t). \tag{3.33}$$

Based on formulas (3.24) and (3.33), we therefore conclude as follows:

1. The operator $U_1(x,t)$ transforms each solution of Eq. (3.14) into a solution of Eq. (3.9), and the operator $U_1^{-1}(x,t)$ performs the inverse transformation.
2. The operator $V_{\text{FH}}^{(0)}(x,t)$ transforms each solution of renormalized Schrödinger equation (3.31) for a free particle into a solution of Eq. (3.14), and the operator $V_{\text{FH}}^{(0)-1}(x,t)$ performs the inverse transformation.
3. Similarly, the operator $V_{\text{FH}}(x,t)$ transforms each solution of renormalized equation (3.31) into a solution of Eq. (3.9), and the operator $V_{\text{FH}}^{-1}(x,t)$ performs the inverse transformation.

We hence have

$$\begin{aligned}
&\psi_{\text{H}}(x,t) = U_1(x,t)\psi_{\text{H}}^{(0)}(x,t), && \psi_{\text{H}}^{(0)}(x,t) = U_1^{-1}(x,t)\psi_{\text{H}}(x,t),\\
&\psi_{\text{H}}^{(0)}(x,t) = V_{\text{FH}}^{(0)}(x,t)\psi_{\text{F}}^{\text{Ren}}(x,t), && \psi_{\text{F}}^{\text{Ren}}(x,t) = V_{\text{FH}}^{(0)-1}(x,t)\psi_{\text{H}}^{(0)}(x,t),\\
&\psi_{\text{H}}(x,t) = V_{\text{FH}}(x,t)\psi_{\text{F}}^{\text{Ren}}(x,t), && \psi_{\text{F}}^{\text{Ren}}(x,t) = V_{\text{FH}}^{-1}(x,t)\psi_{\text{H}}(x,t).
\end{aligned} \tag{3.34}$$

In addition, the Schrödinger operators $\widehat{S}_{\text{F}}^{\text{Ren}}(x,t)$, $\widehat{S}_{\text{H}}^{(0)}(x,t)$, and $\widehat{S}_{\text{H}}(x,t)$ are related to each other by the formulas

$$\begin{aligned}
&\widehat{S}_{\text{H}} = U_1\widehat{S}_{\text{H}}^{(0)}U_1^{-1}, && \widehat{S}_{\text{H}}^{(0)} = U_1^{-1}\widehat{S}_{\text{H}}U_1,\\
&\widehat{S}_{\text{H}}^{(0)} = V_{\text{FH}}^{(0)}\widehat{S}_{\text{F}}^{\text{Ren}}V_{\text{FH}}^{(0)-1}, && \widehat{S}_{\text{F}}^{\text{Ren}} = V_{\text{FH}}^{(0)-1}\widehat{S}_{\text{H}}^{(0)}V_{\text{FH}}^{(0)},\\
&\widehat{S}_{\text{H}} = V_{\text{FH}}\widehat{S}_{\text{F}}^{\text{Ren}}V_{\text{FH}}^{-1}, && \widehat{S}_{\text{F}}^{\text{Ren}} = V_{\text{FH}}^{-1}\widehat{S}_{\text{H}}V_{\text{FH}}.
\end{aligned} \tag{3.35}$$

4. Propagators

Because the propagator at the initial instant is $K(x_2,t_0;x_1,t_0) = \delta(x_2-x_1)$, in analogy to formula (1.2), there is the well-known relation

$$K(x_2,t;x_1,t_0) = \theta(t-t_0)U(x_2,t)\delta(x_2-x_1). \tag{4.1}$$

We first find the propagators for the considered systems (3.1), (3.3), and (3.9) in the x-representation using formula (4.1). In this case, we take into account that the action of the operators $e^{\alpha\partial_x^2}$, $e^{\alpha x\partial_x}$, and $e^{\alpha\partial_x}$ on an arbitrary function $f(x)$ is given by the equalities

$$\begin{aligned}
&e^{\alpha\partial_x^2}f(x) = \frac{1}{\sqrt{4\pi\alpha}}\int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4\alpha}}f(\xi)\,d\xi,\\
&e^{\alpha x\partial_x}f(x) = f(e^{\alpha}x), \qquad e^{\alpha\partial_x}f(x) = f(x+\alpha).
\end{aligned} \tag{4.2}$$

We obtain the following results:

1. The propagator for a free particle with a variable mass is

$$K_{\rm F}(x_2,t;x_1,t_0)=\frac{\theta(t-t_0)}{\sqrt{4\pi i\hbar S_2(t)}}e^{\frac{i(x_2-x_1)^2}{4\hbar S_2(t)}+i\Lambda_0(t)}. \tag{4.3}$$

For $M(t)=m=\text{const}$, we obtain a known formula from (4.3) [2]:

$$K_{\rm F}(x_2,t;x_1,t_0)\big|_{M=m}=\theta(t-t_0)\sqrt{\frac{m}{2\pi i\hbar(t-t_0)}}e^{\frac{im(x_2-x_1)^2}{2\hbar(t-t_0)}+i\Lambda_0(t)}. \tag{4.4}$$

2. The propagator for a free-particle with a renormalized variable mass is

$$K_{\rm F}^{\rm Ren}(x_2,t;x_1,t_0)=\theta(t-t_0)U_{\rm F}^{\rm Ren}(x_2,t)\delta(x_2-x_1)=\frac{\theta(t-t_0)}{\sqrt{4\pi iS(t)}}e^{\frac{i(x_2-x_1)^2}{4S(t)}+i\Lambda_0(t)}. \tag{4.5}$$

3. The propagator for a particle with a variable mass in an alternating homogeneous field is

$$K_{\rm L}(x_2,t;x_1,t_0)=\frac{\theta(t-t_0)}{\sqrt{4\pi i\hbar S_2(t)}}\exp\left\{i\varphi_0(x_2,t)+\frac{i[x_2-x_1-S_1(t)]^2}{4\hbar S_2(t)}+i\Lambda_0(t)\right\}. \tag{4.6}$$

This is the most general expression for the propagator of a quantum particle in an external homogeneous field where both the mass and the force acting on the particle from the field arbitrarily depend on the time. This expression can also be obtained from propagator (4.3) of a free particle using the action of the shift operator $V_{\rm L}(x,t)$ given by (3.5):

$$K_{\rm L}(x_2,t;x_1,t_0)=V_{\rm L}(x_2,t)K_{\rm F}(x_2,t;x_1,t_0)=e^{i\varphi_0(x_2,t)}K_{\rm F}(x_2-S_1(t),t;x_1,t_0). \tag{4.7}$$

Obviously, propagator (4.6) coincides with propagator (4.3) if $F(t)=0$.

4. The propagator of a harmonic oscillator with a variable mass and frequency subject to a variable force is

$$\begin{aligned}K_{\rm H}(x_2,t;x_1,t_0)=&\frac{\theta(t-t_0)}{\sqrt{4\pi iS(t)}}e^{\frac{b(t)}{2}+i\Lambda_0(t)+\frac{i}{\hbar}\sigma_{\rm H}(t)}\times\\&\times\exp\left\{i\left[\frac{M(t)\dot\xi(t)}{\hbar}(x_2-\xi(t))+\alpha(t)(x_2-\xi(t))^2\right]\right\}\times\\&\times\exp\left\{i\frac{[e^{b(t)}(x_2-\xi(t))-x_1]^2}{4S(t)}\right\}.\end{aligned} \tag{4.8}$$

This is the most general expression for the propagator of a harmonic oscillator with a variable mass and frequency subject to a variable force. We note that according to relations (3.33), propagator (4.8) can be obtained from the renormalized free-particle propagator (4.5) of a free particle via the action of the operator $V_{\rm FH}$:

$$\begin{aligned}K_{\rm H}(x_2,t;x_1,t_0)=&V_{\rm FH}(x_2,t)K_{\rm F}^{\rm Ren}(x_2,t;x_1,t_0)=\\=&e^{\frac{b(t)}{2}+\frac{i}{\hbar}\sigma_{\rm H}(t)}\exp\left\{i\left[\frac{M(t)\dot\xi(t)}{\hbar}(x_2-\xi(t))+\alpha(t)(x_2-\xi(t))^2\right]\right\}\times\\&\times K_{\rm F}^{\rm Ren}(e^{b(t)}[x_2-\xi(t)],t;x_1,t_0).\end{aligned} \tag{4.9}$$

We consider particular cases of propagator (4.8):

4.1. Let $M(t) = m = \text{const}$ and $\omega(t) = \omega_0 = \text{const}$. Then the solutions of Eqs. (3.12) and (3.18) with the initial conditions $\xi(t_0) = 0$, $\dot{\xi}(t_0) = 0$, $\eta(t_0) \neq 0$, $\dot{\eta}(t_0) = 0$ are

$$\xi(t) = \frac{1}{m\omega_0} \int_{t_0}^{t} F(t') \sin\omega_0(t - t')\, dt', \qquad \eta(t) = \omega_0 \cos(\omega_0 \tau).$$

In this case, we obtain a known formula for the propagator of a stationary harmonic oscillator subject to a variable force [2].

4.2. Let $\omega(t) = 0$. Because $b(t) = 0$, $\alpha(t) = 0$, $\xi(t) = S_1(t)$, $\eta(t) = 1$, and $S(t) = \hbar S_2(t)$, $\sigma_{\text{H}}(t) = S_1(t)\delta(t) - S_0(t) = \sigma_{\text{L}}(t)$ in this case, propagator (4.8) coincides with propagator (4.6), i.e., $K_{\text{H}}|_{\omega=0} = K_{\text{L}}$.

4.3. Let $F(t) = 0$. In this case, $\xi(t) = 0$, $\sigma_{\text{H}}(t) = 0$, and we obtain the propagator for a harmonic oscillator with a variable mass and frequency from (4.8):

$$K_{\text{H}}^{(0)}(x_2, t; x_1, t_0) = \frac{\theta(t - t_0)}{\sqrt{2\pi i \mu_1(t)}} e^{i\Lambda_0(t)} e^{i[\alpha_1(t)x_2^2 + \beta_1(t)x_2x_1 + \gamma_1(t)x_1^2]}, \tag{4.10}$$

where

$$\mu_1(t) = 2S(t)e^{-b(t)}, \qquad \alpha_1(t) = \alpha(t) + \frac{e^{2b(t)}}{4S(t)}, \qquad \beta_1(t) = -\frac{1}{\mu_1(t)}, \qquad \gamma_1(t) = \frac{1}{4S(t)}. \tag{4.11}$$

Obviously, there is a relation between the propagators of an oscillator and of a free particle with a renormalized mass analogous to relation (4.9),

$$\begin{aligned} K_{\text{H}}^{(0)}(x_2, t; x_1, t_0) &= V_{\text{FH}}^{(0)}(x_2, t) K_{\text{F}}^{\text{Ren}}(x_2, t; x_1, t_0) = \\ &= e^{b(t)/2} e^{i\alpha(t)x_2^2} K_{\text{F}}^{\text{Ren}}(e^{b(t)}x_2, t; x_1, t_0). \end{aligned} \tag{4.12}$$

We also present the relation between propagators (4.8) and (4.10):

$$\begin{aligned} K_{\text{H}}(x_2, t; x_1, t_0) &= U_1(x_2, t) K_{\text{H}}^{(0)}(x_2, t; x_1, t_0) = \\ &= e^{\frac{i}{\hbar}(M(t)\dot{\xi}(t)(x_2 - \xi(t)) + \sigma_{\text{H}}(t))} K_{\text{H}}^{(0)}(x_2 - \xi(t), t; x_1, t_0). \end{aligned} \tag{4.13}$$

Formula (4.10) yields the most general expression for the propagator for a harmonic oscillator with a variable mass and frequency. All the propagators found in [10] can be obtained from this propagator as particular cases. We here consider only three particular cases of formula (4.10):

4.3.1. Let $\omega(t) = 0$. In this case $\alpha(t) = 0$, $b(t) = 0$, $\eta(t) = 1$, and $S(t) = \hbar S_2(t)$. Then

$$\mu_1(t) = 2\hbar S_2(t), \qquad \alpha_1(t) = \gamma_1(t) = \frac{1}{4\hbar S_2(t)}, \qquad \beta_1(t) = -\frac{1}{\mu_1(t)},$$

and propagator (4.10) coincides with propagator (4.3) for a free particle with a variable mass.

4.3.2. Let $M(t) = \omega_0^{-1} e^{2\lambda t}$, $\omega(t) = \omega_0$, $t_0 = 0$, and $\hbar = 1$. This choice corresponds to the model of a damped quantum oscillator: the Caldirola–Kanai model (see, e.g., [10], [28]), which is defined by the Hamiltonian

$$H = \frac{\omega_0}{2}(e^{-2\lambda t}\hat{p}^2 + e^{2\lambda t}\hat{x}^2). \tag{4.14}$$

In this case, from Eqs. (3.17), (3.18), and (3.22), we obtain

$$\begin{aligned} &\eta(t) = e^{-\lambda t} Q(t), \qquad Q(t) = \omega\cos(\omega t) + \lambda\sin(\omega t), \\ &\alpha(t) = -\frac{\omega_0\sin(\omega t)}{2\hbar Q(t)} e^{2\lambda t}, \qquad b(t) = \log\frac{\omega e^{\lambda t}}{Q(t)}, \qquad S(t) = \frac{\hbar\omega_0}{2Q(t)}\sin(\omega t), \end{aligned} \tag{4.15}$$

where $\omega = \sqrt{\omega_0^2 - \lambda^2} > 0$. For coefficients (4.11) in the case of Hamiltonian (4.14), we obtain the expressions

$$\mu_1(t) = \frac{\hbar\omega_0}{\omega} e^{-\lambda t}\sin(\omega t), \qquad \alpha_1(t) = \frac{1}{2\hbar\omega_0}(\omega\cot(\omega t) - \lambda)e^{2\lambda t}, \qquad \gamma_1(t) = \frac{Q(t)}{2\hbar\omega_0\sin(\omega t)},$$

which coincide with the results obtained in [10] in a different way.

4.3.3. Let $M(t) = 1$, $\omega^2(t) = \omega_0^2 + 2\lambda^2/\cosh^2(\lambda t)$, $t_0 = 0$, and $\hbar = 1$. Then we obtain the Hamiltonian of a parametric oscillator [10],

$$H = \frac{1}{2}(\hat{p}^2 + \omega^2(t)\hat{x}^2). \tag{4.16}$$

In this case, the solution of Eq. (3.18) with the initial conditions $\eta(0) = \omega_0$ and $\dot{\eta}(0) = 0$ can be written as

$$\eta(t) = \omega_0\cos(\omega_0 t) - \lambda\sin(\omega_0 t)\tanh(\lambda t). \tag{4.17}$$

Consequently, $b(t) = \log(\omega_0/\eta(t))$, $\alpha(t) = \dot{\eta}(t)/2\hbar\eta(t)$, and the function $S(t)$ is

$$S(t) = \frac{\hbar\omega_0}{2(\omega_0^2 + \lambda^2)\eta(t)}[\omega_0\sin(\omega_0 t) + \lambda\cos(\omega_0 t)\tanh(\lambda t)]. \tag{4.18}$$

Coefficients (4.11) are

$$\begin{aligned} \mu_1(t) &= \frac{\hbar\omega_0}{\omega_0^2 + \lambda^2}[\omega_0\sin(\omega_0 t) + \lambda\cos(\omega_0 t)\tanh(\lambda t)], \\ \alpha_1(t) &= \frac{[\omega_0^2 + \lambda^2\cosh^{-2}(\lambda t)]\cos(\omega_0 t) - \lambda\omega_0\sin(\omega_0 t)\tanh(\lambda t)}{2[\omega_0\sin(\omega_0 t) + \lambda\cos(\omega_0 t)\tanh(\lambda t)]}, \end{aligned} \tag{4.19}$$

and these expressions also coincide with the results in [10].

We note that in view of the limit formula $\lim_{\alpha\to 0}(\pi\alpha)^{-1/2}e^{-x^2/\alpha} = \delta(x)$, it is easy to verify that all the obtained propagators satisfy the required initial condition $\lim_{t\to t_0} K = \delta(x_2 - x_1)$.

5. Invariants: General expressions

Invariants play an important role in studying properties of nonstationary quadratic systems [4], [7]. In this section, we construct invariant operators for the considered systems using the method of evolution operator $U(t)$. By the invariants of a quantum system, we mean time-dependent operators $I(t)$ whose means are independent of time, i.e., $d\bar{I}(t)/dt = 0$. An invariant $I(t)$ commutes with the Schrödinger operator, $[\widehat{S}(t), I(t)] = 0$, and consequently transforms each solution of the Schrödinger equation into another solution of this equation. On the other hand, we know (see, e.g., [7]) that we can use the evolution operator to construct $2N$ independent invariants for any quantum system, where N is the number of degrees of freedom in the system. In our case, there are two independent (basis) invariants $\hat{x}_0(t)$ and $\hat{p}_0(t)$. We construct these invariants using the formulas

$$\hat{x}_0(t) = U(t)\hat{x}U^{-1}(t), \qquad \hat{p}_0(t) = U(t)\hat{p}U^{-1}(t). \tag{5.1}$$

The physical meaning of these invariants is that they are operators of the initial coordinate and momentum. In general, operators (5.1) are linear combinations of the operators $\hat{x}$ and $\hat{p}$ with time-dependent coefficients,

$$\hat{x}_0(t) = e_1(t)\hat{x} + e_2(t)\hat{p} + e_3(t), \qquad \hat{p}_0(t) = d_1(t)\hat{x} + d_2(t)\hat{p} + d_3(t), \tag{5.2}$$

where $\hat{x}_0(t_0) = \hat{x}$ and $\hat{p}_0(t_0) = \hat{p}$, i.e.,

$$e_1(t_0) = d_2(t_0) = 1, \qquad e_2(t_0) = e_3(t_0) = d_1(t_0) = d_3(t_0) = 0. \tag{5.3}$$

It follows from the commutation relation $[\hat{p}_0(t), \hat{x}_0(t)] = [\hat{p}, \hat{x}] = -i\hbar$ that the coefficients in (5.2) satisfy the equality $d_2e_1 - d_1e_2 = 1$, and the equations for these coefficients follow from the commutation relations $[\widehat{S}, \hat{x}_0] = 0$ and $[\widehat{S}, \hat{p}_0] = 0$. If we write the system Hamiltonian as $H = \alpha_2(t)\hat{p}^2 + \beta_2(t)\hat{x}^2 - F(t)\hat{x}$, then these equations become

$$\begin{aligned} &\dot{e}_1 = 2\beta_2 e_2, \qquad \dot{e}_2 = -2\alpha_2 e_1, \qquad \dot{e}_3 = -Fe_2, \\ &\dot{d}_1 = 2\beta_2 d_2, \qquad \dot{d}_2 = -2\alpha_2 d_1, \qquad \alpha_2 d_1 \dot{d}_3 = -Fd_2. \end{aligned} \tag{5.4}$$

We now present the most general forms of the operators $\hat{x}_0(t)$ and $\hat{p}_0(t)$ for the considered quadratic systems explicitly:

1. For a free quantum particle with a variable mass, we have

$$\hat{x}_{0\mathrm{F}}(t) = \hat{x} - 2S_2(t)\hat{p}, \qquad \hat{p}_{0\mathrm{F}}(t) = \hat{p}, \tag{5.5}$$

   i.e., $e_1 = 1$, $e_2 = -2S_2$, $e_3 = 0$, $d_1 = 0$, $d_2 = 1$, and $d_3 = 0$. If $M(t) = m = \text{const}$, then

$$\hat{x}_{0\mathrm{F}} = \hat{x} - \frac{\hat{p}}{m}\tau, \qquad \hat{p}_{0\mathrm{F}} = \hat{p}. \tag{5.6}$$

2. For a particle with a variable mass in an alternating homogeneous field, we have

$$\hat{x}_{0\mathrm{L}}(t) = \hat{x} - 2S_2(t)\hat{p} + 2\delta(t)S_2(t) - S_1(t), \qquad \hat{p}_{0\mathrm{L}}(t) = \hat{p} - \delta(t), \tag{5.7}$$

   i.e., $e_1 = 1$, $e_2 = -2S_2$, $e_3 = 2\delta S_2 - S_1$, $d_1 = 0$, $d_2 = 1$, and $d_3 = -\delta$. If $M(t) = m = \text{const}$, then

$$\hat{x}_{0\mathrm{L}}(t) = \hat{x} - \frac{\hat{p}}{m}\tau + \frac{1}{m}(\delta\tau - \delta_1), \qquad \hat{p}_{0\mathrm{L}}(t) = \hat{p} - \delta. \tag{5.8}$$

3. For a harmonic oscillator with a variable mass and frequency subject to a variable force, we have

$$\begin{aligned} \hat{x}_{0\mathrm{H}}(t) &= -M(t)\dot{a}_2(t)\hat{x} + a_2(t)\hat{p} + M(t)\Delta_2(t), \\ \hat{p}_{0\mathrm{H}}(t) &= -M(t)\dot{a}_1(t)\hat{x} + a_1(t)\hat{p} + M(t)\Delta_1(t), \end{aligned} \tag{5.9}$$

i.e., $e_1 = -M\dot{a}_2$, $e_2 = a_2$, $e_3 = M\Delta_2$, $d_1 = -M\dot{a}_1$, $d_2 = a_1$, and $d_3 = M\Delta_1$, where

$$a_1 = e^{-b(t)} = \frac{\eta(t)}{\eta(0)}, \qquad a_2 = -\frac{2}{\hbar}S(t)a_1(t),$$
$$\Delta_i \equiv \Delta_i(t) = \dot{a}_i(t)\xi(t) - a_i(t)\dot{\xi}(t), \quad i = 1, 2. \tag{5.10}$$

The functions a_1, a_2, Δ_1, and Δ_2 satisfy the initial conditions

$$a_1(t_0) = 1, \qquad \dot{a}_1(t_0) = 0, \qquad a_2(t_0) = 0, \qquad M(t_0)\dot{a}_2(t_0) = -1, \qquad \Delta_1(t_0) = \Delta_2(t_0) = 0.$$

We consider two particular cases of formulas (5.9):

3.1. The Hamiltonian for Caldirola–Kanai oscillator (4.14) subject to a variable force is

$$H = \frac{\hat{p}^2}{2m}e^{-2\lambda t} + \frac{m\omega_0^2}{2}\hat{x}^2 e^{2\lambda t} - F(t)\hat{x}, \tag{5.11}$$

and we obtain ($\tau = t - t_0$ and $\tau' = t' - t_0$ here)

$$\hat{x}_{0\text{H}}(t) = \hat{x}e^{\lambda\tau}\left(\cos(\omega\tau) - \frac{\lambda}{\omega}\sin(\omega\tau)\right) - \frac{\hat{p}}{m\omega}\,e^{-\lambda(t+t_0)}\sin(\omega\tau) + $$
$$+ \frac{1}{m\omega}\int_{t_0}^{t} e^{-\lambda(t'+t_0)}F(t')\sin(\omega\tau')\,dt', \tag{5.12}$$
$$\hat{p}_{0\text{H}}(t) = \hat{x}m\omega_0^2 e^{\lambda(t+t_0)}\frac{\sin\omega\tau}{\omega} + \frac{\hat{p}}{\omega}e^{-\lambda t}Q(t) - \frac{1}{\omega}\int_{t_0}^{t} e^{-\lambda t'}F(t')Q(t')\,dt',$$

where $Q(t') = \omega\cos(\omega\tau') + \lambda\sin(\omega\tau')$. For $\lambda = 0$, formulas (5.12) coincide with the formulas for a stationary oscillator subject to a variable force. For $\omega_0 = 0$, we obtain the operators $\hat{x}_{0\text{L}}(t)$ and $\hat{p}_{0\text{L}}(t)$ (see (5.7)) for a linear potential in the case of a variable mass $M(t) = me^{2\lambda t}$:

$$\hat{x}_{0\text{L}}(t) = \hat{x} - \frac{\hat{p}}{m\lambda}e^{-\lambda(t+t_0)}\sinh(\lambda\tau) + \frac{1}{m\lambda}\int_{t_0}^{t} e^{-\lambda(t'+t_0)}F(t')\sinh(\lambda\tau')\,dt',$$
$$\hat{p}_{0\text{L}}(t) = \hat{p} - \delta(t). \tag{5.13}$$

In the limit case $\lambda \to 0$, we hence obtain operators (5.8) for a linear potential with $M(t) = m =$ const.

3.2. For parametric oscillator (4.17), we have

$$\hat{x}_{0\text{H}}^{(0)}(t) = \frac{\hat{x}}{\omega_0^2 + \lambda^2}\left[\omega_0^2\cos(\omega_0\tau) - \lambda\omega_0\sin(\omega_0\tau)\tanh(\lambda\tau) + \frac{\lambda^2\cos(\omega_0\tau)}{\cosh^2(\lambda\tau)}\right] -$$
$$- \frac{\hat{p}}{m(\omega_0^2 + \lambda^2)}[\omega_0\sin(\omega_0\tau) + \lambda\cos(\omega_0\tau)\tanh(\lambda\tau)],$$
$$\hat{p}_{0\text{H}}^{(0)}(t) = m\hat{x}\left[\omega_0\sin(\omega_0\tau) + \lambda\cos(\omega_0\tau)\tanh(\lambda\tau) + \frac{\lambda^2\sin(\omega_0\tau)}{\omega_0\cosh^2(\lambda\tau)}\right] + \tag{5.14}$$
$$+ \hat{p}\left[\cos(\omega_0\tau) - \frac{\lambda}{\omega_0}\sin(\omega_0\tau)\tanh(\lambda\tau)\right].$$

For $\lambda = 0$, these formulas coincide with the formulas for the stationary oscillator with $F = 0$. In the limit $\omega_0 \to 0$, from (5.14), we obtain the formulas for the oscillator with a variable frequency $\omega^2(t) = 2\lambda^2/\cosh^2(\lambda t)$:

$$\begin{aligned}\hat{x}_{0\text{H}}^{(0)}(t) &= \hat{x}\cosh^{-2}(\lambda\tau) - \frac{\hat{p}}{m\lambda}\tanh(\lambda\tau),\\ \hat{p}_{0\text{H}}^{(0)}(t) &= m\hat{x}\lambda[\tanh(\lambda\tau) + \lambda\tau\cosh^{-2}(\lambda\tau)] + \hat{p}[1 - \lambda\tau\tanh(\lambda\tau)].\end{aligned} \tag{5.15}$$

It is now clear that the most general form of the (either Hermitian or non-Hermitian) nth-order invariant ($n = 1, 2, \dots$) after its expansion in powers of $\hat{x}_0$ and $\hat{p}_0$ is

$$I_n(t) = \sum_{m=1}^{n} Q_{nm}(t) + C_{n0}, \tag{5.16}$$

where

$$Q_{nm}(t) = \sum_{k=0}^{m}[A_{0mk}^{(n)}\hat{x}_0^k(t)\hat{p}_0^{m-k}(t) + B_{0mk}^{(n)}\hat{p}_0^{m-k}(t)\hat{x}_0^k(t)] \tag{5.17}$$

and the coefficients $A_{0mk}^{(n)}$, $B_{0mk}^{(n)}$, and C_{n0} are arbitrary constants. In general, these quantities can be complex. Hence, all invariants $I_n(t)$ ($n = 1, 2, \dots$) can be expressed in terms of basis invariants (5.2). Here, we construct only the linear and quadratic invariants. In the most general case, we have the expressions for them

$$\begin{aligned}I_1(t) &= A_1(t)\hat{p} + B_1(t)\hat{x} + C_1(t),\\ I_2(t) &= A_2(t)\hat{p}^2 + B_2(t)\hat{x}^2 + C_2(t)\hat{p}\hat{x} + \widetilde{C}_2(t)\hat{x}\hat{p} + D_2(t)\hat{p} + E_2(t)\hat{x} + F_2(t).\end{aligned} \tag{5.18}$$

Here, the coefficients of the linear invariant are

$$A_1(t) = A_{10}d_2 + B_{10}e_2, \qquad B_1(t) = A_{10}d_1 + B_{10}e_1, \qquad C_1(t) = A_{10}d_3 + B_{10}e_3 + C_{10}, \tag{5.19}$$

and the coefficients of the quadratic invariant are

$$\begin{aligned}A_2(t) &= A_{20}d_2^2 + B_{20}e_2^2 + (C_{20} + \widetilde{C}_{20})d_2e_2,\\ B_2(t) &= A_2(d_2 \to d_1, e_2 \to e_1),\\ C_2(t) &= A_{20}d_1d_2 + B_{20}e_1e_2 + C_{20}d_2e_1 + \widetilde{C}_{20}d_1e_2,\\ \widetilde{C}_2(t) &= C_2(C_{20} \leftrightarrow \widetilde{C}_{20}),\\ D_2(t) &= 2A_{20}d_2d_3 + 2B_{20}e_2e_3 + (C_{20} + \widetilde{C}_{20})(d_2e_3 + d_3e_2) + D_{20}d_2 + E_{20}e_2,\\ E_2(t) &= D_2(d_2 \to d_1, e_2 \to e_1),\\ F_2(t) &= A_{20}d_3^2 + B_{20}e_3^2 + (C_{20} + \widetilde{C}_{20})d_3e_3 + D_{20}d_3 + E_{20}e_3 + F_{20}.\end{aligned} \tag{5.20}$$

The initial conditions for coefficients (5.19) and (5.20) are $A_1(t_0) = A_{10}$, $A_2(t_0) = A_{20}$, and so on. It is clear that the quadratic invariants of the type I_1^2, $I_1^+I_1$, and so on are particular cases of the invariant I_2 given by (5.18). For instance, if we choose coefficients (5.20) in the form $A_{20} = A_{10}^2$, $B_{20} = B_{10}^2$, $C_{20} = \widetilde{C}_{10} = A_{10}B_{10}$, $D_{20} = 2A_{10}C_{10}$, $E_{20} = 2B_{10}C_{10}$, and $F_{20} = C_{10}^2$, then $I_1^2 = I_2$.

The differential equations for the coefficient functions involved in the invariants I_1 and I_2 follow from the commutation relations $[\widehat{S}, I_1] = 0$ and $[\widehat{S}, I_2] = 0$. We can write these equations as

$$\dot{A}_1 = -2\alpha_2 B_1, \qquad \dot{B}_1 = 2\beta_2 A_1, \qquad \dot{C}_1 = -F A_1 \tag{5.21}$$

and

$$\begin{gathered}\dot{A}_2 = -2\alpha_2 (C_2 + \widetilde{C}_2), \qquad \dot{B}_2 = 2\beta_2 (C_2 + \widetilde{C}_2), \qquad \dot{C}_2 = \dot{\widetilde{C}}_2 = 2\beta_2 A_2 - 2\alpha_2 B_2, \\ \dot{D}_2 = -2F A_2 - 2\alpha_2 E_2, \qquad \dot{E}_2 = -F(C_2 + \widetilde{C}_2) + 2\beta_2 D_2, \qquad \dot{F}_2 = -F D_2.\end{gathered} \tag{5.22}$$

It in turn follows from the first two equations in (5.21) that

$$\alpha_2 \ddot{A}_1 - \dot{\alpha}_2 \dot{A}_1 + 4\alpha_2^2 \beta_2 A_1 = 0.$$

This equation is $\frac{d}{dt}[M(t)\dot{A}_1(t)] = 0$ in the case of a free particle and a particle in a homogeneous field and

$$\frac{d}{dt}[M(t)\dot{A}_1(t)] + M(t)\omega^2(t) A_1(t) = 0$$

in the case of an oscillator. As a consequence of formulas (5.21), we find that linear invariant (5.18) can be expressed only in terms of the coefficient $A_1(t)$:

$$I_1(t) = A_1(t)\hat{p} - M(t)\dot{A}_1(t)\hat{x} - \int_{t_0}^{t} F(t') A_1(t')\, dt',$$

where $(2\alpha_2)^{-1} = M(t)$. Another relation between $A_1(t)$ and $B_1(t)$ also holds:

$$A_1(t) B_1^*(t) - A_1^*(t) B_1(t) = A_{10} B_{10}^* - A_{10}^* B_{10} = \text{const},$$

which can be easily verified using the equality $e_2 \dot{d}_2 - \dot{e}_2 d_2 = 2\alpha_2$. It follows from the definition of the functions $a_1(t)$ and $a_2(t)$ given by (5.10) that they also satisfy Eq. (3.18),

$$\frac{d}{dt}[M(t)\dot{a}_i(t)] + M(t)\omega^2(t) a_i(t) = 0, \quad i = 1, 2,$$

and the relation $M(a_2\dot{a}_1 - a_1\dot{a}_2) = 1$.

6. Conclusion

We have used the evolution operator method to study the physical properties of simple nonstationary quadratic quantum systems, namely, a free particle with a variable mass, a quantum particle with a variable mass in an alternating homogeneous field, and a harmonic oscillator with a variable mass and frequency subject to a variable force. By successive unitary transformations of the Schrödinger equation, we constructed the evolution operators for these systems in an explicit disentangled form. In this regard, we note that in the particular case where the mass $M = m = \text{const}$ and $\Lambda_0(t) = 0$, formula (3.25) was obtained in [29] using a different method: using the Feynman method to disentangle noncommuting operators. But there is a misprint in formula (28′) in [29]: it must be written as $b(t) = i\int_{t_0}^{t} e^{-2ic(t')}\, dt'$.

As emphasized in [23], the evolution operator method for studying quadratic systems turns out to be rather simple and fruitful at the same time; for example, using this method, first, we easily established unitary relations between the considered systems, and second, we found various exact solutions of the equations of motion with an arbitrary time variation for the Hamiltonian parameters (mass $M(t)$, frequency $\omega(t)$, and force $F(t)$), which itself is of research interest.

A unitary relation allows obtaining the wave functions for the harmonic oscillator subject to a force from the appropriate expressions for a free particle with a variable mass. For example, a free particle with a variable mass has the oscillator states [23]

$$\psi_n^{\rm F}(x,t) = N_{n\rm F}\left(\frac{\varepsilon_{\rm F}^*(t)}{\varepsilon_{\rm F}(t)}\right)^{n/2} \exp\left\{-\frac{\lambda_1 x^2}{2\hbar\varepsilon_{\rm F}(t)}\right\} H_n\left(\frac{x}{\sqrt{\hbar}|\varepsilon_{\rm F}(t)|}\right), \tag{6.1}$$

where $N_{n\rm F} = N_{0\rm F}/\sqrt{2^n\, n!}$ is the normalization factor, $N_{0\rm F} = (\pi\hbar)^{-1/4}[\varepsilon_{\rm F}(t)]^{-1/2}$, and $H_n(x)$ are Hermite polynomials. We can then write the appropriate expressions for a nonstationary oscillator with a variable driving force as

$$\psi_n^{\rm H}(x,t) = V_{\rm FH}\psi_n^{\rm Ren\, F}(x,t) = N_{n\rm H}\left(\frac{\varepsilon^{\rm H*}(t)}{\varepsilon^{\rm H}(t)}\right)^{n/2} e^{i(\phi_0 + \hbar^{-1}M(t)\dot\xi(t)[x-\xi(t)])} \times$$
$$\times \exp\left\{i\frac{M(t)\dot\varepsilon^{\rm H}(t)}{2\hbar\varepsilon^{\rm H}(t)}[x-\xi(t)]^2\right\} H_n\left(\frac{x-\xi(t)}{\sqrt{\hbar}|\varepsilon^{\rm H}(t)|}\right), \tag{6.2}$$

where

$$\varepsilon^{\rm H}(t) = a_1(t)\varepsilon_{\rm F}^{\rm Ren}(t) = \lambda_2 a_1(t) - i\lambda_1 a_2(t), \qquad \varepsilon_{\rm F}^{\rm Ren}(t) = \lambda_2 + 2i\lambda_1\hbar^{-1}S(t),$$
$$N_{n\rm H} = \frac{N_{0\rm H}}{\sqrt{2^n\, n!}}, \qquad N_{0\rm H} = (\pi\hbar)^{-1/4}[\varepsilon_{\rm H}(t)]^{-1/2},$$

λ_1 and λ_2 are the complex numbers satisfying the condition $\mathrm{Re}(\lambda_1^*\lambda_2) = 1$, and

$$\psi_n^{\rm Ren\, F}(x,t) = \psi_n^{\rm F}(x,t)\big|_{\varepsilon_{\rm F}\to\varepsilon_{\rm F}^{\rm Ren}}, \qquad \varepsilon_{\rm F} = \varepsilon_{\rm F}^{\rm Ren}\big|_{S\to\hbar S_2}.$$

Functions (6.1) and (6.2) are normalized by the conditions

$$\int_{-\infty}^{\infty} |\psi_n^{\rm F}(x,t)|^2\, dx = 1, \qquad \int_{-\infty}^{\infty} |\psi_n^{\rm H}(x,t)|^2\, dx = 1. \tag{6.3}$$

We emphasize a feature of the unitary relation between a harmonic oscillator and a free particle: the problem of a harmonic oscillator is unitarily equivalent to the problem of not only a free particle but also a free particle with a renormalized mass. Hence, knowing some states of a free particle, we can construct appropriate states of both a particle in a homogeneous field and a harmonic oscillator (whether the latter is subject to a force or not) by applying a unitary operator.

The third result in our paper is that using the evolution operator method, we immediately obtain the most general expressions for the propagators of a free particle, of a particle in an alternating homogeneous field, and of a nonstationary oscillator. Using the unitary relation established above, we express the propagators of both a particle with a variable mass in an alternating homogeneous field and a nonstationary harmonic oscillator in terms of the propagator of a free particle. As particular cases, these general expressions contain results already known in the research literature, which we demonstrated with particular examples. We note that problems with a time-dependent particle mass have long been considered in the literature (see, e.g., [28]). There is reason to believe that such problems are directly related to real physical unstable systems, to the systems with dissipation, and also to real physical processes in an alternating homogeneous gravitational field arising in the early evolutionary stages of the Universe (see [30] and the references therein).

The evolution operator method can be also useful for other quantum mechanical problems that are not specified by quadratic Hamiltonians, of course, if the evolution operators can be constructed in an explicit form. For example, such systems can include a singular oscillator with a variable frequency [5] and a relativistic quantum particle in an alternating homogeneous field described by a finite-difference equation [18]. Other examples can be systems defined by Eq. (2.1) and also some thermal conductivity problems.

We note that for quadratic quantum systems defined by Hamiltonians of the most general form

$$H = A(t)\hat{p}^2 + B(t)\hat{x}^2 + C(t)(\hat{p}\hat{x} + \hat{x}\hat{p}) + D(t)\hat{p} + E(t)\hat{x} + F(t),$$

where the coefficients are real functions of time, the evolution operators can be completely disentangled, and these systems are obviously unitarily equivalent to a free particle but, perhaps, with a mass renormalized in a certain way. This means that we can, in principle, construct all the states of these systems from the known states of a free particle using the unitary operators even if the parameters of the Hamiltonian $A(t)$, $B(t)$, $C(t)$, $D(t)$, $E(t)$, $F(t)$ arbitrarily depend on the time.

We now briefly list advantages of the evolution operator method. If we know the evolution operator, then we can

1. find a solution of an equation with a required property,
2. uniquely find an unlimited number of solutions of an equation with an arbitrary time dependence of the Hamiltonian parameters (there is no problem of defining the phase in this case, unlike in the method of invariants [4]),
3. easily determine the form of propagators,
4. easily determine the form of invariants of any order, and
5. easily establish a unitary relation between the considered systems.

We also note that we can use this method to construct a perturbation theory that is an analogue of the Feynman diagram technique in quantum mechanics.

REFERENCES

1. L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics* [in Russian], Vol. 3, *Quantum Mechanics: Non-Relativistic Theory*, Nauka, Moscow (1989); English transl. prev. ed., Pergamon, Oxford (1977).
2. R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, New York (1965).
3. K. Husimi, "Miscellanea in elementary quantum mechanics. II," *Prog. Theor. Phys.*, **9**, 381–402 (1953).
4. H. R. Lewis and W. B. Riesenfeld, "An exact quantum theory of the time-dependent harmonic oscillator and of a charged particle in a time-dependent electromagnetic field," *J. Math. Phys.*, **10**, 1458–1473 (1969).
5. P. Camiz, A. Gerardi, C. Marchioro, E. Presutti, and E. Scacciatelli, "Exact solution of a time-dependent quantum harmonic oscillator with a singular perturbation," *J. Math. Phys.*, **12**, 2040–2043 (1971).
6. K. B. Wolf, "On time-dependent quadratic quantum Hamiltonians," *SIAM J. Appl. Math.*, **40**, 419–431 (1981).
7. I. A. Malkin and V. I. Man'ko, *Dynamical Symmetries and Coherent States of Quantum Systems* [in Russian], Nauka, Moscow (1979).
8. V. V. Dodonov, V. I. Manko, and O. V. Shakhmistova, "Wigner functions of particle in a time-dependent uniform field," *Phys. Lett. A*, **102**, 295–297 (1984).

9. R. Cordero-Soto and S. K. Suslov, “Time reversal for modified oscillators,” *Theor. Math. Phys.*, **162**, 286–316 (2010).
10. R. Cordero-Soto, E. Suazo, and S. K. Suslov, “Quantum integrals of motion for variable quadratic Hamiltonians,” *Ann. Phys.*, **325**, 1884–1912 (2010); arXiv:0912.4900v9 [math-ph] (2009).
11. D.-Y. Song, “Unitary relations in time-dependent harmonic oscillators,” *J. Phys. A: Math. Gen.*, **32**, 3449–3456 (1999); arXiv:quant-ph/9812038v2 (1998).
12. S. P. Kim, “A class of exactly solved time-dependent quantum oscillators,” *J. Phys. A: Math. Gen.*, **27**, 3927–3936 (1994).
13. J.-Y. Ji, J. K. Kim, and S. P. Kim, “Heisenberg-picture approach to the exact quantum motion of a time-dependent harmonic oscillator,” *Phys. Rev. A*, **51**, 4268–4271 (1995).
14. I. A. Pedrosa, “Exact wave functions of a harmonic oscillator with time-dependent mass and frequency,” *Phys. Rev. A*, **55**, 3219–3221 (1997).
15. M. V. Berry and N. L. Balazs, “Nonspreading wave packets,” *Am. J. Phys.*, **47**, 264–267 (1979).
16. I. Guedes, “Solution of the Schrödinger equation for the time-dependent linear potential,” *Phys. Rev. A*, **63**, 034102 (2001).
17. M. Feng, “Complete solution of the Schrödinger equation for the time-dependent linear potential,” *Phys. Rev. A*, **64**, 034101 (2002); arXiv:quant-ph/0105145v1 (2001).
18. Sh. M. Nagiyev and K. Sh. Jafarova, “Relativistic quantum particle in a time-dependent homogeneous field,” *Phys. Lett. A*, **377**, 747–752 (2013).
19. Sh. M. Nagiyev, “Reexamination of a time-dependent harmonic oscillator,” *Azerb. J. Phys. Fizika*, **22**, 16–23 (2016).
20. Sh. M. Nagiyev, “Wigner function of a relativistic particle in a time-dependent linear potential,” *Theor. Math. Phys.*, **188**, 1030–1037 (2016).
21. A. M. Perelomov and V. S. Popov, “Method of generating functions for a quantum oscillator,” *Theor. Math. Phys.*, **3**, 582–592 (1970).
22. F. J. Dyson, “The S matrix in quantum electrodynamics,” *Phys. Rev.*, **75**, 1736–1755 (1949).
23. Sh. M. Nagiyev, “Using the evolution operator method to describe a particle in a homogeneous alternating field,” *Theor. Math. Phys.*, **194**, 313–327 (2018).
24. K. V. Zhukovsky, “Solving evolutionary-type differential equations and physical problems using the operator method,” *Theor. Math. Phys.*, **190**, 52–68 (2017).
25. K. V. Zhukovsky and G. Dattoli, “Evolution of non-spreading Airy wavepackets in time dependent linear potentials,” *Appl. Math. Comput.*, **217**, 7966–7974 (2017).
26. Sh. M. Nagiyev, “Motion in an alternating quasihomogeneous field and operator identities [in Russian],” *Azerb. J. Phys. Fizika*, **19**, 129–135 (2013).
27. A. L. Rivera, N. M. Atakhishiyev, S. M. Chumakov, and K. B. Wolf, “Evolution under polynomial Hamiltonians in quantum and optical phase spaces,” *Phys. Rev. A*, **55**, 876–889 (1997).
28. H. Dekker, “Classical and quantum mechanics of the damped harmonic oscillator,” *Phys. Rep.*, **80**, 1–110 (1981).
29. V. S. Popov, “Feynman disentangling of noncommuting operators in quantum mechanics,” *JETP*, **101**, 817–829 (2005).
30. V. V. Dodonov and V. I. Man’ko, “Invariants and evolution of nonstationary quantum systems [in Russian],” *Trudy FIAN*, **183**, 71–181 (1987).