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TOWARD AN ANALYTIC PERTURBATIVE SOLUTION FOR THE ABJM QUANTUM SPECTRAL CURVE

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We recently showed how nonhomogeneous second-order difference equations that appear in describing the ABJM quantum spectral curve can be solved using a Mellin space technique. In particular, we provided explicit results for anomalous dimensions of twist-1 operators in the $sl(2)$ sector at arbitrary spin values up to the four-loop order. We showed that the obtained results can be expressed in terms of harmonic sums with additional factors in the form of a fourth root of unity, and the maximum transcendentality principle therefore holds. Here, we show that the same result can also be obtained by directly solving the mentioned difference equations in the space of the spectral parameter u. The solution involves new highly nontrivial identities between hypergeometric functions, which can have various applications. We expect that this method can be generalized both to higher loop orders and to other theories, such as the $\mathcal{N}=4$ supersymmetric Yang–Mills theory.

Keywords: quantum spectral curve, spin chain, anomalous dimension, ABJM model, Baxter equation

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1. Introduction

After the discovery of the AdS/CFT duality [1]–[4], we have recently seen much progress in understanding the integrable structures underlying quantum field theories with extended supersymmetry in dimensions greater than two (see [5]–[11] for a review and introduction). The best-understood theories are the $\mathcal{N}=4$ SYM in four dimensions and the $\mathcal{N}=6$ super Chern–Simons theory (ABJM model) in three dimensions [12]. It was shown that different techniques from the realm of integrable systems, such as the worldsheet and spin-chain S-matrices [13]–[21], the asymptotic Bethe ansatz [22]–[27], the thermodynamic Bethe ansatz (TBA) [28]–[31], and also Y and T systems [32]–[38] are applicable for computing the conformal spectrum of these theories. The integrability-based methods were also used to study the quark–antiquark potential [39]–[42], polygonal Wilson loops at a strong coupling and beyond [43]–[48], eigenvalues of the BFKL kernel [49]–[52], structure constants [53]–[57], correlation functions [53]–[63], one-point functions of

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operators in the defect conformal field theory [64]–[66], and thermal observables such as the Hagedorn temperature of the $\mathcal{N}$=4 SYM [67], [68].

A further detailed study of the TBA equations for the $\mathcal{N}$=4 SYM and ABJM models led to the discovery of the quantum spectral curve (QSC) formulation for these models [69]–[75]. The QSC is an alternative reformulation of the TBA equations in terms of a nonlinear Riemann–Hilbert problem. The numerical solution of the QSC for these theories at a finite coupling applicable even for a complexified spin was presented in [76]–[78]. The iterative procedure for perturbatively solving these Riemann–Hilbert problems for these theories at a weak coupling was given in [79], [80] up to an arbitrary loop order in principle. But the techniques presented there are limited to the situation where the state quantum numbers are explicitly given as some integer numbers. They suffice for reconstructing the full analytic structure of the conserved charges in the corresponding spin chains if a finite basis of functions is known in terms of which they can be written explicitly. If this is not the case, then we can, for example, use a Mellin space technique to solve the QSC equations without assigning specific integer values to state quantum numbers [81]. Here, we want to show that there is in fact a way to solve the QSC equations directly in the space of the spectral parameter u without imposing any restriction on the state quantum numbers. In fact, we here restrict ourself to only the case of twist-1 operators in the $sl(2)$ sector of the ABJM model up to the four-loop order and defer the generalization to higher loops and other states to forthcoming publications. The presented approach, which is still work in progress, also has the potential for generalization to the case of twisted $\mathcal{N}$=4 SYM and ABJM QSCs with different nonpolynomial large spectral parameter asymptotic forms for functions in the corresponding Riemann–Hilbert problems. Moreover, similar ideas should be useful in studying the BFKL regime in the QSC approach [50]–[52], which still uses a perturbative expansion in the coupling constant g and the parameter $w \equiv S+1$ describing the proximity of the Lorentz spin S to -1 such that the ratio g^2/w remains fixed.

This paper is organized as follows. In Sec. 2, we present the set of equations together with the strategy used to find a perturbative solution of the ABJM QSC in the case of twist-1 operators in the $sl(2)$ sector. In Sec. 3, we present the details of solution of the corresponding Baxter equations. In Sec. 4, we discuss our results for anomalous dimensions of twist-1 operators through the four-loop order. Finally, in Sec. 5, we present our conclusion.

2. Perturbative solution of the ABJM QSC

The ABJM model is a three-dimensional $\mathcal{N}$=6 Chern–Simons theory with the product gauge group $U(N)\times\widehat{U}(N)$ at levels $\pm k$. The field content of the theory consists of two gauge fields A_μ and $\hat{A}_\mu$, four complex scalars Y^A, and four Weyl spinors ψ_A. The matter fields transform in the bifundamental representation of the gauge group. The global symmetry group of the ABJM theory for a Chern–Simons level $k>2$ is given by the orthosymplectic supergroup OSp(6|4) [12], [82] and the “baryonic” $U(1)_b$ [82]. Here, as an example, we use anomalous dimensions of twist-1 $sl(2)$-like states given by single-trace operators of the form $\mathrm{tr}[D_+^S(Y^1Y_4^\dagger)]$ (see [83]).

At present, the QSC method is the most advanced method for treating spin-chain spectral problems that arise in studying the $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ duality. In the case of the ABJM model, the QSC formulation was introduced in [73], [74] (also see [80]).

For the perturbative solution of the ABJM QSC, we use the same set of equations as in [80]. They easily follow[1] from the Riemann–Hilbert problem for the $\mathbf{P}\nu$ system and in our case are given by

$$\frac{\nu_1^{[3]}}{\mathbf{P}_1^{[1]}}-\frac{\nu_1^{[-1]}}{\mathbf{P}_1^{[-1]}}-\sigma\left(\frac{\mathbf{P}_0^{[1]}}{\mathbf{P}_1^{[1]}}-\frac{\mathbf{P}_0^{[-1]}}{\mathbf{P}_1^{[-1]}}\right)\nu_1^{[1]}=-\sigma\left(\frac{\mathbf{P}_2^{[1]}}{\mathbf{P}_1^{[1]}}-\frac{\mathbf{P}_2^{[-1]}}{\mathbf{P}_1^{[-1]}}\right)\nu_2^{[1]}, \tag{2.1}$$

[1] See [80] for details.

$$\frac{\nu_2^{[3]}}{\mathbf{P}_1^{[1]}} - \frac{\nu_2^{[-1]}}{\mathbf{P}_1^{[-1]}} + \sigma\left(\frac{\mathbf{P}_0^{[1]}}{\mathbf{P}_1^{[1]}} - \frac{\mathbf{P}_0^{[-1]}}{\mathbf{P}_1^{[-1]}}\right)\nu_2^{[1]} = \sigma\left(\frac{\mathbf{P}_3^{[1]}}{\mathbf{P}_1^{[1]}} - \frac{\mathbf{P}_3^{[-1]}}{\mathbf{P}_1^{[-1]}}\right)\nu_1^{[1]}, \tag{2.2}$$

and

$$\sigma\nu_1^{[2]} = \mathbf{P}_0\nu_1 - \mathbf{P}_2\nu_2 + \mathbf{P}_1\nu_3, \tag{2.3}$$

$$\sigma\nu_2^{[2]} = -\mathbf{P}_0\nu_2 + \mathbf{P}_3\nu_1 + \mathbf{P}_1\nu_4, \tag{2.4}$$

$$\tilde{\mathbf{P}}_2 - \mathbf{P}_2 = \sigma(\nu_3\nu_1^{[2]} - \nu_1\nu_3^{[2]}), \tag{2.5}$$

$$\tilde{\mathbf{P}}_1 - \mathbf{P}_1 = \sigma(\nu_2\nu_1^{[2]} - \nu_1\nu_2^{[2]}), \tag{2.6}$$

$$(\nu_1 + \sigma\nu_1^{[2]})(\mathbf{p}_0 - hx) = \mathbf{p}_2(\nu_2 + \sigma\nu_2^{[2]}) - \mathbf{p}_1(\nu_3 + \sigma\nu_3^{[2]}), \tag{2.7}$$

$$(\nu_2 + \sigma\nu_2^{[2]})(\mathbf{p}_0 + hx) = \mathbf{p}_3(\nu_1 + \sigma\nu_1^{[2]}) + \mathbf{p}_1(\nu_4 + \sigma\nu_4^{[2]}), \tag{2.8}$$

$$(\mathbf{P}_0)^2 = 1 - \mathbf{P}_1\mathbf{P}_4 + \mathbf{P}_2\mathbf{P}_3, \tag{2.9}$$

where $\mathbf{P}_i = (xh)^{-1}\mathbf{p}_i$ and $x \equiv x(u) = (u + \sqrt{u^2 - 4h^2})/2h$ is the Zhukovsky variable used to parameterize the single cut of $\mathbf{P}$ functions on the defining Riemann sheet. It also follows from the analytic structure of the $\nu_i(u)$ functions on the defining Riemann sheet that the combinations of functions

$$\nu_i(u) + \tilde{\nu}_i(u) = \nu_i(u) + \sigma\nu_i^{[2]}(u), \qquad \frac{\nu_i(u) - \tilde{\nu}_i(u)}{\sqrt{u^2 - 4h^2}} = \frac{\nu_i(u) - \sigma\nu_i^{[2]}(u)}{\sqrt{u^2 - 4h^2}}$$

are free of cuts on the whole real axis. Similarly to [79], [80], we seek the solution using the ansatz for $\mathbf{P}(u)$ functions[2]

$$\mathbf{P}_1 = (xh)^{-1}\mathbf{p}_1 = (xh)^{-1}\left(1 + \sum_{k=1}^{\infty}\sum_{l=0}^{\infty} c_{1,k}^{(l)}\frac{h^{2l+k}}{x^k}\right),$$

$$\mathbf{P}_2 = (xh)^{-1}\mathbf{p}_2 = (xh)^{-1}\left(\frac{h}{x} + \sum_{k=2}^{\infty}\sum_{l=0}^{\infty} c_{2,k}^{(l)}\frac{h^{2l+k}}{x^k}\right),$$

$$\mathbf{P}_0 = (xh)^{-1}\mathbf{p}_0 = (xh)^{-1}\left(\sum_{l=0}^{\infty} A_0^{(l)}h^{2l}u + \sum_{l=0}^{\infty} m_0^{(l)}h^{2l} + \sum_{k=1}^{\infty}\sum_{l=0}^{\infty} c_{0,k}^{(l)}\frac{h^{2l+k}}{x^k}\right),$$

$$\mathbf{P}_3 = (xh)^{-1}\mathbf{p}_3 = (xh)^{-1}\left(\sum_{l=0}^{\infty} A_3^{(l)}h^{2l}u^3 + \sum_{j=0}^{2}\sum_{l=0}^{\infty} k_j^{(l)}h^{2l}u^j + \sum_{k=1}^{\infty}\sum_{l=0}^{\infty} c_{3,k}^{(l)}\frac{h^{2l+k}}{x^k}\right),$$

where we take the correct large-u asymptotic behavior of the $\mathbf{P}$ functions into account [73]–[80], playing the role of initial conditions for the above equations depending on the quantum numbers of the operators in which we are interested,

$$\mathbf{P}_a \simeq (A_1u^{-1}, A_2u^{-2}, A_3u^2, A_4u, A_0u^0),$$

$$A_1A_4 = -\frac{1}{3}(\Delta - 1 + S)(\Delta - S)(\Delta + 2 - S)(\Delta + 1 + S),$$

$$A_2A_3 = -\frac{1}{12}(\Delta + S - 2)(\Delta - 1 - S)(\Delta - S + 3)(\Delta + S + 2).$$

[2]The ABJM QSC coupling constant h is a nontrivial function of the ABJM 't Hooft coupling constant λ [26], [84], which scales as $h \sim \lambda$ for a weak coupling constant and as $h \sim \sqrt{\lambda/2}$ for a strong coupling constant.

The anomalous dimension γ in which we are interested is given by $\gamma = \triangle - 1 - S$. The coefficients $A_0^{(l)}$, $A_3^{(l)}$, $c_{i,k}^{(l)}$, $m_j^{(l)}$, and $k_j^{(l)}$ are some functions of only the spin S; otherwise, they are just constants. Here, we also use the gauge freedom[3] to set $A_1 = 1$ and $A_2 = h^2$. The functions analytically continued across the cut are defined as

$$\tilde{\mathbf{P}}_a = \left(\frac{x}{h}\right)^L \tilde{\mathbf{p}}_a, \qquad \tilde{\mathbf{p}}_a = \mathbf{p}_a\big|_{x\to 1/x},$$

and the expansion of the $\nu_i(u)$ functions in terms of the QSC coupling constant h is given by $\nu_i(u) = \sum_{l=0}^{\infty} h^{2l-L}\nu_i^{(l)}(u)$.

3. Solution of Baxter equations

The most complicated part of solving the QSC is solving two inhomogeneous Baxter equations at each perturbation order. To solve these second-order finite-difference equations at arbitrary values of the spin S in [81], we used a Mellin transform technique to convert difference equations to ordinary differential equations. Here, we want to show that these Baxter equations can be more simply solved directly in the space of the spectral parameter u. To iteratively seek the perturbative solution of Eqs. (2.1)–(2.8), we expand Eqs. (2.1) and (2.2) up to h^k and obtain inhomogeneous equations for $q_{1,2}^{(k)} = (\nu_{1,2}^{(k)})^{[-1]}$ in the forms

$$\left(u + \frac{i}{2}\right)q_1^{(k)}(u+i) - i(2S+1)q_1^{(k)}(u) - \left(u - \frac{i}{2}\right)q_1^{(k)}(u-i) = V_1^{(k)}, \tag{3.1}$$

$$\left(u + \frac{i}{2}\right)q_2^{(k)}(u+i) + i(2S+1)q_2^{(k)}(u) - \left(u - \frac{i}{2}\right)q_2^{(k)}(u-i) = V_2^{(k)}. \tag{3.2}$$

Here, $V_1^{(k)}$ depends on $q_{1,2}^{(l)}$, $l < k$, and $V_2^{(k)}$ also depends on $q_1^{(k)}$. The solution of the homogeneous parts of the Baxter equations was described in [81] and is given by

$$q_1^{\text{hom}}(S,u) = \Phi_1^{\text{per}} Q_S(u) + \Phi_1^{\text{ant}} \mathcal{Z}_S(u), \qquad q_2^{\text{hom}}(S,u) = \Phi_2^{\text{ant}} Q_S(u) + \Phi_2^{\text{per}} \mathcal{Z}_S(u),$$

where ($\sigma = (-1)^S$),

$$Q_S(u) = \frac{(-1)^S \Gamma(1/2+iu)}{S!\,\Gamma(1/2+iu-S)}\, {}_2F_1\left(-S, \frac{1}{2}+iu; \frac{1}{2}+iu-S; -1\right),$$

$$\mathcal{Z}_S(u) = i\sigma \sum_{k=0}^{\lfloor\frac{S-1}{2}\rfloor} \frac{1}{S-k} Q_{S-1-2k}(u) + \sigma\eta_{-1}\left(u + \frac{i}{2}\right) Q_S(u),$$

and Φ_i^{per} and Φ_i^{ant} are arbitrary periodic and antiperiodic functions of the spectral parameter u. Otherwise, they are arbitrary functions of the spin S to be determined from the consistency conditions mentioned in the preceding section. We parameterize their dependence on u similarly to [79], [80] with the basis of periodic and antiperiodic combinations of Hurwitz functions defined as

$$\mathcal{P}_k(u) = \eta_k(u) + \operatorname{sgn} k\,(-1)^k \eta_k(i-u) = \operatorname{sgn} k\,\mathcal{P}_k(u+i), \quad k \neq 0, \quad k \in \mathbb{Z},$$

where

$$\eta_a(u) = \sum_{k=0}^{\infty} \frac{(\operatorname{sgn} a)^k}{(u+ik)^{|a|}}.$$

[3]See [80] for details.

The functions $\Phi_a^{\rm per}$ and $\Phi_a^{\rm ant}$ then become

$$\Phi_a^{\rm per}(u) = \phi_{a,0}^{\rm per} + \sum_{j=1}^{\Lambda} \phi_{a,j}^{\rm per} \mathcal{P}_j(u), \qquad \Phi_a^{\rm ant}(u) = \sum_{j=1}^{\Lambda} \phi_{a,j}^{\rm ant} \mathcal{P}_{-j}(u),$$

where Λ is a cutoff dependent on the perturbation theory order.

To find a particular solution of the inhomogeneous Baxter equation, we first rewrite the original Baxter equations in a more convenient form. Substituting the ansatz $q_1^{(k)}(u) = Q_S(u) F_S^{(k)}(u + i/2)$ in Baxter equation (3.1), we obtain

$$-\nabla_-(u Q_S^{[1]} Q_S^{[-1]} \nabla_+ F_S^{(k)}) = Q_S^{[1]} V_1^{(k)[1]},$$

where $\nabla_+ f = f - f^{[2]}$ and $\nabla_- f = f + f^{[2]}$. Introducing the inverse operators $\Psi_\pm$ such that $\nabla_\pm \Psi_\pm f = f$ and solving this difference equation for $F_S^{(k)}(u)$, we obtain

$$F^{(k)} = -\Psi_+ \Bigg(\frac{1}{u Q_S^{[1]} Q_S^{[-1]}} \Psi_-(Q_S^{[1]} V_1^{(k)[1]}) \Bigg).$$

Using the remarkable relation[4] (see its proof in the appendix)

$$\begin{aligned} \frac{1}{u Q_S^{[1]} Q_S^{[-1]}} &= \frac{(-1)^S}{u} + i(-1)^S \sum_{k=0}^{[(S-1)/2]} \frac{1}{S-k} \Bigg(\frac{Q_{S-1-2k}^{[-1]}}{Q_S^{[-1]}} + \frac{Q_{S-1-2k}^{[1]}}{Q_S^{[1]}} \Bigg) = \\ &= (-1)^S \nabla_- \Bigg(\eta_{-1}(u) + i \sum_{k=0}^{[(S-1)/2]} \frac{1}{S-k} \frac{Q_{S-1-2k}^{[-1]}}{Q_S^{[-1]}} \Bigg), \end{aligned} \tag{3.3}$$

we obtain an expression for $F_S^{(k)}(u)$ of the form

$$F_S^{(k)}(u) = -(-1)^S \Psi_+ \Bigg\{ \nabla_- \Bigg(\eta_{-1}(u) + i \sum_{k=0}^{[(S-1)/2]} \frac{1}{S-k} \frac{Q_{S-1-2k}^{[-1]}}{Q_S^{[-1]}} \Bigg) \Psi_-(Q_S^{[1]} V_1^{(k)[1]}) \Bigg\}.$$

Using the relation[5] $\nabla_- f^{[-1]} \Psi_- g^{[1]} = -\nabla_+ (f \Psi_- g)^{[-1]} + (fg)^{[-1]}$, we obtain

$$\begin{aligned} F_S^{(k)}(u) = & -(-1)^S \Psi_+ \Bigg(\frac{1}{u} \Psi_-(Q_S V_1^{(k)})^{[1]} \Bigg) + \\ & + (-1)^S \frac{P_S^{[-1]}}{Q_S^{[-1]}} \Psi_-(Q_S V_1^{(k)})^{[-1]} - (-1)^S \Psi_+ (P_S V_1^{(k)})^{[-1]}, \end{aligned}$$

where

$$P_S(u) = i \sum_{k=0}^{[(S-1)/2]} \frac{1}{S-k} Q_{S-1-2k}(u).$$

Finally, for $q_1^{(k)}(u)$, we obtain

$$\begin{aligned} q_1^{(k)}(u) = & -(-1)^S Q_S \Psi_+ \Bigg(\frac{1}{u + i/2} \Psi_-(Q_S V_1^{(k)})^{[2]} \Bigg) + \\ & + (-1)^S P_S \Psi_-(Q_S V_1^{(k)}) - (-1)^S Q_S \Psi_+ (P_S V_1^{(k)}). \end{aligned}$$

[4] We stress that we in fact initially guessed this identity. It was used in [81] to find a second solution of the homogeneous Baxter equation, where a rigorous proof that it is in fact the desired solution was presented in Appendix B. We note that a similar identity could be written for twist-2 operators.

[5] This relation is easily derived using summation by parts.

Introducing the operator $\mathcal{F}_1^S$ as

$$\mathcal{F}_1^S[f] = -Q_S\Psi_+\left(\frac{1}{u+i/2}\Psi_-(Q_S(-1)^S f)^{[2]}\right) - Q_S\Psi_+(P_S(-1)^S f) + P_S\Psi_-(Q_S(-1)^S f),$$

we obtain the general solution of the first Baxter equation in the form

$$q_1^{(k)} = \mathcal{F}_1^S[V_1^{(k)}] + Q_S\Phi_1^{\mathrm{per},(k)} + \mathcal{Z}_S\Phi_1^{\mathrm{ant},(k)}.$$

Similarly, for the second Baxter equation, we obtain

$$q_2^{(k)} = \mathcal{F}_2^S[V_1^{(k)}] + Q_S\Phi_2^{\mathrm{ant},(k)} + \mathcal{Z}_S\Phi_2^{\mathrm{per},(k)},$$

where

$$\mathcal{F}_2^S[f] = -Q_S\Psi_-\left(\frac{1}{u+i/2}\Psi_+(Q_S(-1)^S f)^{[2]}\right) + Q_S\Psi_-(P_S(-1)^S f) - P_S\Psi_+(Q_S(-1)^S f).$$

It turns out that at least up to the NLO,[6] the right-hand side of the Baxter equations can be transformed into a form containing only Baxter polynomials $Q_S(u)$ for different spin values S and products of Baxter polynomials with the function η_1. For the actions of $\mathcal{F}_{1,2}^S$ on the Baxter polynomials, we find

$$\mathcal{F}_1^{S_1}[Q_{S_2}] = -\frac{i}{2}\frac{Q_{S_1}-Q_{S_2}}{S_1-S_2}, \quad S_1 \neq S_2, \tag{3.4}$$

$$\mathcal{F}_1^S[Q_S] = -\frac{1}{2}Q_S\,\eta_1\left(u+\frac{i}{2}\right) - \frac{i}{2}\sum_{k=1}^{S}\frac{1+(-1)^k}{k}Q_{S-k}, \tag{3.5}$$

$$\mathcal{F}_2^{S_1}[Q_{S_2}] = -\frac{i}{2}\frac{Q_{S_2}}{S_1+S_2+1}. \tag{3.6}$$

In addition, we need one extra rule for $\mathcal{F}_2^S$:

$$\mathcal{F}_2^{S_1}[\eta_1^{[1]}Q_{S_2}] = \frac{1}{2i(S_1+S_2+1)}\left\{\eta_1^{[1]}Q_{S_2} + \mathcal{F}_2^{S_1}[Q_{S_2}^{[2]}] + \mathcal{F}_2^{S_1}[Q_{S_2}^{[-2]}]\right\}. \tag{3.7}$$

The elementary operations needed to transform the inhomogeneous part into the required form are[7]

$$\frac{Q_S}{u\pm i/2} = \frac{(\mp 1)^S}{u\pm i/2} - 2i\sum_{k=1}^{S}(\pm 1)^{k+1}Q_{S-k}\sum_{l=0}^{k-1}\frac{(-1)^l}{S-l}, \tag{3.8}$$

$$Q_S^{[\pm 2]} = Q_S + 2\sum_{k=1}^{S}(\pm 1)^k Q_{S-k}, \tag{3.9}$$

and

$$u\,Q_S = \frac{i}{2}(S+1)Q_{S+1} - \frac{i}{2}SQ_{S-1}. \tag{3.10}$$

[6]The inhomogeneity in this case is polynomial or polynomial times η_1. We will consider the generalization to an arbitrary loop order in one of our future publications.

[7]We will derive Eqs. (3.4)–(3.10) elsewhere, but we prove some of these identities in proving Eq. (3.3) in the appendix.

The results of solving these Baxter equations up to the NLO can be found in [81]. It is instructive to see the types of functions in the solutions for $q_{1,2}$ at different perturbation theory orders. For $q_1^{(0,1)}$, for example, we obtain[8]

$$q_1^{(0)} = \alpha Q_S,$$

$$q_1^{(1)} = 4\alpha B_1(S) Q_S \left\{ \gamma_E + \log 2 - i\eta_1 \left(u + \frac{i}{2} \right) - H_1(S) + i\pi \tanh(\pi u) \right\} +$$

$$+ \alpha \sum_{k=1}^{S} \frac{1 + (-1)^k}{k} (3B_1(S) - B_1(S-k)) Q_{S-k} + \phi_{1,0}^{\text{per}} Q_S + \phi_{1,1}^{\text{per}} \mathcal{P}_1 \left(u + \frac{i}{2} \right) Q_S,$$

where $\alpha^2 = i/4B_1(S)$, $B_1(S) = H_1(S) - H_{-1}(S)$, and[9]

$$\phi_{1,1}^{\text{per}} = -2i\alpha B_1(S),$$

$$\phi_{1,0}^{\text{per}} = \alpha \left\{ \frac{4}{3} B_1(S)^2 + B_2(S) + \frac{3B_3(S) + 2H_3(S) - 2H_{-3}(S)}{3B_1(S)} - 2B_1(S)(1 + 2\log 2) \right\}.$$

The expressions for $q_2^{(0,1)}$ are too long to be presented here and can be found in [81].

Finally, we comment on the generalization of this approach to higher loops. There, we need $\mathcal{F}_{1,2}^S$ images for different products of Hurwitz functions with arbitrary indexes or fractions $1/(u \pm i/2)^a$ with Baxter polynomials Q_S. As our preliminary results show, they all can be obtained algorithmically, and the current procedure hence seems systematic. The details of higher-loop generalization will be the subject of one of our forthcoming publications.

4. Anomalous dimensions

Knowing the solutions of the Baxter equations obtained as described in the preceding section, we follow [81] in determining the coefficients[10] $A_0^{(l)}$, $A_3^{(l)}$, and $c_{i,k}^{(l)}$ in the ansatz for the functions $\mathbf{P}(u)$, and we refer the interested reader to that publication for further details. Up to four loops, we thus obtained the results for the anomalous dimensions of twist-1 operators

$$\gamma(S) = \gamma^{(0)}(S) h^2 + \gamma^{(1)}(S) h^4 + \ldots,$$

where

$$\gamma^{(0)}(S) = 4(\overline{H}_1 + \overline{H}_{-1} - 2\overline{H}_i),$$

$$\gamma^{(1)}(S) = 16\{3\overline{H}_{-2,-1} - 2\overline{H}_{-2,i} - \overline{H}_{-2,1} - \overline{H}_{-1,-2} + 2\overline{H}_{-1,2i} - \overline{H}_{-1,2} - 6\overline{H}_{i,-2} +$$

$$+ 12\overline{H}_{i,2i} - 6\overline{H}_{i,2} - 6\overline{H}_{2i,-1} + 4\overline{H}_{2i,i} + 2\overline{H}_{2i,1} - \overline{H}_{1,-2} + 2\overline{H}_{1,2i} - \overline{H}_{1,2} + 3\overline{H}_{2,-1} -$$

$$- 2\overline{H}_{2,i} - \overline{H}_{2,1} + 2\overline{H}_{-1,i,-1} - 2\overline{H}_{-1,i,1} + 8\overline{H}_{i,-1,-1} - 12\overline{H}_{i,-1,i} + 4\overline{H}_{i,-1,1} -$$

$$- 16\overline{H}_{i,i,-1} + 16\overline{H}_{i,i,i} + 4\overline{H}_{i,1,-1} - 4\overline{H}_{i,1,i} + 2\overline{H}_{1,i,-1} - 2\overline{H}_{1,i,1}\} + 8(H_{-1} - H_1)\zeta_2,$$

[8] See the definition of harmonic sums in the next section.

[9] See Appendix B in [81] for the definition of $B_{2,3}(S)$ sums.

[10] The coefficients $m_j^{(l)}$ and $k_j^{(l)}$ are left undetermined because of the QSC gauge symmetry (see [81] for details).

and we introduce the new sums

$$H_{a,b,\ldots}(S)=\sum_{k=1}^{S}\frac{\mathrm{Re}[(a/|a|)^k]}{k^{|a|}}H_{b,\ldots}(k),\qquad H_{a,\ldots}=H_{a,\ldots}(S),\qquad \overline{H}_{a,\ldots}=H_{a,\ldots}(2S)$$

such that for real indices, these sums reduce to ordinary harmonic sums. Imaginary indices correspond to the generalization of the harmonic sums with the fourth root of unity factor $(e^{i\pi/2})^n$. The obtained expression respects the maximum transcendentality principle [85], [86] and completely agrees with the previously obtained results [87]–[90]. This result could in fact be further rewritten in terms of cyclotomic or S-sums (see [89], [90]) if we would extend the definition of the latter to complex values of the parameters x_i. It can also be expressed in terms of the twisted η-functions in [41].

5. Conclusion

We have shown that multiloop Baxter equations arising in QSC problems can be solved directly in the space of the spectral parameter u by reducing inhomogeneous second-order difference equations with complex hypergeometric functions to a purely algebraic problem. As a particular example, we considered the anomalous dimensions of twist-1 operators in the ABJM model up to the four-loop order. The result is expressible in terms of harmonic sums with imaginary indices and respects the principle of maximum transcendentality. The presented technique can be further generalized both to higher loops and to higher twists of the considered operators.

The procedure seems systematic because all required $\mathcal{F}_{1,2}^{S}$ images for different products of Hurwitz functions with arbitrary indices or fractions $1/(u\pm i/2)^a$ with leading-order Baxter polynomials appearing at different orders of the perturbative expansion in the coupling constant can be obtained algorithmically. Also, similarly to [79], [80], where all operations close on trilinear combinations of rational, η-, and $\mathcal{P}_k$-functions, all our operations close on quadrilinear combinations of rational, η-, $\mathcal{P}_k$-, and Q_S-functions. But the resulting sums that we obtain are written in a form different from generalized harmonic sums, and extra work is needed to reduce them to the latter. Nevertheless, if we already know the basis of the sums in the final answer, then the reduction procedure can be performed by fixing the coefficients in a known function basis. The advantage of the approach under development is that the required number of anomalous dimensions for different spin values can be computed much more quickly. All these details will be the subject of one of our subsequent publications.

The presented techniques should also be applicable for solving twisted $\mathcal{N}$=4 and ABJM QSCs with $\mathbf{P}$ functions having twisted nonpolynomial asymptotic forms at large spectral parameter values. Those models are interesting in connection with the recent progress with so-called fishnet theories [91]–[100]. Moreover, we think that similar ideas should also be useful in studying the BFKL regime in the QSC approach [50]–[52] for the $\mathcal{N}$=4 SYM because the latter uses a perturbative expansion in both the coupling constant g and the parameter $w\equiv S+1$ describing the proximity of the Lorentz spin of the operator S to -1 such that the ratio g^2/w remains fixed.

Appendix: Some identities for Baxter polynomials

We derive identities presented in the main text. We use a powerful approach based on the generating function for Baxter polynomials. It has the form [81]

$$W(x,u)=\sum_{S=0}^{\infty}x^S Q_S(u)=(1-x)^{-1/2+iu}(1+x)^{-1/2-iu}. \tag{A.1}$$

Using this generating function, we can easily establish various linear relations between the Baxter polynomials. In what follows, when writing an identity for the generating function, we imply that it can be easily verified using explicit form (A.1).

First, expanding the identities

$$x\,\partial_x(1\mp x)W\left(x,u\pm\frac{i}{2}\right)=-iux\left[W\left(x,u+\frac{i}{2}\right)+W\left(x,u+\frac{i}{2}\right)\right]$$

in x, we obtain

$$\begin{pmatrix}Q_S^{[1]}\\ Q_S^{[-1]}\end{pmatrix}=\begin{pmatrix}1-iu/S & -iu/S\\ -iu/S & -1-iu/S\end{pmatrix}\begin{pmatrix}Q_{S-1}^{[1]}\\ Q_{S-1}^{[-1]}\end{pmatrix}. \tag{A.2}$$

In particular,

$$Q_S^{[1]}-Q_S^{[-1]}=Q_{S-1}^{[1]}+Q_{S-1}^{[-1]}. \tag{A.3}$$

This identity can be used to prove that

$$Q_S^{[\pm 2]}=Q_S+2\sum_{k=1}^{S}(\pm 1)^k Q_{S-k}$$

by induction (we leave this as an exercise for the reader). Similarly, from

$$(1+x)\,\partial_x(1-x)\frac{W(x,u)-W(x,i/2)}{u-i/2}=-2iW(x,u),$$

we obtain

$$\frac{(Q_S-Q_{S-1})S}{u-i/2}=-2iQ_{S-1}-\frac{(Q_{S-1}-Q_{S-2})(S-1)}{u-i/2},$$

whence by induction we obtain

$$\frac{Q_S-Q_{S-1}}{u-i/2}=\frac{2i}{S}\sum_{k=1}^{S}(-1)^k Q_{S-k}.$$

This formula can in turn be used by induction to prove the identity

$$\frac{Q_S-1}{u-i/2}=2i(-1)^S\sum_{k=1}^{S}(-1)^k(\mathrm{H}_{-1}(S)-\mathrm{H}_{-1}(S-k))Q_{S-k}.$$

Replacing $u\to -u$ and using the symmetry $Q_S(-u)=(-1)^S Q_S(u)$, we obtain the second identity

$$\frac{Q_S-(-1)^S}{u+i/2}=-2i(-1)^S\sum_{k=1}^{S}(\mathrm{H}_{-1}(S)-\mathrm{H}_{-1}(S-k))Q_{S-k}.$$

We can similarly prove the identities

$$\frac{Q_S^{[\pm 1]}-(\pm 1)^S}{u\pm i/2}=-2i\sum_{k=1}^{S}(\pm 1)^{k+1}(\mathrm{H}_1(S)-\mathrm{H}_1(S-k))Q_{S-k}.$$

We now prove some identities quadratic in the Baxter polynomials. First, we have the identity

$$x\partial_x y\partial_y(x+y)(W_x^{[1]}W_y^{[-1]}+W_x^{[-1]}W_y^{[1]})=-iu[y\partial_y+x\partial_x](W_x^{[1]}-W_x^{[-1]})(W_y^{[1]}-W_y^{[-1]}),$$

where $W_x = W(x,u)$. From this identity, we obtain

$$S_1S_2[Q_{S_1}^{[1]}Q_{S_2-1}^{[-1]} + Q_{S_1}^{[-1]}Q_{S_2-1}^{[1]} + Q_{S_1-1}^{[1]}Q_{S_2}^{[-1]} + Q_{S_1-1}^{[-1]}Q_{S_2}^{[1]}] =$$
$$= -iu(S_1+S_2)(Q_{S_1}^{[1]} - Q_{S_1}^{[-1]})(Q_{S_2}^{[1]} - Q_{S_2}^{[-1]}). \qquad \text{(A.4)}$$

In particular, if $S_1 = S_2 = S$, then we have

$$Q_S^{[1]}Q_{S-1}^{[-1]} + Q_S^{[-1]}Q_{S-1}^{[1]} = -\frac{iu}{S}\left(Q_S^{[1]} - Q_S^{[-1]}\right)^2. \qquad \text{(A.5)}$$

We also write the identity

$$Q_S^{[1]}Q_S^{[-1]} + Q_{S-1}^{[1]}Q_{S-1}^{[-1]} = \frac{-iu}{2S}(Q_S^{[1]} - Q_S^{[-1]})(Q_S^{[1]} + Q_S^{[-1]} + Q_{S-1}^{[1]} - Q_{S-1}^{[-1]}), \qquad \text{(A.6)}$$

which can be proved by expressing $Q_S^{[\pm 1]}$ in terms of $Q_{S-1}^{[\pm 1]}$ using Eq. (A.2).

We next prove the identity

$$\frac{1}{2}[Q_S^{[1]} + Q_S^{[-1]} + Q_{S-1}^{[1]} - Q_{S-1}^{[-1]}] =$$
$$= 1 + (-1)^S - iu\left[\frac{Q_{S-1}^{[1]} + Q_{S-1}^{[-1]}}{S} + \sum_{n=1}^{S-1}\frac{1+(-1)^{n+S}}{n}(Q_{n-1}^{[1]} + Q_{n-1}^{[-1]})\right]. \qquad \text{(A.7)}$$

Multiplying this identity by x^{S-1} and summing over S from 1 to ∞, we obtain[11]

$$\frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)[(1+x)W^{[1]} + (1-x)W^{[-1]}] = 1 - iu\int dx\,(W^{[1]} + W^{[-1]}), \qquad \text{(A.8)}$$

where $\int dx \circ$ corresponds to the operator $f(x) \to \int_0^x d\xi\, f(\xi)$. Equation (A.8) is equivalent to Eq. (A.7) and can be explicitly verified by evaluating both sides of the equation and finding that they are both equal to $(1-x)^{iu}(1+x)^{-iu}$.

We can now prove identity (3.3). Multiplying (3.3) by $uQ_S^{[1]}Q_S^{[-1]}$, we obtain

$$Q_S^{[1]}Q_S^{[-1]} = (-1)^S - iu\sum_{n=0}^{S-1}\frac{1+(-1)^n}{2S-n}(Q_S^{[1]}Q_{S-1-n}^{[-1]} + Q_S^{[-1]}Q_{S-1-n}^{[1]}).$$

We prove this identity by induction. The base case is easily verified. We prove the induction step. We find that

$$Q_S^{[1]}Q_S^{[-1]} = Q_S^{[1]}Q_S^{[-1]} + Q_{S-1}^{[1]}Q_{S-1}^{[-1]} - Q_{S-1}^{[1]}Q_{S-1}^{[-1]} \overset{\text{(A.6)}}{=}$$
$$\overset{\text{(A.6)}}{=} \frac{-iu}{2S}(Q_S^{[1]} - Q_S^{[-1]})(Q_S^{[1]} + Q_S^{[-1]} + Q_{S-1}^{[1]} - Q_{S-1}^{[-1]}) - Q_{S-1}^{[1]}Q_{S-1}^{[-1]} \overset{\text{i.h}}{=}$$
$$\overset{\text{i.h.}}{=} -\frac{iu}{2S}(Q_S^{[1]} - Q_S^{[-1]})(Q_S^{[1]} + Q_S^{[-1]} + Q_{S-1}^{[1]} - Q_{S-1}^{[-1]}) +$$
$$+ (-1)^S + iu\sum_{n=0}^{S-2}\frac{1+(-1)^n}{2S-2-n}(Q_{S-1}^{[1]}Q_{S-2-n}^{[-1]} + Q_{S-1}^{[-1]}Q_{S-2-n}^{[1]}) =$$
$$= (-1)^S - iu\sum_{n=0}^{S-1}\frac{1+(-1)^n}{2S-n}(Q_S^{[1]}Q_{S-1-n}^{[-1]} + Q_S^{[-1]}Q_{S-1-n}^{[1]}) + A,$$

[11] In the course of transformations, we must change the order of summation over S and n as $\sum_{S=1}^{\infty}\sum_{n=1}^{S-1} = \sum_{S=2}^{\infty}\sum_{n=1}^{S-1} = \sum_{n=1}^{\infty}\sum_{S=n+1}^{\infty}$.

where "i.h." denotes the induction hypothesis,

$$\begin{aligned}
A &= A_1 + A_2, \\
A_1 &= -\frac{iu}{2S}(Q_S^{[1]} - Q_S^{[-1]})(Q_S^{[1]} + Q_S^{[-1]} + Q_{S-1}^{[1]} - Q_{S-1}^{[-1]}), \\
A_2 &= iu \sum_{n=0}^{S-1} \frac{1+(-1)^n}{2S-n}(Q_S^{[1]}Q_{S-1-n}^{[-1]} + Q_S^{[-1]}Q_{S-1-n}^{[1]}) + \\
&\quad + iu \sum_{n=0}^{S-2} \frac{1+(-1)^n}{2S-2-n}(Q_{S-1}^{[1]}Q_{S-2-n}^{[-1]} + Q_{S-1}^{[-1]}Q_{S-2-n}^{[1]}).
\end{aligned}$$

Our goal is to prove that $A = 0$. Because of Eqs. (A.7) and (A.3), we can write A_1 as

$$A_1 = -\frac{iu}{S}(Q_S^{[1]} - Q_S^{[-1]})\left(1 + (-1)^S - iu\left[\frac{Q_S^{[1]} - Q_S^{[-1]}}{S} + \sum_{n=1}^{S-1} \frac{1+(-1)^{n+S}}{n}(Q_n^{[1]} - Q_n^{[-1]})\right]\right).$$

We now transform A_2. We shift $n \to n-2$ in the second sum and normalize the summation limits by adding/subtracting the terms with $n = 0, 1$ and $n = S$. We then change the summation variable $n \to S - n$ and obtain

$$\begin{aligned}
A_2 = iu \sum_{n=1}^{S} \frac{1+(-1)^{n+S}}{S+n}(Q_S^{[1]}Q_{n-1}^{[-1]} + Q_S^{[-1]}Q_{n-1}^{[1]} + Q_{S-1}^{[1]}Q_n^{[-1]} + Q_{S-1}^{[-1]}Q_n^{[1]}) - \\
- \frac{iu}{S}(Q_{S-1}^{[1]}Q_S^{[-1]} + Q_{S-1}^{[-1]}Q_S^{[1]}) + \frac{iu}{S}(1 + (-1)^S)(Q_{S-1}^{[1]} + Q_{S-1}^{[-1]}).
\end{aligned}$$

The first line can be transformed using Eq. (A.4), and the second line can be transformed using Eqs. (A.3) and (A.5). We obtain

$$\begin{aligned}
A_2 = u^2 \frac{1}{S}(Q_S^{[1]} - Q_S^{[-1]}) \sum_{n=1}^{S} \frac{1+(-1)^{n+S}}{n}(Q_n^{[1]} - Q_n^{[-1]}) + \\
+ \frac{u^2}{S^2}(Q_S^{[1]} - Q_S^{[-1]})^2 + \frac{iu}{S}(1 + (-1)^S)(Q_S^{[1]} - Q_S^{[-1]}).
\end{aligned}$$

Comparing the final expressions for A_1 and A_2, we see that, indeed, $A_1 = -A_2$ and hence $A = 0$.

REFERENCES

1. G. 't Hooft, "A planar diagram theory for strong interactions," *Nucl. Phys. B*, **72**, 461–473 (1974).
2. J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," *Internat. J. Theor. Phys.*, **38**, 1113–1133 (1999); arXiv:hep-th/9711200v3 (1997); "The large N limit of superconformal field theories and supergravity," *Adv. Theor. Math. Phys.*, **2**, 231–252 (1998).
3. S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," *Phys. Lett. B*, **428**, 105–114 (1998); arXiv:hep-th/9802109v2 (1998).
4. E. Witten, "Anti-de Sitter space and holography," *Adv. Theor. Math. Phys.*, **2**, 253–291 (1998); arXiv:hep-th/9802150v2 (1998).
5. N. Beisert et al., "Review of AdS/CFT integrability: An overview," *Lett. Math. Phys.*, **99**, 3–32 (2012); arXiv:1012.3982v5 [hep-th] (2010).

6. D. Bombardelli, A. Cagnazzo, R. Frassek, F. Levkovich-Maslyuk, F. Loebbert, S. Negro, I. M. Szécsényi, A. Sfondrini, S. J. van Tongeren, and A. Torrielli, "An integrability primer for the gauge–gravity correspondence: An introduction," *J. Phys. A: Math. Theor.*, **49**, 320301 (2016); arXiv:1606.02945v2 [hep-th] (2016).
7. S. J. van Tongeren, "Integrability of the $AdS_5 \times S^5$ superstring and its deformations," *J. Phys. A: Math. Theor.*, **47**, 433001 (2014); arXiv:1310.4854v3 [hep-th] (2013).
8. M. de Leeuw, A. C. Ipsen, C. Kristjansen, and M. Wilhelm, "Introduction to integrability and one-point functions in $\mathcal{N}=4$ SYM and its defect cousin," in: *Integrability: From Statistical Systems to Gauge Theory*, Les Houches School of Physics, Les Houches, France (2017); arXiv:1708.02525v1 [hep-th] (2017).
9. N. Gromov, "Introduction to the spectrum of $\mathcal{N}=4$ SYM and the quantum spectral curve," arXiv:1708.03648v1 [hep-th] (2017).
10. S. Komatsu, "Lectures on three-point functions in $\mathcal{N} = 4$ supersymmetric Yang–Mills theory," arXiv: 1710.03853v2 [hep-th] (2017).
11. V. Kazakov, "Quantum spectral curve of γ-twisted $\mathcal{N}=4$ SYM theory and fishnet CFT," *Rev. Math. Phys.*, **30**, 1840010 (2018); arXiv:1802.02160v1 [hep-th] (2018); M.-L. Ge, A. J. Niemi, K. K. Phua, and L. A. Takhtajan, eds., *Ludwig Faddeev Memorial Volume: A Life in Mathematical Physics*, World Scientific, Singapore (2018).
12. O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, "$N=6$ superconformal Chern–Simons-matter theories $M2$-branes and their gravity duals," *JHEP*, **0810**, 091 (2008); arXiv:0806.1218v4 [hep-th] (2008).
13. M. Staudacher, "The factorized S-matrix of CFT/AdS," *JHEP*, **0505**, 054 (2005); arXiv:hep-th/0412188v1 (2004).
14. G. Arutyunov, S. Frolov, and M. Staudacher, "Bethe ansatz for quantum strings," *JHEP*, **0410**, 016 (2004); arXiv:hep-th/0406256v3 (2004).
15. N. Beisert, "The $SU(2|2)$ dynamic S-matrix," *Adv. Theor. Math. Phys.*, **12**, 945–979 (2008); arXiv:hep-th/0511082v4 (2005).
16. N. Beisert, "The analytic Bethe ansatz for a chain with centrally extended $\mathfrak{su}(2|2)$ symmetry," *J. Stat. Mech.*, **0701**, P01017 (2007); arXiv:nlin/0610017v2 (2006).
17. N. Beisert, B. Eden, and M. Staudacher, "Transcendentality and crossing," *J. Stat. Mech.*, **0701**, P01021 (2007); arXiv:hep-th/0610251v2 (2006).
18. R. A. Janik, "The $AdS_5 \times S^5$ superstring worldsheet S-matrix and crossing symmetry," *Phys. Rev. D*, **73**, 086006 (2006); arXiv:hep-th/0603038v2 (2006).
19. G. Arutyunov and S. Frolov, "On $AdS_5 \times S^5$ string S-matrix," *Phys. Lett. B*, **639**, 378–382 (2006); arXiv:hep-th/0604043v2 (2006).
20. G. Arutyunov, S. Frolov, and M. Zamaklar, "The Zamolodchikov–Faddeev algebra for $AdS_5 \times S^5$ superstring," *JHEP*, **0704**, 002 (2007); arXiv:hep-th/0612229v3 (2006).
21. C. Ahn and R. I. Nepomechie, "$\mathcal{N}=6$ super Chern–Simons theory S-matrix and allloop Bethe ansatz equations," *JHEP*, **0809**, 010 (2008); arXiv:0807.1924v2 [hep-th] (2008).
22. J. A. Minahan and K. Zarembo, "The Bethe ansatz for $\mathcal{N}=4$ super Yang–Mills," *JHEP*, **0303**, 013 (2003); arXiv:hep-th/0212208v3 (2002).
23. N. Beisert and M. Staudacher, "The $\mathcal{N}=4$ SYM integrable super spin chain," *Nucl. Phys. B*, **670**, 439–463 (2003); arXiv:hep-th/0307042v3 (2003).
24. N. Beisert and M. Staudacher, "Long-range $\mathfrak{psu}(2;2|4)$ Bethe ansatze for gauge theory and strings," *Nucl. Phys. B*, **727**, 1–62 (2005); arXiv:hep-th/0504190v3 (2005).
25. J. A. Minahan and K. Zarembo, "The Bethe ansatz for superconformal Chern–Simons," *JHEP*, **0809**, 040 (2008); arXiv:0806.3951v6 [hep-th] (2008).
26. D. Gaiotto, S. Giombi, and X. Yin, "Spin chains in $N=6$ superconformal Chern–Simons-matter theory," *JHEP*, **0904**, 066 (2009); arXiv:0806.4589v2 [hep-th] (2008).
27. N. Gromov and P. Vieira, "The all loop AdS_4/CFT_3 Bethe ansatz," *JHEP*, **0901**, 016 (2009); arXiv: 0807.0777v2 [hep-th] (2008).
28. N. Gromov, V. Kazakov, and P. Vieira, "Exact spectrum of anomalous dimensions of planar $\mathcal{N}=4$ supersymmetric Yang–Mills theory," *Phys. Rev. Lett.*, **103**, 131601 (2009); arXiv:0901.3753v3 [hep-th] (2009).

29. D. Bombardelli, D. Fioravanti, and R. Tateo, “Thermodynamic Bethe ansatz for planar AdS/CFT: A proposal,” *J. Phys. A: Math. Theor.*, **42**, 375401 (2009); arXiv:0902.3930v2 [hep-th] (2009).
30. N. Gromov, V. Kazakov, A. Kozak, and P. Vieira, “Exact spectrum of anomalous dimensions of planar $\mathcal{N}=4$ supersymmetric Yang–Mills theory: TBA and excited states,” *Lett. Math. Phys.*, **91**, 265–287 (2010); arXiv: 0902.4458v4 [hep-th] (2009).
31. G. Arutyunov and S. Frolov, “Thermodynamic Bethe ansatz for the $\mathrm{AdS}_5\times S^5$ mirror model,” *JHEP*, **0905**, 068 (2009); arXiv:0903.0141v3 [hep-th] (2009).
32. A. Cavaglia, D. Fioravanti, and R. Tateo, “Extended Y-system for the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence,” *Nucl. Phys. B*, **843**, 302–343 (2011); arXiv:1005.3016v3 [hep-th] (2010).
33. J. Balog and Á. Hegedűs, “$\mathrm{AdS}_5\times S^5$ mirror TBA equations from Y-system and discontinuity relations,” *JHEP*, **1108**, 095 (2011); arXiv:1104.4054v2 [hep-th] (2011).
34. N. Gromov, V. Kazakov, S. Leurent, and Z. Tsuboi, “Wronskian solution for AdS/CFT Y-system,” *JHEP*, **1101**, 155 (2011); arXiv:1010.2720v2 [hep-th] (2010).
35. N. Gromov, V. Kazakov, S. Leurent, and D. Volin, “Solving the AdS/CFT Y-system,” *JHEP*, **1207**, 023 (2012); arXiv:1110.0562v3 [hep-th] (2011).
36. D. Bombardelli, D. Fioravanti, and R. Tateo, “TBA and Y-system for planar $\mathrm{AdS}_4/\mathrm{CFT}_3$,” *Nucl. Phys. B*, **834**, 543–561 (2010); arXiv:0912.4715v1 [hep-th] (2009).
37. N. Gromov and F. Levkovich-Maslyuk, “Y-system, TBA, and quasi-classical strings in $\mathrm{AdS}_4\times\mathbb{CP}^3$,” *JHEP*, **1006**, 088 (2010); arXiv:0912.4911v4 [hep-th] (2009).
38. A. Cavaglià, D. Fioravanti, and R. Tateo, “Discontinuity relations for the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence,” *Nucl. Phys. B*, **877**, 852–884 (2013); arXiv:1307.7587v1 [hep-th] (2013).
39. D. Correa, J. Maldacena, and A. Sever, “The quark anti-quark potential and the cusp anomalous dimension from a TBA equation,” *JHEP*, **1208**, 134 (2012); arXiv:1203.1913v2 [hep-th (2012).
40. N. Drukker, “Integrable Wilson loops,” *JHEP*, **1310**, 135 (2013); arXiv:1203.1617v2 [hep-th] (2012).
41. N. Gromov and F. Levkovich-Maslyuk, “Quantum spectral curve for a cusped Wilson line in $\mathcal{N}=4$ SYM,” *JHEP*, **1604**, 134 (2016); arXiv:1510.02098v1 [hep-th] (2015).
42. N. Gromov and F. Levkovich-Maslyuk, “Quark–anti-quark potential in $\mathcal{N}=4$ SYM,” *JHEP*, **1612**, 122 (2016); arXiv:1601.05679v2 [hep-th] (2016).
43. L. F. Alday, D. Gaiotto, and J. Maldacena, “Thermodynamic bubble ansatz,” *JHEP*, **1109**, 032 (2011); arXiv: 0911.4708v3 [hep-th] (2009).
44. L. F. Alday, J. Maldacena, A. Sever, and P. Vieira, “Y-system for scattering amplitudes,” *J. Phys. A: Math. Theor.*, **43**, 485401 (2010); arXiv:1002.2459v2 [hep-th] (2010).
45. L. F. Alday, D. Gaiotto, J. Maldacena, A. Sever, and P. Vieira, “An operator product expansion for polygonal null Wilson loops,” *JHEP*, **1104**, 088 (2011); arXiv:1006.2788v2 [hep-th] (2010).
46. B. Basso, A. Sever, and P. Vieira, “Spacetime and flux tube S-matrices at finite coupling for $\mathscr{N}=4$ supersymmetric Yang–Mills theory,” *Phys. Rev. Lett.*, **111**, 091602 (2013); arXiv:1303.1396v1 [hep-th] (2013).
47. B. Basso, J. Caetano, L. Cordova, A. Sever, and P. Vieira, “OPE for all helicity amplitudes,” *JHEP*, **1508**, 018 (2015); arXiv:1412.1132v1 [hep-th] (2014).
48. D. Fioravanti, S. Piscaglia, and M. Rossi, “Asymptotic bethe ansatz on the GKP vacuum as a defect spin chain: Scattering particles and minimal area Wilson loops,” *Nucl. Phys. B*, **898**, 301–400 (2015); arXiv:1503.08795v2 [hep-th] (2015).
49. B. Basso, S. Caron-Huot, and A. Sever, “Adjoint BFKL at finite coupling: A shortcut from the collinear limit,” *JHEP*, **1501**, 027 (2015); arXiv:1407.3766v2 [hep-th] (2014).
50. M. Alfimov, N. Gromov, and V. Kazakov, “QCD pomeron from AdS/CFT quantum spectral curve,” *JHEP*, **1507**, 164 (2015); arXiv:1408.2530v4 [hep-th] (2014).
51. N. Gromov, F. Levkovich-Maslyuk, and G. Sizov, “Pomeron eigenvalue at three loops in $\mathcal{N}=4$ supersymmetric Yang–Mills theory,” *Phys. Rev. Lett.*, **115**, 251601 (2015); arXiv:1507.04010v2 [hep-th] (2015).
52. M. Alfimov, N. Gromov, and G. Sizov, “BFKL spectrum of $\mathcal{N}=4$ SYM: non-zero conformal spin,” *JHEP*, **1807**, 181 (2018); arXiv:1802.06908v4 [hep-th] (2018).

53. B. Basso, S. Komatsu, and P. Vieira, “Structure constants and integrable bootstrap in planar $\mathcal{N}$=4 SYM theory,” arXiv:1505.06745v1 [hep-th] (2015).
54. B. Basso, V. Gonçalves, and S. Komatsu, “Structure constants at wrapping order,” *JHEP*, **1705**, 124 (2017); arXiv:1702.02154v1 [hep-th] (2017).
55. Y. Jiang, S. Komatsu, I. Kostov, and D. Serban, “Clustering and the three-point function,” *J. Phys. A: Math. Theor.*, **49**, 454003 (2016); arXiv:1604.03575v2 [hep-th] (2016).
56. I. Balitsky, V. Kazakov, and E. Sobko, “Structure constant of twist-2 light-ray operators in the Regge limit,” *Phys. Rev. D*, **93**, 061701 (2016); arXiv:1506.02038v2 [hep-th] (2015).
57. A. Cavaglià, N. Gromov, and F. Levkovich-Maslyuk, “Quantum spectral curve and structure constants in $\mathcal{N}$=4 SYM: Cusps in the ladder limit,” *JHEP*, **1810**, 060 (2018); arXiv:1802.04237v3 [hep-th] (2018).
58. B. Eden and A. Sfondrini, “Tessellating cushions: Four-point functions in $\mathcal{N}$=4 SYM,” *JHEP*, **1710**, 098 (2017); arXiv:1611.05436v1 [hep-th] (2016).
59. B. Eden, Y. Jiang, D. le Plat, and A. Sfondrini, “Colour-dressed hexagon tessellations for correlation functions and non-planar corrections,” *JHEP*, **1802**, 170 (2018); arXiv:1710.10212v2 [hep-th] (2017).
60. T. Bargheer, J. Caetano, T. Fleury, S. Komatsu, and P. Vieira, “Handling handles I: Nonplanar integrability,” arXiv:1711.05326v1 [hep-th] (2017).
61. T. Fleury and S. Komatsu, “Hexagonalization of correlation functions II: Two-particle contributions,” *JHEP*, **1802**, 177 (2018); arXiv:1711.05327v1 [hep-th] (2017).
62. S. Giombi and S. Komatsu, “Exact correlators on the Wilson loop in $\mathcal{N}$=4 SYM: Localization defect CFT and integrability,” *JHEP*, **1805**, 109 (2018); arXiv:1802.05201v3 [hep-th] (2018).
63. B. Eden, Y. Jiang, M. de Leeuw, T. Meier, D. le Plat, and A. Sfondrini, “Positivity of hexagon perturbation theory,” arXiv:1806.06051v2 [hep-th] (2018).
64. M. de Leeuw, C. Kristjansen, and K. Zarembo, “One-point functions in defect CFT and integrability,” *JHEP*, **1508**, 098 (2015); arXiv:1506.06958v2 [hep-th] (2015).
65. I. Buhl-Mortensen, M. de Leeuw, C. Kristjansen, and K. Zarembo, “One-point functions in AdS/dCFT from matrix product states,” *JHEP*, **1602**, 052 (2016); arXiv:1512.02532v2 [hep-th] (2015).
66. I. Buhl-Mortensen, M. de Leeuw, A. C. Ipsen, C. Kristjansen, and M. Wilhelm, “One-loop one-point functions in gauge-gravity dualities with defects,” *Phys. Rev. Lett.*, **117**, 231603 (2016); arXiv:1606.01886v3 [hep-th] (2016).
67. T. Harmark and M. Wilhelm, “Hagedorn temperature of AdS_5/CFT_4 via integrability,” *Phys. Rev. Lett.*, **120**, 071605 (2018); arXiv:1706.03074v3 [hep-th] (2017).
68. T. Harmark and M. Wilhelm, “The Hagedorn temperature of AdS_5/CFT_4 at finite coupling via the quantum spectral curve,” *Phys. Lett. B*, **786**, 53–58 (2018); arXiv:1803.04416v1 [hep-th] (2016).
69. N. Gromov, V. Kazakov, S. Leurent, and D. Volin, “Quantum spectral curve for planar $\mathcal{N}$=4 super-Yang–Mills theory,” *Phys. Rev. Lett.*, **112**, 011602 (2014); arXiv:1305.1939v2 [hep-th] (2013).
70. N. Gromov, V. Kazakov, S. Leurent, and D. Volin, “Quantum spectral curve for arbitrary state/operator in AdS_5/CFT_4,” *JHEP*, **1509**, 187 (2015); arXiv:1405.4857v3 [hep-th] (2014).
71. V. Kazakov, S. Leurent, and D. Volin, “T-system on T-hook: Grassmannian solution and twisted quantum spectral curve,” *JHEP*, **1612**, 044 (2016); arXiv:1510.02100v2 [hep-th] (2015).
72. C. Marboe and D. Volin, “The full spectrum of AdS_5/CFT_4 I: Representation theory and one-loop Q-system,” *J. Phys. A: Math. Theor.*, **51**, 165401 (2018); arXiv:1701.03704v3 [hep-th] (2017).
73. A. Cavaglià, D. Fioravanti, N. Gromov, and R. Tateo, “Quantum spectral curve of the $\mathcal{N}$=6 supersymmetric Chern–Simons theory,” *Phys. Rev. Lett.*, **113**, 021601 (2014); arXiv:1403.1859v2 [hep-th] (2014).
74. D. Bombardelli, A. Cavaglià, D. Fioravanti, N. Gromov, and R. Tateo, “The full quantum spectral curve for AdS_4/CFT_3,” *JHEP*, **1709**, 140 (2017); arXiv:1701.00473v4 [hep-th] (2017).
75. R. Klabbers and S. J. van Tongeren, “Quantum spectral curve for the eta-deformed AdS5xS5 superstring,” *Nucl. Phys. B*, **925**, 252–318 (2017); arXiv:1708.02894v2 [hep-th] (2017).
76. N. Gromov, F. Levkovich-Maslyuk, and G. Sizov, “Quantum spectral curve and the numerical solution of the spectral problem in AdS_5/CFT_4,” *JHEP*, **1606**, 036 (2016); arXiv:1504.06640v3 [hep-th] (2015).

77. Á. Hegedűs and J. Konczer, "Strong coupling results in the AdS_5/CFT_4 correspondence from the numerical solution of the quantum spectral curve," *JHEP*, **1608**, 061 (2016); arXiv:1604.02346v1 [hep-th] (2016).
78. D. Bombardelli, A. Cavaglià, R. Conti, and R. Tateo, "Exploring the spectrum of planar AdS_4/CFT_3 at finite coupling," *JHEP*, **1804**, 117 (2018); arXiv:1803.04748v2 [hep-th] (2018).
79. C. Marboe and D. Volin, "Quantum spectral curve as a tool for a perturbative quantum field theory," *Nucl. Phys. B*, **899**, 810–847 (2015); arXiv:1411.4758v2 [hep-th] (2014).
80. L. Anselmetti, D. Bombardelli, A. Cavaglià, and R. Tateo, "12 loops and triple wrapping in ABJM theory from integrability," *JHEP*, **1510**, 117 (2015); arXiv:1506.09089v2 [hep-th] (2015).
81. R. N. Lee and A. I. Onishchenko, "ABJM quantum spectral curve and Mellin transform," *JHEP*, **1805**, 179 (2018); arXiv:1712.00412v2 [hep-th] (2017).
82. M. A. Bandres, A. E. Lipstein, and J. H. Schwarz, "Studies of the ABJM theory in a formulation with manifest SU(4) R-symmetry," *JHEP*, **0809**, 027 (2008); arXiv:0807.0880v2 [hep-th] (2008).
83. T. Klose, "Review of AdS/CFT integrability chapter IV.3: N=6 Chern–Simons and STRINGS on $AdS_4\times\mathbb{CP}_3$," *Lett. Math. Phys.*, **99**, 401–423 (2012); arXiv:1012.3999v5 [hep-th] (2010).
84. G. Grignani, T. Harmark, and M. Orselli, "The $SU(2)\times SU(2)$ sector in the string dual of $\mathscr{N}=6$ superconformal Chern–Simons theory," *Nucl. Phys. B*, **810**, 115–134 (2009); arXiv:0806.4959v4 [hep-th] (2008).
85. A. V. Kotikov and L. N. Lipatov, "DGLAP and BFKL equations in the N=4 supersymmetric gauge theory," *Nucl. Phys. B*, **661**, 19–61 (2003); Erratum, **685**, 405–407 (2004); arXiv:hep-ph/0208220v3 (2002).
86. A. V. Kotikov, L. N. Lipatov, A. I. Onishchenko, and V. N. Velizhanin, "Three loop universal anomalous dimension of the Wilson operators in $\mathcal{N}$=4 SUSY Yang–Mills model," *Phys. Lett. B*, **595**, 521–529 (2004); Erratum, **632**, 754–756 (2006); arXiv:hep-th/0404092v5 (2004).
87. M. Beccaria and G. Macorini, "QCD properties of twist operators in the $\mathscr{N}=6$ Chern–Simons theory," *JHEP*, **0906**, 008 (2009); arXiv:0904.2463v3 [hep-th] (2009).
88. M. Beccaria, F. Levkovich-Maslyuk, and G. Macorini, "On wrapping corrections to GKP-like operators," *JHEP*, **1103**, 001 (2011); arXiv:1012.2054v2 [hep-th] (2010).
89. J. Ablinger, J. Blümlein, and C. Schneider, "Analytic and algorithmic aspects of generalized harmonic sums and polylogarithms," *J. Math. Phys.*, **54**, 082301 (2013); arXiv:1302.0378v1 [math-ph] (2013).
90. J. Ablinger, J. Blümlein, and C. Schneider, "Harmonic sums and polylogarithms generated by cyclotomic polynomials," *J. Math. Phys.*, **52**, 102301 (2011); arXiv:1105.6063v1 [math-ph] (2011).
91. A. B. Zamolodchikov, "'Fishing-net' diagrams as a completely integrable system," *Phys. Lett. B*, **97**, 63–66 (1980).
92. D. Chicherin, S. Derkachov, and A. P. Isaev, "Conformal group: R-matrix and star-triangle relation," *JHEP*, **1304**, 020 (2013); arXiv:1206.4150v2 [math-ph] (2013).
93. Ö. Gürdoğan and V. Kazakov, "New integrable 4D quantum field theories from strongly deformed planar $\mathcal{N}$=4 supersymmetric Yang–Mills theory," *Phys. Rev. Lett.*, **117**, 201602 (2016); Addendum, **117**, 259903 (2016); arXiv:1512.06704v3 [hep-th] (2015).
94. J. Caetano, O. Gürdoğan, and V. Kazakov, "Chiral limit of $\mathcal{N}$=4 SYM and ABJM and integrable Feynman graphs," arXiv:1612.05895v3 [hep-th] (2016).
95. D. Chicherin, V. Kazakov, F. Loebbert, D. Müller, and D.-l. Zhong, "Yangian symmetry for bi-scalar loop amplitudes," *JHEP*, **1805**, 003 (2017); arXiv:1704.01967v1 [hep-th] (2017).
96. B. Basso and L. J. Dixon, "Gluing ladder Feynman diagrams into fishnets," *Phys. Rev. Lett.*, **119**, 071601 (2017); arXiv:1705.03545v2 [hep-th] (2017).
97. N. Gromov, V. Kazakov, G. Korchemsky, S. Negro, and G. Sizov, "Integrability of conformal fishnet theory," *JHEP*, **1801**, 095 (2017); arXiv:1706.04167v2 [hep-th] (2017).
98. D. Chicherin, V. Kazakov, F. Loebbert, D. Müller, and D.-l. Zhong, "Yangian symmetry for fishnet Feynman graphs," *Phys. Rev. D*, **96**, 121901 (2017); arXiv:1708.00007v1 [hep-th] (2017).
99. D. Grabner, N. Gromov, V. Kazakov, and G. Korchemsky, "Strongly γ-deformed $\mathcal{N}$=4 SYM as an integrable CFT," *Phys. Rev. Lett.*, **120**, 111601 (2018); arXiv:1711.04786v3 [hep-th] (2017).
100. V. Kazakov and E. Olivucci, "Bi-scalar integrable CFT at any dimension," *Phys. Rev. Lett.*, **121**, 131601 (2018); arXiv:1801.09844v3 [hep-th] (2018).