Theoretical and Mathematical Physics, **197**(3): 1779–1796 (2018)

HIGHER HIROTA DIFFERENCE EQUATIONS AND THEIR REDUCTIONS

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We previously proposed an approach for constructing integrable equations based on the dynamics in associative algebras given by commutator relations. In the framework of this approach, evolution equations determined by commutators of (or similarity transformations with) functions of the same operator are compatible by construction. Linear equations consequently arise, giving a base for constructing nonlinear integrable equations together with the corresponding Lax pairs using a special dressing procedure. We propose an extension of this approach based on introducing higher analogues of the famous Hirota difference equation. We also consider some (1+1)-dimensional discrete integrable equations that arise as reductions of either the Hirota difference equation itself or a higher equation in its hierarchy.

Keywords: integrability, commutator identity, Hirota difference equation, higher integrable equation, reduction

DOI: 10.1134/S0040577918120085

To the memory of Ludwig Dmitrievich Faddeev

1. Introduction

Here, we introduce and consider higher integrable generalizations of the famous Hirota difference equation (HDE). It was introduced in [1], [2] in the bilinear form as an equation for the τ-function,

$$\tau^{(1)}(m)\tau^{(2,3)}(m)+\tau^{(2)}(m)\tau^{(3,1)}(m)+\tau^{(3)}(m)\tau^{(1,2)}(m)=0, \tag{1.1}$$

where $\tau(m)=\tau(m_1,m_2,m_3)$ is a function depending on three integers (independent variables) $m_1,m_2,m_3\in\mathbb{Z}$. Here and hereafter, superscripts 1, 2, and 3 in parenthesis denote unit shifts of the variable with the corresponding index:

$$\begin{aligned}&\tau^{(1)}(m)=\tau(m_1+1,m_2,m_3), \qquad \tau^{(2)}(m)=\tau(m_1,m_2+1,m_3),\\ &\tau^{(2,3)}(m)=\tau(m_1,m_2+1,m_3+1), \qquad \text{and so on.}\end{aligned} \tag{1.2}$$

The HDE is a (2+1)-dimensional integrable equation to which many papers are devoted (see, e.g., [2]–[11] and the references therein) because it generates many known integrable partial difference, difference–differential, and differential equations via various continuous limits and (1+1)-dimensional integrable reductions. In particular, such equations include the standard and modified Korteweg–de Vries (KdV) and

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This research is supported by a grant from the Russian Science Foundation (Project No. 14-50-00005).

Prepared from an English manuscript submitted by the author; for the Russian version, see *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 197, No. 3, pp. 444–463, December, 2018. Original article submitted February 22, 2018.

Kadomtsev–Petviashvili equations and the two-dimensional Toda lattice, sine-Gordon, and Benjamin–Ono equations, etc. Therefore, the HDE is often considered a fundamental integrable system.

The τ-function is well known as a tool for studying nonlinear integrable equations but is not convenient for studying direct and inverse scattering problems. Therefore, in the literature (see, e.g., [6]), the function $v(m) = v(m_1, m_2, m_3)$ was introduced by the equalities

$$v^{(1)}(m) - v^{(3)}(m) = \frac{\tau^{(1,3)}(m)\tau(m)}{\tau^{(1)}(m)\tau^{(3)}(m)}, \tag{1.3}$$

$$v^{(2)}(m) - v^{(1)}(m) = \frac{\tau^{(1,2)}(m)\tau(m)}{\tau^{(2)}(m)\tau^{(1)}(m)}, \tag{1.4}$$

which allows rewriting (1.1) in the form (see notation (1.2))

$$v^{(1,2)}(v^{(1)} - v^{(2)}) + v^{(2,3)}(v^{(2)} - v^{(3)}) + v^{(3,1)}(v^{(3)} - v^{(1)}) = 0. \tag{1.5}$$

In these terms, the Lax pair for the HDE is given by any two of the three equations

$$\varphi^{(2)} = \varphi^{(1)} + (v^{(2)} - v^{(1)})\varphi, \tag{1.6a}$$

$$\varphi^{(3)} = \varphi^{(2)} + (v^{(3)} - v^{(2)})\varphi, \tag{1.6b}$$

$$\varphi^{(1)} = \varphi^{(3)} + (v^{(1)} - v^{(3)})\varphi, \tag{1.6c}$$

which emphasizes the symmetry of the HDE with respect to all three variables m_i. At the same time, we can regard Eq. (1.5) as an evolution equation, where, for example, m_1 and m_2 play the role of space variables and m_3 is the time variable.

Here, in deriving higher generalizations of the HDE, we follow [12]–[14], where nonlinear integrable equations were derived as a result of a special procedure for dressing the corresponding linear equations. Hence, the initial step in our approach is to derive these linear equations, in a sense, equations for the scattering data. For this, we use the following construction. Let A and B be two arbitrary elements of an associative algebra over $\mathbb{C}$. We introduce the dependence of B on discrete or continuous parameters via the commutators of some functions of A with B. We consider the term "commutator" in both senses: in the algebraic sense for continuous parameters and in the group sense (similarity transformation) for discrete parameters. We interpret these parameters as independent variables (times) yielding mutually commuting flows on the algebra, continuous or discrete. The linear evolution equation is then a result of a commutator identity specific for the chosen functions of A holding on an arbitrary associative algebra.

This paper is structured as follows. In Sec. 2, we use the example of the HDE to demonstrate the main topics of our approach (see [13] for more details). In Sec. 3, we consider some reductions of this (2+1)-dimensional integrable system to (1+1)-integrable systems. In Sec. 4. we give a generic construction of higher equations of the HDE type, including a discrete analogue of the Zakharov–Shabat system. In Sec. 5, we consider an example of such a higher HDE. In Sec. 6, we give some (1+1)-dimensional reductions of this equation. In Sec. 7, we present concluding remarks.

2. The HDE as an example of a general construction

2.1. Commutator identity and a linear equation on an associative algebra. Here, we follow [13] to illustrate details of our approach. We start with the simplest case of the HDE itself. Let there be an associative algebra with a unit I, and for some element A in this algebra, let there exist inverse elements $(A - a_i I)^{-1}$ for some constants $a_1, a_2, a_3 \in \mathbb{C}$ ($a_1 \neq a_2 \neq a_3 \neq a_1$). Let B be another element of this algebra. We introduce the dependence of B on three discrete variables m_1, m_2, and m_3 belonging to $\mathbb{Z}$ by the equalities (see notation (1.2))

$$\begin{aligned} B^{(1)} &= (A - a_1 I) B (A - a_1 I)^{-1}, \\ B^{(2)} &= (A - a_2 I) B (A - a_2 I)^{-1}, \\ B^{(3)} &= (A - a_3 I) B (A - a_3 I)^{-1}. \end{aligned} \tag{2.1}$$

Then $B(m)$ satisfies the linear difference equation

$$a_{12}\{B^{(12)} + B^{(3)}\} + \text{cycle}\{1,2,3\} = 0, \tag{2.2}$$

where the differences

$$a_{ij} = a_i - a_j \tag{2.3}$$

are nonzero. Equation (2.2) follows from the commutator identity

$$\begin{aligned} a_{12}\{(A - a_1)(A - a_2) B (A - a_1)^{-1}(A - a_2)^{-1} + {} \\ + (A - a_3) B (A - a_3)^{-1}\} + \text{cycle}(1,2,3) = 0, \end{aligned} \tag{2.4}$$

which holds on an arbitrary associative algebra. Equation (2.2) is a linearized version of HDE (1.5) (see (2.27) below or [13]).

2.2. Operator realization and dressing procedure. In [13], we used an analogy to pseudodifferential operators to formulate a dressing procedure. We consider (infinite) matrices $F, G, \ldots$ and let T denote the shift matrix $T_{m_1,m_1'} = \delta_{m_1,m_1'+1}$. For any matrix $F = \{F_{ij}\}_{i,j\in\mathbb{Z}}$, we introduce $f_n(m_1) = F_{m_1,m_1-n}$, and F can hence be written as

$$F = \sum_{n\in\mathbb{Z}} f_n T^n, \tag{2.5}$$

where all matrices $f_n = \operatorname{diag}\{f_n(m_1)\}_{m_1\in\mathbb{Z}}$ are diagonal and are hence mutually commuting matrices. We note that this is a leading consideration only; we therefore do not discuss convergence of the above series. With this accuracy, we uniquely assign each matrix F its symbol

$$\widetilde{F}(m_1, z) = \sum_{n\in\mathbb{Z}} f_n(m_1) z^n, \tag{2.6}$$

where $m_1 \in \mathbb{Z}$ and $z = z_{\Re} + i z_{\Im} \in \mathbb{C}$. It is easy to see that the standard product of matrices F and G in terms of their symbols becomes

$$\widetilde{FG}(m_1, z) = \oint_{|\zeta|=1} \frac{d\zeta}{2\pi i\zeta} \widetilde{F}(m_1, z\zeta) \sum_{m_1'\in\mathbb{Z}} \zeta^{m_1 - m_1'} \widetilde{G}(m_1', z). \tag{2.7}$$

As useful examples, we mention that for the unit and shift matrices I and T, we have $i_n(m_1) = \delta_{m_1,0}$ and $t_n(m_1) = \delta_{m_1,1}$. Hence, by the above definition, we have the symbols

$$\widetilde{I}(m, z) = 1, \qquad \widetilde{T}(m, z) = z. \tag{2.8}$$

Relation (2.5) shows that we use an analogue of the normal order: all shift operators are placed to the right of multiplication operators. In particular, let F be a diagonal matrix, which means its symbol is independent of z: $\widetilde{F}(m_1,z)\equiv\widetilde{F}(m_1)$ (see (2.5)). By (2.7), we then have

$$\widetilde{FG}(m_1,z)=\widetilde{F}(m_1)\widetilde{G}(m_1,z) \tag{2.9}$$

for any operator G. Hence, operators with z-independent symbols play the role of multiplication operators. Conversely, let G be a function of only the shift operator, i.e., by (2.5), its symbol is independent of the discrete variable: $\widetilde{G}(m_1,z)\equiv\widetilde{G}(z)$. By (2.7), we then have

$$\widetilde{FG}(m_1,z)=\widetilde{F}(m_1,z)\widetilde{G}(z) \tag{2.10}$$

for an arbitrary F. The similarity transformation by the operator T, as follows from (2.7) and (2.10), gives a shift of the discrete variable:

$$\widetilde{TFT^{-1}}(m_1,z)=\widetilde{F}(m_1+1,z),\quad \text{i.e., } TFT^{-1}=F^{(1)}, \tag{2.11}$$

where we use notation (1.2). This relation is essential for the construction below.

In what follows, we consider a set of "pseudomatrix" operators $F,G,\dots$ given by their symbols $\widetilde{F},\widetilde{G},\dots$ with the above composition law. We impose the condition that these symbols are tempered distributions in their variables or Fourier coefficients of distributions. But in the generic case, we do not expect any relation of type (2.5) for these operators with matrices; in particular, we do not expect any analyticity property of the symbols of operators with respect to the variable z. Therefore, we can introduce operations on this set that are well defined in terms of symbols but have no analogue on the set of matrices. In particular, in [13], we defined the $\bar{\partial}$-differentiation operation $F\to\bar{\partial}F$:

$$(\widetilde{\bar{\partial}F})(m,z)=\frac{\partial\widetilde{F}(m,z)}{\partial\bar{z}}. \tag{2.12}$$

This derivative is a measure of the departure of the operator symbol from analyticity and hence also gives a measure of the departure of operator from the infinite matrix, i.e., from the case where the series in (2.5) converges. In particular, the unit and shift operators, as follows from (2.8), satisfy

$$\bar{\partial}I=0,\qquad \bar{\partial}T=0. \tag{2.13}$$

2.3. Dressing procedure. We regard operators A and B as operators of the above kind with symbols $\widetilde{A}$ and $\widetilde{B}$. The dependence of the symbol of B on m_1, $B^{(1)}=(A-a_1)B(A-a_1)^{-1}$, is exactly the same as under similarity transformation (2.11) using the operator T. We can therefore set

$$A=T+a_1,\quad \text{i.e., } \widetilde{A}(m,z)=z+a_1. \tag{2.14}$$

In accordance with (2.1), we then have

$$\begin{aligned} &B^{(1)}=TBT^{-1},\qquad B^{(2)}=(T+a_{12})B(T+a_{12})^{-1},\\ &B^{(3)}=(T+a_{13})B(T+a_{13})^{-1}, \end{aligned} \tag{2.15}$$

where we use notation (1.2) and (2.3). The symbol $\widetilde{B}(m_1,m_2,m_3,z)$ of B now depends on the three discrete variables $m_1,m_2,m_3\in\mathbb{Z}$ in addition to the variable z. In what follows, we use the notation $\widetilde{B}(m,z)$, setting

$m = \{m_1, m_2, m_3\}$. The dependence on m_2 and m_3 does not affect (2.7), where the product of symbols must be considered pointwise with respect to these variables. Therefore,

$$(FG)^{(i)} = F^{(i)} G^{(i)}, \quad i = 1, 2, 3, \tag{2.16}$$

where this equality follows from (2.7) for $i = 1$.

Because of the m-dependence of the operator B specified in (2.15), we can represent its symbol as

$$\widetilde{B}(m, z) = \oint_{|\zeta|=1} \frac{d\zeta}{2\pi i\zeta} \zeta^{m_1} \left(\frac{z\zeta + a_{12}}{z + a_{12}}\right)^{m_2} \left(\frac{z\zeta + a_{13}}{z + a_{13}}\right)^{m_3} b(\zeta, z), \tag{2.17}$$

where $b(\zeta, z)$ is some function. It was shown in [13] that to avoid growth of this symbol with respect to m_2 and m_3, it suffices to set all a_j to be real and the support of the function $b(\zeta, z)$ to be accumulated on the surface $\zeta = \bar{z}/z$. For the symbol of B, we hence have the representation

$$\widetilde{B}(m, z) = \left(\frac{\bar{z}}{z}\right)^{m_1} \left(\frac{\bar{z} + a_{12}}{z + a_{12}}\right)^{m_2} \left(\frac{\bar{z} + a_{13}}{z + a_{13}}\right)^{m_3} f(z), \tag{2.18}$$

where $f(z)$ is an arbitrary function of $z \in \mathbb{C}$.

We introduce the main object of our construction, the *dressing operator* K with the symbol $\widetilde{K}(m, z)$, as the solution of the $\bar{\partial}$-problem

$$\bar{\partial} K = KB, \qquad \lim_{z\to\infty} \widetilde{K}(m, z) = 1, \tag{2.19}$$

where the product in the right-hand side is understood in the sense of (2.7). In what follows, we assume the unique solvability of this problem, which is crucial for our construction but not essential for its results (see Sec. 7). The dependence of K on the m variables is introduced via the same $\bar{\partial}$-problem:

$$\bar{\partial} K^{(j)} = K^{(j)} B^{(j)}, \qquad \lim_{z\to\infty} \widetilde{K^{(j)}}(m, z) = 1, \quad j = 1, 2, 3, \tag{2.20}$$

where we take (2.16) into account.

The compatibility of the evolution equations for the operator B is obvious by construction. By (2.16) and (2.20), we then have $\bar{\partial} K^{(i,j)} = K^{(i,j)} B^{(i,j)}$ and $\bar{\partial} K^{(j,i)} = K^{(j,i)} B^{(i,j)}$ for any $i, j = 1, 2, 3$. Therefore, the difference $K^{(i,j)} - K^{(j,i)}$ satisfies the $\bar{\partial}$ equation in (2.19), but with zero asymptotic behavior. Hence, this difference is equal to zero because we assumed the unique solvability. We consider consequences of equality (2.15) for the operator K. We note that this operator, like any operator of the considered class, satisfies (2.11),

$$K^{(1)} = TKT^{-1}, \tag{2.21}$$

which is compatible with (2.20) for $j = 1$ because of the first equality in (2.15). We now consider $j = 2$. From (2.13) and (2.15), we derive

$$\bar{\partial}(K^{(2)}(T + a_{12})) = (K^{(2)}(T + a_{12}))B,$$

and the product $K^{(2)}(T + a_{12})$ hence satisfies the same $\bar{\partial}$ equation but with a linear asymptotic growth at large z. And again by the assumption that problem (2.19) is uniquely solvable, there exists a multiplication operator X, an operator whose symbol is independent of z, such that $K^{(2)}(T + a_{12}) = (T + X)K$. Specifying the asymptotic condition in (2.19) by the next term of the expansion,

$$K = I + uT^{-1} + \dots, \quad z \to \infty, \tag{2.22}$$

where the ellipsis denotes terms with symbols decaying faster than z^{-1} and where u is a multiplication operator, because of (2.21), we obtain

$$K^{(2)}(T+a_{12}) = K^{(1)}T + (a_{12}+u^{(2)}-u^{(1)})K. \tag{2.23}$$

An analogous consideration shows that the evolution in m_3 is given by the equation

$$K^{(3)}(T+a_{13}) = K^{(1)}T + (a_{13}+u^{(3)}-u^{(1)})K. \tag{2.24}$$

Because of (2.9) and (2.10), relations (2.22) and (2.23) in terms of symbols become

$$(z+a_{12})\widetilde{K}^{(2)}(m,z) = z\widetilde{K}^{(1)}(m,z) + (v^{(2)}(m)-v^{(1)}(m))\widetilde{K}(m,z), \tag{2.25a}$$

$$(z+a_{13})\widetilde{K}^{(3)}(m,z) = z\widetilde{K}^{(1)}(m,z) + (v^{(3)}(m)-v^{(1)}(m))\widetilde{K}(m,z), \tag{2.25b}$$

and $z \in \mathbb{C}$ hence plays the role of the spectral parameter. Here, for brevity, we introduce the new dependent variable

$$v(m) = u(m) - m_1a_1 - m_2a_2 - m_3a_3. \tag{2.26}$$

Equations (2.25) are compatible by construction, and directly verifying this compatibility shows that the function $v(m)$ satisfies (1.5), i.e., the HDE. With the substitution of (2.26), this equation becomes

$$u^{(12)}(u^{(2)}-u^{(1)}+a_{12}) + a_{12}u^{(3)} + \text{cycle} = 0, \tag{2.27}$$

and the original Eq. (2.2) is its linearized version. We note that while the constants a_i are absent from (1.5), by (2.26), they determine the asymptotic behavior of $v(m)$: this function grows linearly with m at infinity. In [13], we demonstrated that this asymptotic behavior resolves the problem of the ill definedness of the Cauchy problem for (1.5).

To pass to the standard notation, we introduce the *Jost solution*

$$\varphi(m,z) = \widetilde{K}(m,z)z^{m_1}(z+a_{12})^{m_2}(z+a_{13})^{m_3}. \tag{2.28}$$

Then (2.25a) and (2.25b) reduce to (1.6a) and (1.6c), i.e., to the Lax pair for the HDE. The Jost solution also simplifies writing $\bar{\partial}$-problem (2.19). To show this, we note that because of (2.7) and (2.18), we can write problem (2.19) in terms of the symbol of the dressing operator as

$$\frac{\partial\widetilde{K}(m,z)}{\partial\bar{z}} = \widetilde{K}(m,\bar{z})\left(\frac{\bar{z}}{z}\right)^{m_1}\left(\frac{\bar{z}+a_{12}}{z+a_{12}}\right)^{m_2}\left(\frac{\bar{z}+a_{13}}{z+a_{13}}\right)^{m_3} f(z), \tag{2.29}$$

which because of (2.28) gives the standard form of the $\bar{\partial}$-problem with conjugation:

$$\frac{\partial\varphi(m,z)}{\partial\bar{z}} = \varphi(m,\bar{z})f(z). \tag{2.30}$$

Representation (2.29) demonstrates the degeneracy of the dispersion law, a property well known [15] to be directly related to integrability.

3. The (1+1)-dimensional reductions of the HDE

In [12]–[14] and [16], we demonstrated that the approach based on commutator relations leads to integrable equations in (2+1) dimensions. To obtain (1+1)-dimensional integrable systems, we must perform reductions. Following the idea of our approach, we start by constructing reductions of linear equation (2.2) for B and then apply the dressing procedure to obtain nonlinear integrable systems. In this case, dimensional reduction is thus understood as a relation between values of the operator B given by some shifts of the independent variables m_i. Such a relation must be compatible with (2.18) and must preserve the dependence of B on two independent variables. Because of (2.18), it is easy to see that any such reduction leads to an equation for the spectral parameter z: it must belong to a some curve on $\mathbb{C}$. This is possible only if the function $f(z)$ in (2.18) and then $\widetilde{B}(m,z)$ itself have a support on this curve, which we here regard as proportionality to a corresponding δ-function for simplicity. But (2.29) then means that the symbol $\widetilde{K}(m,z)$ is an analytic function outside this curve, and inverse problem (2.19) must hence be replaced with the standard Riemann–Hilbert problem. We do not go into these details here because if we have a reduction in the above sense, then we can derive the reduced Lax pair and the nonlinear integrable system directly from the (2+1)-dimensional ones.

It is natural to assume that the simplest reduction of the HDE is given by $B^{(i)} = B$ for some i, i.e., by the condition that the symbol $\widetilde{B}(m,z)$ is independent of one of the variables m_1, m_2, or m_3. But representation (2.18) shows that this is not a reduction in our sense, because it is possible only if $z_{\Im} = 0$, which cancels the dependence of $\widetilde{B}(m,z)$ on all other independent variables. Therefore, to obtain nontrivial reductions, we must seek more complicated relations on the independent variables. We note that the trivial condition $z_{\Im} = 0$ appears in all possible reductions, and we hence omit it below.

3.1. The reduction $B^{(2)} = B^{(-1)}$. We start with the condition $B^{(2)} = B^{(-1)}$, where the superscript (-1) denotes the shift $m_1 \to m_1 - 1$ in the function argument. In terms of symbols, this reduction gives

$$\widetilde{B}(m_1, m_2, m_3, z) = \widetilde{B}(m_1 - m_2, 0, m_3, z), \tag{3.1}$$

which by (2.18) is possible only if $z_{\Re} = -a_{12}/2$ (we omit the trivial case $z_{\Im} = 0$). Setting

$$a_2 = -a_1 \tag{3.2}$$

for simplicity here, we see that the above reduction requires the proportionality of a symbol B to the δ-function $\delta(z_{\Re} + a_1)$, and by (2.18), we hence have

$$\widetilde{B}(m_1, 0, m_3, z) = \left(\frac{a_1 + iz_{\Im}}{a_1 - iz_{\Im}}\right)^{m_1} \left(\frac{a_3 + iz_{\Im}}{a_3 - iz_{\Im}}\right)^{m_3} b(z_{\Im})\delta(z_{\Re} + a_1), \tag{3.3}$$

where $b(z_{\Im})$ is an arbitrary function of its argument (scattering data). The operator B with this symbol obviously satisfies the equation

$$a_{13}(B^{(1,3)} - B) + (a_1 + a_3)(B^{(1)} - B^{(3)}) = 0, \tag{3.4}$$

and the corresponding reduction of the original Eq. (2.2) gives

$$a_{13}(B^{(1,3)} - B) + (a_1 + a_3)(B^{(1)} - B^{(3)}) = [a_{13}(B^{(1,3)} - B) + (a_1 + a_3)(B^{(1)} - B^{(3)})]^{-1}.$$

Both sides of this equation are independent of m_1, and (3.4) hence arises as a result of summing it.

We emphasize that because of (2.19), the symbol $\widetilde{K}(m,z)$ of the dressing operator is an analytic function of $z \in \mathbb{C}$ in the half-planes $z_{\Re} \gtrless -a_1$ (see the discussion at the beginning of this section).

Because of (2.7), (2.19), and (3.1), we also obtain $K^{(2)} = K^{(-1)}$, i.e.,

$$\widetilde{K}(m_1, m_2, m_3, z) = \widetilde{K}(m_1 - m_2, 0, m_3, z), \quad z \in \mathbb{C}.$$

Therefore, Eq. (2.25b) of the Lax pair is unchanged, and for (2.25a), we have

$$(z + 2a_1)\widetilde{K}^{(-1)}(m,z) = z\widetilde{K}^{(1)}(m,z) + (v^{(-1)}(m) - v^{(1)}(m))\widetilde{K}(m,z). \tag{3.5}$$

Because of (2.22), we also have $u(m_1, m_2, m_3, z) = u(m_1 - m_2, 0, m_3, z)$. Relation (3.2) gives the same dependence of $v(m)$ on $m_1 - m_2$ and m_3. Because of this specific dependence on m, we must modify definition (2.28) of the Jost solution:

$$\psi(m_1 - m_2, m_3, \lambda) = \widetilde{K}(m,z) z^{m_1 - m_2} (z + a_{13})^{m_3}, \tag{3.6}$$

where we set

$$\lambda = z + a_1, \tag{3.7}$$

which in fact is the symbol of the operator A (see (2.15)). Therefore, now setting $m_2 = 0$, we write

$$v(m) \equiv v(m_1, m_3) = v(m_1, m_3, 0) - a_1 m_1 - a_3 m_3, \tag{3.8}$$

and Eq. (1.6c) hence remains unchanged, $\psi^{(3)} = \psi^{(1)} + (v^{(3)} - v^{(1)})\psi$, while (3.5) and the Lax pair itself become

$$\psi^{(1)} = (v^{(1)} - v^{(-1)})\psi + (\lambda - a_1^2)\psi^{(-1)}, \tag{3.9}$$

$$\psi^{(3)} = (v^{(3)} - v^{(-1)})\psi + (\lambda - a_1^2)\psi^{(-1)}, \tag{3.10}$$

where $\psi^{(1)}$ in the second equality was replaced with (3.9).

The compatibility equation for this pair can be derived either directly or as a reduction of (1.5) and is

$$(v^{(1,3)} - v)(v^{(3)} - v^{(1)}) = (v^{(-1,3)} - v)(v^{(3)} - v^{(-1)}), \tag{3.11}$$

which we can rewrite as

$$\{(v^{(3)} - v^{(-1)})(v^{(3)} - v^{(1)})^{(-1)}\}^{(1)} = (v^{(3)} - v^{(-1)})(v^{(3)} - v^{(1)})^{(-1)}. \tag{3.12}$$

The multiplication operator in the right-hand side (or left-hand side) thus has a symbol independent of m_1. Taking (3.8) and the decay of $u(m)$ as $m_1 \to \infty$ into account, we obtain

$$v^{(3)}(m) - v^{(\pm 1)}(m) \to \pm a_{13}. \tag{3.13}$$

Therefore, (3.12) gives

$$(v^{(1,3)} - v)(v^{(3)} - v^{(1)}) = a_3^2 - a_1^2, \tag{3.14}$$

where we shift $m_1 \to m_1 + 1$. Equation (3.14) is known as the discrete potential KdV equation. It was derived in [17], [18] and was discussed in detail in the literature together with its non-Abelian generalizations (see [19] and the references therein). We derived this equation here as an example of dimensional reduction in the framework of our approach.

3.2. The reduction $B^{(3)} = B^{(1,2)}$. The (1+1)-dimensional reduction $B^{(3)} = B^{(1,2)}$ of the HDE preserves its specific property: symmetry with respect to the independent variables. Let

$$a_3 = a_1 + a_2 \tag{3.15}$$

for simplicity. Then by (2.18), this reduction means that z must satisfy the condition

$$|z - a_2|^2 = a_1 a_2. \tag{3.16}$$

In other words, the symbol $\widetilde{B}(m,z)$ must be proportional to the δ-function on circle (3.16). Hence, $a_1 a_2 > 0$ here, and the symbol of the dressing operator is analytic inside and outside circle (3.16). We also note that because of this reduction, the symbols of B and K satisfy the conditions

$$\begin{aligned} \widetilde{B}(m_1, m_2, m_3, z) &= \widetilde{B}(m_1 + m_3, m_2 + m_3, 0, z), \\ \widetilde{K}(m_1, m_2, m_3, z) &= \widetilde{K}(m_1 + m_3, m_2 + m_3, 0, z). \end{aligned} \tag{3.17}$$

Consequently, by (2.22), we have the same dependence of $u(m)$ on m_i, and by (2.26) and (3.15), this also holds for the function v,

$$v(m_1, m_2, m_3) = v(m_1 + m_3, m_2 + m_3, 0). \tag{3.18}$$

We see that Eq. (2.23) is unchanged under this reduction and (2.24) reduces to

$$(z - a_2)\widetilde{K}^{(1,2)}(m,z) = z\widetilde{K}^{(1)}(m,z) + (v^{(1,2)}(m) - v^{(1)}(m))\widetilde{K}(m,z),$$

where now $m_3 = 0$. We here introduce the Jost solution via the equation (cf. (2.28))

$$\psi(m_1, m_2, k) = \widetilde{K}(m_1, m_2, 0, z) z^{m_1} (z + a_{12})^{m_2}, \tag{3.19}$$

where

$$k = \frac{2}{a_{12}} \left(\frac{a_1}{z + a_{12}} - \frac{a_2}{z} \right) \tag{3.20}$$

is the spectral parameter. Finally, for the Lax pair, we obtain

$$\psi^{(2)} - \psi^{(1)} = (v^{(2)} - v^{(1)})\psi, \tag{3.21}$$

$$k\psi^{(1,2)} = \psi^{(1)} + \psi^{(2)} + (2v^{(1,2)} - v^{(1)} - v^{(2)})\psi, \tag{3.22}$$

and the corresponding nonlinear integrable equation is

$$(v^{(1,2)}(v^{(2)} - v) + vv^{(2)})^{(1)} = (v^{(1,2)}(v^{(1)} - v) + vv^{(1)})^{(2)}, \tag{3.23}$$

which is an equation of a (1+1)-dimensional chain with discrete time evolution, symmetric with respect to both independent variables.

3.3. The reduction $B^{(3)} = B^{(-1,-2)}$. The reduction $B^{(3)} = B^{(-1,-2)}$ is another that leads to a symmetric chain. Repeating the same arguments as above, we obtain

$$\widetilde{B}(m,z) = \widetilde{B}(m_1 - m_3, m_2 - m_3, 0, z), \tag{3.24}$$

which means $\widetilde{B}(m,z)$ is proportional to the δ-function on the hyperbola given by the equation $3(z_{\Re} + a_1)^2 - z_{\Im}^2 = (a_1 + a_2)^2 - a_1 a_2$. Omitting other details, we present only the corresponding nonlinear equation:

$$v^{(1,2)}(v^{(1)} - v^{(2)}) - v^{(-1,-2)}(v^{(-1)} - v^{(-2)}) = v^{(1)}v^{(-2)} - v^{(-1)}v^{(2)}. \tag{3.25}$$

Analogous reductions are considered below as an example of the higher HDE.

4. Higher Hirota difference equations

An obvious way to introduce new discrete independent variables into the HDE is to increase the number of evolution equations of type (2.1), i.e., to the discrete variables $\{m_1, m_2, m_3\}$, to add as many other variables $\{m_4, m_5, \dots\}$ as we want such that the dynamics with respect to any of them is given by $B^{(i)} = (A - a_i)B(A - a_i)^{-1}$, where $a_4, a_5, \dots$ are different (real) parameters. All these evolutions are mutually compatible and are compatible with the original variables, but their definition shows that for any i, j, and k, we have an analogue of (2.2) (also see (2.3)):

$$a_{ij}\{B^{(ij)} + B^{(k)}\} + \text{cycle}\{i, j, k\} = 0.$$

Mimicking the construction in Sec. 2, we then find that with respect to any three variables m_i, m_j, and m_k, the function $u(m_1, \dots)$ defined in (2.22) and the function $v(m) = u(m) - \sum_i a_i m_i$ (cf. (2.26)) satisfy the same HDE. Therefore, this "extension" is trivial and can be interesting only for studying symmetries of the HDE (see, e.g., [20]).

Therefore, to obtain higher analogues of the HDE, we must consider higher analogues of similarity transformations (2.1). Let $p_i = p_i(T)$, $i = 1, 2, 3$, be polynomials in the operator T of the orders n_i with constant coefficients, i.e., symbols $\tilde{p}_i(m, z) = p_i(z)$ are polynomials in $z \in \mathbb{C}$. We also assume that all these polynomials have simple and pairwise different zeros and that the coefficients of the highest powers are equal to I. As before, we consider an operator B with the symbol $\widetilde{B}(m_1, m_2, m_3, z)$ depending on discrete variables $m_i \in \mathbb{Z}$, but the dependence on these variables is now given by

$$B^{(i)} = p_i B p_i^{-1}, \quad i = 1, 2, 3, \tag{4.1}$$

instead of (2.1). We note that because of the condition on the polynomials p_i, we can write each of them as

$$p_i(T) = \prod_{j=1}^{n_i} (T - x_{ij}), \tag{4.2}$$

and the shift in the ith variable by (4.1) is hence equivalent to the n_i shifts in the sense of (2.1). Nevertheless, deriving evolution equations (4.1) using such multidimensional reductions is very complicated even in the linear case, and we therefore construct nonlinear equations directly based on (4.1). To be consistent with the operator approach developed in Sec. 2.2, we choose

$$p_1(T) = T. \tag{4.3}$$

The dressing operator K is defined by the same $\bar{\partial}$-problem (2.19), and its dependence on m_i is given by (4.1). As before, under the assumption that (2.19) is uniquely solvable, there then exist polynomials $P_i(T)$ such that

$$K^{(i)} p_i = P_i K, \quad i = 1, 2, 3. \tag{4.4}$$

We write

$$p_i(T) = \sum_{j=0}^{n_i} y_{ij} T^j, \qquad P_i(T) = \sum_{j=0}^{n_i} Y_{ij} T^j, \tag{4.5}$$

where $y_{i,n_i} = Y_{i,n_i} \equiv 1$, all y_{ij} are constants, and Y_{ij} are multiplication operators, $\widetilde{Y}_{ij}(m, z) = \widetilde{Y}_{ij}(m)$. Then (4.4) becomes

$$K^{(i)} \sum_{j=0}^{n_i} y_{ij} T^j = \sum_{j=0}^{n_i} Y_{ij} K^{(1\times j)} T^j, \quad i = 1, 2, 3. \tag{4.6}$$

Here, we introduce the notation (cf. (2.11))

$$\widetilde{K}^{(1\times j)}(m_1, m_2, \ldots, z) = \widetilde{K}(m_1 + j, m_2, \ldots, z). \tag{4.7}$$

Equation (4.6) can be simplified by writing it in terms of the Jost solution (cf. (2.28))

$$\varphi(m, z) = \widetilde{K}(m, z) p_1(z)^{m_1} p_2(z)^{m_2} p_3(z)^{m_3}, \tag{4.8}$$

which by (2.8), (4.6), and (4.7) gives

$$\varphi^{(i)}(m_1, m_2, m_3, z) = \sum_{j=0}^{m_i} Y_{i,j}(m) \varphi(m_1 + j, m_2, m_3, z). \tag{4.9}$$

The representation of the symbol of B follows from (4.1) by analogy to (2.17):

$$\widetilde{B}(m, z) = \oint_{|\zeta|=1} \frac{d\zeta}{2\pi i \zeta} \zeta^{m_1} \left(\frac{p_2(\zeta z)}{p_2(z)}\right)^{m_2} \left(\frac{p_3(\zeta z)}{p_3(z)}\right)^{m_3} b(\zeta, z),$$

where $b(\zeta, z)$ is some function. To prevent increase of the symbol with m_2 or m_3, we impose the condition $|p_i(\zeta z)| = |p_i(z)|$. Moreover, for simplicity, we assume that the polynomials $p_i(z)$ have real coefficients and $b(\zeta, z) = \delta_c(\zeta z/\bar{z}) f(z)$, where $\delta_c(\zeta)$ is the δ-function on the unit contour and $f(z)$ is an arbitrary function of $z \in \mathbb{C}$. By analogy to (2.18), we then obtain the representation

$$\widetilde{B}(m, z) = \left(\frac{\bar{z}}{z}\right)^{m_1} \left(\frac{p_2(\bar{z})}{p_2(z)}\right)^{m_2} \left(\frac{p_3(\bar{z})}{p_3(z)}\right)^{m_3} f(z). \tag{4.10}$$

Repeating the arguments in Sec. 2.2, from the assumption that problem (2.19) is uniquely solvable, we find that evolution equations (4.4) (or (4.6)) are compatible:

$$K^{(i,j)} = K^{(j,i)} \tag{4.11}$$

for any i and j. This compatibility allows deriving a discrete version of the Zakharov–Shabat system [21]. Indeed, because of (2.16) and (4.4), we have

$$K^{(i,j)} p_i p_j = P_i^{(j)} K^{(j)} p_j = P_i^{(j)} P_j K, \quad i, j = 1, 2, 3. \tag{4.12}$$

Taking into account that the polynomials p_i and p_j with constant coefficients commute (see (2.10)), we find that the left-hand side of (4.12) is symmetric with respect to i and j by (4.11). The right-hand side the gives

$$P_i^{(j)} P_j = P_j^{(i)} P_i, \quad i, j = 1, 2, 3. \tag{4.13}$$

Conversely, (4.11) follows from (4.13).

Discrete version (4.13) of the Zakharov–Shabat system allows deriving evolution equations for the coefficient functions of the polynomials P_i. First, if i or j equals unity in (4.13), then this equality becomes an identity because of (2.11) and (4.3) (cf. (2.21)). We now let $i, j \neq 1$. Then substituting $P_i(T)$ from (4.5), we move all operators T to the right using (2.11). It is then clear that both sides of (4.13) are polynomials in T of the order $n_2 + n_3$ with unit leading coefficients. Equating coefficients of equal powers of T, we obtain exactly $n_2 + n_3$ equations:

$$\sum_{k=\max\{l-n_j,0\}}^{\min\{l,n_i\}} (Y_{ik}^{(j)} Y_{ik}^{(1\times k)} - Y_{j,m-k}^{(i)} Y_{ik}^{(1\times(l-k))}) = 0, \quad l = 0, 1, \ldots, n_2 + n_3 - 1, \tag{4.14}$$

where we use notation (4.7) and the value $l = n_2+n_3$ is omitted because it is trivial. On the other hand, we must note that the functions Y_{ij} are not independent, because by (4.6), these n_2+n_3 coefficient functions are given in terms of $n=\max\{n_2, n_3\}$ coefficient functions of the asymptotic decomposition of the dressing operator,

$$K = I + \sum_{j=1}^{n} k_j T^{-j} + o(T^{-n}), \tag{4.15}$$

where the symbols of k_j are independent of z, $\tilde{k}_j(m,z) = k_j(m)$, and $o(T^{-n})$ is understood in the sense of symbols, i.e., as $o(z^{-n})$. By construction, system (4.14) remains compatible after such a substitution and gives a nonlinear integrable system.

The simplest example of this construction is given by the choice $n_2 = n_3 = 1$, whence $P_2 = T + Y_{2,0}$ and $P_3 = T + Y_{3,0}$. Then (4.14) reduces to the system

$$Y_{2,0}^{(1)} + Y_{3,0}^{(2)} = Y_{3,0}^{(1)} + Y_{2,0}^{(3)}, \qquad Y_{3,0}^{(2)} Y_{2,0} = Y_{2,0}^{(3)} Y_{3,0}. \tag{4.16}$$

If we introduce $Y_{2,0} = v^{(2)} - v^{(1)}$ and $Y_{3,0} = v^{(3)} - v^{(1)}$ (cf. (2.23), (2.24), and (2.26)), then the first equality in (4.16) becomes an identity, and the second equality gives exactly HDE (1.5).

Remark. In the above construction, the order of the coefficients y_{ij} and Y_{ij} never changed, which means that relation (4.13) also holds in the non-Abelian case, where these coefficients are not mutually commuting [14].

5. An example of the higher Hirota difference equation

In this section, we consider an example of a higher equation closest to the HDE. Let the dynamics of the operator B in (4.1) be given via the polynomials

$$p_1(T) = T, \qquad p_2(T) = T + a_{12}, \qquad p_3(T) = (T + a_1)^2 - a_3^2, \tag{5.1}$$

where (4.3) is taken into account and a_1, a_2, and a_3^2 are real constants, $a_{12} = a_1 - a_2 \neq 0$, $a_3 \neq 0, \pm a_2$. Let $\Delta_i B = B^{(i)} - B$ denote the first difference of the operators. Then B satisfies the difference equation

$$[(\Delta_1 a_1 - \Delta_2 a_2)^2 - a_3^2(\Delta_1 - \Delta_2)^2]\Delta_3 B = a_{12}\Delta_1\Delta_2(a_{12}\Delta_1\Delta_2 + 2\Delta_1 a_1 - 2\Delta_2 a_2)B, \tag{5.2}$$

which follows from the corresponding commutator identity. It can also be verified directly because (4.10) here becomes

$$\widetilde{B}(m,z) = \left(\frac{\bar{z}}{z}\right)^{m_1} \left(\frac{\bar{z} + a_{12}}{z + a_{12}}\right)^{m_2} \left(\frac{(\bar{z} + a_1)^2 - a_3^2}{(z + a_1)^2 - a_3^2}\right)^{m_3} f(z). \tag{5.3}$$

The dressing operator K is defined as always by (2.19), and by (4.5), we hence have

$$P_2(T) = T + Y_{20}, \tag{5.4}$$

$$P_3(T) = T^2 + Y_{31}T + Y_{30}, \tag{5.5}$$

where the symbols of Y_{ij} are independent of z. By (4.6), the Lax pair is then given in the form

$$K^{(2)}(T + a_{12}) = K^{(1)}T + Y_{20}K, \tag{5.6}$$

$$K^{(3)}[(T + a_1)^2 - a_3^2] = K^{(1,1)}T^2 + Y_{31}K^{(1)}T + Y_{30}K, \tag{5.7}$$

where the coefficients by (4.13), (5.4), and (5.5) satisfy

$$
\begin{aligned}
&Y_{31}^{(1)} + Y_{20}^{(3)} = Y_{20}^{(1,1)} + Y_{31}^{(2)},\\
&Y_{30}^{(1)} + Y_{20}^{(3)} Y_{31} = Y_{30}^{(2)} + Y_{31}^{(2)} Y_{20}^{(1)},\\
&Y_{20}^{(3)} Y_{30} = Y_{30}^{(2)} Y_{20}.
\end{aligned} \tag{5.8}
$$

Taking the symmetry of this reduction with respect to m_1 and m_2 into account, we can justifiably rewrite (5.7) in the explicitly symmetric form using (5.6). We thus obtain

$$K^{(2)}(A - a_2) = K^{(1)}(A - a_1) + Y_{20} K, \tag{5.9}$$

$$
\begin{aligned}
K^{(3)}[A^2 - a_3^2] = {}& K^{(1,2)}(A - a_1)(A - a_2) + \\
&+ X_{31}(K^{(1)}(A - a_1) + K^{(2)}(A - a_2)) + X_{30} K,
\end{aligned} \tag{5.10}
$$

where we again use (2.14) for symmetry and where the new coefficients are

$$X_{31} = \frac{1}{2}(Y_{31} - Y_{20}^{(1)}), \qquad X_{30} = Y_{30} + X_{31} Y_{20}. \tag{5.11}$$

In these terms, relations (5.8) also take a symmetric form:

$$Y_{20}^{(3)} = Y_{20}^{(1,2)} + 2X_{31}^{(2)} - 2X_{31}^{(1)}, \tag{5.12}$$

$$2Y_{20}^{(3)} X_{31} = X_{20}^{(2)} - X_{30}^{(1)} + X_{31}^{(2)} Y_{20}^{(2)} + X_{31}^{(1)} Y_{20}^{(1)}, \tag{5.13}$$

$$2Y_{20}^{(3)} X_{30} = [X_{30}^{(2)} + X_{30}^{(1)} + X_{31}^{(2)} Y_{20}^{(2)} - X_{31}^{(1)} Y_{20}^{(1)}] Y_{20}. \tag{5.14}$$

The coefficients Y_{ij} (or X_{ij}) must be defined by substituting asymptotic expansion (4.15) in (5.6) and (5.7) (or (5.9) and (5.10)). To preserve the abovementioned symmetry, we here use the last two equations, and instead of (4.15), we write the expansion in the form

$$K = I + uA^{-1} + wA^{-2} + \dots \tag{5.15}$$

(see (2.14)), where the symbols of u and w depend only on the variables m. We omit the computation details here, and to present the results, we introduce the functions

$$v(m) = u(m) - m_1 a_1 - m_2 a_2, \tag{5.16}$$

$$
\begin{aligned}
f(m) = {}& w(m) - (m_1 a_1 + m_2 a_2) u(m) + \\
&+ \frac{1}{2}(m_1 a_1 + m_2 a_2)^2 - \frac{m_1 a_1^2}{2} - \frac{m_2 a_2^2}{2} - m_3 a_3^2.
\end{aligned} \tag{5.17}
$$

Substituting (5.15) in (5.9) and (5.10), we then obtain

$$Y_{20} = v^{(2)} - v^{(1)}, \tag{5.18a}$$

$$f^{(2)} - f^{(1)} = Y_{20} v, \tag{5.18b}$$

$$X_{31} = \frac{1}{2}(v^{(3)} - v^{(1,2)}), \tag{5.18c}$$

$$X_{30} = f^{(3)} - f^{(1,2)} - X_{31}(v^{(1)} + v^{(2)}). \tag{5.18d}$$

The three functions Y_{20}, X_{30}, and X_{31} are thus given in terms of the two functions v and f and must satisfy the three Eqs. (5.12)–(5.14). As noted above, this system is compatible. In particular, it is easy to verify that (5.12) and (5.13) become identities because of (5.18a)–(5.18c) and that (5.18d) reduces to

$$2v^{(2,3)}[f^{(3)} - f^{(1,2)} - v^{(3)}v^{(2)} + v^{(2)}v^{(1,2)}] - 2v^{(1,3)}[f^{(3)} - f^{(1,2)} - v^{(3)}v^{(1)} + v^{(1)}v^{(1,2)}] = \\ = (v^{(2)} - v^{(1)})[(f^{(2)} - v^{(2)}v)^{(3)} + (f^{(1)} - v^{(1)}v)^{(3)} - (f^{(2)} - v^{(2)}v)^{(1,2)} - (f^{(1)} - v^{(1)}v)^{(1,2)}], \quad (5.19)$$

which gives one equation for two functions. These functions are not independent, because by (5.18a) and (5.18b), we have

$$f^{(2)} - v^{(2)}v = f^{(1)} - v^{(1)}v. \quad (5.20)$$

Equations (5.19) and (5.20) are equations of an integrable system, which gives an example of a higher HDE. This system follows as the compatibility condition for Lax pair (5.9), (5.10), which in terms of the Jost solution (cf. (4.8))

$$\varphi(m, z) = \widetilde{K}(m, z)z^{m_1}(z + a_{12})^{m_2}[(z + a_1)^2 - a_3^2]^{m_3} \quad (5.21)$$

is

$$\varphi^{(2)} = \varphi^{(1)} + (v^{(2)} - v^{(1)})\varphi, \quad (5.22)$$

$$\varphi^{(3)} = \varphi^{(1,2)} + (v^{(3)} - v^{(1,2)})\frac{\varphi^{(1)} + \varphi^{(2)}}{2} + f^{(3)} - f^{(1,2)} - \frac{1}{2}(v^{(1)} + v^{(2)})(v^{(3)} - v^{(1,2)})]\varphi, \quad (5.23)$$

where we use (5.18). Omitting details, we note that because of (5.22), we can write (5.23) as

$$\varphi^{(3)} = \varphi^{(1,1)} + (v^{(3)} - v^{(1,1)})\varphi^{(1)} + [f^{(3)} - f^{(1,2)} - v^{(1)}(v^{(3)} - v^{(1,2)})]\varphi, \quad (5.24)$$

which together with (5.22) gives an equivalent Lax pair.

6. Reductions of system (5.19), (5.20)

6.1. The reduction $B^{(3)} = B$. System (5.19), (5.20) also admits (1+1)-dimensional reductions, and the discussion at the beginning of Sec. 3 is applicable here with the only difference that the curve on $\mathbb{C}$ is now determined by (5.3) instead of (2.18). Therefore, although system (5.19), (5.20) differs from the HDE only by the dependence on the variable m_3, the reduction $B^{(3)} = B$ is here nontrivial. Indeed, because of (5.3), this reduction means that the symbol $\widetilde{B}(m, z)$ is nonzero if $(\bar{z} + a_1)^2 = (z + a_1)^2$, i.e., $z_{\Re} = -a_1$, and the function $f(z)$ in (5.3) must therefore be proportional to $\delta(z_{\Re} + a_1)$:

$$\widetilde{B}(m, z) = \left(\frac{a_1 + iz_{\Im}}{a_1 - iz_{\Im}}\right)^{m_1}\left(\frac{a_1 + iz_{\Im}}{a_1 - iz_{\Im}}\right)^{m_2}\delta(z_{\Re} + a_1)r(z_{\Im}). \quad (6.1)$$

Inverse problem (2.19) then not only shows that the dressing operator is independent of m_3 but also shows that its symbol $\widetilde{K}(m_1, m_2, z)$ is an analytic function of z if $z_{\Re} \neq -a_1$.

To obtain the reduced Lax pair and a nonlinear equation, we note that the coefficients of asymptotic expansion (5.15) are independent of m_3, i.e., $u(m) = u(m_1, m_2)$ and $w(m) = w(m_1, m_2)$. Correspondingly, by (5.16) and (5.17), we have $v(m) = v(m_1, m_2)$ and $f(m) = g(m_1, m_2) - a_3^3 m_3$, where $g(m_1, m_2) = f(m_1, m_2, 0)$. Substituting these relations in (5.19) and (5.20), we obtain the nonlinear integrable system

$$(g^{(2)} - v^{(2)}v)^{(1,2)} + (g^{(1)} - v^{(1)}v)^{(1,2)} - (g^{(2)} - v^{(2)}v)^{(1)} - (g^{(1)} - v^{(1)}v)^{(2)} - \\ - (g^{(2)} - v^{(2)}v) - (g^{(1)} - v^{(1)}v) + 2g - 2v(v^{(1)} + v^{(2)}) = 0, \quad (6.2)$$

$$g^{(2)} - v^{(2)}v = g^{(1)} - v^{(1)}v. \quad (6.3)$$

Taking into account that the symbol $\widetilde{K}(m,z)$ is now independent of m_3, we define the Jost solution by the equality (see (5.21))

$$\psi(m_1,m_2,z+a_1)=\widetilde{K}(m,z)z^{m_1}(z+a_{12})^{m_2}\equiv\frac{\varphi(m,z)}{[(z+a_1)^2-a_3^2]^{m_3}}. \tag{6.4}$$

From (5.23) and (5.21), we thus obtain the reduced Lax pair

$$\psi^{(1,1)}+(v-v^{(1,1)})\psi^{(1)}+[g-g^{(1,2)}-v^{(1)}(v-v^{(1,2)})]\psi=\lambda^2\psi, \tag{6.5}$$

$$\psi^{(2)}=\psi^{(1)}+(v^{(2)}-v^{(1)})\psi, \tag{6.6}$$

where we use the spectral parameter $\lambda=z+a_1$ (see (3.7)).

6.2. The reduction $B^{(2)}=B^{(-1)}$. The reduction $B^{(2)}=B^{(-1)}$ is completely analogous to the reduction considered in Sec. 3.1. We again have (3.1), which by (5.3) means that here $\widetilde{B}(m,z)\sim\delta(z_\Re+a_{12}/2)$, where $a_{12}=a_1-a_2$, i.e.,

$$\widetilde{B}(m,z)=\left(\frac{a_{12}/2+iz_\Im}{a_{12}/2-iz_\Im}\right)^{m_1-m_2}\left(\frac{((a_1+a_2)/2-iz_\Im)^2-a_3^2}{((a_1+a_2)/2+iz_\Im)^2-a_3^2}\right)^{m_3}\delta(z_\Re+a_{12}/2)r(z_\Im).$$

The symbol of the dressing operator is analytic if $z_\Re\neq-a_{12}/2$. Moreover, we have $\widetilde{K}(m,z)=\widetilde{K}(m_1-m_2,0,m_3,z)$, $u(m)=u(m_1-m_2,0,m_3)$, and $w(m)=w(m_1-m_2,0,m_3)$, but we here do not use condition (3.2) to preserve the dependence on m_3. Correspondingly, we must use the functions $u(m)$ and $w(m)$ but not $v(m)$ and $f(m)$ in (5.16) and (5.17). The Jost solution is here defined as

$$\psi(m_1-m_2,m_3,z+a_1)=\widetilde{K}(m,z)z^{m_1}[(z+a_1)^2-a_3^2]^{m_3}\equiv\frac{\varphi(m,z)}{[z(z+a_{12})]^{m_2}},$$

and the first equation of the Lax pair derived from (5.23) and (5.16) is hence

$$\psi^{(1,1)}-(u^{(1,1)}-u+a_{12})\psi^{(1)}=(\lambda-a_1)(\lambda-a_2)\psi. \tag{6.7}$$

We skip the further construction because it is obvious, very cumbersome, and not instructive.

6.3. The reduction $B^{(3)}=B^{(1,2)}$. Here, as in Sec. 3.2, we have relations (3.17), which because of (5.3) means that to satisfy the reduction $B^{(3)}=B^{(1,2)}$ and to preserve the dependence on m_1 and m_2, the symbol $\widetilde{B}(m,z)$ must have support on the circle $(a_1+a_2)|z+a_1|^2-2(a_3^2+a_1a_2)(z+a_1)_\Re=(a_1+a_2)a_3^2$ (cf. (3.16)). We assume that $a_1+a_2\neq0$, and to simplify the presentation, we set

$$a_3^2=-a_1a_2<0, \tag{6.8}$$

and for the symbol of B, we hence obtain the representation

$$\widetilde{B}(m,z)=\left(\frac{\bar z}{z}\right)^{m_1+m_3}\left(\frac{\bar z+a_{12}}{z+a_{12}}\right)^{m_2+m_3}\delta(|z+a_1|^2-a_1a_2)r(z). \tag{6.9}$$

The functions $u(m)$ and $w(m)$, being coefficients of expansion (5.15) of the symbol $\widetilde{K}$, also satisfy the equations $u^{(3)}(m)=u^{(1,2)}(m)$ and $w^{(3)}(m)=w^{(1,2)}(m)$. Hence, because of (5.17), $w(m)$ disappears from (5.18d), evolution equation (5.19), and Lax pair (5.24), (5.22). Subject to the dependence of the dressing operator on m, $\widetilde{K}(m,z) = \widetilde{K}(m_1+m_3, m_2+m_3, 0, z)$, the Jost solution must be defined as (cf. (5.21) and (6.4))

$$\psi(m_1+m_3, m_2+m_3, z+a_1) = \widetilde{K}(m,z) z^{m_1+m_3}(z+a_{12})^{m_2+m_3} \equiv$$
$$\equiv \left(\frac{z(z+a_{12})}{(z+a_1)^2 - a_3^2}\right)^{m_3} \varphi(m,z), \tag{6.10}$$

and the Lax pair hence becomes

$$k\psi^{(1,1)} = (1 - k(q^{(2)} - q^{(1)})^{(1)})\psi^{(1)} + (q^{(2)} - q)^{(1)}\psi, \tag{6.11}$$

$$\psi^{(2)} = \psi^{(1)} + (q^{(2)} - q^{(1)})\psi, \tag{6.12}$$

where the spectral parameter k is now defined as

$$k = \frac{z+a_1}{z(z+a_{12})}. \tag{6.13}$$

In (6.11) and (6.12), we also introduce a new dependent variable (cf. (5.16))

$$q(m_1, m_2) = u(m_1, m_2) + m_1 a_2 + m_2 a_1. \tag{6.14}$$

Because of this notation, Eq. (5.19) under this reduction gives

$$(q^{(2)} - q)^{(1,2)}(q^{(1)} - q)^{(2)} = (q^{(1)} - q)^{(1,2)}(q^{(2)} - q)^{(1)}, \tag{6.15}$$

and it is easy to verify that this nonlinear equation is the compatibility condition for Lax pair (6.11), (6.12). Similarly to (3.23), this equation is symmetric with respect to both independent variables.

7. Conclusion

We have considered a method for deriving nonlinear (difference) integrable equations and their Lax pairs. Our construction is not free of assumptions: first, we assumed that $\bar{\partial}$-problem (2.19) is uniquely solvable and that asymptotic expansions (2.22) and (4.15) exist. These assumptions are extremely essential for our derivation. On the other hand, verifying the compatibility of the equations for the derived Lax pairs is a purely algebraic operation, which needs no assumptions and leads to an integrable nonlinear equation. For example, the higher HDE, i.e., system (5.19), (5.20), is the compatibility condition for (5.22) and (5.23), which can be verified directly.

In [12]–[14] and [16], we noted that integrability of the nonlinear equations is ultimately related to the existence of commutator identities of type (2.4) for the HDE and of type (5.2) for its higher analogue (considered in Sec. 5). These identities lead to the existence of linear equations for the operator B generated by similarity transformations of type (2.15), (4.1), or (5.1) in our case or by commutator relations in the case of differential equations. In all considered cases, these transformations are given using functions of the shift operator T (the differentiation operator D for differential equations), but a specific property of the derived linear equations is that they do not contain T (see (2.2) and (5.4)). This ensures the existence of linearized versions of the derived integrable equations. Indeed, inverse problem (2.19) linearized with respect to B because of (2.22) gives

$$u(m) = -\frac{1}{\pi^2}\int d^2z\, \widetilde{B}(m,z) + \ldots, \tag{7.1}$$

which proves the above statement.

In the general situation considered in (4.1), the existence of the corresponding commutator identity is equivalent to existence of a polynomial $Q(x_1, x_2, x_3)$ such that

$$Q(\mathrm{Ad}_1, \mathrm{Ad}_2, \mathrm{Ad}_3) = 0, \tag{7.2}$$

where

$$\mathrm{Ad}_i\, B = p_i(T) B p_i(T)^{-1}, \quad i = 1, 2, 3, \tag{7.3}$$

denotes the adjoint action of T on the associative algebra discussed in the introduction. Here, B is an arbitrary element of this algebra, but if we switch on its dependence on the variables m_i using (4.1), $B^{(i)} = \mathrm{Ad}_i\, B$, then by (7.2), we obtain a closed linear equation for $B(m_1, m_2, m_3)$ (cf. (5.2)). This argument and the construction presented in this paper show that it is natural to suppose that the only linear difference equations in (2+1) dimensions that can be lifted to nonlinear integrable ones are those that can be represented in the form of commutator identities. We note that relation (4.3) was not used in this discussion.

In the case where three polynomials p_i are chosen, the problem of the existence of a corresponding commutator identity reduces to the problem of the existence of a polynomial Q in three variables such that the equality

$$Q\biggl(\frac{p_1(x)}{p_1(y)}, \frac{p_2(x)}{p_2(y)}, \frac{p_3(x)}{p_3(y)}\biggr) \equiv 0 \tag{7.4}$$

holds for any x and y in $\mathbb{C}$. While the solution of this problem is unknown to us, the fact that our construction in Sec. 4 had no obstructions shows that such a polynomial can exist. This argument is not a proof, of course, because our approach here was not free of assumptions that can be quite essential for it. On the other hand, the commutator identities should exist in a more general situation where we deal with meromorphic functions instead of polynomials p_i and where their coefficients do not commute with B (the non-Abelian case; see, e.g., [15]).

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