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ENTANGLEMENT OF MULTIPARTITE FERMIONIC COHERENT STATES FOR PSEUDO-HERMITIAN HAMILTONIANS

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We study the entanglement of multiqubit fermionic pseudo-Hermitian coherent states (FPHCSs) described by anticommutative Grassmann numbers. We introduce pseudo-Hermitian versions of well-known maximally entangled pure states, such as Bell, GHZ, Werner, and biseparable states, by integrating over the tensor products of FPHCSs with a suitable choice of Grassmannian weight functions. As an illustration, we apply the proposed method to the tensor product of two- and three-qubit pseudo-Hermitian systems. For a quantitative characteristic of entanglement of such states, we use a measure of entanglement determined by the corresponding concurrence function and average entropy.

Keywords: pseudo-Hermitian, entanglement, pseudo-fermionic coherent state, pseudo-Bell state, pseudo-GHZ state, pseudo-Werner state

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1. Introduction

The theoretical self-consistency of quantum information theory has recently increased as a result of introducing several outstanding results. The most important result was the entanglement phenomena of quantum states [1], formulated in robust theoretical schemes and experimentally verified using several tests [2]–[9]. In fact, entanglement is the most interesting and simultaneously the strangest feature of quantum physics. The idea of entanglement starts from the apparent conflict between the superposition principle and the nonseparability of the related quantum states. It happens when a state of two or more subsystems of a composite quantum system cannot be factored into pure local states of the subsystems. In other words, an entangled state can be used to steer a distant particle into one state of a set of states with a certain probability.

Recent research in theoretical physics and quantum optics revealed the importance of coherent states. They can be used to encode quantum information on continuous variables [10]. While the entanglement of the bosonic $su(2)$ and $su(1,1)$ coherent states, as the nonorthogonal states that play an important role in quantum cryptography and quantum information processing, has been widely investigated [11]–[17], the entanglement properties of multipartite fermionic coherent states remain a challenging problem in quantum information theory, even from the theoretical standpoint [18]–[22]. Fermionic coherent states are defined as eigenstates of the annihilation operator with Grassmannian eigenvalues [23]–[26].

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On the other hand, the interest in non-Hermitian Hamiltonians with real spectra has grown in the last decade [25], [27]–[34]. Considering the results of various numerical studies, Bender and his collaborators [28], [29] found certain examples of one-dimensional non-Hermitian Hamiltonians with real spectra. Because these Hamiltonians were invariant under PT transformations, their spectral properties were linked to their PT symmetry. Mostafazadeh later introduced the notion of pseudo-Hermiticity as an alternative possible approach for a non-Hermitian operator to admit a real spectrum [33]–[35]. Moreover, pseudo-Hermitian Hamiltonians were introduced to study some specific effects in condensed matter physics [36], [37].

The entanglement of Grassmannian coherent states for multipartite n-level Hermitian systems were recently investigated in [20]. For this, the tensor product $|\theta\rangle = |0\rangle - \theta|1\rangle$ of one-mode fermionic coherent states (i.e., $|\theta_1\rangle|\theta_2\rangle$) was considered as represented in terms of the standard basis ($|0\rangle, |1\rangle$) and anticommuting Grassmann numbers $\theta_i\theta_j = -\theta_j\theta_i$. This rule is justified in the context of quantum field theory, where, for example, the tensor product of two one-particle states is a two particle state and so on. The authors of [20] found standard maximally entangled Bell, GHZ, and Werner states by integrating over the tensor product of two-mode, three-mode, and multimode fermionic coherent states with appropriate weight functions.

Our goal here is to extend the presented method to pseudo-Hermitian systems. For them instead of the standard basis, we use two sets $\{|\psi_0\rangle, |\psi_1\rangle\}$ and $\{|\phi_0\rangle, |\phi_1\rangle\}$ of basis states, which are the respective eigenstates of H and $H^\dagger$. As a result, we have two possible fermionic pseudo-Hermitian coherent states (FPHCSs) $|\theta\rangle = |\psi_0\rangle - \theta|\psi_1\rangle$ and $|\tilde{\theta}\rangle = |\phi_0\rangle - \theta|\phi_1\rangle$.

If two pure correlated or uncorrelated quantum systems A and B are given, then the state of the bipartite composite system $|\psi\rangle_{AB}$ is the sum of the tensor products $|i\rangle_A \otimes |j\rangle_B$:

$$|\psi\rangle_{AB} = \sum_{i,j=0,1} a_{ij}|i\rangle_A \otimes |j\rangle_B. \tag{1.1}$$

In this case, we use the fact that, as in standard quantum information theory, a Bloch vector (or qubit) $|\psi\rangle = a|0\rangle + |1\rangle$ is represented in the standard basis $\{|0\rangle, |1\rangle\}$. This basis is used to construct entangled states such as

Bell states $\dfrac{|00\rangle \pm |11\rangle}{\sqrt{2}}$ and $\dfrac{|01\rangle \pm |10\rangle}{\sqrt{2}}$,

GHZ states $\dfrac{|000\rangle \pm |111\rangle}{\sqrt{2}}$,

Werner state $\dfrac{|001\rangle + |010\rangle + |011\rangle}{\sqrt{3}}$,

and so on. These states and their extension are useful resources in quantum information processing in the sense that some tasks including quantum teleportation [38], quantum cryptography [39], remote state preparation [40], and quantum communication [41] can be accomplished using entangled states. Mathematically, a closely related extension is to reconstruct the entangled states with

- the sum $\sum(\,\cdot\,)$ replaced with the integral $\int(\,\cdot\,)\,d\theta_1\,d\theta_2$,
- the coefficients a_{ij} replaced with the Grassmann weight function $w(\theta_1, \theta_2)$, and
- the basis $|i\rangle_A|j\rangle_B$ replaced with the Grassmannian coherent states $|\theta_1\rangle|\theta_2\rangle$.

In this case, the extension of Eq. (1.1) contains both the pseudo-Hermiticity and multimode fermionic coherent states that involve Grassmann (anticommuting) integration variables. Establishing the integral approach for pseudo-quantum mechanics, we set the stage for introducing functional field integral methods for spin systems explored in the context of many-body theories [42].

The paper is divided into two main parts. The first part is devoted to constructing different families of a pseudo-Hermitian version of well-known maximally entangled pure states such as Bell, GHZ, Werner, and pseudo-biseparable states by integrating over the tensor product of FPHCSs of two and three one-qubit pseudo-Hermitian system with a suitable choice of Grassmannian weight functions. In Sec. 2, we briefly introduce pseudo-Hermitian quantum mechanics. In Sec. 3 for a two-level system, we use the results for the generalized Grassmannian pseudo-Hermitian coherent state [26] to present the FPHCSs as a special case of generalized Grassmannian pseudo-Hermitian coherent states. In Sec. 4, we construct pseudo-Hermitian versions of Bell, Werner, and GHZ states.

In the second part, in Sec. 5, we use the measures of concurrence and average entropy to quantify the respective entanglement of the pseudo-Bell states and GHZ and Werner states and discuss the results compared with Hermitian maximally entangled pure states. Concluding remarks are contained in Sec. 6.

2. Pseudo-Hermitian Hamiltonians and a biorthonormal eigenbasis

The Schrödinger equation with complex potentials but a real spectrum has been studied intensively by different methods. The pioneering papers [27]–[34] initiated the investigation of PT-symmetric systems, and Mostafazadeh subsequently introduced a more general class of pseudo-Hermitian models [33], [34]. Following the second approach, we let $H\colon \mathcal{H} \to \mathcal{H}$ be a linear operator acting in a Hilbert space $\mathcal{H}$ and $\eta\colon \mathcal{H} \to \mathcal{H}$ be a linear Hermitian automorphism (invertible transformation). The η-pseudo-Hermitian adjoint of H is then defined by $H^\sharp = \eta^{-1}H^\dagger\eta$. We say that H is pseudo-Hermitian with respect to η or simply η-pseudo-Hermitian if $H^\sharp = H$. The eigenvalues of a pseudo-Hermitian Hamiltonian H are either real or complex-conjugate pairs, and in the nondegenerate case, we have the relations

$$H^\dagger = \eta H \eta^{-1}. \tag{2.1}$$

For diagonalizable operators H with a discrete spectrum, there exists a complete biorthonormal eigenbasis $\{|\psi_i\rangle, |\phi_i\rangle\}$ such that

$$\begin{aligned} &H|\psi_i\rangle = E_i|\psi_i\rangle, \qquad H^\dagger|\phi_i\rangle = \bar{E}_i|\phi_i\rangle, \\ &\langle\phi_i|\psi_j\rangle = \delta_{ij}, \qquad \sum_i |\psi_i\rangle\langle\phi_i| = \sum_i |\phi_i\rangle\langle\psi_i| = I. \end{aligned} \tag{2.2}$$

For a given pseudo-Hermitian H, there are infinitely many η satisfying Eq. (2.1). Nevertheless, these can be expressed in terms of a complete biorthonormal basis of H. In the nondegenerate case, the explicit form of η and its inverse satisfying Eq. (2.1) are

$$\begin{aligned} &\eta = \sum_i |\phi_i\rangle\langle\phi_i|, \qquad \eta^{-1} = \sum_i |\psi_i\rangle\langle\psi_i|, \\ &|\phi_i\rangle = \eta|\psi_i\rangle, \qquad\quad |\psi_i\rangle = \eta^{-1}|\phi_i\rangle. \end{aligned} \tag{2.3}$$

Everywhere in this paper, we assume that the pseudo-Hamiltonian H and consequently the transformation η act in a two-dimensional Hilbert space.

3. Fermionic pseudo-Hermitian coherent states

3.1. Grassmannian variables. The basic properties of Grassmann variables were discussed in [43]–[46]. For our purpose here, we review the properties of this algebra generated by the variables $(\theta_1, \ldots, \theta_n)$ by definition having the properties

$$\theta_i^2 = 0, \qquad \theta_i\theta_j = -\theta_j\theta_i, \quad j = 1, \ldots, n. \tag{3.1}$$

Analogous rules also apply to the Hermitian conjugate of θ, $\theta^\dagger = \bar{\theta}$:

$$\bar{\theta}_i^2 = 0, \qquad \bar{\theta}_i\bar{\theta}_j = -\bar{\theta}_j\bar{\theta}_i, \quad i, j = 1, \ldots, n. \tag{3.2}$$

Similarly, $\theta_i\bar{\theta}_j = -\bar{\theta}_j\theta_i$. Any linear combination of θ_i with complex coefficients is called a Grassmann number. In other words, the Taylor expansion of a Grassmann function is

$$g(\theta_1, \ldots, \theta_n) = c_0 + \sum_{i=1} c_i\theta_i + \sum_{i,j} c_{i,j}\theta_i\theta_j + \ldots,$$

where c_0, c_i, and $c_{i,j}$ are complex numbers. For instance, $e^{\theta_1\theta_2} = 1+\theta_1\theta_2$. The integration and differentiation over complex Grassmann variables are given by the Berezin rules:

$$\begin{aligned}
&\int d\theta f(\theta) = \frac{\partial f(\theta)}{\partial\theta}, \\
&\int d\theta = 0, \qquad \int d\theta\,\theta = 1, \qquad \int d\bar{\theta} = 0, \qquad \int d\bar{\theta}\,\bar{\theta} = 1, \\
&\frac{\partial}{\partial\theta}\theta = 1, \qquad \frac{\partial}{\partial\theta}1 = 0, \qquad \frac{\partial}{\partial\bar{\theta}}\bar{\theta} = 1, \qquad \frac{\partial}{\partial\bar{\theta}}1 = 0, \qquad \frac{\partial^2}{\partial\theta^2} = 0, \qquad \frac{\partial^2}{\partial\bar{\theta}^2} = 0.
\end{aligned} \tag{3.3}$$

To compute the integral of any function over the Grassmann algebra, we need the relations

$$\begin{aligned}
&\theta\, d\bar{\theta} = -d\bar{\theta}\,\theta, \qquad \bar{\theta}\, d\theta = -d\theta\,\bar{\theta}, \qquad \theta\, d\theta = -d\theta\,\theta, \qquad \bar{\theta}\, d\bar{\theta} = -d\bar{\theta}\,\bar{\theta}, \\
&d\theta d\bar{\theta} = -d\bar{\theta}\, d\theta, \qquad \theta\bar{\theta} = -\bar{\theta}\theta.
\end{aligned} \tag{3.4}$$

3.2. Coherent states. Following [25], [26], we can construct the pseudo-fermionic coherent states for the two-level pseudo-Hermitian Hamiltonian. Here, we outline the main results. Considering the biorthonormality of pseudo-Hermitian systems, we can define two pairs of annihilation and creation operators corresponding to the respective biorthonormal eigenstates ($|\psi_i\rangle$ and $|\phi_i\rangle$) as

$$\begin{aligned}
&b := |\psi_0\rangle\langle\phi_1|, \qquad &&\tilde{b} = \eta b\eta^{-1} = |\phi_1\rangle\langle\psi_0|, \\
&b^\sharp := \eta^{-1}b^\dagger\eta = |\psi_1\rangle\langle\phi_0|, \qquad &&\tilde{b}^{\sharp\prime} = \eta'^{-1}b^\dagger\eta = |\phi_0\rangle\langle\psi_1|,
\end{aligned} \tag{3.5}$$

where $\eta'^{-1} = \eta$ and $bb^\sharp + b^\sharp b = I$. We can then construct two families of coherent states for the two-level pseudo-Hermitian Grassmannian system in terms of $|\psi_k\rangle$ and $|\phi_k\rangle$. The FPHCSs corresponding to $|\psi_k\rangle$ and $|\phi_k\rangle$ respectively denoted by $|\theta\rangle$ and $|\tilde{\theta}\rangle$ by definition are the eigenstates of the annihilation operators b and $\tilde{b}$,

$$b|\theta\rangle = \theta|\theta\rangle, \qquad \tilde{b}|\tilde{\theta}\rangle = \theta|\tilde{\theta}\rangle, \tag{3.6}$$

and up to normalization factors are

$$|\theta\rangle = |\psi_0\rangle - \theta|\psi_1\rangle, \qquad |\tilde{\theta}\rangle = |\phi_0\rangle - \theta|\phi_1\rangle. \tag{3.7}$$

We can use the explicit forms of the two families of FPHCS and the characteristic biorthonormality of pseudo-Hermitian systems to identify the possible integrals of $|\theta\rangle$ and $|\tilde{\theta}\rangle$, i.e., $|\theta\rangle\langle\tilde{\theta}|$ and $|\tilde{\theta}\rangle\langle\theta|$, over the measure $d\bar{\theta}\, d\theta\, w(\theta, \bar{\theta})$, which leads to the resolution of the identity

$$\int d\bar{\theta}\, d\theta\, w(\theta, \bar{\theta})|\theta\rangle\langle\tilde{\theta}| = \int d\bar{\theta}\, d\theta\, w(\theta, \bar{\theta})|\tilde{\theta}\rangle\langle\theta| = I, \tag{3.8}$$

where $w(\theta, \bar{\theta}) = 1 + \theta\bar{\theta}$. Equation (3.8) is called a bi-over-completeness relation. To compute the weight function, we require satisfaction of the quantization relations between the biorthonormal eigenstates $|\psi_k\rangle$ and $|\phi_k\rangle$ $(k = 0, 1)$ and the Grassmannian variables θ and $\bar{\theta}$

$$\begin{aligned}
\theta|\psi_k\rangle &= (-1)^{k-1}|\psi_k\rangle\theta, & \bar{\theta}\langle\psi_k| &= (-1)^{k-1}\langle\psi_k|\bar{\theta},\\
\theta\langle\psi_k| &= (-1)^{k-1}\langle\psi_k|\theta, & \bar{\theta}|\psi_k\rangle &= (-1)^{k-1}|\psi_k\rangle\bar{\theta},\\
\theta|\phi_k\rangle &= (-1)^{k-1}|\phi_k\rangle\theta, & \bar{\theta}\langle\phi_k| &= (-1)^{k-1}\langle\phi_k|\bar{\theta},\\
\theta\langle\phi_k| &= (-1)^{k-1}\langle\phi_k|\theta, & \bar{\theta}|\phi_k\rangle &= (-1)^{k-1}|\phi_k\rangle\bar{\theta}.
\end{aligned} \tag{3.9}$$

The above discussion makes it clear that neither the integral of $|\theta\rangle\langle\theta|$ nor the integral of $|\tilde{\theta}\rangle\langle\tilde{\theta}|$ over the measure $d\bar{\theta}\, d\theta\, w(\theta, \bar{\theta})$ is normalized:

$$\int d\bar{\theta}\, d\theta\, w(\theta, \bar{\theta})|\theta\rangle\langle\theta| \neq I, \qquad \int d\bar{\theta}\, d\theta\, w(\theta, \bar{\theta})|\tilde{\theta}\rangle\langle\tilde{\theta}| \neq I. \tag{3.10}$$

We can show that fermionic coherent states (3.7) remain coherent at all times if the time evolution of the initial states governed by the Hamiltonian is also an eigenstate of the lowering operators.

4. Maximally pseudo-entangled states

We consider a fermionic system in which the particles can go to n-mode channels. For this, we consider the tensor product of n one-mode FPHCSs, each of which governed by a pseudo-Hermitian Hamiltonians. For simplicity, we consider $n = 2, 3$. The case of arbitrary n is straightforward. We now introduce a pseudo-Hermitian version of the well-known maximally entangled pure two- and three-qubit states, such as the Bell, GHZ, and Werner states [47] by integrating over the tensor product of FPHCSs with a suitable choice of Grassmannian weight functions.

4.1. Pseudo-Bell-like states. We start with a non-normalized pseudo-Hermitian version of standard Bell states,

$$|\Psi^{\pm}\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}, \qquad |\Phi^{\pm}\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}, \tag{4.1}$$

i.e.,

$$|B_1^-\rangle = |\psi_0\rangle|\psi_1\rangle - |\psi_1\rangle|\psi_0\rangle. \tag{4.2}$$

To obtain this state, we consider the tensor product of two one-mode FPHCSs with the same Grassmann numbers:

$$|\theta\rangle|\theta\rangle = |\psi_0\rangle|\psi_0\rangle + \theta(|\psi_0\rangle|\psi_1\rangle - |\psi_1\rangle|\psi_0\rangle). \tag{4.3}$$

As mentioned above, this method becomes understandable in the context of quantum field theory. To obtain Eq. (4.3), we use the explicit form of $|\theta\rangle$ given by Eq. (3.7).

For the next step, our task is to find the appropriate weight function $w(\theta)$ such that the integration over Grassmann numbers θ leads to the Eq. (4.2). For this, we let

$$\int d\theta\, w(\theta)|\theta\rangle|\theta\rangle = |B_1^-\rangle. \tag{4.4}$$

Setting $w(\theta) = c_0 + c_1\theta$ in (4.4) yields $c_0 = 1$ and $c_1 = 0$, and the appropriate weight function then becomes $w(\theta) = 1$.

Considering the tensor product of $|\theta\rangle|\tilde{\theta}\rangle$, $|\tilde{\theta}\rangle|\theta\rangle$, and $|\tilde{\theta}\rangle|\tilde{\theta}\rangle$ with $w(\theta) = 1$, we can also construct the other forms of pseudo-Bell states as

$$|B_2^-\rangle = \int d\theta\, |\theta\rangle|\tilde{\theta}\rangle = |\psi_0\rangle|\varphi_1\rangle - |\psi_1\rangle|\varphi_0\rangle. \tag{4.5}$$

So far, we have been concerned with the tensor product of two one-mode FPHCSs with the same Grassmannian numbers θ and obtained the pseudo-Hermitian versions of $|\Psi^-\rangle$. To establish the other pseudo-Bell states, we must consider the tensor product of FPHCSs with different Grassmann numbers, i.e.,

$$|\theta_1\rangle|\theta_2\rangle = |\psi_0\rangle|\psi_0\rangle + \theta_2|\psi_0\rangle|\psi_1\rangle - \theta_1|\psi_1\rangle|\psi_0\rangle + \theta_1\theta_2|\psi_1\rangle|\psi_1\rangle. \tag{4.6}$$

In this case the general form of the weight function is $w(\theta_1, \theta_2) = c_0 + c_1\theta_1 + c_2\theta_2 + c_3\theta_1\theta_2$. The task is to find $w(\theta_1, \theta_2)$ such that in addition to $|B_i^-\rangle$ given above, we obtain the other three families of pseudo-Bell states. We let $|B_i^+\rangle$ and $|B_i'^{\pm}\rangle$ denote these three families. We summarize the results in Table 1. For example, the pseudo-Bell state $|\phi_0\rangle|\psi_1\rangle + |\phi_1\rangle|\psi_0\rangle$ can be obtained by taking the tensor product of $|\tilde{\theta}_1\rangle|\theta_2\rangle$ with $w(\theta_1, \theta_2) = -\theta_1 - \theta_2$.

Table 1

State	FPHCS	Weight function	Pseudo-Bell state
$\lvert B_1^{\pm}\rangle$	$\lvert\theta_1\rangle\lvert\theta_2\rangle$	$-(\theta_1 \pm \theta_2)$	$\lvert\psi_0\rangle\lvert\psi_1\rangle \pm \lvert\psi_1\rangle\lvert\psi_0\rangle$
$\lvert B_2^{\pm}\rangle$	$\lvert\theta_1\rangle\lvert\tilde{\theta}_2\rangle$	$-(\theta_1 \pm \theta_2)$	$\lvert\psi_0\rangle\lvert\phi_1\rangle \pm \lvert\psi_1\rangle\lvert\phi_0\rangle$
$\lvert B_3^{\pm}\rangle$	$\lvert\tilde{\theta}_1\rangle\lvert\theta_2\rangle$	$-(\theta_1 \pm \theta_2)$	$\lvert\phi_0\rangle\lvert\psi_1\rangle \pm \lvert\phi_1\rangle\lvert\psi_0\rangle$
$\lvert B_4^{\pm}\rangle$	$\lvert\tilde{\theta}_1\rangle\lvert\tilde{\theta}_2\rangle$	$-(\theta_1 \pm \theta_2)$	$\lvert\phi_0\rangle\lvert\phi_1\rangle \pm \lvert\phi_1\rangle\lvert\phi_0\rangle$
$\lvert B_1'^{\pm}\rangle$	$\lvert\theta_1\rangle\lvert\theta_2\rangle$	$-(\theta_1\theta_2 \pm 1)$	$\lvert\psi_0\rangle\lvert\psi_0\rangle \pm \lvert\psi_1\rangle\lvert\psi_1\rangle$
$\lvert B_2'^{\pm}\rangle$	$\lvert\theta_1\rangle\lvert\tilde{\theta}_2\rangle$	$-(\theta_1\theta_2 \pm 1)$	$\lvert\psi_0\rangle\lvert\phi_0\rangle \pm \lvert\psi_1\rangle\lvert\phi_1\rangle$
$\lvert B_3'^{\pm}\rangle$	$\lvert\tilde{\theta}_1\rangle\lvert\theta_2\rangle$	$-(\theta_1\theta_2 \pm 1)$	$\lvert\phi_0\rangle\lvert\psi_0\rangle \pm \lvert\phi_1\rangle\lvert\psi_1\rangle$
$\lvert B_4'^{\pm}\rangle$	$\lvert\tilde{\theta}_1\rangle\lvert\tilde{\theta}_2\rangle$	$-(\theta_1\theta_2 \pm 1)$	$\lvert\phi_0\rangle\lvert\phi_0\rangle \pm \lvert\phi_1\rangle\lvert\phi_1\rangle$
$\lvert B_1^-\rangle$	$\lvert\theta\rangle\lvert\theta\rangle$	1	$\lvert\psi_0\rangle\lvert\psi_1\rangle - \lvert\psi_1\rangle\lvert\psi_0\rangle$
$\lvert B_2^-\rangle$	$\lvert\theta\rangle\lvert\tilde{\theta}\rangle$	1	$\lvert\psi_0\rangle\lvert\phi_1\rangle - \lvert\psi_1\rangle\lvert\phi_0\rangle$
$\lvert B_3^-\rangle$	$\lvert\tilde{\theta}\rangle\lvert\theta\rangle$	1	$\lvert\phi_0\rangle\lvert\psi_1\rangle - \lvert\phi_1\rangle\lvert\psi_0\rangle$
$\lvert B_4^-\rangle$	$\lvert\tilde{\theta}\rangle\lvert\tilde{\theta}\rangle$	1	$\lvert\phi_0\rangle\lvert\phi_1\rangle - \lvert\phi_1\rangle\lvert\phi_0\rangle$

Non-normalized pseudo-Bell states and the corresponding weight functions.

4.2. Pseudo-GHZ and pseudo-Werner states. We construct a pseudo version of the GHZ and Werner states,

$$|GHZ^{\pm}\rangle = \frac{|000\rangle \pm |111\rangle}{\sqrt{2}}, \qquad |W\rangle = \frac{|100\rangle + |010\rangle + |001\rangle}{\sqrt{3}}, \tag{4.7}$$

which are widely used in quantum information theory. To construct a three-qubit pseudo-GHZ state, we must consider the tensor product of three one-mode FPHCSs with different Grassmann numbers. They can take one of the eight forms

$$
\begin{array}{cccc}
|\theta_1\rangle|\theta_2\rangle|\theta_3\rangle, & |\tilde\theta_1\rangle|\theta_2\rangle|\theta_3\rangle, & |\theta_1\rangle|\tilde\theta_2\rangle|\theta_3\rangle, & |\theta_1\rangle|\theta_2\rangle|\tilde\theta_3\rangle, \\
|\tilde\theta_1\rangle|\tilde\theta_2\rangle|\theta_3\rangle, & |\tilde\theta_1\rangle|\theta_2\rangle|\tilde\theta_3\rangle, & |\theta_1\rangle|\tilde\theta_2\rangle|\tilde\theta_3\rangle, & |\tilde\theta_1\rangle|\tilde\theta_2\rangle|\tilde\theta_3\rangle.
\end{array} \tag{4.8}
$$

As an example, we consider the state

$$
|G_1^\pm\rangle = \int d\theta_1\, d\theta_2\, d\theta_3\, w^\pm(\theta_1,\theta_2,\theta_3)|\theta_1\rangle|\theta_2\rangle|\theta_3\rangle = |\psi_0\rangle|\psi_0\rangle|\psi_0\rangle \pm |\psi_1\rangle|\psi_1\rangle|\psi_1\rangle, \tag{4.9}
$$

where the weight functions are

$$
w^\pm(\theta_1,\theta_2,\theta_3) = \theta_3\theta_2\theta_1 \pm 1. \tag{4.10}
$$

It is easy to verify that this function is suitable for each of the states $|G_i^\pm\rangle$, $i = 1, \ldots, 8$. We summarize the results for non-normalized pseudo-GHZ states in Table 2.

Table 2

State	FPHCS	Weight function	Pseudo-GHZ state
$\|G_1^\pm\rangle$	$\|\theta_1\rangle\|\theta_2\rangle\|\theta_3\rangle$	$\theta_3\theta_2\theta_1 \pm 1$	$\|\psi_0\rangle\|\psi_0\rangle\|\psi_0\rangle \pm \|\psi_1\rangle\|\psi_1\rangle\|\psi_1\rangle$
$\|G_2^\pm\rangle$	$\|\tilde\theta_1\rangle\|\theta_2\rangle\|\theta_3\rangle$	$\theta_3\theta_2\theta_1 \pm 1$	$\|\varphi_0\rangle\|\psi_0\rangle\|\psi_0\rangle \pm \|\varphi_1\rangle\|\psi_1\rangle\|\psi_1\rangle$
$\|G_3^\pm\rangle$	$\|\theta_1\rangle\|\tilde\theta_2\rangle\|\theta_3\rangle$	$\theta_3\theta_2\theta_1 \pm 1$	$\|\psi_0\rangle\|\varphi_0\rangle\|\psi_0\rangle \pm \|\psi_1\rangle\|\varphi_1\rangle\|\psi_1\rangle$
$\|G_4^\pm\rangle$	$\|\theta_1\rangle\|\theta_2\rangle\|\tilde\theta_3\rangle$	$\theta_3\theta_2\theta_1 \pm 1$	$\|\psi_0\rangle\|\psi_0\rangle\|\varphi_0\rangle \pm \|\psi_1\rangle\|\psi_1\rangle\|\varphi_1\rangle$
$\|G_5^\pm\rangle$	$\|\tilde\theta_1\rangle\|\tilde\theta_2\rangle\|\theta_3\rangle$	$\theta_3\theta_2\theta_1 \pm 1$	$\|\varphi_0\rangle\|\varphi_0\rangle\|\psi_0\rangle \pm \|\varphi_1\rangle\|\varphi_1\rangle\|\psi_1\rangle$
$\|G_6^\pm\rangle$	$\|\tilde\theta_1\rangle\|\theta_2\rangle\|\tilde\theta_3\rangle$	$\theta_3\theta_2\theta_1 \pm 1$	$\|\varphi_0\rangle\|\psi_0\rangle\|\varphi_0\rangle \pm \|\varphi_1\rangle\|\psi_1\rangle\|\varphi_1\rangle$
$\|G_7^\pm\rangle$	$\|\theta_1\rangle\|\tilde\theta_2\rangle\|\tilde\theta_3\rangle$	$\theta_3\theta_2\theta_1 \pm 1$	$\|\psi_0\rangle\|\varphi_0\rangle\|\varphi_0\rangle \pm \|\psi_0\rangle\|\varphi_1\rangle\|\varphi_0\rangle$
$\|G_8^\pm\rangle$	$\|\tilde\theta_1\rangle\|\tilde\theta_2\rangle\|\tilde\theta_3\rangle$	$\theta_3\theta_2\theta_1 \pm 1$	$\|\varphi_0\rangle\|\varphi_0\rangle\|\varphi_0\rangle \pm \|\varphi_1\rangle\|\varphi_1\rangle\|\varphi_1\rangle$

Non-normalized pseudo-GHZ states and the corresponding weight functions.

To construct pseudo-Werner states, we can use the tensor product of FPHCSs with three either different or the same Grassmann numbers. Below, we give one example $\mathcal{W}$ and $\mathcal{W}'$ for each of the two categories.

For the tensor product of the FPHCSs with different Grassmann numbers, we have

$$
\begin{aligned}
|\mathcal{W}_1\rangle = \int d\theta_1\, d\theta_2\, d\theta_3\, w_1(\theta_1,\theta_2,\theta_3)|\theta_1\rangle|\theta_2\rangle|\theta_3\rangle = \\
= |\psi_0\rangle|\psi_0\rangle|\psi_1\rangle + |\psi_0\rangle|\psi_1\rangle|\psi_0\rangle + |\psi_1\rangle|\psi_0\rangle|\psi_0\rangle,
\end{aligned} \tag{4.11}
$$

where

$$
w(\theta_1,\theta_2,\theta_3) = \theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3. \tag{4.12}
$$

Similarly, for the same Grassmann numbers, we have

$$
|\mathcal{W}_1'\rangle = \int d\theta\, w(\theta)\, |\theta\rangle|\theta\rangle|\theta\rangle = -|\psi_0\rangle|\psi_0\rangle|\psi_1\rangle + |\psi_0\rangle|\psi_1\rangle|\psi_0\rangle - |\psi_1\rangle|\psi_0\rangle|\psi_0\rangle, \tag{4.13}
$$

where the appropriate weight function is $w'(\theta) = 1$.

As can be seen from Table 3, for a given tensor product of three different one-mode FPHCSs, for example, $|\theta_1\rangle|\theta_2\rangle|\theta_3\rangle$, depending on the selection of the weight function, there are eight pseudo-Werner states. We emphasize that although we constructed the category $\mathcal{W}'$ in terms of FPHCSs with the same Grassmann numbers, we could also obtain the same result with the different Grassmann numbers, which would in turn yield the weight function $w = -\theta_1\theta_2 + \theta_1\theta_3 - \theta_2\theta_3$.

Table 3

State	FPHCS	Weight function	Pseudo-Werner state
$\|W_1^{(i)}\rangle$	$\|\theta_1\rangle\|\theta_2\rangle\|\theta_3\rangle$	$\pm\theta_1\theta_2 \pm \theta_1\theta_3 \pm \theta_2\theta_3$	$\pm\|\psi_0\rangle\|\psi_0\rangle\|\psi_1\rangle \pm \|\psi_0\rangle\|\psi_1\rangle\|\psi_0\rangle \pm \|\psi_1\rangle\|\psi_0\rangle\|\psi_0\rangle$
$\|W_2^{(i)}\rangle$	$\|\tilde\theta_1\rangle\|\theta_2\rangle\|\theta_3\rangle$	$\pm\theta_1\theta_2 \pm \theta_1\theta_3 \pm \theta_2\theta_3$	$\pm\|\varphi_0\rangle\|\psi_0\rangle\|\psi_1\rangle \pm \|\varphi_0\rangle\|\psi_1\rangle\|\psi_0\rangle \pm \|\varphi_1\rangle\|\psi_0\rangle\|\psi_0\rangle$
$\|W_3^{(i)}\rangle$	$\|\theta_1\rangle\|\tilde\theta_2\rangle\|\theta_3\rangle$	$\pm\theta_1\theta_2 \pm \theta_1\theta_3 \pm \theta_2\theta_3$	$\pm\|\psi_0\rangle\|\varphi_0\rangle\|\psi_1\rangle \pm \|\psi_0\rangle\|\varphi_1\rangle\|\psi_0\rangle \pm \|\psi_1\rangle\|\varphi_0\rangle\|\psi_0\rangle$
$\|W_4^{(i)}\rangle$	$\|\theta_1\rangle\|\theta_2\rangle\|\tilde\theta_3\rangle$	$\pm\theta_1\theta_2 \pm \theta_1\theta_3 \pm \theta_2\theta_3$	$\pm\|\psi_0\rangle\|\psi_0\rangle\|\varphi_1\rangle \pm \|\psi_0\rangle\|\psi_1\rangle\|\varphi_0\rangle \pm \|\psi_1\rangle\|\psi_0\rangle\|\varphi_0\rangle$
$\|W_5^{(i)}\rangle$	$\|\tilde\theta_1\rangle\|\tilde\theta_2\rangle\|\theta_3\rangle$	$\pm\theta_1\theta_2 \pm \theta_1\theta_3 \pm \theta_2\theta_3$	$\pm\|\varphi_0\rangle\|\varphi_0\rangle\|\psi_1\rangle \pm \|\varphi_0\rangle\|\varphi_1\rangle\|\psi_0\rangle \pm \|\varphi_1\rangle\|\varphi_0\rangle\|\psi_0\rangle$
$\|W_6^{(i)}\rangle$	$\|\tilde\theta_1\rangle\|\theta_2\rangle\|\tilde\theta_3\rangle$	$\pm\theta_1\theta_2 \pm \theta_1\theta_3 \pm \theta_2\theta_3$	$\pm\|\varphi_0\rangle\|\psi_0\rangle\|\varphi_1\rangle \pm \|\varphi_0\rangle\|\psi_1\rangle\|\varphi_0\rangle \pm \|\varphi_1\rangle\|\psi_0\rangle\|\varphi_0\rangle$
$\|W_7^{(i)}\rangle$	$\|\theta_1\rangle\|\tilde\theta_2\rangle\|\tilde\theta_3\rangle$	$\pm\theta_1\theta_2 \pm \theta_1\theta_3 \pm \theta_2\theta_3$	$\pm\|\psi_0\rangle\|\varphi_0\rangle\|\varphi_1\rangle \pm \|\psi_0\rangle\|\varphi_1\rangle\|\varphi_0\rangle \pm \|\psi_1\rangle\|\varphi_0\rangle\|\varphi_0\rangle$
$\|W_8^{(i)}\rangle$	$\|\tilde\theta_1\rangle\|\tilde\theta_2\rangle\|\tilde\theta_3\rangle$	$\pm\theta_1\theta_2 \pm \theta_1\theta_3 \pm \theta_2\theta_3$	$\pm\|\varphi_0\rangle\|\varphi_0\rangle\|\varphi_1\rangle \pm \|\varphi_0\rangle\|\varphi_1\rangle\|\varphi_0\rangle \pm \|\varphi_1\rangle\|\varphi_0\rangle\|\varphi_0\rangle$
$\|W'_1\rangle$	$\|\theta\rangle\|\theta\rangle\|\theta\rangle$	1	$-\|\psi_0\rangle\|\psi_0\rangle\|\psi_1\rangle + \|\psi_0\rangle\|\psi_1\rangle\|\psi_0\rangle - \|\psi_1\rangle\|\psi_0\rangle\|\psi_0\rangle$
$\|W'_2\rangle$	$\|\tilde\theta\rangle\|\theta\rangle\|\theta\rangle$	1	$-\|\varphi_0\rangle\|\psi_0\rangle\|\psi_1\rangle + \|\varphi_0\rangle\|\psi_1\rangle\|\psi_0\rangle - \|\varphi_1\rangle\|\psi_0\rangle\|\psi_0\rangle$
$\|W'_3\rangle$	$\|\theta\rangle\|\tilde\theta\rangle\|\theta\rangle$	1	$-\|\psi_0\rangle\|\varphi_0\rangle\|\psi_1\rangle + \|\psi_0\rangle\|\varphi_1\rangle\|\psi_0\rangle - \|\psi_1\rangle\|\varphi_0\rangle\|\psi_0\rangle$
$\|W'_4\rangle$	$\|\theta\rangle\|\theta\rangle\|\tilde\theta\rangle$	1	$-\|\psi_0\rangle\|\psi_0\rangle\|\varphi_1\rangle + \|\psi_0\rangle\|\psi_1\rangle\|\varphi_0\rangle - \|\psi_1\rangle\|\psi_0\rangle\|\varphi_0\rangle$
$\|W'_5\rangle$	$\|\tilde\theta\rangle\|\tilde\theta\rangle\|\theta\rangle$	1	$-\|\varphi_0\rangle\|\varphi_0\rangle\|\psi_1\rangle + \|\varphi_0\rangle\|\varphi_1\rangle\|\psi_0\rangle - \|\varphi_1\rangle\|\varphi_0\rangle\|\psi_0\rangle$
$\|W'_6\rangle$	$\|\tilde\theta\rangle\|\theta\rangle\|\tilde\theta\rangle$	1	$-\|\varphi_0\rangle\|\psi_0\rangle\|\varphi_1\rangle + \|\varphi_0\rangle\|\psi_1\rangle\|\varphi_0\rangle - \|\varphi_1\rangle\|\psi_0\rangle\|\varphi_0\rangle$
$\|W'_7\rangle$	$\|\theta\rangle\|\tilde\theta\rangle\|\tilde\theta\rangle$	1	$-\|\psi_0\rangle\|\varphi_0\rangle\|\varphi_1\rangle + \|\psi_0\rangle\|\varphi_1\rangle\|\varphi_0\rangle - \|\psi_1\rangle\|\varphi_0\rangle\|\varphi_0\rangle$
$\|W'_8\rangle$	$\|\tilde\theta\rangle\|\tilde\theta\rangle\|\tilde\theta\rangle$	1	$-\|\varphi_0\rangle\|\varphi_0\rangle\|\varphi_1\rangle + \|\varphi_0\rangle\|\varphi_1\rangle\|\varphi_0\rangle - \|\varphi_1\rangle\|\varphi_0\rangle\|\varphi_0\rangle$

Non-normalized pseudo-Werner states and the corresponding weight functions: the superscript (i) refers to combinations of symbols $(+,+,+)$, $(+,+,-)$, $\dots$, $(-,-,-)$, denoting the respective weight functions $\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3$, $\theta_1\theta_2+\theta_1\theta_3-\theta_2\theta_3$, $\dots$, $-\theta_1\theta_2-\theta_1\theta_3-\theta_2\theta_3$.

4.3. Pseudo-biseparable states. Here, we use FPHCSs to construct pseudo-biseparabile states. Depending on how a given state is divided into two parts, there exists a partial entanglement of their subsystems. If a pure state $|ABC\rangle$ involves three subsystems A, B, and C, then one part, A for example, can be separable while the other two parts, B and C, are entangled.

As an illustration, we consider the examples

$$\begin{gathered}\int d\theta_1\,d\theta_2\,d\theta_3\,(\theta_1\theta_2 \pm \theta_1\theta_3)|\theta_1\rangle|\theta_2\rangle|\theta_3\rangle = |\psi_0\rangle_{(1)} \otimes |B_1^{\pm}\rangle_{(2,3)},\\ \int d\theta_1\,d\theta_2\,d\theta_3\,(\theta_3\theta_2\theta_1 \mp \theta_1)|\theta_1\rangle|\theta_2\rangle|\theta_3\rangle = |\psi_0\rangle_{(1)} \otimes |B_1'^{\pm}\rangle_{(2,3)},\\ \int d\theta_1\,d\theta_2\,d\theta_3\,(\theta_1\theta_2 \mp \theta_3\theta_2)|\theta_1\rangle|\theta_2\rangle|\theta_3\rangle = |\psi_0\rangle_{(2)} \otimes |B_1^{\pm}\rangle_{(1,3)}.\end{gathered} \tag{4.14}$$

The first two examples show that the partition $(2,3)$ is a pseudo-Bell state and is separable with respect to partition 1. The same holds for the partitions $(1,3)$ and 2 in the last example. These examples make it clear that we can find different pseudo-biseparable states just by considering the integration over the tensor product $|\theta_1\rangle|\theta_2\rangle|\theta_3\rangle$ using different weight functions. But we note that the family W' does not lead to any pseudo-biseparable states.

5. Entanglement of multipartite pseudo-Hermitian states

In this section, we study the entanglement of pseudo-Bell states using the concurrence function and pseudo-GHZ and pseudo-Werner states using the average entropy. For this, we consider the two-level pseudo-Hermitian Hamiltonians

$$H_i = \begin{pmatrix} r_i e^{i\beta_i} & s_i \\ t_i & r_i e^{-i\beta_i} \end{pmatrix}, \quad i = 1, 2, 3, \tag{5.1}$$

where i denotes the ith system and r_i, s_i, t_i, and β_i are real numbers. This Hamiltonian is non-Hermitian but has real eigenvalues if $s_i t_i > r_i^2 \sin^2 \beta_i$ [48]. We assume that the systems are in four- and eight-dimensional Hilbert spaces and are governed by $H_1 \otimes H_2$ and $H_1 \otimes H_2 \otimes H_3$. The biorthonormal eigenstates of H_i and $H_i^\dagger$ are

$$\begin{aligned} |\psi_0\rangle^{(i)} &= \frac{1}{\sqrt{2}\cos\alpha_i}\begin{pmatrix} e^{i\alpha_i/2} \\ e^{-i\alpha_i/2} \end{pmatrix}, \qquad & |\varphi_0\rangle^{(i)} &= \frac{1}{\sqrt{2}\cos\alpha_i}\begin{pmatrix} e^{-i\alpha_i/2} \\ e^{i\alpha_i/2} \end{pmatrix}, \\ |\psi_1\rangle^{(i)} &= \frac{1}{\sqrt{2}\cos\alpha_i}\begin{pmatrix} e^{-i\alpha_i/2} \\ -e^{i\alpha_i/2} \end{pmatrix}, \qquad & |\varphi_1\rangle^{(i)} &= \frac{1}{\sqrt{2}\cos\alpha_i}\begin{pmatrix} e^{i\alpha_i/2} \\ -e^{-i\alpha_i/2} \end{pmatrix}, \end{aligned} \tag{5.2}$$

where $\sin\alpha_i = (r_i/\sqrt{s_i t_i}\,)\sin\beta_i$. The matrix $\eta = \sum_i |\varphi_i\rangle\langle\varphi_i|$ of the (pseudo-)metric operator for the Hamiltonian H_i has the form

$$\eta = \frac{1}{\cos^2\alpha_i}\begin{pmatrix} 1 & -i\sin\alpha_i \\ i\sin\alpha_i & 1 \end{pmatrix}. \tag{5.3}$$

We next consider the pseudo-Bell states.

5.1. Entanglement of pseudo-Bell states. It is well known that the entanglement of a two-qubit state $|\psi\rangle$ can be expressed as a concurrence function [49], [50]

$$\mathcal{C}(|\psi\rangle) \equiv |\langle\psi|\sigma_y \otimes \sigma_y|\psi^*\rangle|, \tag{5.4}$$

where σ_y is the Pauli y matrix $\left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ and $|\psi^*\rangle$ is the complex conjugate of $|\psi\rangle$. We use the concurrence function to describe the entanglement of the pseudo-Bell state quantitatively.

After the pseudo-Bell states in Table 1 are normalized and the explicit forms of $|\psi_k\rangle^{(i)}$ and $|\varphi_k\rangle^{(i)}$ ($k = 0, 1$) given by (5.2) are substituted, the corresponding concurrences become

$$\begin{aligned} \mathcal{C}(|B_1^-\rangle) = \mathcal{C}(|B_4^-\rangle) &= \left|\frac{\cos\alpha_1\cos\alpha_2}{1 - \sin\alpha_1\sin\alpha_2}\right|, \\ \mathcal{C}(|B_2^-\rangle) = \mathcal{C}(|B_3^-\rangle) &= \left|\frac{\cos\alpha_1\cos\alpha_2}{1 + \sin\alpha_1\sin\alpha_2}\right|. \end{aligned} \tag{5.5}$$

We focused on the lower third of Table 1. Similar reasoning applies to the other pseudo-Bell states. Hence, the concurrence of $|B_j^-\rangle$s is a periodic function in the parameters α_1 and α_2 with the period π, i.e., $\mathcal{C}(\alpha_1, \alpha_2) = \mathcal{C}(\alpha_1 + m\pi, \alpha_2 + m\pi)$, where $m \in \mathbb{Z}$. Equations (5.5) show that in both the cases $\mathcal{C}_{\max} = 1$ and $\mathcal{C}_{\min} = 0$, the maximum and minimum values of (5.5) become $\alpha_1 = \alpha_2 = m\pi$ and $\alpha_1 = \alpha_1 = (2m+1)\pi/2$. In the special case $\alpha_1 = \alpha_2 = \alpha$, Eq. (5.5) is

$$\mathcal{C}(|B_1^-\rangle) = \mathcal{C}(|B_4^-\rangle) = 1, \qquad \mathcal{C}(|B_2^-\rangle) = \mathcal{C}(|B_3^-\rangle) = \frac{\cos^2\alpha}{1 + \sin^2\alpha}. \tag{5.6}$$

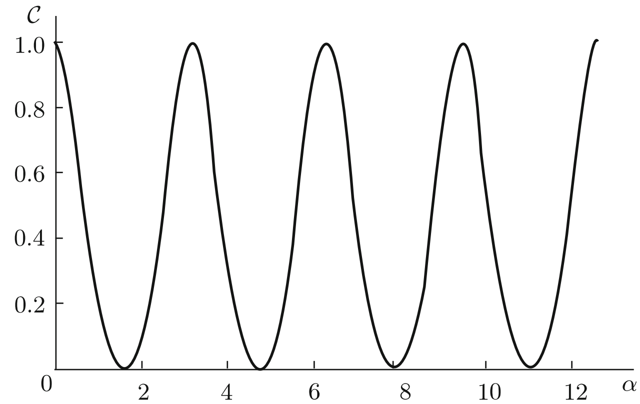

Fig. 1. Concurrence of $|B_2^-\rangle$ and $|B_3^-\rangle$ as function of the parameter α.

It should be no surprise that we obtain $\mathcal{C}$ for $|B_1^-\rangle$ and $|B_4^-\rangle$ independent of the parameter α because these states for $\alpha_1 = \alpha_2 = \alpha$ reduce to the standard Bell state $|\Psi^-\rangle$ up to the total phase $e^{-i\pi}$,

$$|B_1\rangle^- = \frac{|\psi_0\rangle|\psi_1\rangle - |\psi_1\rangle|\psi_0\rangle}{\|\,|B_1^-\rangle\,\|} = -\frac{|01\rangle - |10\rangle}{\sqrt{2}},$$
$$|B_4\rangle^- = \frac{|\varphi_0\rangle|\varphi_1\rangle - |\varphi_1\rangle|\varphi_0\rangle}{\|\,|B_4^-\rangle\,\|} = -\frac{|01\rangle - |10\rangle}{\sqrt{2}}. \tag{5.7}$$

In contrast, the concurrence of the states $|B_2^-\rangle$ and $|B_3^-\rangle$ depends on α (see Fig. 1).

A simple calculation shows that for $\alpha_1 = \alpha_2 = \alpha$, the pseudo-Bell states

$$|B_2'^-\rangle = |B_3'^-\rangle = |\Psi^+\rangle, \qquad |B_1'^+\rangle = |B_4'^+\rangle = |\Phi^+\rangle, \qquad |B_2^+\rangle = |B_3^+\rangle = |\Phi^-\rangle \tag{5.8}$$

reduce to standard Bell states.

We consider two special cases that are interesting from the standpoint of dipole interaction.

1. If $st = r^2 \sin^2\beta$, then $\mathcal{C}(|B_2^-\rangle) = \mathcal{C}(|B_3^-\rangle) = 0$.
2. If $r = \delta/2$, $\beta = -\pi/2$, and $t = s$, then the Hamiltonian reduces to

$$H_{1,2} = \frac{1}{2}\begin{pmatrix} -i\delta & 2s \\ 2s & i\delta \end{pmatrix}. \tag{5.9}$$

This Hamiltonian arises in the interaction of a two-level atom with an electromagnetic field where the real constant δ is the decay rate for the upper and lower levels and the quantity s characterizes the radiation–atom interaction matrix element between the levels described in the interaction picture with the rotating weight approximation [25], [51], [52]. In this case, the concurrence in terms of s and δ is

$$\mathcal{C}(|B_2^-\rangle) = \mathcal{C}(|B_3^-\rangle) = \frac{4s^2 - \delta^2}{4s^2 + \delta^2}. \tag{5.10}$$

Because $\sin\alpha = -\delta/2s$, we have $4s^2 - \delta^2 \geq 0$, which ensures that the concurrence is nonnegative. Concurrence (5.10) for the intervals $1 \leq s \leq 2$ and $-2 \leq \delta \leq 2$ is shown in Fig. 2.

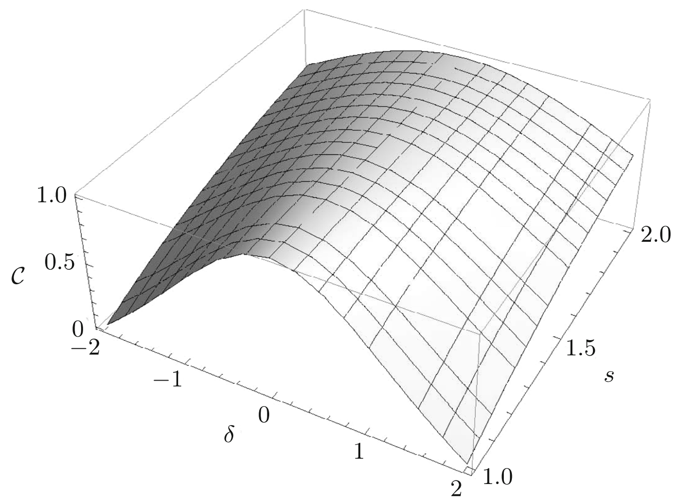

Fig. 2. Concurrence of $|B_2^-\rangle$ and $|B_3^-\rangle$ in terms of the parameters δ and s: it can be seen that the concurrences of the states for the points (δ, s) with $\delta = 0$ are equal to one and they are maximally entangled.

5.2. Entanglement of pseudo-GHZ and pseudo-Werner states. For the next step, we quantitatively describe the entanglement of pseudo-GHZ and pseudo-Werner states. We consider the average entropy $\langle S_{\rm L}\rangle$, which is a good measure of entanglement,

$$\langle S_{\rm L}\rangle = \binom{N}{n}^{-1} \sum_{A_n} S_{\rm L}^{(A_n;B_{N-n})}. \tag{5.11}$$

We define it in terms of the linear entropy [53]

$$S_{\rm L}^{(A_n;B_{N-n})} = \frac{d}{d-1}(1 - {\rm Tr}_{A_n}[\,\rho_{A_n}]^2), \qquad \rho_{A_n} = {\rm Tr}_{B_{N-n}}[\rho], \tag{5.12}$$

where $d = \min\{2^n, 2^{N-n}\}$ is the dimension of the reduced density matrix ρ_{A_n}. We note that although the linear entropy and von Neumann entropy [54] are similar measures of state mixing, the linear entropy is easier to calculate because it does not require diagonalizing the density matrix. The linear entropy can range from zero (a completely pure state) to one (a completely mixed state).

Based on the entanglement measure defined as average entropy, as examples, we investigate the entanglement of the normalized states $|G_1^+\rangle$, $|\mathcal{W}_7^{(+,+,+)}\rangle$, and $|\mathcal{W}_6^{(-,+,-)}\rangle$ (the latter are denoted by W_7 and W_6 for simplicity). We have

$$\begin{aligned}
|G_1^+\rangle &= \frac{|\varphi_0\rangle|\varphi_0\rangle|\varphi_0\rangle \pm |\varphi_1\rangle|\varphi_1\rangle|\varphi_1\rangle}{\|\,|G_1^+\rangle\,\|},\\
|\mathcal{W}_7\rangle &= \frac{|\psi_0\rangle|\varphi_0\rangle|\varphi_1\rangle + |\psi_0\rangle|\varphi_1\rangle|\varphi_0\rangle + |\psi_1\rangle|\varphi_0\rangle|\varphi_0\rangle}{\|\,|W_7\rangle\,\|},\\
|\mathcal{W}_6\rangle &= \frac{-|\varphi_0\rangle|\psi_0\rangle|\varphi_1\rangle + |\varphi_0\rangle|\psi_1\rangle|\varphi_0\rangle - |\varphi_1\rangle|\psi_0\rangle|\varphi_0\rangle}{\|\,|W_6\rangle\,\|}.
\end{aligned} \tag{5.13}$$

In accordance with Eq. (5.11), the average entropy of the normalized $|G_1^+\rangle$ is

$$\langle S_{\rm L}\rangle_{(G_1^+)} = \frac{5 + \cos 2\alpha_2 - 2\sin^2\alpha_1(1 + \cos^2\alpha_3 \sin^2\alpha_2) + (\cos 2\alpha_1 \cos 2\alpha_2 - 3)\sin^2\alpha_3}{6}.$$

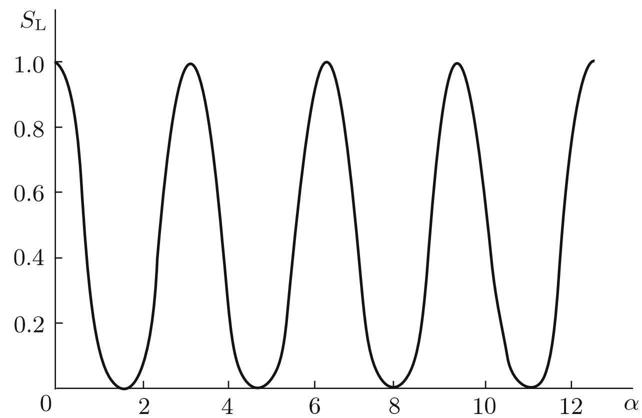

Fig. 3. The average entropy of all pseudo-GHZ states as a function of the parameter α.

Direct calculations show that the average entropy of all the pseudo-GHZ states are the same.

As before, we consider the quantum states with $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, which yields

$$\langle S_{\rm L}\rangle_{(G_1^+)} = \frac{1}{2}\cos^4\alpha(3 - \cos 2\alpha). \tag{5.14}$$

The average entropy of the state $|G_1^+\rangle$ as a function of the parameter α is shown in Fig. 3. The maximum and minimum values of the average entropy for pseudo-GHZ states are attained for $\alpha = k\pi$ and $\alpha = (2k+1)\pi/2$.

As another example, we consider the normalized $|\mathcal{W}_7\rangle$. In this case, the average entropies in the cases of different and identical α_i are

$$\begin{aligned}\langle S_{\rm L}\rangle_{(\mathcal{W}_7)} = {} & \frac{1}{3(2\sin\alpha_2\sin\alpha_3 - 2\sin\alpha_1(\sin\alpha_2 + \sin\alpha_3) + 3)^2} \times \\ & \times \big(2(\cos 2\alpha_1 + \cos 2\alpha_2 + 2)\cos 2\alpha_3 + \cos 2(\alpha_1 - \alpha_2) + \\ & + \cos 2(\alpha_1 + \alpha_2) + 4\cos 2\alpha_1 + 4\cos 2\alpha_2 + 6\big),\end{aligned} \tag{5.15}$$

$$\langle S_{\rm L}\rangle_{(\mathcal{W}_7)} = \frac{8\cos^4\alpha}{(\cos 2\alpha + 2)^2}. \tag{5.16}$$

The average entropy of $|\mathcal{W}_7\rangle$ in the case of identical α_i is shown in Fig. 4. It is in the interval $0 \leq \langle S_{\rm L}(\alpha)\rangle_{(\mathcal{W}_7)} \leq 8/9$, and the upper and lower bounds are attained for $\alpha = k\pi$ and $\alpha_k = (2k+1)\pi/2$.

Finally, as the last example, we study the average entropy of the normalized state $|\mathcal{W}_6\rangle$. Taking Eq. (5.11) into account, we derive the expression

$$\begin{aligned}\langle S_{\rm L}\rangle_{(\mathcal{W}_6)} = {} & \frac{1}{3(2\sin\alpha_2\sin\alpha_3 + 2\sin\alpha_1(\sin\alpha_2 + \sin\alpha_3) + 3)^2} \times \\ & \times \big(2(\cos 2\alpha_1 + \cos 2\alpha_2 + 2)\cos 2\alpha_3 + \cos 2(\alpha_1 - \alpha_2) + \\ & + \cos 2(\alpha_1 + \alpha_2) + 4\cos 2\alpha_1 + 4\cos 2\alpha_2 + 6\big).\end{aligned} \tag{5.17}$$

It is easy to verify that in the case of $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, this equation reduces to

$$\langle S_{\rm L}\rangle_{(\mathcal{W}_6)} = \frac{8\cos^4\alpha}{9(\cos 2\alpha - 2)^2}. \tag{5.18}$$

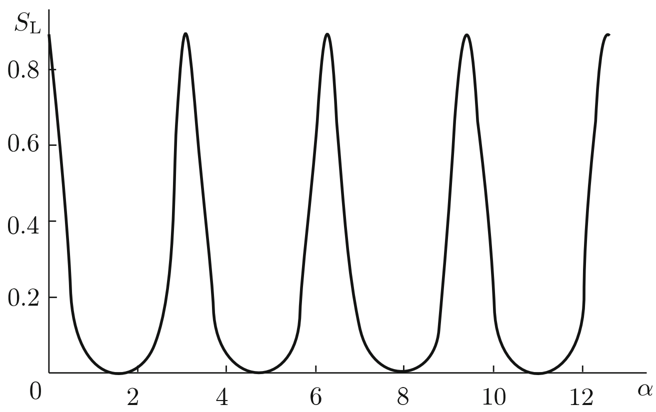

Fig. 4. The average entropy of $|\mathcal{W}_7\rangle$ as a function of α: the lower bound 0 and upper bound 8/9 correspond to separable and maximally entangled pseudo-Werner states $|\mathcal{W}_7\rangle$.

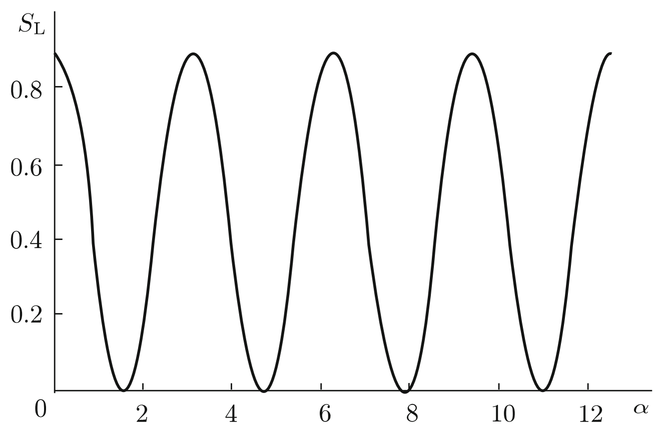

Fig. 5. The average entropy of $|W_6\rangle$ as a function of α.

The average entropy of $|\mathcal{W}_6\rangle$ in terms of the parameter α is shown in Fig. 5. As in the preceding case, the maximum average entropy is exactly equal to the average entropy of the entangled states described by the standard Hermitian Hamiltonian.

The method presented above can be extended to multipartite n-level systems.

6. Conclusion

We have constructed a pseudo-Hermitian version of the well-known maximally entangled pure states such as Bell, GHZ, Werner, and biseparable states by integrating over the tensor product of one-mode FPHCSs and using a suitable Grassmannian weight function. As clarifying examples, we considered the biorthonormal eigenstates of the pseudo-Hermitian Hamiltonian that appears in the interaction of a two-level atom with an electromagnetic field.

To quantify the entanglement of these pseudo-states, we used the concurrence function for two-qubit (pseudo-Bell) states and the average linear entropy for three-qubit (pseudo-GHZ and pseudo-Werner) states. We found that for $\alpha_1 = \alpha_2 = \alpha$, the pseudo-Bell states $|B_1^-\rangle$ and $|B_4^-\rangle$ are the same as the standard Bell state $|\Psi^-\rangle$ up to the phase factor $e^{-i\pi}$. Similarly, $|B_2'^-\rangle$ and $|B_3'^-\rangle$ are the same as $|\Psi^-\rangle$, $|B_1'^+\rangle$ and $|B_4'^+\rangle$ are the same as $|\Phi^+\rangle$, and $|B_2^+\rangle$ and $|B_3^+\rangle$ are the same as $|\Phi^-\rangle$.

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