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INVERSE SCATTERING PROBLEM FOR THE SCHRÖDINGER EQUATION WITH AN ADDITIONAL QUADRATIC POTENTIAL ON THE ENTIRE AXIS

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We consider the Schrödinger equation with an additional quadratic potential on the entire axis and use the transformation operator method to study the direct and inverse problems of the scattering theory. We obtain the main integral equations of the inverse problem and prove that the basic equations are uniquely solvable.

Keywords: Schrödinger equation, oscillator, reflection coefficient, inverse scattering problem, basic equation

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1. Introduction

The problem of a quantum oscillator was an essential problem solved by Heisenberg in the framework of matrix mechanics and by Schrödinger in the language of wave mechanics. The problem of describing the oscillatory motions of atoms in molecules and crystals reduces to solving precisely this problem (see [1]). A "quantized" electromagnetic field is equivalent to a system of oscillators.

The inverse spectral problem consisting in reconstructing one-dimensional Schrödinger operators with an additional oscillator and the same discrete spectrum was studied in [2], [3]. The most complete solution of the inverse problem for a perturbed oscillator of the form $Ty = -y'' + x^2y + q(x)y$, where $q(x)$ is a real potential and $q'(x), xq(x) \in L_2(-\infty, +\infty)$, was given in [4].

Here, we study the direct and inverse scattering problem for the Schrödinger equation with an additional quadratic potential

$$- y'' - x^2y + q(x)y = \lambda y, \quad -\infty < x < +\infty, \tag{1}$$

where the real potential $q(x)$ is a smooth function and satisfies the condition

$$\int_{-\infty}^{\infty} (1 + x^4)e^{2x^2}|q(x)|\,dx < \infty. \tag{2}$$

We note that the inverse scattering problem in the case without an additional quadratic potential, i.e., for the equation $-y'' + q(x)y = \lambda y$ with a rapidly decreasing potential $q(x)$, has been studied in detail by many authors (see [5]–[7] and the references therein). In the presence of an additional quadratic potential,

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the continuous spectrum of the unperturbed equation is associated with eigenfunctions that in contrast to an exponential function do not have a multiplicative property, and it was therefore necessary to modify several classical arguments in [5]–[7].

We note that the scattering problem for Eq. (1) defined on a half-line was considered in [8]. But the eigenfunctions of the continuous spectrum of the unperturbed equation, which were used in that paper, are unsuitable for Eq. (1) defined on the entire axis because infinite poles of the transmission coefficients appear. Here, we express the eigenfunctions of the continuous spectrum of the unperturbed equation in terms of the parabolic cylinder functions $D_\nu(z)$.

We note that the inverse scattering problem for the one-dimensional Schrödinger equations with different increasing potentials was studied in [9]–[13]. In contrast to [9], [10], [12], we here obtain both Marchenko-type integral equations, which allow drawing conclusions about the behavior of the reconstructed potential at both ends.

Several problems of the spectral theory of the one-dimensional Schrödinger equation were studied in [14]–[16].

The direct scattering problem for a multidimensional Schrödinger operator of the form $H = -\Delta + x_1^2 + q(x)$, $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$, was considered in [17].

2. Preliminary consideration of the unperturbed equation

We consider the unperturbed equation

$$-y'' - x^2 y = \lambda y, \quad -\infty < x < +\infty. \tag{3}$$

It is known (see [18], [19]) that Eq. (3) has a solution $\phi_0(x,\lambda)$ that can be represented as $\phi_0(x,\lambda) = D_{i\lambda/2-1/2}(\sqrt{2}\,e^{i\pi/4}x)$, where $U(a,x) = D_{-a-1/2}(x)$ is a parabolic cylinder function, which is a solution of the equation

$$-y'' + \frac{x^2}{4} y = -ay.$$

The behavior of the function $D_\nu(z)$ for large values of $|z|$ and a fixed value of ν is determined by the asymptotic formulas [19] as $z \to \infty$

$$D_\nu(z) \sim z^\nu e^{-z^2/4}, \qquad |\arg z| < \frac{3\pi}{4},$$

$$D_\nu(z) \sim e^{-z^2/4} z^\nu - \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i\pi\nu} e^{z^2/4} z^{-\nu-1}, \quad \frac{\pi}{4} < \arg z < \frac{5\pi}{4}.$$

These formulas imply that

$$\phi_0(x,\lambda) = D_{i\lambda-1/2}\big(\sqrt{2}\,e^{-i\pi/4}x\big) \sim e^{-ix^2/2}\big(\sqrt{2}\,e^{-i\pi/4}x\big)^{i\lambda-1/2}, \quad x \to +\infty.$$

The function in the right-hand side is of the order $O(x^{-(1/2)-\operatorname{Im}\lambda})$, and $\phi_0(x,\lambda)$ hence belongs to $L_2(0,\infty)$ for $\operatorname{Im}\lambda > 0$. Because Eq. (3) does not change under replacing x with $-x$, the function $\phi_0(-x,\lambda)$ is also a solution of this equation. Obviously, the function $\phi_0(-x,\lambda)$ belongs to $L_2(-\infty,0)$ for $\operatorname{Im}\lambda > 0$.

On the other hand, the known expressions [18]

$$U\left(-\frac{i\lambda}{2}, 0\right) = \frac{2^{(i\lambda-1)/4}\sqrt{\pi}}{\Gamma(3/4 - i\lambda/4)}, \qquad U'_x\left(-\frac{i\lambda}{2}, 0\right) = -\frac{2^{(i\lambda+1)/4}\sqrt{\pi}}{\Gamma(1/4 - i\lambda/4)}, \tag{4}$$

imply that for real λ, the two solutions $\{\phi_0(x,\lambda), \overline{\phi_0(x,\lambda)}\}$ of Eq. (3) are linearly independent, and their Wronskian is given by the formula

$$W\{\phi_0(x,\lambda), \overline{\phi_0(x,\lambda)}\} = -i\sqrt{2}\, e^{\pi\lambda/4}.$$

For real values of λ, the solutions $\phi_0(\pm x, \lambda)$ are bounded, which corresponds to the continuous spectrum of problem (3). The constraint formula

$$\phi_0(-x,\lambda) = a_0(\lambda)\overline{\phi_0(x,\lambda)} + b_0(\lambda)\phi_0(x,\lambda) \tag{5}$$

holds on the spectrum, and the functions $t_0(\lambda) = 1/a_0(\lambda)$ and $r_0(\lambda) = b_0(\lambda)/a_0(\lambda)$ have the meaning of transmission and reflection coefficients in scattering theory. It follows from (5) that the coefficients $a_0(\lambda)$ and $b_0(\lambda)$ satisfy the normalization condition

$$|a_0(\lambda)|^2 - |b_0(\lambda)|^2 = 1.$$

Moreover, formulas (4) and (5) imply the relations

$$\begin{aligned} a_0(\lambda) &= \frac{W\{\phi_0(x,\lambda), \phi_0(-x,\lambda)\}}{W\{\phi_0(x,\lambda), \overline{\phi_0(x,\lambda)}\}} = i\frac{\sqrt{2\pi}\, e^{-i\pi/4}}{e^{\pi\lambda/4}\Gamma(1/2 - i\lambda/2)}, \\ b_0(\lambda) &= \frac{W\{\phi_0(-x,\lambda), \overline{\phi_0(x,\lambda)}\}}{W\{\phi_0(x,\lambda), \overline{\phi_0(x,\lambda)}\}} = ie^{-\pi\lambda/2}. \end{aligned} \tag{6}$$

These formulas show that $a_0(\lambda)$ and $b_0(\lambda)$ can be continued analytically to the upper half-plane $\operatorname{Im}\lambda \geq 0$.

Following Titchmarsh [20], we further deduce that the functions $\psi_1(x,\lambda)$ and $\psi_2(x,\lambda)$, defined up to a factor in the general theory [20], respectively coincide with $\phi_0(x,\lambda)$ and $\phi_0(-x,\lambda)$. The functions $\phi_0(x,\lambda)$ and $\phi_0(-x,\lambda)$ hence serve as eigenfunctions of the continuous spectrum of Eq. (3). For any function $h(x) \in L_2(-\infty,\infty)$, we therefore have

$$h(x) = \int_{-\infty}^{\infty} h(y)\left\{\frac{1}{\sqrt{2}\,\pi}\int_{-\infty}^{\infty}\left[\overline{\phi_0(\pm x,\lambda)} + r_0(\lambda)\phi_0(\pm x,\lambda)\right]\phi_0(\pm y,\lambda)\omega_0(\lambda)\,d\lambda\right\}dy,$$

where $\omega_0(\lambda) = e^{-\pi\lambda/4}$. In particular, if $h(x)$ takes real values, then these relations become

$$h(x) = \int_{-\infty}^{\infty} h(y)\left\{\frac{1}{\sqrt{2}\,\pi}\int_{-\infty}^{\infty}\operatorname{Re}\{[\overline{\phi_0(\pm x,\lambda)} + r_0(\lambda)\phi_0(\pm x,\lambda)]\phi_0(\pm y,\lambda)\}\omega_0(\lambda)\,d\lambda\right\}dy.$$

We therefore have the expansion formulas for the eigenfunctions of the continuous spectrum of Eq. (3)

$$\frac{1}{\sqrt{2}\,\pi}\int_{-\infty}^{\infty}\operatorname{Re}\{[\overline{\phi_0(\pm x,\lambda)} + r_0(\lambda)\phi_0(\pm x,\lambda)]\phi_0(\pm y,\lambda)\}\omega_0(\lambda)\,d\lambda = \delta(x-y), \tag{7}$$

where δ is the Dirac delta function.

3. Direct scattering problem

We now describe the scattering problem for Eq. (1). We consider the eigenvalue problem for Eq. (1) in the class of functions $y = y(x)$ bounded on the entire axis. The real values of the energy λ are associated with the continuous spectrum of problem (1). The eigenfunctions of the continuous spectrum are determined by asymptotic conditions at $\pm\infty$ in x,

$$f_\pm(x,\lambda) = \phi_0(\pm x,\lambda) + o(1), \quad x \to \pm\infty.$$

It follows from [8] that the solutions $f_\pm(x,\lambda)$ satisfy triangular representations, which demonstrate the scattering effect,

$$f_\pm(x,\lambda) = \phi_0(\pm x,\lambda) \pm \int_x^{\pm\infty} K^\pm(x,t)\phi_0(\pm t,\lambda)\,dt. \tag{8}$$

The kernels $K^\pm(x,t)$ are real and satisfy the relations

$$K^\pm(x,t) = O\biggl(\int_{(x+t)/2}^{\pm\infty} |q(s)|\,ds\biggr), \quad x+t \to \pm\infty, \qquad K^\pm(x,x) = \pm\frac{1}{2}\int_x^{\pm\infty} q(s)\,ds. \tag{9}$$

According to (8) and (9), the solutions $f_\pm(x,\lambda)$ admit analytic continuations to the half-plane $\operatorname{Im}\lambda > 0$. Moreover, because the potential $q(x)$ is real and estimate (9) holds on the continuous spectrum, the pairs of solutions $\{f_\pm(x,\lambda), \overline{f_\pm(x,\lambda)}\}$ are linearly independent because their Wronskian $W\{f_\pm(x,\lambda), \overline{f_\pm(x,\lambda)}\}$ is equal to $\mp i\sqrt{2}\,e^{\pi\lambda/4}$. Therefore, for real λ, we have

$$f_-(x,\lambda) = a(\lambda)\overline{f_+(x,\lambda)} + b(\lambda)f_+(x,\lambda), \tag{10}$$

$$f_+(x,\lambda) = a(\lambda)\overline{f_-(x,\lambda)} - \overline{b(\lambda)}f_-(x,\lambda), \tag{11}$$

which implies that the transition matrix has the form $T(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ \overline{b(\lambda)} & \overline{a(\lambda)} \end{pmatrix}$. It follows from (10) and (11) that

$$|a(\lambda)|^2 - |b(\lambda)|^2 = 1, \tag{12}$$

i.e., the transition matrix is unimodular, $\det T(\lambda) = 1$. The quantities $r^+(\lambda) = b(\lambda)/a(\lambda)$ and $r^-(\lambda) = -\overline{b(\lambda)}/a(\lambda)$ are respectively called the right and left reflection coefficients. Formula (10) implies that

$$\begin{aligned} a(\lambda) &= \frac{i}{\sqrt{2}}e^{-\pi\lambda/4}W\{f_+(x,\lambda), f_-(x,\lambda)\}, \\ b(\lambda) &= -\frac{i}{\sqrt{2}}e^{-\pi\lambda/4}W\{\overline{f_+(x,\lambda)}, f_-(x,\lambda)\}. \end{aligned} \tag{13}$$

Using (8), (9), and (13), we find that the relations

$$a(\lambda) = a_0(\lambda)[1 + O(\lambda^{-1/2})], \qquad b(\lambda) = b_0(\lambda)[1 + O(\lambda^{-1/2})] \tag{14}$$

hold as $\lambda \to \infty$. It follows from (13) that $a(\lambda)$ admits an analytic continuation to the upper half-plane $\operatorname{Im}\lambda > 0$ and is continuous on the real axis. Moreover, $a(\lambda)$ has no zeros. Indeed, $a(\lambda)$ does not vanish on the real axis by normalization condition (12). We now assume that $a(\lambda_0) = 0$, $\operatorname{Im}\lambda_0 > 0$. It then follows from (13) that the solutions $f_+(x,\lambda_0)$ and $f_-(x,\lambda_0)$ are linearly dependent and self-adjoint problem (1) would formally have a complex eigenvalue, which is impossible.

We note that analogously to formulas (7), we have formulas for the expansion of the continuous spectrum of Eq. (1) in eigenfunctions:

$$\frac{1}{\sqrt{2}\,\pi}\int_{-\infty}^{\infty}\mathrm{Re}\{[\overline{f_\pm(x,\lambda)}+r^\pm(\lambda)f_\pm(x,\lambda)]f_\pm(y,\lambda)\}\omega_0(\lambda)\,d\lambda=\delta(x-y). \tag{15}$$

The transition matrix $T(\lambda)$ completely describes the behavior of the eigenfunctions of the continuous spectrum of problem (1). We stress that all the information about the matrix $T(\lambda)$ is indeed contained in one of the reflection coefficients. Indeed, we assume that the coefficient $r^+(\lambda)$, for example, is given. Taking normalization condition (12) into account, as in [6], [7], we obtain

$$a(z)=a_0(z)\exp\biggl\{-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log(1-|r^+(\lambda)|^2)-\log(1-|r_0(\lambda)|^2)}{\lambda-z}\,d\lambda\biggr\},\quad \mathrm{Im}\,z>0. \tag{16}$$

The coefficients $b(\lambda)$ and $r^-(\lambda)$ can now be constructed using the formulas

$$b(\lambda)=a(\lambda)r^+(\lambda),\qquad r^-(\lambda)=-\overline{r^+(\lambda)}\frac{\overline{a(\lambda)}}{a(\lambda)}. \tag{17}$$

To complete this section, we note that relations (12)–(14) imply the main properties of the reflection coefficients. We rewrite them as the following condition.

Condition 1. *The functions $r^\pm(\lambda)$ are continuous on the real axis and satisfy the relations*

$$|r^\pm(\lambda)|<1,\qquad r^\pm(\lambda)-r_0(\lambda)=O(|\lambda|^{-1/2})\cdot\begin{cases}e^{-\pi\lambda/2}, & \lambda\to+\infty,\\ 1, & \lambda\to-\infty.\end{cases}$$

The coupling formula $r^-(\lambda)/r^+(\lambda)=-\overline{a(\lambda)}/a(\lambda)$ holds, where $a(\lambda)$ is determined by formula (14).

4. Inverse scattering problem

The inverse scattering problem for Eq. (1) is to reconstruct the potential $q(x)$ from one of the reflection coefficients. The basic Gelfand–Levitan–Marchenko integral equations play an important role in solving the inverse problem.

Theorem 1. *For each fixed x, the functions $K^\pm(x,y)$ in representation* (8) *satisfy the integral equations*

$$F^\pm(x,y)+K^\pm(x,y)\pm\int_x^{\pm\infty}K^\pm(x,t)F^\pm(t,y)\,dt=0,\quad \pm y>\pm x, \tag{18}$$

where

$$F^\pm(x,y)=\frac{1}{\sqrt{2}\,\pi}\,\mathrm{Re}\int_{-\infty}^{\infty}\bigl(r^\pm(\lambda)-r_0(\lambda)\bigr)\phi_0(\pm x,\lambda)\phi_0(\pm y,\lambda)\omega_0(\lambda)\,d\lambda. \tag{19}$$

Proof. To derive Eqs. (18), we use expansion formulas (7) and (15). For definiteness, we consider the case with the plus sign. It follows from the well-known properties of the transformation operators (see, e.g., [7]) and representations (8) that

$$\phi_0(y,\lambda)=f_+(y,\lambda)+\int_y^{\infty}K(y,t)f_+(t,\lambda)\,dt,$$

where the kernel $K(y,t)$ satisfies a relation similar to (9). For $y > x$, the last formula with regard to (15) then implies

$$\frac{1}{\sqrt{2}\,\pi}\int_{-\infty}^{\infty}\mathrm{Re}\{[\,\overline{f_+(x,\lambda)}+r^+(\lambda)f_+(x,\lambda)]\phi_0(y,\lambda)\}\omega_0(\lambda)\,d\lambda =$$

$$= \delta(x-y)+\int_{-\infty}^{y}K(y,t)\biggl(\frac{1}{\sqrt{2}\,\pi}\int_{-\infty}^{\infty}\mathrm{Re}\{[\,\overline{f_+(x,\lambda)}+r^+(\lambda)f_+(x,\lambda)]f_+(t,\lambda)\}\omega_0(\lambda)\,d\lambda\biggr)\,dt =$$

$$= \delta(x-y)+\int_{-\infty}^{y}K(y,t)\delta(x-t)\,dt = \delta(x-y)+K(y,x) = \delta(x-y). \tag{20}$$

On the other hand, using (7) and (8), we obtain

$$\frac{1}{\sqrt{2}\,\pi}\int_{-\infty}^{\infty}\mathrm{Re}\{[\,\overline{f_+(x,\lambda)}+r^+(\lambda)f_+(x,\lambda)]\phi_0(y,\lambda)\}\omega_0(\lambda)\,d\lambda =$$

$$= \frac{1}{\sqrt{2}\,\pi}\int_{-\infty}^{\infty}\mathrm{Re}\{[\,\overline{\phi_0(x,\lambda)}+r_0^+(\lambda)\phi_0(x,\lambda)]\phi_0(y,\lambda)\}\omega_0(\lambda)\,d\lambda +$$

$$+\int_{x}^{\infty}K^+(x,t)\biggl\{\frac{1}{\sqrt{2}\,\pi}\int_{-\infty}^{\infty}\mathrm{Re}\{[\,\overline{\phi_0(t,\lambda)}+r_0^+(\lambda)\phi_0(t,\lambda)]\phi_0(y,\lambda)\}\omega_0(\lambda)\,d\lambda\biggr\}dt +$$

$$+\frac{1}{\sqrt{2}\,\pi}\int_{-\infty}^{\infty}\mathrm{Re}\{[r^+(\lambda)-r_0^+(\lambda)]\phi_0(x,\lambda)\phi_0(y,\lambda)\}\omega_0(\lambda)\,d\lambda +$$

$$+\int_{x}^{\infty}K^+(x,t)\biggl\{\frac{1}{\sqrt{2}\,\pi}\int_{-\infty}^{\infty}\mathrm{Re}\{[r^+(\lambda)-r_0^+(\lambda)]\phi_0(t,\lambda)\phi_0(y,\lambda)\}\omega_0(\lambda)\,d\lambda\biggr\}dt =$$

$$= \delta(x-y)+\int_{x}^{\infty}K^+(x,t)\delta(t-y)\,dt +$$

$$+\frac{1}{\sqrt{2}\,\pi}\int_{-\infty}^{\infty}\mathrm{Re}\{[r^+(\lambda)-r_0^+(\lambda)]\phi_0(x,\lambda)\phi_0(y,\lambda)\}\omega_0(\lambda)\,d\lambda +$$

$$+\int_{x}^{\infty}K^+(x,t)\biggl\{\frac{1}{\sqrt{2}\,\pi}\int_{-\infty}^{\infty}\mathrm{Re}\{[r^+(\lambda)-r_0^+(\lambda)]\phi_0(t,\lambda)\phi_0(y,\lambda)\}\omega_0(\lambda)\,d\lambda\biggr\}\,dt =$$

$$= \delta(x-y)+K^+(x,y)+F^+(x,y)+\int_{x}^{\infty}K^+(x,t)F^+(t,y)\,dt.$$

Comparing this relation with (20), we obtain (17). The theorem is proved.

The basic equations permit solving the inverse problem as follows. Using basic equation (18) as in [10], [21], we establish the following property of the scattering functions $F^{\pm}(x,y)$.

Condition 2. *The functions $F^{\pm}(x,y)$ defined by formulas (19) are continuously differentiable and satisfy the relations*

$$|F^{\pm}(x,y)| \le C_{\pm}(a), \quad \pm x \ge a, \quad \pm y \ge a, \qquad \biggl|\int_{a}^{\pm\infty}\sup_{\pm x>a}|F^{\pm}(x,t)|\,dt\biggr| < \infty,$$

$$\lim_{N\to\pm\infty}\biggl|\int_{N}^{\pm\infty}\sup_{\pm(x-a)\ge 0}|F^{\pm}(x,y)|\,dy\biggr| = 0,$$

$$(1+|x_1|)\sup_{\pm t\geq 0}\left|\int_{x_1}^{\infty}|F^{\pm}_{x_i}(x_1+t,x_2)|\,dx_2\right|\leq C_i^{\pm}(a),\quad \pm(x_1-a)\geq 0,$$

$$\left|\int_a^{\pm\infty}(1+x^4)\left|\frac{d}{dx}F^{\pm}(x,x)\right|dx\right|<C_0^{\pm}(a),$$

$$\lim_{h\to 0}\sup_{\pm(z-x)\geq 0}\left|\int_x^{\pm\infty}|F^{\pm}(z,y+h)-F^{\pm}(z,y)|\,dy\right|=0.$$

Theorem 2. *If Conditions* 1 *and* 2 *are satisfied, then for each fixed* x*, Eqs.* (18) *have unique solutions* $K^{\pm}(x,\,\cdot\,)\in L_p(x,\pm\infty)$*,* $p=1,2$*.*

Proof. Condition 2 implies that Eqs. (18) are generated by completely continuous operators. Therefore, by the Fredholm alternative, the sought solution of (18) exists and is unique in the space $L_p(x,\infty)$, $p=1,2$, if the homogeneous equation has no nontrivial solutions in $L_p(x,\infty)$, $p=1,2$.

Without loss of generality, we consider the case with the plus sign. We assume that for some x, the homogeneous equation

$$h(y)+\int_x^{\infty}F^{+}(y,t)h(t)\,dt=0,\quad \pm y>\pm x, \tag{21}$$

has a nontrivial solution in $L_2(x,\infty)$. Because the kernel $F^{+}(x,y)$ is real, we can assume that $h(y)$ takes real values. Then

$$\int_x^{\infty}|h(y)|^2\,dy+\int_x^{\infty}\int_x^{\infty}F^{+}(y,t)h(t)h(y)\,dt\,dy=0.$$

Substituting the functions $F^{+}(x,y)$ given by (19) in the last relation, we obtain

$$\int_x^{\infty}|h(y)|^2\,dy+\frac{1}{2\sqrt{2}\,\pi}\int_{-\infty}^{\infty}\big(r^{+}(\lambda)-r_0^{+}(\lambda)\big)\left[\int_x^{\infty}\phi_0(t,\lambda)h(t)\,dt\right]\left[\int_x^{\infty}\phi_0(y,\lambda)h(y)\,dy\right]\omega_0(\lambda)\,d\lambda+$$
$$+\frac{1}{2\sqrt{2}\,\pi}\int_{-\infty}^{\infty}\big(\overline{r^{+}(\lambda)}-\overline{r_0^{+}(\lambda)}\big)\left[\int_x^{\infty}\overline{\phi_0(t,\lambda)}h(t)\,dt\right]\left(\int_x^{\infty}\overline{\phi_0(y,\lambda)}h(y)\,dy\right]\omega_0(\lambda)\,d\lambda=0.$$

We rewrite this relation as

$$\int_x^{\infty}|h(y)|^2\,dy+\frac{1}{2\sqrt{2}\,\pi}\int_{-\infty}^{\infty}r^{+}(\lambda)\left[\int_x^{\infty}\phi_0(t,\lambda)h(t)\,dt\right]\left[\int_x^{\infty}\phi_0(y,\lambda)h(y)\,dy\right]\omega_0(\lambda)\,d\lambda+$$
$$+\frac{1}{2\sqrt{2}\,\pi}\int_{-\infty}^{\infty}\overline{r^{+}(\lambda)}\left[\int_x^{\infty}\overline{\phi_0(t,\lambda)}h(t)\,dt\right]\left[\int_x^{\infty}\overline{\phi_0(y,\lambda)}h(y)\,dy\right]\omega_0(\lambda)\,d\lambda-$$
$$-\frac{1}{2\sqrt{2}\,\pi}\int_{-\infty}^{\infty}r_0^{+}(\lambda)\left[\int_x^{\infty}\phi_0(t,\lambda)h(t)\,dt\right]\left[\int_x^{\infty}\phi_0(y,\lambda)h(y)\,dy\right]\omega_0(\lambda)\,d\lambda-$$
$$-\frac{1}{2\sqrt{2}\,\pi}\int_{-\infty}^{\infty}\overline{r_0^{+}(\lambda)}\left[\int_x^{\infty}\overline{\phi_0(t,\lambda)}h(t)\,dt\right]\left[\int_x^{\infty}\overline{\phi_0(y,\lambda)}h(y)\,dy\right]\omega_0(\lambda)\,d\lambda=0. \tag{22}$$

On the other hand, by virtue of expansion formula (7), we have

$$\int_x^\infty |h(y)|^2\,dy = \int_{-\infty}^\infty \left[\frac{1}{\sqrt{2}\,\pi}\int_x^\infty h(t)\phi_0(t,\lambda)\,dt\right]\times$$
$$\times\left[\int_x^\infty h(y)\big(\overline{\phi_0(y,\lambda)} + r_0^+(\lambda)\phi_0(y,\lambda)\big)\,dy\right]\omega_0(\lambda)\,d\lambda,$$

$$\int_x^\infty |h(y)|^2\,dy = \int_{-\infty}^\infty \left[\frac{1}{\sqrt{2}\,\pi}\int_x^\infty h(t)\overline{\phi_0(t,\lambda)}\,dt\right]\times$$
$$\times\left[\int_x^\infty h(y)\big(\overline{\phi_0(y,\lambda)} + \overline{r_0^+(\lambda)}\,\overline{\phi_0(y,\lambda)}\big)\,dy\right]\omega_0(\lambda)\,d\lambda,$$

whence it follows that

$$\int_x^\infty |h(y)|^2\,dy - \frac{1}{2\sqrt{2}\,\pi}\int_{-\infty}^\infty r_0^+(\lambda)\left[\int_x^\infty h(y)\phi_0(y,\lambda)\,dy\right]^2\omega_0(\lambda)\,d\lambda -$$
$$-\frac{1}{2\sqrt{2}\,\pi}\int_{-\infty}^\infty \overline{r_0^+(\lambda)}\left[\int_x^\infty h(y)\overline{\phi_0(y,\lambda)}\,dy\right]^2\omega_0(\lambda)\,d\lambda = \frac{1}{\sqrt{2}\,\pi}\int_{-\infty}^\infty\left|\int_x^\infty h(y)\phi_0(y,\lambda)\,dy\right|^2 d\lambda.$$

Taking these formulas into account in (22) and setting

$$H(\lambda) = \int_x^\infty h(y)\phi_0(y,\lambda)\,dy,$$

we obtain

$$\int_{-\infty}^\infty |H(\lambda)|^2\omega_0(\lambda)\,d\lambda + \frac{1}{2}\int_{-\infty}^\infty r^+(\lambda)H^2(\lambda)\omega_0(\lambda)\,d\lambda + \frac{1}{2}\int_{-\infty}^\infty \overline{r^+(\lambda)}\,\overline{H^2(\lambda)}\omega_0(\lambda)\,d\lambda = 0.$$

Taking this relation into account, we conclude that

$$\int_{-\infty}^\infty \big(1-|r^+(\lambda)|\big)|H(\lambda)|^2\omega_0(\lambda)\,d\lambda \le 0,$$

whence it follows that

$$H(\lambda) = \int_x^\infty h(y)\phi_0(y,\lambda)\,dy \equiv 0$$

because $1-|r^+(\lambda)| > 0$ and $\omega_0(\lambda) = e^{-\pi\lambda/4} > 0$ for all real values of λ. It follows from the last identity that $h(y)\equiv 0$.

Therefore, homogeneous equation (21) has only a trivial solution in $L_2(x,\infty)$. The uniqueness of the solution in $L_1(x,\infty)$ follows because any solution $h(y)$ of Eq. (21) in $L_1(x,\infty)$ belongs to $L_2(x,\infty)$. The proof of the last assertion is precisely the same as in [21]. The theorem is proved.

In conclusion, we note that Conditions 1 and 2 obtained above are sufficient for uniquely reconstructing the potential $q(x)$ in the class

$$\int_{-\infty}^\infty \big(1+|x|^4\big)|p(x)|\,dx < \infty$$

from the right reflection coefficient. Indeed, let the right reflection coefficient $r^+(\lambda)$ be given. Using formulas (15) and (16), we obtain the functions $a(\lambda)$ and $r^-(\lambda)$. By formulas (19), we determine the functions $F^{\pm}(x,y)$. Under Conditions 1 and 2, basic equations (18) have unique solutions $K^{\pm}(x,y)$. As in [6], [7], [9], [10], we can then prove that the functions $f_{\pm}(x,\lambda)$ defined by (8) are solutions of the equations

$$-f''_{\pm}(x,\lambda)+[-x^2+q^{\pm}(x)]f_{\pm}(x,\lambda)=\lambda f_{\pm}(x,\lambda),$$

where the potentials $q^{\pm}(x)$ are given by

$$q^{\pm}(x)=\mp 2\frac{dK^{\pm}(x,x)}{dx}$$

and satisfy the inequalities

$$\left|\int_a^{\pm\infty}(1+|x|^4)|q^{\pm}(x)|\,dx\right|<\infty$$

for each a. Further, as in [6], [7], [21], we prove that, for real λ, the functions $f_{\pm}(x,\lambda)$ are related by

$$\frac{1}{a(\lambda)}f_-(x,\lambda)=\overline{f_+(x,\lambda)}+r^+(\lambda)f_+(x,\lambda),$$

$$\frac{1}{a(\lambda)}f_+(x,\lambda)=\overline{f_-(x,\lambda)}+r^-(\lambda)f_-(x,\lambda),$$

which implies that $q^+(x)=q^-(x)$.

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